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Estimates for distributions of Hölder semi-norms of random processes from $\mathbb{F}_\psi(\Omega)$ spaces, defined on the interval $[0,\infty)$

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Abstract In the present article we study properties of random processes from the Banach spaces $\mathbb{F}_\psi(\Omega)$. Estimates are obtained for distributions of semi-norms of sample functions of processes from $\mathbb{F}_\psi(\Omega)$ spaces, defined on the infinite interval $[0,\infty)$, in Hölder spaces.

Keywords Random Processes, $\mathbb{F}_\psi(\Omega)$ Spaces of Random Variables, Moduli of Continuity, Hölder Semi-norms

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1. Introduction

Let $(\mathbb{T},\,\rho)$ be some metric space. Consider a random process $X=\{X(t),\,t\in\mathbb{T}\}$ such that the following inequality holds

$$\limsup_{\varepsilon\downarrow 0}\frac{\sup\limits_{\substack{0<\rho(t,s)\le\varepsilon\\ t,s\in\mathbb{T}}}|X(t)-X(s)|}{f(\varepsilon)}\le 1$$

with probability 1. Here the function f must be a modulus of continuity for the process X.

A space of functions with moduli of continuity $f(\varepsilon)$ is the Hölder space and the functional

$$\sup_{\substack{0<\rho(t,s)\le\varepsilon\\ t,s\in\mathbb{T}}}\frac{|X(t)-X(s)|}{f(\rho(t,s))}$$

is a semi-norm in the Hölder space. In the following we deal with estimates of distributions of Hölder semi-norms of sample functions of random processes $X=\{X(t),\,t\in[0,\infty)\}$ belonging to $\mathbb{F}_\psi(\Omega)$ spaces, i.e. probabilities

$$\mathsf{P}\left\{\sup_{\substack{0<|t-s|\le\varepsilon\\ t,s\in[0,\infty)}}\frac{|X(t)-X(s)|}{f(|t-s|)}>x\right\}.$$

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Such estimates and assumptions under which semi-norms of sample functions of processes from $\mathbb{F}_\psi(\Omega)$ spaces, defined on a compact space, satisfy the Hölder conditions were obtained in [11]. Also estimates for distributions of supremum of the increments of processes belonging to $\mathbb{F}_\psi(\Omega)$ spaces were investigated by Mlavets [14]. Similar results were provided for Gaussian processes, defined on a compact, by Dudley [3]. Kozachenko [9] generalized Dudley's results for random processes belonging to Orlicz spaces (see also [2, 17]). Talagrand [15] found necessary and sufficient conditions for sample path continuity or boundedness of Gaussian stochastic processes. L^p moduli of continuity for a wide class of continuous Gaussian processes were obtained by Marcus and Rosen [12]. Kozachenko et al. [8] studied the Lipschitz continuity of generalized sub-Gaussian processes and provided estimates for distributions of Lipschitz norms of such processes. But all these problems were not considered yet for processes, defined on an infinite interval. Only for $L_p(\Omega)$ processes, defined on an infinite interval, estimates for distributions of semi-norms of these processes and assumptions under which semi-norms of sample functions of these processes satisfy Hölder conditions were obtained by Zatula [16].

The theory of sample path properties of non-stationary Gaussian processes based on concepts of the entropy and majorizing measures is now well studied. For an accessible introduction to these concepts and to the general theory of continuity, boundedness and suprema distributions for real-valued Gaussian processes, we refer to Adler [1].

The Hölder continuity of random processes is applicable to problems of approximating random functions and studying the rate of approximation. In particular, Kamenshchikova and Yanevich [6] investigated an approximation of stochastic processes belonging to spaces $L_p(\Omega)$ by trigonometric sums in the space $L_q[0, 2\pi]$. Mathé [13] provided the rate of convergence of multivariate Bernstein polynomials on the class of Hölder continuous random functions. Also there is a number of works devoted to the study of the approximation of Hölder continuous set-valued functions by Bernstein polynomials. Among them, Kels and Dyn [7] obtained estimates of the approximations of functions whose values are formed by random sets.

2. Preliminary results

Below we provide definitions of random variables and processes, belonging to $\mathbb{F}_\psi(\Omega)$ spaces, and auxiliary results to be used in subsequent results.

Let $\psi(u) > 0,\ u \geq 1$ be some increasing function such that $\psi(u) \to \infty$ as $u \to \infty$.

Definition 1 ([10])
A random variable ξ *belongs to the space* $\mathbb{F}_\psi(\Omega)$ if

$$\sup_{u\geq 1} \frac{(\mathsf{E}|\xi|^u)^{1/u}}{\psi(u)} < \infty.$$

It is proved in the paper [4] (see also [10]) that $\mathbb{F}_\psi(\Omega)$ is a Banach space with respect to the norm

$$\|\xi\|_\psi = \sup_{u\geq 1} \frac{(\mathsf{E}|\xi|^u)^{1/u}}{\psi(u)}.$$

Here are some examples of random variables belonging to $\mathbb{F}_\psi(\Omega)$ spaces.

Example 1
A random variable ξ such that satisfies $|\xi| < C$ with probability one, where $C > 0$ is some constant, belongs to any $\mathbb{F}_\psi(\Omega)$ space and

$$\|\xi\|_\psi = \sup_{u\geq 1} \frac{\left(\mathsf{E}\,|\xi|^u\right)^{1/u}}{\psi(u)} \leq \sup_{u\geq 1} \frac{(C^u)^{1/u}}{\psi(u)} = \sup_{u\geq 1} \frac{C}{\psi(u)} = \frac{C}{\psi(1)}.$$

Example 2
A normally distributed random variable $\xi \sim N(0,1)$ belongs to the $\mathbb{F}_\psi(\Omega)$ space with the function $\psi(u) = u^{1/2}$ because $\sqrt[2l]{\mathsf{E}\,|\xi|^{2l}} = \sqrt[2l]{\frac{(2l)!}{2^l l!}} \sim l^{1/2}$ as $l \geq 1$.

Properties of random variables and processes from $\mathbb{F}_\psi(\Omega)$ spaces were considered in detail in [10]. Henceforth we will consider the spaces $\mathbb{F}_\psi(\Omega)$, which have the following property.

Let ξ_1, ..., ξ_n be random variables belonging to the space $\mathbb{F}_\psi(\Omega)$. Denote $\eta_n = \max\limits_{1\le k\le n} |\xi_k|$, $a_n = \max\limits_{1\le k\le n} \|\xi_k\|_\psi$.

Definition 2 ([11])
$\mathbb{F}_\psi(\Omega)$ space has *property Z* if there are monotone non-decreasing function $z(x) > 0$, monotone increasing function $U(n)$ and a real number $x_0 > 0$ such that for any sequence of random variables $(\xi_k,\ k = \overline{1,n})$ from $\mathbb{F}_\psi(\Omega)$ space, $\forall x > x_0$ and for all $n \ge 2$ the following inequality is performed:

$$\mathsf{P}\{\eta_n > x \cdot a_n \cdot U(n)\} \le \frac{1}{n}\exp\{-z(x)\}.$$

Below are some examples of the spaces $\mathbb{F}_\psi(\Omega)$ which have property Z.

Theorem 1 ([11])
Let $\psi(u) = u^\alpha$, $\alpha > 0$. Then the following inequality holds $\forall x > x_1$, $x_1 = \max\left\{\frac{1}{(\ln 3)^\alpha}, \left(\frac{2e\ln 3}{\alpha(\ln 3 - 1)}\right)^\alpha\right\}$:

$$\mathsf{P}\left\{\eta_n > x \cdot a_n \cdot (\ln(n+2))^\alpha\right\} \le \frac{1}{n}\exp\left\{-\frac{\alpha}{e}x^{1/\alpha}\right\},$$

i.e. $z(x) = \frac{\alpha}{e}x^{1/\alpha}$, $U(n) = (\ln(n+2))^\alpha$.

Theorem 2 ([11])
Let $\psi(u) = e^{\alpha u^\beta}$, $\alpha > 0$, $\beta > 0$. Then the following inequality holds $\forall x \ge x_2$, $x_2 = \exp\left\{\left(\frac{\ln 3}{b\sqrt[\beta]{2}(\ln 3 - 1)}\right)^{\frac{2\beta}{\beta+1}}\right\}$, $b = \frac{\beta}{\alpha^{1/\beta}} \cdot (\beta+1)^{-\frac{\beta+1}{\beta}}$:

$$\mathsf{P}\left\{\eta_n > x \cdot a_n \cdot \exp\left\{(\ln(n+2))^{\frac{2\beta}{\beta+1}}\right\}\right\} \le \frac{1}{n}\exp\left\{-\frac{\beta}{\alpha^{1/\beta}}\left(\frac{2}{\beta+1}\right)^{\frac{\beta+1}{\beta}}(\ln x)^{\frac{\beta+1}{2\beta}}\right\},$$

i.e. $z(x) = \frac{\beta}{\alpha^{1/\beta}} \cdot \left(\frac{2}{\beta+1}\right)^{\frac{\beta+1}{\beta}}(\ln x)^{\frac{\beta+1}{2\beta}}$, $U(n) = \exp\left\{(\ln(n+2))^{\frac{2\beta}{\beta+1}}\right\}$.

Let $(\mathbb{T},\ \rho)$ be some metric space.

Definition 3 ([2])
The metric massiveness $N_{(\mathbb{T},\,\rho)}(u) := N(u)$ is the minimal number of closed balls (defined with respect to the metric ρ) of radius u that cover $\mathbb{T}$.

Definition 4 ([10])
We say that a random process $X = (X(t),\ t \in \mathbb{T})$ belongs to the space $\mathbb{F}_\psi(\Omega)$ if random variables $X(t)$ belong to $\mathbb{F}_\psi(\Omega)$ for all $t \in \mathbb{T}$.

Definition 5 ([2])
A function $q = \{q(t),\ t \in \mathbb{R}\}$ is called *a modulus of continuity* if $q(t) \ge 0$, $q(0) = 0$ and $q(t+s) \le q(t) + q(s)$ for $t > 0$ and $s > 0$.

Definition 6 ([5])
A function $v(x)$ satisfies *Hölder condition* with exponent $\alpha \in (0,1]$ if the following value is finite:

$$[v]_{\alpha,\mathbb{T}} = \sup_{\substack{t,s\in\mathbb{T}\\ t\ne s}} \frac{|v(t)-v(s)|}{|t-s|^\alpha}.$$

This value is an α^{th}-Hölder semi-norm of the function v. The Hölder space $C^{0,\alpha}(\overline{\mathbb{T}})$ consists of all continuous functions which satisfy the Hölder condition with exponent α in $\mathbb{T}$.

Remark 1
The space $C^{0,\alpha}(\overline{\mathbb{T}})$, where $\mathbb{T}$ is a bounded space, is a Banach space with respect to the norm

$$\|v\|_{C^{0,\alpha}(\overline{\mathbb{T}})} = \sup_{\mathbb{T}} |v| + [v]_{\alpha,\mathbb{T}}.$$

In further investigations we will deal with the generalization of the concept of Hölder semi-norm $[v]_{\alpha,\mathbb{T}}$ in the space $C^{0,\alpha}(\overline{\mathbb{T}})$. Consider a value

$$[v]_{q,\rho,\mathbb{T}} = \sup_{\substack{t,s\in\mathbb{T}\\ t\neq s}} \frac{|v(t)-v(s)|}{q(\rho(t,s))},$$

where ρ is a metric in the space $\mathbb{T}$, and $q = \{q(t), t \in \mathbb{T}\}$ is a modulus of continuity such that $\exists\, \alpha \in (0,1]$ $\forall t, s \in \mathbb{T}, t \neq s : q(\rho(t,s)) \leq |t-s|^{\alpha}$. If the value $[v]_{q,\rho,\mathbb{T}'}$ is finite for all $\mathbb{T}' \subset \mathbb{T}$ then $v \in C^{0,\alpha}(\overline{\mathbb{T}})$.

The following result is the theorem on the estimation of distributions of the Hölder semi-norms and the moduli of the continuity of random processes from $\mathbb{F}_{\psi}(\Omega)$ spaces of random variables, defined on a compact.

Theorem 3 ([11])
Let $(\mathbb{T}, \rho)$ be some compact metric space. Consider a separable random process $X = (X(t),\, t \in \mathbb{T})$ belonging to the Banach space $\mathbb{F}_{\psi}(\Omega)$, which has the property Z with functions $U(n)$ and $z(x)$ for $x_0 > 0$.

Assume that there is a monotonically increasing continuous function $\sigma = \{\sigma(h),\, h \geq 0\}$ such that $\sigma(0) = 0$ and

$$\sup_{\rho(t,s)\leq h} \|X(t) - X(s)\|_{\psi} \leq \sigma(h).$$

Let $N(\varepsilon) = N_{\rho}(\mathbb{T},\, \varepsilon)$ be the metric massiveness of the space $(\mathbb{T}, \rho)$, $\varepsilon_0 = \sigma^{(-1)}\left(\sup_{t,s\in\mathbb{T}} \rho(t,s)\right)$, where $\sigma^{(-1)}(h)$ is an inverse function to a function $\sigma(h)$, and let $\forall\, \varepsilon > 0$:

$$g_B(\varepsilon) = \int_0^{\sigma(\varepsilon)} U(B^2 N^2(\sigma^{(-1)}(t)))dt < \infty.$$

Then for $x > x_0$, $\varepsilon \in (0, \varepsilon_0)$ and $B > 1$ the following inequality holds true

$$\mathsf{P}\left\{ \sup_{0<\rho(t,s)\leq\varepsilon} \frac{|X(t)-X(s)|}{(6+4\sqrt{2})f_B(\rho(t,s)) + (5+2\sqrt{6})g_B(\rho(t,s))} > x \right\} \leq \frac{2B(2B+1)}{(B^2-1)N(\varepsilon)} \cdot \exp\{-z(x)\},$$

where $f_B(\varepsilon) = \int\limits_0^{\sigma(\varepsilon)} U(BN(\sigma^{(-1)}(t)))dt,\ \varepsilon > 0.$

Definition 7 ([11])
$\mathbb{F}_{\psi}(\Omega)$ space has *property* Z_1 if it has the property Z with functions $z(x)$ and $U(n)$ for $x > x_0$, and if there is such a constant $b_0 > 0$ that $\forall n \geq 1$:

$$U(n^2) \leq b_0\, U(n).$$

3. Estimates for distributions of Hölder semi-norms of random processes from $\mathbb{F}_{\psi}(\Omega)$ spaces, defined on the interval $[0,\infty)$

Now we formulate and prove the main result, which is based on Theorem 3.

Theorem 4
Consider a separable random process $X = \{X(t), t \in [0,\infty)\}$ belonging to the Banach space $\mathbb{F}_{\psi}(\Omega)$, which

has the property Z with functions $U(n)$ and $z(x)$ for $x_0 > 0$. Let $[0,\infty) = \bigcup_{i=0}^{\infty} A_i$, where $A_i = [a_i, a_{i+1}]$, $\{a_i, i = 0, 1, ..., \infty\}$ is some increasing sequence, $a_0 = 0$. Denote $\alpha_i = a_{i+1} - a_i$, $D_i = [a_i, a_{i+1} + \theta]$, $\theta \in \left(0, \min_{i\geq 0} \alpha_i\right)$.

Assume that there are monotonically increasing continuous functions $\sigma_i = \{\sigma_i(h), h \geq 0\}$ such that $\sigma_i(0) = 0, i = 0, 1, ...$ and $\forall i = 0, 1, ...$:

$$\sup_{\substack{\rho(t,s)\leq h \\ t,s\in D_i}} \|X(t) - X(s)\|_\psi \leq \sigma_i(h), \qquad 0 < h < \alpha_i + \theta. \tag{1}$$

Let $N_i(\varepsilon)$ be metric massivenesses of intervals D_i, $i = 0, 1, ...$ with respect to the metric $\rho(t, s) = |t - s|$, $t, s \in [0, \infty)$. Also let

$$\varepsilon_0 = \min_{i\geq 0} \left\{ \sigma_i^{(-1)} \left(\sup_{t,s\in D_i} \rho(t, s) \right) \right\} = \min_{i\geq 0} \left\{ \sigma_i^{(-1)}(\alpha_i + \theta) \right\},$$

where $\sigma_i^{(-1)}(h)$ is an inverse function to a function $\sigma_i(h)$, $i = 0, 1, ...,$ and $\forall i = 0, 1, ...,$ $\forall \varepsilon > 0$:

$$g_{B,i}(\varepsilon) = \int_0^{\sigma_i(\varepsilon)} U(B^2 N_i^2(\sigma_i^{(-1)}(t)))dt < \infty; \quad f_{B,i}(\varepsilon) = \int_0^{\sigma_i(\varepsilon)} U(BN_i(\sigma_i^{(-1)}(t)))dt.$$

Denote

$$w_{B,i}(t, s) = (6 + 4\sqrt{2}) f_{B,i}(|t - s|) + (5 + 2\sqrt{6}) g_{B,i}(|t - s|),\ t, s \in D_i,$$

and $w_B(t, s)$ is such a function that

$$w_B(t, s) = \{w_{B,i}(t, s)\,|\, t, s \in A_i \text{ or } \min\{t, s\} \in A_i,\ \max\{t, s\} \in A_{i+1}\}.$$

Then for all $x > x_0$, $\theta \in \left(0, \min_{i\geq 0} \alpha_i\right)$ and $\varepsilon \in (0, \min\{\varepsilon_0, \theta\})$ under the condition that $\sum_{i=0}^{\infty} \frac{1}{\alpha_i} < \infty$ the following inequality holds true:

$$\mathsf{P} \left\{ \sup_{\substack{0<|t-s|\leq\varepsilon \\ t,s\in[0,\infty)}} \frac{|X(t) - X(s)|}{w_B(t, s)} > x \right\} \leq M(B, \varepsilon) \cdot \exp\{-z(x)\} \cdot \sum_{i=0}^{\infty} \frac{1}{\alpha_i + \varepsilon},$$

where $M(B, \varepsilon) = \frac{4\varepsilon B(2B+1)}{B^2-1}$.

PROOF. According to Theorem 3, for $x > x_0$, $\varepsilon \in \left(0, \sigma_i^{(-1)}(\alpha_i + \theta)\right)$ and $\forall i = 0, 1, ...$ the following inequality holds

$$\mathsf{P} \left\{ \sup_{\substack{0<|t-s|\leq\varepsilon \\ t,s\in D_i}} \frac{|X(t) - X(s)|}{w_{B,i}(t, s)} > x \right\} \leq \frac{2B(2B + 1)}{(B^2 - 1)N_i(\varepsilon)} \cdot \exp\{-z(x)\}, \tag{2}$$

where

$$w_{B,i}(t, s) = (6 + 4\sqrt{2}) \int_0^{\sigma_i(|t-s|)} U(BN_i(\sigma_i^{(-1)}(t)))dt + (5 + 2\sqrt{6}) \int_0^{\sigma_i(|t-s|)} U(B^2 N_i^2(\sigma_i^{(-1)}(t)))dt.$$

Since $\forall i = 0, 1, \dots$:

$$\frac{\alpha_i + \theta}{2\varepsilon} \le N_i(\varepsilon) \le \frac{\alpha_i + \theta}{2\varepsilon} + 1,$$

then $\forall i = 0, 1, \dots$:

$$\frac{1}{N_i(\varepsilon)} \le \frac{2\varepsilon}{\alpha_i + \theta}. \tag{3}$$

Inequalities (2) and (3) imply that for $\theta \in \left(0, \min\limits_{i\ge 0} \alpha_i\right)$, $\varepsilon \in (0, \min\{\varepsilon_0, \theta\})$ and $x > x_0$ under the condition $\sum\limits_{i=0}^{\infty} \frac{1}{\alpha_i} < \infty$ the following inequality holds:

$$\mathsf{P}\left\{\sup_{\substack{0<|t-s|\le\varepsilon\\ t,s\in[0,\infty)}} \frac{|X(t)-X(s)|}{w_B(t,s)} > x\right\} = \mathsf{P}\left\{\sup_{\substack{0<|t-s|\le\varepsilon\\ t,s\in \bigcup\limits_{i\ge 0} D_i}} \frac{|X(t)-X(s)|}{w_B(t,s)} > x\right\} \le$$

$$\le \sum_{i=0}^{\infty} \mathsf{P}\left\{\sup_{\substack{0<|t-s|\le\varepsilon\\ t,s\in D_i}} \frac{|X(t)-X(s)|}{w_{B,i}(t,s)} > x\right\} \le \sum_{i=0}^{\infty} \frac{2B(2B+1)}{(B^2-1)N_i(\varepsilon)} \cdot \exp\{-z(x)\} \le$$

$$= \frac{4\varepsilon B(2B+1)}{B^2-1} \cdot \exp\{-z(x)\} \cdot \sum_{i=0}^{\infty} \frac{1}{\alpha_i + \theta}.$$

Since $\theta > \varepsilon$, then, substituting ε instead of θ in the last inequality, we obtain the statement of the theorem. □

Remark 2
Let all the assumptions of Theorem 4 be fulfilled. Also let $\forall i = 0, 1, \dots$: $\sigma_i(h) = \sigma(h)$, i.e. the condition (1) takes on the following form:

$$\sup_{\substack{|t-s|\le h\\ t,s\in D_i}} \|X(t)-X(s)\|_\psi \le \sigma(h), \qquad 0 < h < \alpha_i + \theta,\ i = 0, 1, \dots$$

Then $g_{B,i}(\varepsilon) = \int\limits_0^{\sigma(\varepsilon)} U(B^2 N_i^2(\sigma^{(-1)}(t)))dt$, $f_{B,i}(\varepsilon) = \int\limits_0^{\sigma(\varepsilon)} U(BN_i(\sigma^{(-1)}(t)))dt$, $\varepsilon > 0$, and

$$\varepsilon_0 = \min_{i\ge 0}\left\{\sigma^{(-1)}(\alpha_i + \theta)\right\} = \sigma^{(-1)}\left(\min_{i\ge 0} \alpha_i + \theta\right),$$

because $\sigma^{(-1)}(h)$ is a non-decreasing function.

Corollary 1
Let all the assumptions of Theorem 4 be fulfilled and the space $\mathbb{F}_\psi(\Omega)$ has the property Z_1. In this case

$$w_{B,i}(t,s) = (6+4\sqrt{2})f_{B,i}(|t-s|) + (5+2\sqrt{6})g_{B,i}(|t-s|) \le$$
$$\le (6+4\sqrt{2}+b_0(5+2\sqrt{6}))f_{B,i}(|t-s|) := v_{B,i}(t,s),\ t,s\in D_i,$$

and

$$w_B(t,s) \le v_B(t,s) = \{v_{B,i}(t,s)\,|\,t,s\in A_i \text{ or } \min\{t,s\}\in A_i,\ \max\{t,s\}\in A_{i+1}\}.$$

Therefore for $x > x_0$, $\varepsilon \in \left(0, \min\left\{\sigma^{(-1)}\left(\min\limits_{i\ge 0}\alpha_i + \theta\right), \theta\right\}\right)$ and under the condition that $\sum\limits_{i=0}^{\infty} \frac{1}{\alpha_i} < \infty$ the following inequality holds

$$\mathsf{P}\left\{\sup_{\substack{0<|t-s|\le\varepsilon \\ t,s\in[0,\infty)}} \frac{|X(t)-X(s)|}{v_B(t,s)} > x\right\} \le M(B,\varepsilon)\cdot\exp\{-z(x)\}\cdot\sum_{i=0}^{\infty}\frac{1}{\alpha_i+\varepsilon}.$$

Theorem 5
Let all the assumptions of Corollary 1 be fulfilled. If the function $\psi(u)=u^{\alpha}$, $\alpha>0$ and for all $i=0,1,\ldots$ functions $\sigma_i(h)=dh^{\varkappa}$, $h,\varkappa,d>0$ then for $x>x_1$, $B>1$, $\mu<\frac{\varkappa}{\alpha}$, $\varepsilon\in\left(0,\min\left\{\frac{1}{\sqrt[\varkappa]{d}}\sqrt[\varkappa]{\min_{i\ge0}\alpha_i+\theta},\,\theta\right\}\right)$ and under the condition that $\sum_{i=0}^{\infty}\frac{1}{\alpha_i}<\infty$ the following inequality holds true

$$\mathsf{P}\left\{\sup_{\substack{0<|t-s|\le\varepsilon \\ t,s\in[0,\infty)}} \frac{|X(t)-X(s)|}{v_B(t,s)} > x\right\} \le M(B,\varepsilon)\cdot\exp\left\{-\frac{\alpha}{e}x^{1/\alpha}\right\}\cdot\sum_{i=0}^{\infty}\frac{1}{\alpha_i+\varepsilon},$$

where $v_B(t,s)=\{v_{B,i}(t,s)\,|\,t,s\in A_i \text{ or } \min\{t,s\}\in A_i,\,\max\{t,s\}\in A_{i+1}\}$,

$$v_{B,i}(t,s)=C_1\cdot\frac{\left(\frac{B(\alpha_i+\theta)}{2}+(B+1)\cdot|t-s|\right)^{\alpha\mu}}{\mu^{\alpha}}\cdot\frac{\varkappa d}{\varkappa-\alpha\mu}\cdot|t-s|^{\varkappa-\alpha\mu},$$

$C_1=\left(6+4\sqrt{2}+(5+2\sqrt{6})\cdot 2^{\alpha}\right)$.

PROOF. The inverse function to the function $\sigma(h)$ is $\sigma^{(-1)}(h)=\sqrt[\varkappa]{\frac{h}{d}}$. According to Theorem 1, the space $\mathbb{F}_{\psi}(\Omega)$ has the property Z with functions $U(n)=(\ln(n+2))^{\alpha}$ and $z(x)=\frac{\alpha}{e}x^{1/\alpha}$ for $x>x_1$. Therefore, functions $f_{B,i}(\varepsilon)$ and $g_{B,i}(\varepsilon)$ take the following form:

$$f_{B,i}(\varepsilon)=\int_0^{d\varepsilon^{\varkappa}} U\left(BN_i\left(\sqrt[\varkappa]{\frac{t}{d}}\right)\right)dt=\int_0^{d\varepsilon^{\varkappa}}\left(\ln\left(BN_i\left(\sqrt[\varkappa]{\frac{t}{d}}\right)+2\right)\right)^{\alpha}dt;$$

$$g_{B,i}(\varepsilon)=\int_0^{d\varepsilon^{\varkappa}} U\left(B^2N_i^2\left(\sqrt[\varkappa]{\frac{t}{d}}\right)\right)dt=\int_0^{d\varepsilon^{\varkappa}}\left(\ln\left(B^2N_i^2\left(\sqrt[\varkappa]{\frac{t}{d}}\right)+2\right)\right)^{\alpha}dt,$$

and $g_{B,i}(\varepsilon)\le 2^{\alpha}f_{B,i}(\varepsilon)$.

In this case, the space $\mathbb{F}_{\psi}(\Omega)$ has the property Z_1 with $b_0=2^{\alpha}$. According to Corollary 1 and Remark 2, for $\varepsilon\in(0,\min\{\varepsilon_0,\theta\})$,

$$\varepsilon_0=\sigma^{(-1)}\left(\min_{i\ge0}\alpha_i+\theta\right)=\frac{1}{\sqrt[\varkappa]{d}}\sqrt[\varkappa]{\min_{i\ge0}\alpha_i+\theta},$$

$B>1$, $x>x_1$ and under the condition that $\sum_{i=0}^{\infty}\frac{1}{\alpha_i}<\infty$ the following inequality holds

$$\mathsf{P}\left\{\sup_{\substack{0<|t-s|\le\varepsilon \\ t,s\in[0,\infty)}} \frac{|X(t)-X(s)|}{v_B(t,s)} > x\right\} \le M(B,\varepsilon)\cdot\exp\left\{-\frac{\alpha}{e}x^{1/\alpha}\right\}\cdot\sum_{i=0}^{\infty}\frac{1}{\alpha_i+\varepsilon},$$

where $v_B(t,s)=\{C_1\cdot f_{B,i}(|t-s|)\,|\,t,s\in A_i \text{ or } \min\{t,s\}\in A_i,\,\max\{t,s\}\in A_{i+1}\}$.

Using the inequality for the metric massiveness

$$N_i\left(\sqrt[\varkappa]{\frac{p}{d}}\right)\le\frac{\alpha_i+\theta}{2}\sqrt[\varkappa]{\frac{d}{p}}+1,$$

we can limit the function $f_{B,i}(\varepsilon)$ above:

$$f_{B,i}(\varepsilon) \le \int\limits_0^{d\varepsilon^{\varkappa}} \left(\ln\left(B\cdot\left(\frac{\alpha_i+\theta}{2}\sqrt[\varkappa]{\frac{d}{p}}+1\right)+2\right)\right)^{\alpha} dp.$$

The last integral can be estimated and calculated. Therefore under the condition $\mu < \frac{\varkappa}{\alpha}$ we have

$$\int\limits_0^{d\varepsilon^{\varkappa}} \left(\ln\left(B\cdot\left(\frac{\alpha_i+\theta}{2}\sqrt[\varkappa]{\frac{d}{p}}+1\right)+2\right)\right)^{\alpha} dp \le \frac{\left(\frac{B(\alpha_i+\theta)}{2}+\varepsilon(B+1)\right)^{\alpha\mu}}{\mu^{\alpha}}\cdot\frac{\varkappa d}{\varkappa-\alpha\mu}\cdot\varepsilon^{\varkappa-\alpha\mu}.$$

Finally, in accordance with Corollary 1 and Remark 2, we get the statement of the theorem. □

Example 3

Consider a stationary process $X=\{X(t),\, t\in[0,\infty)\}$ that belongs to the space $\mathbb{F}_{\psi}(\Omega)$, $\psi(u)=u^{\alpha}$, $\alpha>0$, with $\mathsf{E}X(t)=0$, $\mathsf{D}X(t)=1$ and a covariance function of the following form

$$R(t,s)=\int\limits_0^{\infty}\cos((t-s)p)f(p)dp,$$

where $f(p)$, $p\in[0,\infty)$ is some function. Let's find a restriction on this function such that the conditions of Theorem 5 are satisfied for the process X.

Using known inequality $|\sin x|\le|x|^{\varkappa}$, $0<\varkappa\le 1$, for any $t,s\in[0,\infty)$ the following inequality holds

$$\mathsf{E}(X(t)-X(s))^2 = 2\int\limits_0^{\infty}(1-\cos((t-s)p))\,f(p)dp \le 4\cdot 2^{2\varkappa}(t-s)^{2\varkappa}\int\limits_0^{\infty}p^{2\varkappa}f(p)dp.$$

Therefore

$$\|X(t)-X(s)\|_{\psi}\le D(\varkappa)\cdot(t-s)^{\varkappa},$$

where $D(\varkappa)=2^{\varkappa+1}\sqrt{\int\limits_0^{\infty}p^{2\varkappa}f(p)dp}$. Thus, for any function f and a constant $\varkappa\in(0,1]$ such that $\int\limits_0^{\infty}p^{2\varkappa}f(p)dp<\infty$ the condition (1) of Theorem 4 is satisfied with the functions $\sigma_i(h)=\sigma(h)=D(\varkappa)\cdot h^{\varkappa}$, $h>0$, $i=0,1,...$

Now let $\{a_i, i=0,1,...,\infty\}$ is some increasing sequence such that $\sum\limits_{i=0}^{\infty}\frac{1}{a_{i+1}-a_i}<\infty$ and $a_0=0$, and $\theta\in\left(0,\min\limits_{i\ge0}\alpha_i\right)$. Consider a process $Y(t)=\frac{X(t)}{c(t)}$, where $c(t)$, $t\in[0,\infty)$ is some monotonically increasing function such that

$$c(a_i)\ge(\alpha_i+\theta)^{\alpha\mu},\quad i=0,1,... \tag{4}$$

and $\forall\, t,s\in[0,\infty)$ there are such $b>0$ and $\gamma\in(0,\varkappa+\alpha\mu)$ that the following inequality holds

$$|c(t)-c(s)|\le b|t-s|^{\gamma}. \tag{5}$$

Since $\forall\, t,s\in[0,\infty)$:

$$\begin{aligned}\|Y(t)-Y(s)\|_{\psi} &= \left\|\frac{X(t)}{c(t)}-\frac{X(s)}{c(t)}+\frac{X(s)}{c(t)}-\frac{X(s)}{c(s)}\right\|_{\psi}\le\\ &\le\frac{1}{|c(t)|}\cdot\|X(t)-X(s)\|_{\psi}+\|X(s)\|_{\psi}\cdot\left|\frac{1}{c(t)}-\frac{1}{c(s)}\right|\le\\ &\le\frac{D(\varkappa)|t-s|^{\varkappa}}{|c(t)|}+\left|\frac{c(t)-c(s)}{c(t)c(s)}\right|,\end{aligned}$$

then implying inequality (5) we have that $\forall\, t,s\in[0,\infty)$:

$$\|Y(t)-Y(s)\|_\psi \le \frac{D(\varkappa)|t-s|^\varkappa}{|c(t)|}+\frac{b|t-s|^\gamma}{|c(t)c(s)|}=|t-s|^\varkappa\cdot\frac{1}{|c(t)|}\left(D(\varkappa)+\frac{b|t-s|^{\gamma-\varkappa}}{|c(s)|}\right).$$

Thus, implying inequality (4), $\forall i=0,1,...$ and $h\in(0,\alpha_i+\theta)$:

$$\sup_{\substack{|t-s|\le h\\ t,s\in D_i}}\|Y(t)-Y(s)\|_\psi\le h^\varkappa\cdot\frac{1}{c(a_i)}\left(D(\varkappa)+\frac{b(\alpha_i+\theta)^{\gamma-\varkappa}}{c(a_i)}\right)\le$$

$$\le h^\varkappa\cdot\frac{1}{c(a_i)}\left(D(\varkappa)+\frac{b(\alpha_i+\theta)^{\gamma-\varkappa}}{(\alpha_i+\theta)^{\alpha\mu}}\right)\le\frac{D(\varkappa)+b}{c(a_i)}\cdot h^\varkappa.$$

According to Theorem 5, the modulus of continuity $v_B(t,s)$ of the process X takes the following form:

$$v_B(t,s)=\{v_{B,i}(t,s)\,|\,t,s\in A_i \text{ or } \min\{t,s\}\in A_i,\ \max\{t,s\}\in A_{i+1}\},$$

$$v_{B,i}(t,s)=C_1\cdot\frac{\left(\frac{B(\alpha_i+\theta)}{2}+(B+1)\cdot|t-s|\right)^{\alpha\mu}}{\mu^\alpha}\cdot\frac{\varkappa D(\varkappa)}{\varkappa-\alpha\mu}\cdot|t-s|^{\varkappa-\alpha\mu}.$$

A similar form has a modulus of continuity of the process Y with a difference in constants. Implying inequality (4), we have that for the functions $v_{B,i}$, $i=0,1,...$ of a modulus of continuity of the process Y the following holds

$$C_1\cdot\frac{\left(\frac{B(\alpha_i+\theta)}{2}+(B+1)\cdot|t-s|\right)^{\alpha\mu}}{\mu^\alpha}\cdot\frac{\varkappa}{\varkappa-\alpha\mu}\cdot\frac{D(\varkappa)+b}{c(a_i)}\cdot|t-s|^{\varkappa-\alpha\mu}\le$$

$$\le C_1\cdot\left(\frac{\frac{B(\alpha_i+\theta)}{2}+(B+1)\cdot|t-s|}{\mu^{1/\mu}(\alpha_i+\theta)}\right)^{\alpha\mu}\cdot\frac{\varkappa(D(\varkappa)+b)}{\varkappa-\alpha\mu}\cdot|t-s|^{\varkappa-\alpha\mu}\le$$

$$\le C_1\cdot\left(\frac{B}{2\mu^{1/\mu}}+\frac{(B+1)\cdot|t-s|}{\mu^{1/\mu}\left(\min\limits_{i\ge0}\alpha_i+\theta\right)}\right)^{\alpha\mu}\cdot\frac{\varkappa(D(\varkappa)+b)}{\varkappa-\alpha\mu}\cdot|t-s|^{\varkappa-\alpha\mu}.$$

Thus, according to Theorem 5, for $\varepsilon\in\left(0,\min\left\{\frac{1}{\sqrt[\varkappa]{D(\varkappa)+b}}\left(\min\limits_{i\ge0}\alpha_i+\theta\right)^{\frac{\alpha\mu+1}{\varkappa}},\theta\right\}\right)$, $x>x_1$, $B>1$, $\mu<\frac{\varkappa}{\alpha}$ and under the condition $\sum\limits_{i=0}^\infty\frac{1}{\alpha_i}<\infty$ the following inequality holds

$$\mathsf{P}\left\{\sup_{\substack{0<|t-s|\le\varepsilon\\ t,s\in[0,\infty)}}\frac{|Y(t)-Y(s)|}{v_B(t,s)}>x\right\}\le M(B,\varepsilon)\cdot\exp\left\{-\frac{\alpha}{e}x^{1/\alpha}\right\}\cdot\sum_{i=0}^\infty\frac{1}{\alpha_i+\varepsilon},$$

where

$$v_B(t,s)=C_1\cdot\left(\frac{B}{2\mu^{1/\mu}}+\frac{(B+1)\cdot|t-s|}{\mu^{1/\mu}\left(\min_{i\ge0}\alpha_i+\theta\right)}\right)^{\alpha\mu}\cdot\frac{\varkappa(D(\varkappa)+b)}{\varkappa-\alpha\mu}\cdot|t-s|^{\varkappa-\alpha\mu}.$$

Theorem 6
Let all the assumptions of Theorem 4 be fulfilled. If the function $\psi(u)=e^{\alpha u^\beta}$, $\alpha>0$, $\beta>0$ and for all $i=0,1,...$ functions $\sigma_i(h)=\sigma(h)=dh^\varkappa$, $h,\varkappa,d>0$ then for $\varepsilon\in\left(0,\min\left\{\frac{1}{\sqrt[\varkappa]{d}}\sqrt[\varkappa]{\min\limits_{i\ge0}\alpha_i+\theta},\,\theta\right\}\right)$, $B>1$, $\beta\in(0,1)$, $x\ge x_2$ and under the condition $\sum\limits_{i=0}^\infty\frac{1}{\alpha_i}<\infty$ the following inequality holds true

$$\mathsf{P}\left\{\sup_{\substack{0<|t-s|\le\varepsilon\\ t,s\in[0,\infty)}}\frac{|X(t)-X(s)|}{v_B(t,s)}>x\right\}\le M(B,\varepsilon)\cdot\exp\left\{-\frac{\beta}{\alpha^{1/\beta}}\left(\frac{2}{\beta+1}\right)^{\frac{\beta+1}{\beta}}(\ln x)^{\frac{\beta+1}{2\beta}}\right\}\cdot\sum_{i=0}^{\infty}\frac{1}{\alpha_i+\varepsilon},$$

where $v_B(t,s)=\{v_{B,i}(t,s)\,|\,t,s\in A_i \text{ or } \min\{t,s\}\in A_i,\ \max\{t,s\}\in A_{i+1}\}$,

$$v_{B,i}(t,s)=m_{B,i}\left(\min\left\{\frac{1}{\sqrt[\varkappa]{d}}\sqrt[\varkappa]{\min_{i\ge 0}\alpha_i+\theta},\,\theta\right\}\right)\cdot|t-s|^{\varkappa},$$

$$m_{B,i}(\varepsilon)=d\left((6+4\sqrt{2})\cdot\exp\left\{\left(\ln\left(\sqrt[\varkappa]{d}\left(\frac{B(\alpha_i+\theta)}{2}+\varepsilon(B+2)\right)\right)\right)^{\frac{2\beta}{\beta+1}}\right\}+\right.$$
$$\left.+(5+2\sqrt{6})\cdot\exp\left\{\left(\ln\left(B^2\sqrt[\varkappa]{d}(\alpha_i+\theta)\left(\frac{\sqrt[\varkappa]{d}(\alpha_i+\theta)}{4}+1\right)+\max\{1,\varepsilon\sqrt[\varkappa]{d}\}\cdot(B^2+2)\right)\right)^{\frac{2\beta}{\beta+1}}\right\}\right).$$

PROOF. The inverse function to the function $\sigma(h)$ is $\sigma^{(-1)}(h)=\sqrt[\varkappa]{\frac{h}{d}}$. According to Theorem 2, the space $\mathbb{F}_\psi(\Omega)$ has the property Z with functions $z(x)=\frac{\beta}{\alpha^{1/\beta}}\left(\frac{2}{\beta+1}\right)^{\frac{\beta+1}{\beta}}(\ln x)^{\frac{\beta+1}{2\beta}}$ and $U(n)=\exp\left\{(\ln(n+2))^{\frac{2\beta}{\beta+1}}\right\}$ for $x>x_2$. Therefore, functions $f_{B,i}(\varepsilon)$ and $g_{B,i}(\varepsilon)$ take the following form:

$$f_{B,i}(\varepsilon)=\int_0^{d\varepsilon^{\varkappa}}U\left(BN_i\left(\sqrt[\varkappa]{\frac{t}{d}}\right)\right)dt=\int_0^{d\varepsilon^{\varkappa}}\exp\left\{\left(\ln\left(BN_i\left(\sqrt[\varkappa]{\frac{t}{d}}\right)+2\right)\right)^{\frac{2\beta}{\beta+1}}\right\}dt;$$
$$g_{B,i}(\varepsilon)=\int_0^{d\varepsilon^{\varkappa}}U\left(B^2N_i^2\left(\sqrt[\varkappa]{\frac{t}{d}}\right)\right)dt=\int_0^{d\varepsilon^{\varkappa}}\exp\left\{\left(\ln\left(B^2N_i^2\left(\sqrt[\varkappa]{\frac{t}{d}}\right)+2\right)\right)^{\frac{2\beta}{\beta+1}}\right\}dt.$$

According to Theorem 4 and Remark 2, for $x\ge x_2$, $\varepsilon\in(0,\min\{\varepsilon_0,\theta\})$, $\varepsilon_0=\frac{1}{\sqrt[\varkappa]{d}}\sqrt[\varkappa]{\min_{i\ge0}\alpha_i+\theta}$, $B>1$ and under the condition that $\sum_{i=0}^{\infty}\frac{1}{\alpha_i}<\infty$ the following inequality holds

$$\mathsf{P}\left\{\sup_{\substack{0<|t-s|\le\varepsilon\\ t,s\in[0,\infty)}}\frac{|X(t)-X(s)|}{v_B(t,s)}>x\right\}\le M(B,\varepsilon)\cdot\exp\left\{-\frac{\beta}{\alpha^{1/\beta}}\left(\frac{2}{\beta+1}\right)^{\frac{\beta+1}{\beta}}(\ln x)^{\frac{\beta+1}{2\beta}}\right\}\cdot\sum_{i=0}^{\infty}\frac{1}{\alpha_i+\varepsilon},$$

where $v_B(t,s)=\{v_{B,i}(t,s)\,|\,t,s\in A_i \text{ or } \min\{t,s\}\in A_i,\ \max\{t,s\}\in A_{i+1}\}$,

$$v_{B,i}(t,s)=(6+4\sqrt{2})f_{B,i}(|t-s|)+(5+2\sqrt{6})g_{B,i}(|t-s|).$$

Using the inequality for the metric massiveness

$$N_i\left(\sqrt[\varkappa]{\frac{u}{d}}\right)\le\frac{\alpha_i+\theta}{2}\sqrt[\varkappa]{\frac{d}{u}}+1,$$

we can limit functions $f_{B,i}(\varepsilon)$ and $g_{B,i}(\varepsilon)$ above:

$$f_{B,i}(\varepsilon)\le\int_0^{d\varepsilon^{\varkappa}}\exp\left\{\left(\ln\left(B\cdot\left(\frac{\alpha_i+\theta}{2}\sqrt[\varkappa]{\frac{d}{t}}+1\right)+2\right)\right)^{\frac{2\beta}{\beta+1}}\right\}dt=:I_{1,B,i};$$

$$g_{B,i}(\varepsilon) \le \int_0^{d\varepsilon^{\varkappa}} \exp\left\{\left(\ln\left(B^2 \cdot \left(\frac{\alpha_i+\theta}{2}\sqrt[\varkappa]{\frac{d}{t}}+1\right)^2+2\right)\right)^{\frac{2\beta}{\beta+1}}\right\} dt =: I_{2,B,i}.$$

Integrals $I_{1,B,i}$ and $I_{2,B,i}$ can be estimated and calculated. Therefore for $\beta \in (0,1)$ we have

$$I_{1,B,i} \le d\varepsilon^{\varkappa} \cdot \exp\left\{\left(\ln\left(\sqrt[\varkappa]{d}\left(\frac{B(\alpha_i+\theta)}{2}+\varepsilon(B+2)\right)\right)\right)^{\frac{2\beta}{\beta+1}}\right\};$$

$$I_{2,B,i} \le \exp\left\{\left(\ln\left(B^2\sqrt[\varkappa]{d}(\alpha_i+\theta)\left(\frac{\sqrt[\varkappa]{d}(\alpha_i+\theta)}{4}+1\right)+B^2+2\right)\right)^{\frac{2\beta}{\beta+1}}\right\}+$$

$$+(d\varepsilon^{\varkappa}-1)\cdot\exp\left\{\left(\ln\left(B^2\sqrt[\varkappa]{d}(\alpha_i+\theta)\left(\frac{\sqrt[\varkappa]{d}(\alpha_i+\theta)}{4}+1\right)+\varepsilon\sqrt[\varkappa]{d}(B^2+2)\right)\right)^{\frac{2\beta}{\beta+1}}\right\} \le$$

$$\le d\varepsilon^{\varkappa}\cdot\exp\left\{\left(\ln\left(B^2\sqrt[\varkappa]{d}(\alpha_i+\theta)\left(\frac{\sqrt[\varkappa]{d}(\alpha_i+\theta)}{4}+1\right)+\max\{1,\varepsilon\sqrt[\varkappa]{d}\}\cdot(B^2+2)\right)\right)^{\frac{2\beta}{\beta+1}}\right\}.$$

Using the notation $m_{B,i}(\varepsilon)$ from the statement of the theorem, the following inequality holds true for $\varepsilon \in \left(0, \min\left\{\frac{1}{\sqrt[\varkappa]{d}}\sqrt[\varkappa]{\min_{i\ge 0}\alpha_i+\theta},\,\theta\right\}\right)$:

$$m_{B,i}(\varepsilon) \le m_{B,i}\left(\min\left\{\frac{1}{\sqrt[\varkappa]{d}}\sqrt[\varkappa]{\min_{i\ge 0}\alpha_i+\theta},\,\theta\right\}\right).$$

Finally, in accordance with Theorem 4 and Remark 2, we get the statement of the theorem. □

Consider an example to this theorem.

Example 4

Let $X=\{X(t), t\in[0,\infty)\}$ be some non-stationary process that belongs to $\mathbb{F}_{\psi}(\Omega)$ space with the function $\psi(u)=e^{\alpha u^{\beta}}$, $\alpha>0$, $\beta>0$, and a covariance function of which has the form

$$R(t,s)=\int_0^{\infty} f(t,p)f(s,p)dp,$$

where $\exists\, \frac{\partial f(t,p)}{\partial t}$, $t\in[0,\infty)$, and $\left|\frac{\partial f(t,p)}{\partial t}\right| < \frac{|c_i(p)|}{\alpha_i+\theta}$, $t\in D_i$, $p\in[0,\infty)$.

For any $t,s\in D_i$, $\varkappa\in(0,1]$ and $\mu\in(\min\{t,s\},\max\{t,s\})$ the following inequality holds

$$\mathsf{E}(X(t)-X(s))^u = \int_0^{\infty}|f(t,p)-f(s,p)|^u dp =$$

$$= \int_0^{\infty}|f(t,p)-f(s,p)|^{u\varkappa}|f(t,p)-f(s,p)|^{u(1-\varkappa)}dp \le$$

$$\le |t-s|^{u\varkappa}\int_0^{\infty}|f'(\mu,p)|^{u\varkappa}\cdot(f(t,p)-f(s,p))^{u(1-\varkappa)}dp \le$$

$$\le |t-s|^{u\varkappa}\int_0^{\infty}\frac{|c_i(p)|^{u\varkappa}}{(\alpha_i+\theta)^{u\varkappa}}\cdot(f(t,p)-f(s,p))^{u(1-\varkappa)}dp.$$

We use the following inequality. For any $a > 0$ and $b > 0$ the following is true:

$$(a+b)^r \leq c_r(a^r + b^r),$$

where

$$c_r = \begin{cases} 2^{r-1} & \text{for } r \geq 1; \\ 1 & \text{for } r \in [0,1). \end{cases}$$

Then we get for $t, s \in D_i$:

$$\mathsf{E}(X(t)-X(s))^u \leq \frac{c_r|t-s|^{u\varkappa}}{(\alpha_i+\theta)^{u\varkappa}} \int_0^\infty |c_i(p)|^{u\varkappa} \cdot \left(|f(t,p)|^{u(1-\varkappa)} + |f(s,p)|^{u(1-\varkappa)}\right) dp \leq$$

$$\leq \frac{2c_r|t-s|^{u\varkappa}}{(\alpha_i+\theta)^{u\varkappa}} \int_0^\infty |c_i(p)|^{u\varkappa} \cdot \sup_{t\in D_i} |f(t,p)|^{u(1-\varkappa)} dp.$$

Since in this case $r = u(1-\varkappa) \in [0, u)$, then $c_r \in [1, 2^{u-1})$, from where

$$\mathsf{E}(X(t)-X(s))^u \leq \frac{2^u|t-s|^{u\varkappa}}{(\alpha_i+\theta)^{u\varkappa}} \int_0^\infty |c_i(p)|^{u\varkappa} \cdot \sup_{t\in D_i} |f(t,p)|^{u(1-\varkappa)} dp.$$

Therefore

$$\|X(t)-X(s)\|_\psi = \sup_{u\geq 1} \frac{\left(\mathsf{E}(X(t)-X(s))^u\right)^{1/u}}{\psi(u)} \leq$$

$$\leq \frac{2|t-s|^{\varkappa}}{(\alpha_i+\theta)^{\varkappa}} \cdot \sup_{u\geq 1} \frac{\left(\int_0^\infty |c_i(p)|^{u\varkappa} \cdot \sup_{t\in D_i} |f(t,p)|^{u(1-\varkappa)} dp\right)^{1/u}}{\psi(u)}.$$

Denote

$$C_0 = 2 \cdot \sup_{u\geq 1} \frac{\left(\int_0^\infty |c_i(p)|^{u\varkappa} \cdot \sup_{t\in D_i} |f(t,p)|^{u(1-\varkappa)} dp\right)^{1/u}}{e^{\alpha u^\beta}}.$$

Thus, according to Theorem 6, for $x \geq x_2$, $\varepsilon \in \left(0, \min\left\{\frac{1}{\sqrt[\varkappa]{C_0}}\left(\min_{i\geq 0}\alpha_i+\theta\right)^{1+\frac{1}{\varkappa}}, \theta\right\}\right)$, $\beta \in (0,1)$, $B > 1$ and under the condition that $\sum_{i=0}^{\infty} \frac{1}{\alpha_i} < \infty$ the following inequality holds

$$\mathsf{P}\left\{\sup_{\substack{0<|t-s|\leq\varepsilon \\ t,s\in[0,\infty)}} \frac{|X(t)-X(s)|}{v_B(t,s)} > x\right\} \leq M(B,\varepsilon) \cdot \exp\left\{-\frac{\beta}{\alpha^{1/\beta}}\left(\frac{2}{\beta+1}\right)^{\frac{\beta+1}{\beta}} (\ln x)^{\frac{\beta+1}{2\beta}}\right\} \cdot \sum_{i=0}^{\infty} \frac{1}{\alpha_i+\varepsilon},$$

where

$$v_B(t,s) = \frac{C_0}{\left(\min_{i\geq 0}\alpha_i+\theta\right)^{\varkappa}} \cdot m_B\left(\min\left\{\frac{1}{\sqrt[\varkappa]{C_0}}\left(\min_{i\geq 0}\alpha_i+\theta\right)^{1+\frac{1}{\varkappa}}, \theta\right\}\right) \cdot |t-s|^{\varkappa},$$

$$m_B(\varepsilon) = (6+4\sqrt{2}) \cdot \exp\left\{ \left(\ln \left(\sqrt[\varkappa]{C_0} \left(\frac{B}{2} + \frac{\varepsilon(B+2)}{\min\limits_{i\geq 0} \alpha_i + \theta} \right) \right) \right)^{\frac{2\beta}{\beta+1}} \right\} +$$

$$+ (5+2\sqrt{6}) \cdot \exp\left\{ \left(\ln \left(B^2 \sqrt[\varkappa]{C_0} \left(\frac{\sqrt[\varkappa]{C_0}}{4} + 1 \right) + \max\left\{ 1, \frac{\varepsilon \sqrt[\varkappa]{C_0}}{\min\limits_{i\geq 0} \alpha_i + \theta} \right\} \cdot (B^2+2) \right) \right)^{\frac{2\beta}{\beta+1}} \right\}.$$

4. Conclusion

In this article we analyse estimations of distributions of random processes from $\mathbb{F}_\psi(\Omega)$ spaces. Definitions and some properties of random variables and processes from $\mathbb{F}_\psi(\Omega)$ spaces are given. Estimates for distributions of Hölder semi-norms of processes from $\mathbb{F}_\psi(\Omega)$ spaces, defined on an infinite interval, are obtained.

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