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On The Packing k-Coloring of Some Family Trees

Arika Indah Kristiana [1,*], Sri Moeliyana Citra [1], Dafik [1], Ridho Alfarisi [2], Robiatul Adawiyah [1]

[1] *Department of Mathematics Education, Universitas Jember, Indoensia*
[2] *Department of Mathematics, Universitas Jember, Indoensia*

Abstract All graphs in this paper are simple and connected. Let $G = (V, E)$ be a graph where $V(G)$ is nonempty of vertex set of G and $E(G)$ is possibly empty set of unordered pairs of elements of $V(G)$. The distance from u to v in G is the length of a shortest path joining them, denoted by $d(u, v)$. For some positive integer k, a function $c : V(G) \to \{1, 2, ...k\}$ is called packing $k-$coloring if any two not adjacent vertices u and v, $c(u) = c(v) = i$ and $d(u, v) \geq i + 1$. The minimum number k such that the graph G has a packing $k-$coloring is called the packing chromatic number, denoted by $\chi_\rho(G)$. In this paper, we investigate the packing chromatic number of some family trees, namely centipede, firecracker, broom, double star and banana tree graphs.

Keywords Packing Coloring, Packing Chromatic Number, Tree

AMS 2010 subject classifications 05C78

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1. Introduction

A graph G is a pair of sets (V, E) where $V(G)$ is nonempty and $E(G)$ is a (possibly empty) set of unordered pairs of elements of $V(G)$. The elements of $V(G)$ are called the vertices of G and the elements of $E(G)$ are called the edges of G. Sometimes we write $V(G)$ for the vertices of G and $E(G)$ for the edges of G [12]. The cardinality of the vertex set of a graph G is called the order of G while the cardinality of its edge set is the size of G [6]. If x and y are vertices of a graph G, we say x is adjacent to y if there is an edge between x and y. We denote such an edge by xy [6] while the minimum length of a path will be called the distance between u and v and denoted by $d(u, v)$ [7]. A subgraph H of G is a graph H such that $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$ [12]. An isomorphism between two graphs G_1 and G_2 is a bijection $f : V_1 \to V_2$ between the vertices of the graphs such that $\{a, b\}$ is an edge in G_1 if and only if $\{f(a), f(b)\}$ is an edge in G_2. Two graphs are isomorphic denoted by $G_1 \cong G_2$ [13].

A coloring of a graph G is an assignment of colors (elements of some set) to the vertices (or edges) of a graph G so that adjacent vertices (or edges) are assigned distinct colors [5]. The history of graph coloring dated 1852 by Augustus De Morgan. The problem of determining whether the countries of any map on plane can be colored with four or fewer colors such that adjacent countries are colored differently, originated and grew in the minds of mathematicians. The problem is represented using a planar graph. The first originator on the coloring problem was Francis Guthrie. Then, this problem is answered by the four-color theorem which states that ”The chromatic number of a planar graph is not more than four colors” [10].

There are three types of coloring of a graph namely, vertex, edge and region coloring. A vertex coloring is a coloring of vertices of graph such that every two adjacent vertices are colored differently. The smallest number of

*Correspondence to: Arika Indah Kristiana (Email: arika.fkip@unej.ac.id). Department of Mathematics Education, Universitas Jember, Indoensia.

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colors needed to get a vertex coloring is called the chromatic number of graph, written $\chi(G)$ [13]. An edge coloring is a coloring of edges graph such that every two adjacent edges are colored differently. The smallest number of colors needed to get an edge coloring is called the chromatic index of graph, written $\chi'(G)$ [13]. A region coloring is a coloring of regions of graph such that every two adjacent region adjacent are colored differently [5].

In this paper, we discuss one of various types of vertex coloring on a graph, i.e. a packing $k-$coloring. The concept of a packing $k-$coloring was introduced by Goddard, et.al under the name broadcast coloring. Goddart introduced the concept of packing k-coloring from area of frequency assignment in wireless networks [20]. This results in that two stations with the same frequency assignment to be placed at a certain distance so that two broadcasts do not experience interference from the broadcast reception process and can maximize the strength of the broadcast signal. In the future discussion, we need some definitions as follows.

Definition 1
[11] Let G be a connected graph and k be an integer, $k \geq 1$. A packing $k-$coloring of a graph G is a mapping $\pi : V(G) \to \{1, 2, ...k\}$ such that any two vertices for color i are at distance least $i + 1$.

Definition 2
[3] The packing chromatic number $\chi_\rho(G)$ of a graph G is the smallest integer k such that that the vertex set of G can be partitioned into packings with pairwise different widths.

Definition 3
[16] A graph G is a tree if G contains no cycles and has $n - 1$ edges.

Definition 4
[4] A centipede graph is a graph of $2n$ vertices obtained by joining the bottoms of n-copies of the path graph P_2 laid in a row with edge and it is denoted by Cp_n.

Definition 5
[19] A firecracker graph obtained by the concatenation of n $k - stars$ by linking one leaf from each.

Definition 6
[18] A broom graph consists of a path P with d vertices, together with $(n - d)$ pendant vertices all adjacent to the same end vertex of P.

Definition 7
[5] A double star is a tree of diameter 3. Thus, if G is a double star, then G contains exactly two vertices that are not end-vertices, called the central vertices of G.

Definition 8
[9] A banana tree consists of a vertex v joined to one leaf of any number of stars.

In this paper, we will use the following lemma and proposition as a lower bound of the packing chromatic number. [20] Assume that the subgraph G is H. Then $\chi_\rho(H) \leq \chi_\rho(G)$.

Proposition 1
[20] $\chi_\rho(P_n) = \begin{cases} 2, \ 2 \leq n \leq 3 \\ 3, \ n \geq 4 \end{cases}$

There are several results of researchs of the packing coloring that have been studied, Goddard et.al determine the maximum broadcast chromatic number of graph [20]; Brešar et.al proved packing chromatic number of cartesian products, hexagonal lattice and trees [3]; Finbow and Rall proved packing chromatic number of some lattice [1]; William and Roy studied the certain graphs [2]; Roy has found packing chromatic number of certain fan and wheel related graphs [17]; Rajalakshmi and Venkatachalam proved packing k-coloring of middle, total, central, line graph of double wheel graph [11]; Joedo, et.al studied packing k-coloring of edge corona product [8]. Then Ariningtyas, et.al studied packing $k-$coloring of unicyclic graph [14].

2. Results and Discussion

In this paper, we study the packing chromatic number of some family trees, namely centipede, firecracker, broom, double star and banana tree graph. We describe the result in the following.

Theorem 1
Let n be a positive integer at least 2,

$$\chi_\rho(Cp_n) = \begin{cases} 3, & \text{if } n = 2,3 \\ 4, & \text{if } 4 \leq n \leq 7 \\ 5, & \text{if } n \geq 8 \end{cases}$$

Proof. Let the centipede graph Cp_n has vertex set $V(Cp_n) = \{x_i | 1 \leq i \leq n\} \cup \{y_i | 1 \leq i \leq n\}$ and edge set $E(Cp_n) = \{x_i y_i | 1 \leq i \leq n\} \cup \{x_i x_{i+1} | 1 \leq i \leq n-1\}$. For the illustration of centipede graph Cp_n, see Figure 1.

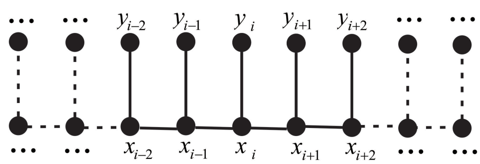

Figure 1. Centipede graph Cp_n.

To show the exact value of the packing chromatic number on graph Cp_n, we divide into three cases as follows.
Cases 1: for $n = 2, 3$.
Sub Cases 1: for $n = 2$
We know that the graph Cp_2 is isomorphic to graph P_4. It follows by Proposition 1, $\chi_\rho(Cp_2) = \chi_\rho(P_4) = 3$.
Sub Cases 2: for $n = 3$
To prove that the packing chromatic number of graph Cp_3 is 3. First, it will be shown $\chi_\rho(Cp_3) \geq 3$. Suppose that $\chi_\rho(Cp_3) < 3$ for some packing $2-$coloring c. Consider two possibilities:

1. For two adjacent vertices, it should be colored with the different color i.e. $c(x_1) \neq c(x_2)$ and $c(x_1) \neq c(y_1)$. If the vertex x_1 is colored by color 1, it means that $c(x_2) = 2$. Suppose that $c(y_1) = 2$ and $d(x_2, y_1) = 2$ means that it is not allowed by Definition 1.
2. For two adjacent vertices, it should be colored with the different color i.e. $c(x_2) \neq c(x_1)$ and $c(x_2) \neq c(x_3)$. If the vertex x_1 is colored by color 2, it means that $c(x_2) = 1$. Suppose that $c(x_3) = 2$ and $d(x_1, x_3) = 2$, means that it is not allowed by Definition 1.

So, we can not have $\chi_\rho(Cp_3) < 3$. The lower bound of the packing chromatic number of Cp_3 is at least 3 or $\chi_\rho(Cp_3) \geq 3$. Then, it will be shown that $\chi_\rho(Cp_3) \leq 3$. Define a packing $3-$coloring $c : V(Cp_3) \rightarrow \{1, 2, 3\}$ by rule:

$$c(x_i)=\begin{cases} 1, \text{ for } i = 1 \text{ or } 3 \\ 3, \text{ for } i = 2 \end{cases}$$

$c(y_i) = 2$, for $1 \leq i \leq 3$
It is clearly that for every two vertices u and v of the color i has the distance at least $i + 1$. We obtain the upper bound $\chi_\rho \leq 3$. Combine the lower bound and upper bound, we conclude that $\chi_\rho(Cp_n) = 3$ for $n = 2, 3$.
Cases 2: for $4 \leq n \leq 7$.
It will be shown $\chi_\rho(Cp_n) \geq 4$. Suppose that $\chi_\rho(Cp_n) = 3$ for some packing $3-$coloring c. There is no two adjacent vertices that recieve the same color. It must be that $c(x_i) \neq c(y_i)$, $c(x_i) \neq c(x_i) \neq c(x_{i+1})$ and $c(x_i) \neq c(x_{i-1})$. If

the vertex x_i is colored by color 1, based on definition of vertex coloring, $c(x_{i-1}) = 2$ and $c(x_{i+1}) = 3$. Suppose that $c(y_i) = 2$ and $d(y_i, x_{i-1}) = 2$, it contradics by Definition 1. And suppose that $c(y_i) = 3$ and $d(y_i, x_{i+1}) = 2$, it contradics by Definition 1. So, it is imposible we have $\chi_\rho(Cp_n) < 4$. We get the lower bound $\chi(Cp_n) \geq 4$. To prove the upper bound of $\chi_\rho(Cp_n)$ we define a packing $4-$coloring $c : V(Cp_n) \to \{1, 2, 3, 4\}$ by formula:

$$c(x_i)=\begin{cases} 1, \text{ for } i = 1, 3, 5 \text{ or } 7 \\ 3, \text{ for } i = 2 \text{ or } 6 \\ 4, \text{ for } i = 4 \end{cases}$$

$c(y_i) = 2$, for $1 \leq i \leq n$

Obviously, for every two vertices u and v has the condition $c(u) = c(v) = i$ and $d(u, v) \leq i + 1$. We obtain $\chi_\rho(Cp_n) \leq 4$. Since we have $\chi_\rho(Cp_n) \geq 4$, it can be concluded that $\chi_\rho(Cp_n) = 4$ for $4 \leq n \leq 7$.

Cases 3: for $n \geq 8$.

Suppose that $\chi_\rho(Cp_n) = 5$ for some packing $4-$coloring c. Every two adjacent vertices u and v has to in the different colors. It must be $c(x_i) \neq c(y_i)$, $c(x_i) \neq c(x_i) \neq c(x_{i+1})$ and $c(x_i) \neq c(x_{i-1})$. If the vertex x_i is colored by color 3, based on definition of vertex coloring, $c(x_{i-1}) = 1$, $c(x_{i+1}) = 1$ and $c(y_i) = 1$ or $c(y_i) = 2$. Assume that the vertex x_{i-1} is colored by color 1. It requires $c(x_{i-2}) = 2$ and $c(y_{i-1}) = 4$. Therefore, $c(x_{i+1}) = 1$. Then we obtain $c(y_{i+1}) = 4$ and suppose that $c(x_{i+2}) = 4$ $d(x_{i+2}, y_{i-1}) = 4$ means that it is not suitable the definition of a packing coloring. So, it is imposible $\chi_\rho(Cp_n) < 5$. We get the lower bound $\chi(Cp_n) \geq 5$. To prove the upper bound of $\chi_\rho(Cp_n)$, use a packing $5-$coloring $c : V(Cp_n) \to \{1, 2, 3, 4, 5\}$ defined by:

$$c(x_i) = \begin{cases} 1, \text{ for } i \text{ odd}, 1 \leq i \leq n \\ 3, \text{ for } i \equiv 2 (\text{mod } 4),\ 1 \leq i \leq n \\ 4, \text{ for } i \equiv 4 (\text{mod } 8),\ 1 \leq i \leq n \\ 5, \text{ for } i \equiv 0 (\text{mod } 8),\ 1 \leq i \leq n \end{cases}$$

$c(y_i) = 2$, for $1 \leq i \leq n$.

By this coloring, for every two vertices u and v where $c(u) = c(v) = i$, then $d(u, v) \leq i + 1$. We obtain $\chi_\rho(Cp_n) \leq 5$. Since we have $\chi_\rho(Cp_n) \geq 5$, it can be concluded that $\chi_\rho(Cp_n) = 5$ for $n \geq 8$. For the illustrations of a packing $k-$coloring of centipede graph Cp_8, see Figure 2.

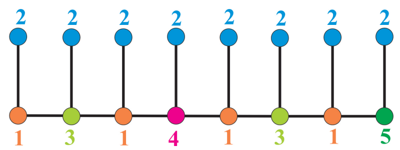

Figure 2. A packing coloring of centipede graph Cp_8.

Theorem 2

Let $m \geq 2$ and $n \geq 3$ where m and n be a positive integer,

$$\chi_\rho(F_{m,n}) = \begin{cases} 3, & \text{if } m = 2, 3 \\ 4, & \text{if } 4 \leq m \leq 7 \\ 5, & \text{if } m \geq 8 \end{cases}$$

Proof. Let the firecracker graph $F_{m,n}$ is the graph with vertex set $V(F_{m,n}) = \{x_i | 1 \leq i \leq m\} \cup \{y_i | 1 \leq i \leq m\} \cup \{y^i_j | 1 \leq i \leq m,\ 1 \leq j \leq n-1\}$ and edge set $E(F_{m,n}) = \{x_i x_{i+1} | 1 \leq i \leq m-1\} \cup \{x_i y_i | 1 \leq i \leq m\} \cup \{y_i y^i_j | 1 \leq i \leq m,\ 1 \leq j \leq n-1\}$. To show the lower bound we need the fact that $F_{m,n} \supseteq Cp_n$. Using Lemma 1.1, we obtain $\chi_\rho(F_{m,n}) \geq \chi_\rho(Cp_n)$. Then, for the upper bound we divide into three cases.

Cases 1: for $m = 2, 3$.

Define a coloring $c : V(F_{m,n}) \to \{1, 2, 3\}$ defined by:

$c(x_i)=\begin{cases}1, \text{ for } i=1 \text{ or}, 3\\ 3, \text{ for } i=2\end{cases}$
$c(y_i)=2$, for $1\le i\le m$
$c(y_j^i)=1$, for $1\le i\le m$ and $1\le j\le n-1$
Clearly, this coloring is a packing $k-$coloring because for any two vertices u and v of the same color i, the distance of both is at least $i+1$. Therefore, we get $\chi_\rho(F_{m,n})\le 3$.
Cases 2: for $4\le m\le 7$.
In this case, use a packing $4-$coloring $c:V(F_{m,n})\to\{1,2,3,4\}$ by rule:
$c(x_i)=\begin{cases}1, \text{ for } i=1,3,5 \text{ or } 7\\ 3, \text{ for } i=2 \text{ or } 6\\ 4, \text{ for } i=4\end{cases}$
$c(y_i)=2$, for $1\le i\le m$
$c(y_j^i)=1$, for $1\le i\le m$ and $1\le j\le n-1$
By this coloring, it is clearly that two vertices u and v of the color i has the distance at least $i+1$. We obtain $\chi_\rho(F_{m,n})\le 4$.
Cases 3: for $m\ge 8$
Put a packing $5-$coloring $c:V(F_{m,n})\to\{1,2,3,4,5\}$ by formula:
$c(x_i)=\begin{cases}1, \text{ for } i \text{ odd}, 1\le i\le m\\ 3, \text{ for } i\equiv 2(\text{mod } 4),\ 1\le i\le m\\ 4, \text{ for } i\equiv 4(\text{mod } 8),\ 1\le i\le m\\ 5, \text{ for } i\equiv 0(\text{mod } 8)\ ,1\le i\le m\end{cases}$
$c(y_i)=2$, for $1\le i\le m$
$c(y_j^i)=1$, for $1\le i\le m$ and $1\le j\le n-1$
By the packing $k-$coloring above, we get two vertices u and v of the color i and their has the distance at least $i+1$. We get $\chi_\rho(F_{m,n})\le 5$. Combine the lower bound and upper bound, we conclude that $\chi_\rho(F_{m,n})=\chi_\rho(Cp_n)$. For the illustrations of a packing $k-$coloring of firecracker graph $F_{6,4}$, see Figure 3.

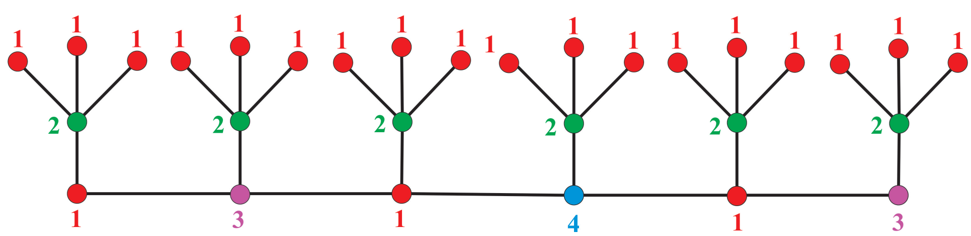

Figure 3. A packing coloring of firecricker graph $F_{6,4}$.

Theorem 3
Let d and n be positive integers with $d\ge 3$ and $n-d\ge 3$, $\chi_\rho(B_d^n)=3$.

Proof. Let the broom graph B_d^n has vertex set $V(B_d^n)=\{u_i|1\le i\le d\}\cup\{v_j|1\le j\le n-d\}$ and edge set $E(B_d^n)=\{u_iu_{i+1}|1\le i\le d-1\}\cup\{u_iv_j|i=d,1\le j\le n-d\}$. We have $\chi_\rho(P_{d+1})=3$ by Proposition 1 and the fact $B_d^n\supseteq P_{d+1}$. It follows by Lemma 1.1 that $\chi_\rho(B_d^n)\ge\chi_\rho(P_{d+1})$. Thus, we get $\chi_\rho(B_d^n)\ge\chi_\rho(P_{d+1})=3$. To determine the upper bound of $\chi_\rho(B_d^n)$, make a coloring function $c:V(B_d^n)\to\{1,2,3\}$ by formula:
$c(u_i)=\begin{cases}2,1,3,1,2,1,3,1,..., \text{ for } d\equiv 0,1,3(\text{mod } 4),\ 1\le i\le d-1\\ 3,1,2,1,3,1,2,1,..., \text{ for } d\equiv 2(\text{mod } 4),\ 1\le i\le d-1\end{cases}$
$c(u_d)=\begin{cases}2,\ \text{for } d\equiv 0,1,2(\text{mod } 4),\ d\ge 3\\ 3,\ \text{for } d\equiv 3(\text{mod } 4),\ d\ge 3\end{cases}$
$c(v_j)=1$, for $1\le j\le n-d$.

Obviuosly, for two vertices u and v which is $c(u) = c(v) = i$ and $d(u, v) \geq i + 1$. Therefore, c is a packing $k-$coloring on B_d^n. Now, we get $\chi_\rho(B_d^n) \leq 3$. Based on, the results of the lower bound and upper bound, we deduce that $\chi_\rho((B_d^n) = 3$. For the illustrations of a packing $k-$coloring of broom graph B^{10}_{7}, see Figure 4.

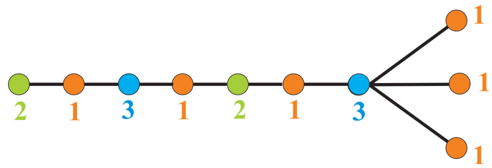

Figure 4. A packing coloring of broom graph B^{10}_{7}.

Theorem 4
Let m and n be positive integers at least 2, $\chi_\rho(S_{m,n}) = 3$.

Proof. Let the double star graph $S_{m,n}$ has vertex set $V(S_{m,n}) = \{x\} \cup \{x_i|1 \leq i \leq m\} \cup \{y\} \cup \{y_j|1 \leq j \leq n\}$ and edge set $E(S_{m,n}) = \{xx_i|1 \leq i \leq m\} \cup \{yy_j|1 \leq j \leq n\} \cup \{xy\}$. First, we will show that the lower bound. Since $S_{m,n} \supseteq Cp_2$ and $\chi_\rho(Cp_2) = 3$ in Theorem 2.1 then based on Lemma 1.1 we have $\chi_\rho(S_{m,n}) \geq 3$. Next, define a coloring $c : V(S_{m,n}) \to \{1, 2, 3\}$ defined by:
$c(x_i) = 1$, for $1 \leq i \leq m$
$c(y_j) = 1$, for $1 \leq j \leq n$
$c(x) = 2$
$c(y) = 3$.
It is clearly that for every two vertices u and v of the color i has the distance at least $i + 1$. So, c is a packing $k-$coloring on $S_{m,n}$. We obtain the upper bound of $\chi_\rho(S_{m,n}) \leq 3$. Combine the lower and upper bound, it can be concluded that $\chi_\rho(S_{m,n}) = 3$ for $m \geq 2$ and $n \geq 2$. For the illustrations of a packing $k-$coloring of double star graph $S_{5,5}$, see Figure 5.

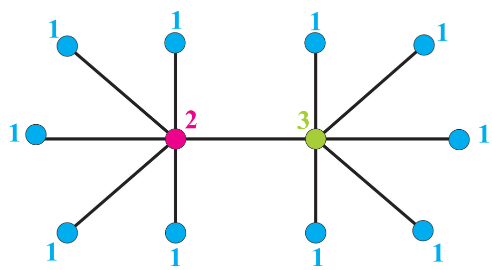

Figure 5. A packing coloring of double star graph $S_{5,5}$.

Theorem 5
Let m and n be positive integers with $m \geq 2$ and $n \geq 3$, $\chi_\rho(B_{m,n}) = 3$.

Proof. Let the banana tree graph has vertex set $V(B_{m,n}) = \{x^j|1 \leq j \leq m\} \cup \{x_i^j|1 \leq i \leq n, 1 \leq j \leq m\} \cup \{y\}$ and edge set $E(B_{m,n}) = \{x^j x_i^j|1 \leq i \leq n, 1 \leq j \leq m\} \cup \{x_i^j y|1 \leq j \leq m,\ i = n\}$. Consider B_3^n as a subgraph $B_{m,n}$. Theorem 2.3 give us the result $\chi_\rho(B_3^n) = 3$. Apply Lemma 1.1 to get $\chi_\rho(B_{m,n}) \geq 3$. The upper bound of $\chi_\rho(B_{m,n})$ can be determined by using a packing $3-$coloring $c : V(B_{m,n}) \to \{1, 2, 3\}$ defined by:
$c(x^j) = 2$, for $1 \leq j \leq m$
$c(x_i^j) = 1$, for $1 \leq j \leq m$ for $1 \leq i \leq n$
$c(y) = 3$.
Clearly, for every two vertices u and v where $c(u) = c(v) = i$ then $d(u, v) \geq i + 1$. Hence, we obtain $\chi_\rho(B_{m,n}) \leq$

3. Since we have $\chi_\rho(B_{m,n}) \geq 3$ and $\chi_\rho(B_{m,n}) \leq 3$, it can be conclude that $\chi_\rho(B_{m,n}) = 3$ for $n \geq 3$ and $m \geq 2$. For the illustrations of a packing $k-$coloring of banana tree graph $B_{4,5}$, see Figure 6.

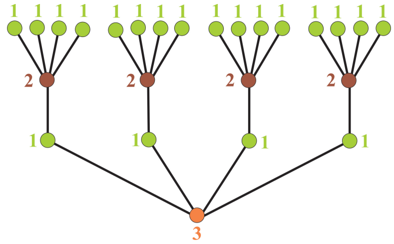

Figure 6. A packing coloring of banana tree graph $B_{4,5}$.

3. Concluding Remarks

In this paper, we have studied the packing chromatic number of some family trees, namely centipede, firecracker, broom, double star and banana tree graph. However, the packing chromatic number for another family graph is still open.

Open Problem 1
Determine the chromatic number of another family graph for the packing $k-$coloring.

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