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A Condition-based Maintenance Policy in Chance Space

Somayyeh Shahraki Dehsoukhteh, and Mostafa Razmkhah *

Department of Statistics, Faculty of Mathematical Sciences, Ferdowsi University of Mashhad, Mashhad, Iran

Abstract A condition-based maintenance policy is considered for a deteriorating system including both of preventive and corrective maintenance actions. The gamma process is used to model stochastic degradation in the probability space. Although, the cost of preventive maintenance is considered as an uncertain variable due to incomplete information, and its distribution is estimated based on the opinions of some experts using the Delphi method. The optimal policy is determined by minimizing the expected cost rate function. Since in this function, there are both random variables discussing in a probability space, and an uncertain variable, which is considered in an uncertain space, we have to study the optimal policy in a chance space which is a combination of probability and uncertain spaces. The proposed methodology is explained in an illustrative example. Finally, the results are applied to a real data set.

Keywords Chance ordering, Cost function, Gamma process, Optimal inspection interval, Uncertainty

AMS 2010 subject classifications 90B25, 90C70

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1. Introduction

Maintenance plays an important role in industry. Recent developments in the industry have led to intense competition among manufacturers to provide high-quality products. Considering this issue, manufacturers must increase the quality and reliability of their production systems to achieve success. System failure creates heavy costs for manufacturers. In addition, replacing the entire system is expensive and it is unreasonable to do after each failure. Therefore, it is very important to provide maintenance solutions to keep the system in optimal operating conditions. In fact, optimal maintenance solutions increase the operating time of the system and improve its reliability. To read more about the types of maintenance policies, one may refer to [4, 11].

All systems and equipment in various branches of industry such as production lines, aviation and shipping systems are subject to degradation. Therefore, many failure mechanisms can be interpreted using degradation models. The level of erosion of a system or its components can be measured continuously or at different inspection times. Many researchers have used the theory of stochastic processes, especially gamma and Wiener processes, to develop degradation models. To study about the mathematical features of the gamma process, one may refer to [2]. Noortwijk [15] used the gamma process in maintenance. Pandey et al. [18] discussed a maintenance analysis based on gamma stochastic process.

In recent years, condition-based maintenance (CBM) policies based on degradation models have been developed. A CBM is a preventive maintenance strategy that relies on the monitoring, or periodic or aperiodic inspection of assets or equipment to determine when a maintenance action is necessary. It involves the use of sensors and other monitoring equipment to collect data on the performance of equipment. The CBM policies end up in a more efficient and cost-effective maintenance actions against performing maintenance on a fixed schedule or when an

*Correspondence to: Mostafa Razmkhah (Email: razmkhah_m@um.ac.ir).

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equipment is failed because of unexpected downtime and emergency repairs. Li et al. [6] did a comprehensive review on CBM policies.

Maintenance policies are mainly categorized into two major classes: corrective maintenance (CM) that is performed whenever the system fails, and preventive maintenance (PM) which is an action performed to a system which is still operating but may function in unsatisfactory working conditions. In fact, a PM can reduce the possibility of a sudden failure of a system and make a system function properly. So, the PM is economic, especially when the failures are serious and will result in heavy losses. Recently, Wei et al. [24] studied a continuous flow manufacturing system with two machines subjected to condition-based PM. Also, Peng et al. [19] considered a manufacturer performing PM on a product according to a one- or two-dimensional policy.

In this paper, a condition-based maintenance policy is considered for a system that its degradation level is inspected periodically, and a PM or CM action is done if the system degradation level reaches or exceeds the thresholds M or L $(M < L)$, respectively, at some inspection time. The gamma process is used to model stochastic degradation. The expected cost rate criterion is used to obtain the optimal maintenance policy. The cost of CM is considered to be fixed. However, the cost of PM is assumed to have different uncertain values depending on the opinion of each expert or repairman. Note that the decisions are usually made in a state of uncertainty in the real world. In other words, in practice, we encounter with phenomena whose results cannot be predicted in advance. How to model uncertainty is an important research topic not only in the field of mathematics and statistics but also in the field of science and engineering. If there are sufficient information including enough frequencies about a phenomenon, the probability theory is the best tool to model the uncertainty. In many situations, the information may be insufficient, vague, imprecise or contradictory. Various types of incomplete information lead to various types of uncertainty. Various theories such as fuzzy theory [23, 26], possibility theory [1, 27] and uncertainty theory [7, 8] have been presented to investigate the variables and systems that are not sufficiently and accurately known. All these theories are placed in a general framework called measures of uncertainty. The fuzzy measures are used in dealing with uncertainty caused by imprecise information. The theory of possibility is used in dealing with uncertainty caused by incomplete information. The Uncertainty theory covers all aspects of uncertain information. Uncertainty theory is inferred with three basic concepts ”uncertain measure”, ”uncertain variable” and ”uncertainty distribution”. The uncertain measure is a set function to measure the degree of belief of an uncertain event. An uncertain variable is a function that is defined on an uncertain space to express imprecise quantities. Uncertainty distribution is used to describe uncertain variables in an easy way. In general, uncertainty distribution plays the role of an interface between uncertainty theory and experts. Yao and Zhou [25] proposed a concept of uncertain random renewal process, and applied the methodology to a block replacement policy. Zhang et al. [28] developed belief reliability metric to evaluate reliability of uncertain random systems which are affected by both aleatory and epistemic uncertainties. Recently, Shahraki et al. [20] discussed the block replacement policy and used the experts’ judgments to estimate the parameters of failure time distribution. In the situations that we encounter with both probability and uncertainty spaces simultaneously in analysing a system or a phenomenon, we should use the chance theory suggested by Liu [9].

As previously mentioned, in our proposed maintenance policy, the system degradation is a random process which is discussed in a probability space, while the PM cost is considered as an uncertain variable due to insufficient frequencies. Hence, we have to use the chance theory to get the accurate analysis; otherwise, the accuracy of the results wouldn’t be guaranteed.

To determine the uncertainty distribution of the PM cost, we use the subjective judgment of some experts. One of the important methods to determine the uncertainty distribution is the Delphi method. Accepting that group experience is more valid than individual experience, the Delphi method is a process mostly used in research and economics, that aims collecting opinions on a particular research question or specific topic in order to reach a consensus. The premise of this method is that pooled intelligence can enhance individual judgement. In practice, the researcher chooses a panel of experts, and develops a series of iterative questionnaires. Panellists reply anonymously to the iterative questionnaires, where every questionnaire sent represent a round. In the first round, experts evaluate the issues independently and express their opinions based on their knowledge and personal experience. The anonymity of people is essential in the Delphi method. This means that no one knows who else is participating in this survey. In the second round, the feedback from the previous round is provided to the experts so that they can re-evaluate the same items and make new judgments about changing their opinions. The opinions of experts tend to be compatible when they see the opinions of other experts. Finally, a composite survey can be obtained according to the opinion of all experts. For more details about the Delphi method, one may refer to [16, 17, 22].

The rest of this paper is organized as follows. In Section 2, some preliminaries are presented regarding the gamma process, and uncertainty concepts. The maintenance policy is described in Section 3, and some main results are presented. In Section 4, we compare the chance order of the cost function for two systems. In Section 5, the method of determining the uncertainty distribution according to the Delphi method is explained. A real data set is used to illustrate the proposed method in Section 6, and the optimal inspection period is determined. Finally, some conclusions are expressed in Section 7. The potential challenges in implementing the proposed CBM policy in practical settings are also discussed in this section.

2. Preliminaries

In this section, some fundamental concepts about Gamma degradation process, uncertain variables and chance space are presented. These are all useful for defining the proposed maintenance policy and obtaining optimal scheme.

2.1. Gamma processes

Gamma process is a stochastic process that can be used as an effective method to model gradual damage such as wear, corrosion, erosion and deterioration. Let $\alpha(t)$ be a non-decreasing, right-continuous, real-valued function for $t \geq 0$, with $\alpha(0) = 0$. The gamma process with shape function $\alpha(t) > 0$ and scale parameter $\beta > 0$, denoted by $Ga(\alpha(t), \beta)$, is a continuous-time stochastic process $\{X(t); t \geq 0\}$ with the following properties:
(1) $X(0) = 0$ with probability one;
(2) $X(s) - X(t) \sim Ga(\alpha(s) - \alpha(t), \beta)$, for all $s > t \geq 0$.
(3) $X(t)$ has independent increments, i.e., $X(s_2) - X(s_1)$ is independent of $X(t_2) - X(t_1)$, for $t_1 < t_2 < s_1 < s_2$.
The probability density function (pdf) of $X(t)$ is given by:

$$g(x; \alpha(t), \beta) = \frac{x^{\alpha(t)-1} e^{-\frac{x}{\beta}}}{\beta^{\alpha(t)} \Gamma(\alpha(t))}, \quad x > 0, \tag{1}$$

where $\Gamma(\alpha) = \int_0^\infty y^{\alpha-1} e^{-y} dy$ stands for the complete gamma function.
If degradation of a system is monitored during time, failure time T may be defined as the first instant the degradation reaches the pre-specified threshold level L. If the degradation of the system during time follows a gamma process with the pdf (1), the cumulative distribution function (cdf) of the system failure time is

$$\begin{aligned} Pr(T \leq t) &= Pr(X(t) \geq L) \\ &= \int_L^\infty g(x; \alpha(t)) dx \\ &= \frac{\Gamma(\alpha(t), \frac{L}{\beta})}{\Gamma(\alpha(t))} := \overline{G}(L; \alpha(t), \beta), \end{aligned} \tag{2}$$

where $\Gamma(\alpha, x) = \int_x^\infty t^{\alpha-1} e^{-t} dt$ is the incomplete gamma function for $x > 0$ and $\alpha > 0$.

2.2. Uncertain variables

The uncertainty theory as a branch of mathematics was founded by Liu [7] and refined by Liu [8]. Let Θ be a nonempty set, and $\mathcal{L}$ a σ-algebra over Θ. Each element Λ in $\mathcal{L}$ is called an event. A set function $\mathcal{M} : \mathcal{L} \to [0, 1]$ is called an uncertain measure if it satisfies the following axioms (Liu [7]):

(1) (Normality axiom) $\mathcal{M}\{\Theta\} = 1$ for the universal set Θ.
(2) (Duality axiom) $\mathcal{M}\{\Lambda\} + \mathcal{M}\{\Lambda^c\} = 1$ for any event Λ.
(3) (Subadditivity axiom) For every countable sequence of events $\Lambda_1, \Lambda_2, \ldots,$

$$\mathcal{M}\{\bigcup_{i=1}^{\infty} \Lambda_i\} \leq \sum_{i=1}^{\infty} \mathcal{M}\{\Lambda_i\}.$$

The triplet $(\Theta, \mathcal{L}, \mathcal{M})$ is called an uncertainty space.
(4) (Product axiom) Let $(\Theta_j, \mathcal{L}_j, \mathcal{M}_j)$ be uncertainty spaces for $j = 1, 2, \ldots$. The product uncertain measure $\mathcal{M}$ is also an uncertain measure satisfying

$$\mathcal{M}\{\prod_{j=1}^{\infty} \Lambda_j\} = \bigwedge_{j=1}^{\infty} \mathcal{M}_j\{\Lambda_j\}.$$

where Λ_j is an arbitrarily chosen event from $\mathcal{L}_j$, for $j = 1, 2, \ldots,$ and $\bigwedge_{j=1}^{\infty}$ is the minimum operator.

Definition 1
(Liu [7]) An uncertain variable η is a function from an uncertainty space $(\Theta, \mathcal{L}, \mathcal{M})$ to the set of real numbers such that $\{\eta \in B\}$ is an event for any Borel set B of real numbers. Further, the uncertainty distribution Ψ of an uncertain variable η is defined by

$$\Psi_\eta(x) = \mathcal{M}(\eta \leq x).$$

for any real number x.

Definition 2
(Liu [8]) The expected value of an uncertain variable η with regular uncertainty distribution Ψ is defined by

$$E[\eta] = \int_0^1 \Psi^{-1}(\alpha) d\alpha. \tag{3}$$

where $\Psi^{-1}(\alpha)$ is the inverse uncertainty distribution of η.

2.3. Chance space

Uncertainty and randomness are two basic types of indeterminacy. Chance theory was pioneered by Liu [9] for modeling complex systems with not only uncertainty but also randomness. This subsection will recall from Liu [9] the concepts of chance measure, uncertain random variable, chance distribution, and expected value.

Let $(\Theta, \mathcal{L}, \mathcal{M})$ be an uncertainty space, and let $(\Omega, \mathcal{A}, Pr)$ be a probability space. Then, the product space

$$(\Theta, \mathcal{L}, \mathcal{M}) \times (\Omega, \mathcal{A}, Pr) = (\Theta \times \Omega, \mathcal{L} \times \mathcal{A}, \mathcal{M} \times Pr)$$

is a chance space, where $\Theta \times \Omega$ is the set of all ordered pairs of the form (γ, ω), where $\gamma \in \Theta$ and $\omega \in \Omega$, $\mathcal{L} \times \mathcal{A}$ is the product σ-algebra, and $\mathcal{M} \times Pr$ is the product measure.

Definition 3
The chance measure of an event $\Xi \in \mathcal{L} \times \mathcal{A}$ is defined as

$$Ch\{\Xi\} = \int_0^1 Pr\{\omega \in \Omega | \mathcal{M}\{\gamma \in \Theta | (\gamma, \omega) \in \Xi\} \geq x\} dx.$$

satisfying normality, duality and monotonicity properties, i.e.,
(1) $Ch\{\Theta \times \Omega\} = 1,$
(2) $Ch\{\Xi\} + Ch\{\Xi^c\} = 1$ for any event Ξ,
(3) $Ch\{\Xi_1\} \leq Ch\{\Xi_2\}$ for any events $\Xi_1 \subset \Xi_2$,
respectively.

Definition 4
An uncertain random variable ξ is a measurable function from a chance space $(\Theta, \mathcal{L}, \mathcal{M}) \times (\Omega, \mathcal{A}, Pr)$ to the set of real numbers such that $\xi \in B$ is an event in $\mathcal{L} \times \mathcal{A}$ for any Borel set B.

Theorem 1
Let ξ be an uncertain random variable on the chance space $(\Theta, \mathcal{L}, \mathcal{M}) \times (\Omega, \mathcal{A}, Pr)$, and let B be a Borel set. Then, $\{\xi \in B\}$ is an uncertain random event with chance measure

$$Ch\{\xi \in B\} = \int_0^1 Pr\{\omega \in \Omega | M\{\gamma \in \Theta | \xi(\gamma, \omega) \in B\} \geq x\} dx.$$

Definition 5
The chance distribution and the expected value of an uncertain random variable ξ are defined by

$$\Phi_\xi(x) = Ch\{\xi \leq x\}, \quad x \in \Re$$

and

$$E[\xi] = \int_0^{+\infty} Ch\{\xi \geq x\} dx - \int_{-\infty}^0 Ch\{\xi \leq x\} dx,$$

respectively, provided that at least one of two integrals is finite.

Liu [10] showed that if X is a random variable defined on the probability space $(\Omega, \mathcal{A}, Pr)$ and η is an uncertain variable defined on the uncertainty space $(\Theta, \mathcal{L}, \mathcal{M})$, then, the summation $X + \eta$ is an uncertain random variable. Further, for any real numbers a and b, we have

$$E[aX + b\eta] = aE[X] + bE[\eta]. \tag{4}$$

In the next section, the maintenance policy used in this paper is first described in details. Then, the expected cost rate function is constructed.

3. Maintenance policy

Consider a deteriorating system that starts operating at time zero without any degradation and is subjected to changes in random evolving environments. The system is inspected at period times $\tau, 2\tau, 3\tau, \ldots$, where τ is the inspection time interval. If the degradation at the kth inspection time is less than M $(i.e., X(k\tau) < M)$, no action is taken. If it is realized that the degradation level exceeds a threshold M but not reaches L, where $M < L$, i.e., $M \leq X(k\tau) < L$ a PM is performed at time $k\tau$. However, the system is considered to be failed if the amount of its degradation reaches or exceeds the threshold level L. Suppose that N represents the number of inspections until the system fails. That is,

$$N = \inf\{k; X(k\tau) \geq L\}. \tag{5}$$

At the failure time $T = N\tau$, the system is replaced by a new one, that is, a CM is performed, and a cycle of maintenance is completed. A summary of the proposed maintenance policy is presented in Figure 1 that shows that the system started working at time zero with no degradation level. In Figure1(a), the degradation of the system takes place between the M and L levels at time $j\tau$. So, a PM is performed at this time. Figure1(b) shows that the degradation of the system reaches zero at time $j\tau$. Then, at time $k\tau$, it is realized that the degradation of the system is again placed between M and L, and another PM repair is done for the system. So, the degradation level reaches zero again at time $k\tau$, which is seen in Figure1(c). Further, it is observed that the degradation level of the system is lower than M before the inspection time $(N-1)\tau$, while it is suddenly exceeds L at the next inspection time $N\tau$; hence, a CM is done at this time.

To determine the objective expected cost rate, let us denote the costs of each inspection and CM by c_I and c_F, respectively. The PM cost of a system somehow depends on the repairmen, so we assume that the PM repair

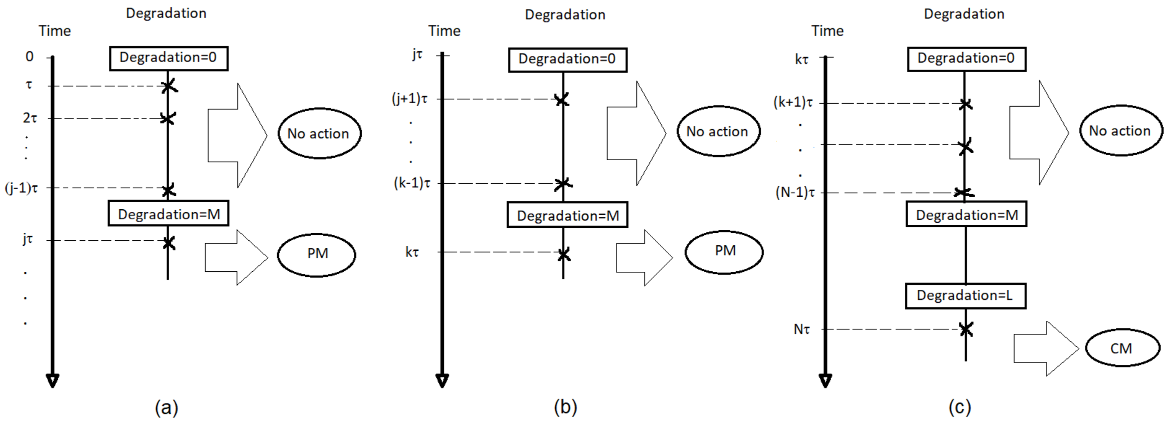

Figure 1. A summary of proposed maintenance policy.

cost is an uncertain variable. Let us denote the number of PMs occurred in one cycle by N_P, where the ith $(i = 1, 2, \cdots, N_P)$ one costs η_i. Also, assume η_i's are iid uncertain variables. It is reasonable to assume that $c_I < \eta_i < c_F$.

Note that in a cycle of operation, one CM, N inspections and N_p PMs each with costs c_F, c_I and $\eta_i (1 \leq i \leq N_P)$, respectively, are performed. Therefore, the total cost in one cycle of the proposed maintenance policy is

$$C(T) = Nc_I + \sum_{i=1}^{N_P} \eta_i + c_F, \tag{6}$$

that is obviously a combination of two random variables (N and N_P) and some uncertain variables $(\eta_1, \cdots, \eta_{N_P})$. So, $C(T)$ is an uncertain random variable.

In our policy, it is of interest to find the optimal τ that minimizes the long-run expected cost per time unit. By elementary renewal theory, it is known that our objective function is equal to the ratio of the expected cost on only one renewal cycle, over the expected length of that cycle. Therefore, the expected cost rate is

$$ECR(\tau) = \frac{E(C(T))}{E(T)} = \frac{E(C(T))}{\tau E(N)}, \tag{7}$$

where $C(T)$ is as defined in (6). Using (4), we have

$$\begin{aligned} E(C(T)) &= E\left(Nc_I + \sum_{i=1}^{N_P} \eta_i + c_F\right) \\ &= c_I E(N) + E\left(\sum_{i=1}^{N_P} \eta_i\right) + c_F. \end{aligned} \tag{8}$$

Moreover, from Theorem 5 of Yao and Zhou [25], we get

$$E\left(\sum_{i=1}^{N_P} \eta_i\right) = E(N_P)E(\eta_1). \tag{9}$$

So, by substituting (9) in (8), we have

$$E(C(T)) = c_I E(N) + E(N_P)E(\eta_1) + c_F.$$

Therefore, the expected cost rate (7) can be rewritten as follows

$$ECR(\tau) = \frac{c_I E(N) + E(N_P)E(\eta_1) + c_F}{\tau E(N)}. \tag{10}$$

Now, we need to calculate the expected values of N, N_P and η_1. If the uncertainty distribution of η_1 is known, its expected value is easily calculated. However, the uncertainty distribution is usually unknown and needs to be estimated. Toward this end, we use the Delphi method, where the procedure is illustrated in Section 5. The expected values of N and N_P are given in Lemmas 1 and 2, respectively. Eventually, the optimal inspection time interval τ^* is derived by minimizing the expected cost rate (10). The proposed procedure is investigated on a real data set in Section 6.

Lemma 1
Assume the degradation follows a gamma process. The expected number of total inspections in a cycle is given by

$$E(N) \quad = \quad \Sigma_{n=1}^{\infty} n \int_0^L \overline{G}(L-x; \Delta\alpha(n\tau), \beta) g(x; \alpha((n-1)\tau), \beta) dx,$$

where $g(\cdot)$ and $\overline{G}(\cdot)$ are as defined in (1) and (2), respectively. Also, $\Delta\alpha(n\tau) = \alpha(n\tau) - \alpha((n-1)\tau)$.

Proof. First of all note that according to the monotone path property of gamma process, we get

$$X(\tau) \leq X(2\tau) \leq \cdots \leq X(N\tau),$$

with probability one. So, using (5) and the properties of gamma process, the probability mass function (pmf) of the random variable N at point n ($n = 1, 2, ...$), is obtained as

$$\begin{aligned}
\varphi(n) :&= Pr(N = n) \\
&= Pr\big(X((n-1)\tau) < L, X(n\tau) \geq L\big) \\
&= \int_0^L Pr\big(X(n\tau) > L | X((n-1)\tau) = x\big) g\big(x; \alpha(n-1)\tau, \beta\big) dx \\
&= \int_0^L Pr\big(X(n\tau) - X((n-1)\tau) > L - x\big) g\big(x; \alpha(n-1)\tau, \beta\big) dx \\
&= \int_0^L \bar{G}\big(L - x; \Delta\alpha(n\tau), \beta\big) g\big(x; \alpha(n-1)\tau, \beta\big) dx,
\end{aligned} \tag{11}$$

where the last equality is obtained using the fact that the increment $\Delta X(n\tau) = X(n\tau) - X((n-1)\tau)$ follows $Ga(\Delta\alpha(n\tau), \beta)$ process. So, using the formula $E(N) = \Sigma_{n=1}^{\infty} n P(N = n)$, the proof is complete. □

Remark 1
It is obvious that in the special case of $\alpha(\tau) = \alpha\tau$, the increment $\Delta X(n\tau)$ follows $Ga(\alpha\tau, \beta)$ distribution for all $n \geq 1$. That is, in this case, the increments $\Delta X(\tau), \Delta X(2\tau), \cdots$ are all identically distributed.

Lemma 2
The expected number of total PMs in a cycle is given by

$$E(N_P) \quad = \quad \Sigma_{n=1}^{\infty} \varphi(n) \bigg(\Sigma_{i=1}^{n-1} (\bar{G}(M; \alpha(i\tau), \beta) - \bar{G}(L; \alpha(i\tau), \beta)) \bigg),$$

where $\bar{G}(M; \alpha(i\tau), \beta)$ and $\varphi(n)$ are as defined in (2) and (11), respectively. Moreover, for any sequence of constant value, $\{a_i; i \geq 1\}$, we have $\sum_{i=1}^{0} a_i = 0$.

Proof. Note that N_P can be written as $N_P = \Sigma_{i=1}^{N-1} I_i$, where $I_i = 1$ if a PM is performed at the ith inspection time, and $I_i = 0$ otherwise. In addition, N is independent of $I_1, I_2, ..., I_{N-1}$. Therefore, to calculate $E(N_P)$, we get

$$\begin{aligned} E(N_P) &= E(\Sigma_{i=1}^{N-1} I_i) \\ &= E[E(\Sigma_{i=1}^{N-1} I_i|N)] \\ &= \Sigma_{n=1}^{\infty} E\big(\Sigma_{i=1}^{n-1} I_i|N=n\big)\varphi(n) \\ &= \Sigma_{n=1}^{\infty}\left(\Sigma_{i=1}^{n-1} E(I_i)\right)\varphi(n) \\ &= \Sigma_{n=1}^{\infty}\left(\Sigma_{i=1}^{n-1} Pr(M \leq X(i\tau) < L)\right)\varphi(n). \end{aligned}$$

Hence, the proof is complete. □

In the next section, we investigate some ordering properties of two systems with different PM costs, and various number of inspections and PMs. The results are useful to compare the total maintenance cost of the systems.

4. Maintenance cost ordering

As stated in the previous section, the system maintenance cost is a function of random variables and some uncertain variables. So, the maintenance cost of the system is an uncertain random variable. Comparing and ordering of uncertain random variables give a guideline to make decisions in uncertain random environments. Therefore, we compare the maintenance cost of two different systems in this section. Consider system I with PM costs η_i's, the number of inspections N and number of PMs N_p, and take into account system II with similar variables η'_i, N' and N'_p, respectively. Our goal is to compare the chance order of the cost function (6) for two systems I and II that are given by

$$C(T) = Nc_I + \sum_{i=1}^{N_P} \eta_i + c_F \text{ and } C'(T) = N'c_I + \sum_{i=1}^{N'_P} \eta'_i + c_F,$$

respectively. For this purpose, we first recall the following definitions and theorem from [3, 13, 21].

Definition 6
The random variable τ_1 is said to be stochastically smaller than the random variable τ_2 in random dominance, denoted by $\tau_1 \preceq_{st} \tau_2$, if

$$Pr(\tau_1 > t) \leq Pr(\tau_2 > t) \text{ for any } t.$$

Definition 7
The uncertain variable η_1 is said to be smaller than the uncertain variable η_2 in uncertain dominance, denoted by $\eta_1 \preceq_{un} \eta_2$, if

$$\mathcal{M}(\eta_1 > t) \leq \mathcal{M}(\eta_2 > t) \text{ for any } t. \tag{12}$$

Definition 8
The uncertain random variable ξ_1 is said to be smaller than the uncertain random variable ξ_2 in chance ordering, denoted by $\xi_1 \preceq_{ch} \xi_2$, if

$$Ch(\xi_1 > t) \leq Ch(\xi_2 > t) \text{ for any } t.$$

Theorem 2
Let $\tau_1, \cdots, \tau_m$ and $\tau'_1, \cdots, \tau'_m$ be two sets of independent random variables, and let $\eta_1, \cdots, \eta_n$ and $\eta'_1, \cdots, \eta'_n$ be two sets of independent uncertain variables. If $\tau_i \preceq_{st} \tau'_i$ $(i = 1, 2, \cdots m)$ and $\eta_j \preceq_{un} \eta'_j$ $(j = 1, 2, \cdots, n)$, then

$$f(\tau_1, \cdots, \tau_m, \eta_1, \cdots \eta_n) \preceq_{ch} f(\tau'_1, \cdots, \tau'_m, \eta'_1, \cdots \eta'_n),$$

where $f : \mathcal{R}^{m+n} \longrightarrow \mathcal{R}$ is strictly monotone function component-wise.

Now, we compare the total cost of the proposed systems in the next theorem.

Theorem 3
If $N \preceq_{st} N'$, $N_p \preceq_{st} N'_p$, and $\eta_i \preceq_{un} \eta'_i$ $(i = 1, 2, \dots)$, then

$$C(T) \preceq_{ch} C'(T).$$

Proof. Note that $\sum_{i=1}^{N_P} \eta_i$ and $\sum_{i=1}^{N_P} \eta'_i$ are uncertain random variables. Hence, using Theorem 2 and that $N_p \preceq_{st} N'_p$, we get

$$\sum_{i=1}^{N_P} \eta_i \preceq_{ch} \sum_{i=1}^{N_P} \eta'_i \preceq_{ch} \sum_{i=1}^{N'_P} \eta'_i. \tag{13}$$

Let us denote by $\Phi_\xi(\cdot)$ the chance distribution function of any uncertain random variable ξ. Also, denote the cdf of the random variable N by $\Upsilon_N(\cdot)$. For the chance distribution function of the uncertain random variables $C(T)$, we have

$$\begin{aligned}
\Phi_{C(T)}(t) &= Ch(C(T) \leq t) \\
&= Ch(Nc_I + \sum_{i=1}^{N_P} \eta_i + c_F \leq t) \\
&= \int_{-\infty}^{+\infty} Ch(\sum_{i=1}^{N_P} \eta_i \leq t - c_F - yc_I \mid N = y) d\Upsilon_N(y) \\
&= \int_{-\infty}^{+\infty} \Phi_{\sum_{i=1}^{N_P} \eta_i}(t - c_F - yc_I) d\Upsilon_N(y),
\end{aligned}$$

where the last equality is deduced from the independence of N and N_P. Using (13) and Definition 8, $\Phi_{\sum_{i=1}^{N_P} \eta_i}(\cdot) \geq \Phi_{\sum_{i=1}^{N'_P} \eta'_i}(\cdot)$. Therefore,

$$\Phi_{C(T)}(t) \geq \int_{-\infty}^{+\infty} \Phi_{\sum_{i=1}^{N'_P} \eta'_i}(t - c_F - yc_I) d\Upsilon_N(y).$$

On the other hand, from $N \preceq_{st} N'$ and using Definition 6, we get $\Upsilon_N(\cdot) \geq \Upsilon_{N'}(\cdot)$. Hence,

$$\begin{aligned}
\Phi_{C(T)}(t) &\geq \int_{-\infty}^{+\infty} \Phi_{\sum_{i=1}^{N'_P} \eta'_i}(t - c_F - yc_I) d\Upsilon_{N'}(y) \\
&= \Phi_{C'(T)}(t) \text{ for any } t.
\end{aligned}$$

So, from Definition 8, the proof is complete. □

Now, let $X(t)$ and $X'(t)$ show the degradation of systems I and II at time t, which follow $\Gamma(\alpha\tau, \beta)$ and $\Gamma(\alpha'\tau, \beta)$ processes, respectively. Since the number of PMs in the proposed maintenance policy is less than the number of inspections, it is reasonable to assume that the conditional distribution of N_P given $N = n$, is a truncated poisson process with rate λ such that $N_P \leq n - 1$. Similarly, assume that N'_P given $N' = n$ obeys a truncated poisson process with rate λ', such that $N'_P \leq n - 1$. Further, let η_i and η'_i be independent uncertain variables with linear uncertainty distribution functions $\mathcal{L}(0, 1)$ and $\mathcal{L}(2, 3)$, respectively, such that

$$\Psi_\eta(x) = \begin{cases} 0, & x < 0, \\ x, & 0 \leq x < 1, \\ 1, & x \geq 1, \end{cases} \quad \text{and} \quad \Psi_{\eta'}(x) = \begin{cases} 0, & x < 2, \\ x - 2, & 2 \leq x < 3, \\ 1, & x \geq 3. \end{cases}$$

Therefore,

$$\Psi_\eta(x) - \Psi_{\eta'}(x) = \begin{cases} 0, & x < 0, \\ x, & 0 \leq x < 1, \\ 1, & 1 \leq x < 2, \\ 3 - x, & 2 \leq x < 3, \\ 0, & x \geq 3, \end{cases}$$

It is obvious that $\Psi_\eta(x) - \Psi_{\eta'}(x) \geq 0$ for any real number x. Hence, using Definition 7, $\eta \preceq_{un} \eta'$.

In the following theorem, we express some sufficient conditions on the basis of the degradation parameter and poisson rate of PM actions to compare the total costs of the two proposed systems.

Theorem 4
If $\alpha' < \alpha$ and $\lambda < \lambda'$, then

$$C(T) \preceq_{ch} C'(T).$$

Proof. Using (11), for all positive integer values of y, we get

$$\begin{aligned} S(y) &= \Upsilon_N(y) - \Upsilon_{N'}(y) \\ &= \sum_{n=1}^{y} \int_0^L \Big[\bar{G}\big(L-x;\alpha\tau,\beta\big) g\big(x;\alpha(n-1)\tau,\beta\big) - \bar{G}\big(L-x;\alpha'\tau,\beta\big) g\big(x;\alpha'(n-1)\tau,\beta\big) \Big] dx, \end{aligned}$$

where $\Upsilon_N(\cdot)$ stands for the cdf of N. For other values of y, $S(y) = 0$. Using numerical methods, it can be shown that when $\alpha' < \alpha$, $S(y) > 0$ for all possible values of y and given value of β. Therefore, according to Definition 6, we get $N \preceq_{st} N'$. In addition, the cdf of N_p may be derived as

$$\begin{aligned} \Upsilon_{N_p}(y) &= \sum_{x=0}^{y} Pr(N_p = x) \\ &= \sum_{x=0}^{y} \sum_{n=1}^{\infty} Pr\left(N_P = x \mid N = n\right) \varphi(n) \\ &= \sum_{x=1}^{y} \sum_{n=x+1}^{\infty} \frac{(\lambda(n-1)\tau)^x e^{-\lambda(n-1)\tau}/x!}{\sum_{i=0}^{n-1} (\lambda(n-1)\tau)^i e^{-\lambda(n-1)\tau}/i!} \varphi(n), \end{aligned}$$

where $\varphi(n)$ is as derived in (11). Similarly, the cdf of N'_P may be obtained. It is not difficult to show that for given β,

$$\Upsilon_{N_p}(y) \geq \Upsilon_{N'_p}(y), \quad \text{for all } y,$$

as long as $\alpha' < \alpha$ and $\lambda < \lambda'$. This means that $N_p \preceq_{st} N'_p$. Since all sufficient conditions of Theorem 3 are hold, it is concluded that $C(T) \preceq_{ch} C'(T)$. Hence, the proof is complete. □

Using the approach presented in this section, two different systems with individual PM costs and degradation processes may be compared based on total maintenance costs. But, in order to find the optimal maintenance policy, we need to know the uncertainty distribution of PM costs of the proposed policy. To do this in mind, the process of determining the uncertainty distribution is explained in the next section.

5. Uncertainty distribution of PM cost

Since the cost of PM actions are uncertain variable, the process of determining the uncertainty distribution based on the opinions and expertise of experts in the relevant field is stated in this section. A questionnaire is designed to collect experimental data from experts. For each expert, the distribution of experimental uncertainty is first determined (see, Subsection 5.1). Then, by using the Delphi method, the combined uncertainty distribution is obtained according to the opinions of all experts (the details are presented in Subsection 5.2). Finally, an example is presented in Subsection 5.3 to illustrate the computational process of obtaining uncertainty distribution of PM cost.

5.1. Expert's experimental data

In order to obtain expert experimental data through a questionnaire, one or more experts (for example, system repairmen) should be asked to answer questions about the PM cost individually. Toward this end, we ask an expert about the possible value x of the PM cost η, and the degree of their belief that η is less than x. Denoting the expert's degree of belief by α, an expert's experimental datum is (x, α). More precisely, we first ask an expert how much do you think the minimum cost of each PM action is. Suppose the answer we receive from the expert is x_1, then we get the expert's experimental datum $(x_1, 0)$. Then, we ask what do you think the maximum cost of each PM action is. When the answer we receive is x_n, the experimental datum is $(x_n, 1)$. To get the intermediate cost data and belief values, we ask the following question. What is the possible cost of a PM action in your opinion? If the answer is x_i, we would ask the expert about the degree of their belief that the real cost is less than x_i. Assuming that the answer is α_i, the experimental datum is (x_i, α_i). By repeating improved questions n times, experimental data for an expert are obtained as

$$(x_1, \alpha_1), (x_2, \alpha_2), \dots, (x_n, \alpha_n). \tag{14}$$

Assuming the lowest degree of belief is 0 and the highest degree of belief is 1, the rest of the values are arranged in ascending order between 0 and 1. For more details, interested readers may refer to Chapter 4 of Liu [7]. That is, the experimental data in (14) satisfy the conditions $x_1 < x_2 < \cdots < x_n$, and $0 \le \alpha_1 \le \alpha_2 \le \cdots \le \alpha_n \le 1$. Liu [8] presented the experimental uncertainty distribution as

$$\Psi(x) = \begin{cases} 0, & x \le x_1, \\ \alpha_i + \frac{(\alpha_{i+1}-\alpha_i)(x-x_i)}{(x_{i+1}-x_i)}, & x_i \le x \le x_{i+1}, 1 \le i < n, \\ 1, & x \ge x_n. \end{cases} \tag{15}$$

Hence, using (3), the expected value of η is

$$E(\eta) \quad = \quad \frac{\alpha_1 + \alpha_2}{2} x_1 + \Sigma_{i=2}^{n} \frac{\alpha_{i+1} - \alpha_{i-1}}{2} x_i + (1 - \frac{\alpha_{n-1} + \alpha_n}{2}) x_n.$$

If there are M experts, it is sensible to have M experimental uncertainty distributions $\Psi_1(x), \dots, \Psi_M(x)$. So, it is of interest to estimate the uncertainty distribution of PM costs according to the opinion of all experts. This is done by combining uncertain statistics and the Delphi method, which emphasizes the use of the group experience of experts.

5.2. Delphi method

The Delphi method is a structured communication technique, originally developed as a systematic, interactive forecasting method that relies on a panel of experts. The work steps of the Delphi method are such that a researcher first selects a group of experts who have sufficient knowledge and experience on the subject under investigation. In the next step, the researcher explains the problem he wants to solve comprehensively and accurately to the experts. Then, according to Algorithm 1 below, experts' opinions are asked in several rounds to reach a consensus. Here, we perform the Delphi method to obtain the uncertainty distribution $\Psi(x)$ of PM cos η.

Algorithm 1.

Step 1. The M domain experts provide their experimental data based on the first round of the questionnaire,

$$(x_{ij}, \alpha_{ij}), \quad i = 1, \dots, M, \ j = 1, 2, \dots, n_i.$$

where x_{ij} stands for the jth possible value provided by the ith expert, and α_{ij} denotes the ith expert's belief degree that η is less then x_{ij}.

Step 2. For $i = 1, \dots, M$, use the ith expert's experimental data $(x_{i1}, \alpha_{i1}), (x_{i2}, \alpha_{i2}), \dots, (x_{in_i}, \alpha_{in_i})$ to obtain the uncertainty distributions Ψ_i.

Step 3. Denote by n the number of all various values of PM cost represented by all experts. Precisely, denote by $x_1 < x_2 < \cdots < x_n$ the sorted union of all PM costs stated by the experts. That is, $\bigcup_{i=1}^{M} \bigcup_{k=1}^{n_i} \{x_{ik}\} =$

Table 1. Experts' experimental data about PM cost

Expert 1	(40,0.40)	(50,0.60)	(60,0.85)	(70,0.95)	(80,1)
Expert 2	(40,0.30)	(50,0.65)	(60,0.80)	(70,0.90)	(80,1)
Expert 3	(45,0.40)	(55,0.50)	(65,0.65)	(75,0.95)	(80,0.975)
Expert 4	(45,0.45)	(50,0.65)	(60,0.80)	(70,0.85)	(80,1)
Expert 5	(50,0.30)	(60,0.70)	(70,0.85)	(80,0.95)	
Expert 6	(45,0.25)	(55,0.60)	(65,0.80)	(75,0.90)	(80,0.95)

Table 2. Values of $\Psi_i(x)$ calculated for various values of x

	x								
	40	45	50	55	60	65	70	75	80
$\Psi_1(x)$	0.4	0.5	0.60	0.725	0.85	0.90	0.95	0.975	1
$\Psi_2(x)$	0.3	0.475	0.65	0.725	0.80	0.85	0.90	0.95	1
$\Psi_3(x)$	0	0.40	0.45	0.50	0.575	0.65	0.80	0.95	0.975
$\Psi_4(x)$	0	0.45	0.65	0.725	0.80	0.825	0.85	0.925	1
$\Psi_5(x)$	0	0	0.30	0.50	0.70	0.775	0.85	0.90	0.95
$\Psi_6(x)$	0	0.25	0.425	0.60	0.70	0.80	0.85	0.90	0.95

$\{x_1, \dots, x_n\}$. For $j = 1, 2, \dots, n$, compute

$$\overline{\alpha}_j = \frac{1}{M}\sum_{i=1}^{M} \Psi_i(x_j), \tag{16}$$

and

$$d_j = \frac{1}{M}\sum_{i=1}^{M} (\Psi_i(x_j) - \overline{\alpha}_j)^2. \tag{17}$$

Step 4. If d_j is less than a given level $\varepsilon > 0$ for all j, then go to Step 5. Otherwise, all domain experts receive the summary $\overline{\alpha}_j$ obtained in Step 3 in addition to the reasons of other experts for their opinions. Accordingly, a set of revised experimental data are provided by going to Step 2.

Step 5. Use the integrated data

$$(x_1, \overline{\alpha}_1), (x_2, \overline{\alpha}_2), \dots, (x_n, \overline{\alpha}_n)$$

to obtain the uncertainty distribution Ψ according to equation (15).

5.3. Explanatory example

Here, we determine the uncertainty distribution of PM cost with the help of experts' opinions and the Delphi method in an example. Toward this end, assume that six experts are invited to analyse the PM cost in a completely hypothetical way. Each expert estimates the cost of PM repair as well as degree of belief based on her/his own knowledge and expertise. Following the steps described in Algorithm 1, we consider $\varepsilon = 0.03$. We also assume the weight of all experts to be the same. The experimental data of the experts about PM cost are as reported in Table 1.

By reviewing the opinions of all experts presented in Table 1, the various estimated PM cost by the experts are 40, 45, 50, 55, 60, 65, 70, 75 and 80. Based on these data, six experimental uncertainty distributions $\Psi_i(x)$, $1 \le i \le 6$, may derived according to equation (15). The results are reported in Table 2.

According to the Step 3 of Algorithm 1 and with the help of (16) and (17), we calculate the values of $\overline{\alpha}_j$ and d_j. The results are presented in Table 3.

By checking the values of $d_1, \dots, d_9$ in Table 3 and comparing them with $\varepsilon = 0.03$, we see that $d_2 > 0.03$. So, we should provide the feedback of the first round of opinions to the experts so that each expert can re-estimate

Table 3. Values of $\overline{\alpha}_j$ and d_j.

j	1	2	3	4	5	6	7	8	9
x_j	40	45	50	55	60	65	70	75	80
$\overline{\alpha}_j$	0.1167	0.3458	0.5125	0.6292	0.7375	0.8000	0.8667	0.9333	0.9792
d_j	0.0281	0.0305	0.0170	0.0103	0.0083	0.0060	0.0022	0.0007	0.0005

Table 4. Revised experimental data about PM cost.

Expert1	(40,0.40)	(50,0.60)	(60,0.85)	(70,0.95)	(80,1)				
Expert2	(40,0.30)	(50,0.65)	(60,0.80)	(70,0.90)	(80,1)				
Expert3	(40,0.25)	(45,0.40)	(55,0.50)	(65,0.65)	(75,0.95)	(80,0.975)			
Expert4	(40,0.30)	(45,0.50)	(50,0.65)	(60,0.80)	(70,0.85)	(80,1)			
Expert5	(40,0.15)	(45,0.30)	(50,0.45)	(55,0.55)	(60,0.70)	(70,0.85)	(80,0.95)		
Expert6	(40,0.15)	(45,0.30)	(50,0.5)	(55,0.60)	(60,0.70)	(65,0.80)	(70,0.85)	(75,0.90)	(80,0.95)

Table 5. Value of revised $\Psi_i(x)$ calculated for various values of x.

	40	45	50	55	60	65	70	75	80
$\Psi_1(x)$	0.4	0.5	0.60	0.725	0.85	0.90	0.95	0.975	1
$\Psi_2(x)$	0.3	0.475	0.65	0.725	0.80	0.85	0.90	0.95	1
$\Psi_3(x)$	0.25	0.0.40	0.45	0.50	0.575	0.65	0.80	0.95	0.975
$\Psi_4(x)$	0.30	0.50	0.65	0.725	0.80	0.825	0.85	0.925	1
$\Psi_5(x)$	0.15	0.30	0.45	0.55	0.70	0.775	0.85	0.90	0.95
$\Psi_6(x)$	0.15	0.30	0.50	0.60	0.70	0.80	0.85	0.90	0.95

Table 6. Revised values of $\overline{\alpha}_j$ and d_j.

j	1	2	3	4	5	6	7	8	9
x_j	40	45	50	55	60	65	70	75	80
$\overline{\alpha}_j$	0.2583	0.4125	0.5500	0.6375	0.7375	0.8000	0.8667	0.9333	0.9792
d_j	0.0078	0.0074	0.0075	0.0085	0.0083	0.0060	0.0022	0.0007	0.0005

the PM cost and her/his degree of belief based on the opinions of other experts. The revised experimental data of experts about PM cost are reported in Table 4.

Similarly, from the data presented in Table 4, six revised experimental uncertainty distributions may derived according to equation (15). By reviewing the opinions of all experts, all the revised estimated PM cost by 6 experts are 40, 45, 50, 55, 60, 65, 70, 75 and 80. The values of revised $\Psi_i(x)$ are reported in Table 5.

Similarly, we calculate the revised values of $\overline{\alpha}_j$ and d_j. The results are presented in Table 6. From Table 6, it is observed that $d_j < \varepsilon = 0.03$, for $j = 1, \ldots, 6$, so, and we no longer need to provide feedback on the opinions to the experts. Hence, the integrated experts' experimental data $(x_i, \overline{\alpha}_i), i = 1, \ldots, n$, are obtained as (40,0.2583), (45,0.4125), (50,0.5500), (55,0.6375), (60,0.7375), (65,0.800), (70,0.8667), (75,0.9333) and (80,0.9792).

Based on these data, the experimental uncertainty distribution of PM cost is obtained as follows:

$$\Psi(x) = \begin{cases} 0, & x \leq 40, \\ 0.0308x - 0.9753, & 40 \leq x \leq 45, \\ 0.0275x - 0.825, & 45 \leq x \leq 50, \\ 0.0155x - 0.225, & 50 \leq x \leq 55, \\ 0.02x - 0.4725, & 55 \leq x \leq 60, \\ 0.0125x - 0.0125, & 60 \leq x \leq 65, \\ 0.0133x - 0.0671, & 65 \leq x \leq 70, \\ 0.0133x - 0.0657, & 70 \leq x \leq 75, \\ 0.0092x + 0.2448, & 75 \leq x \leq 80, \\ 1, & x \geq 80. \end{cases} \tag{18}$$

Now, the value of $E(\eta)$ may be obtained according to equation (16). Using the empirical distribution (18), we get $E(\eta) = 52.21875$. Our goal is to obtain the optimal inspection time by minimizing the cost function (10). In this function, the value of $E(\eta)$ is unknown, and since η is an uncertain variable, its distribution was estimated according to the Delphi method based on experts' opinions. By placing the calculated value of $E(\eta)$ in equation (10), we can minimize this function with respect to τ and get the optimal value τ^*. Using this value, the optimal inspection time and minimum cost may be calculated for any real example.

6. Application to a real data set

Here, we use the fatigue crack growth data to investigate the proposed method in this paper. This data set contains 21 sample paths that was previously used by some authors such as Hudak et al. [5] and Lu and Meeker [12]. These data have been measured at predetermined inspection times $t_0 = 0, t_1 = 0.01, ..., t_{12} = 0.12$ million cycles that are considered as degradation data. Let us denote the degradations by $D_i(t_j), i = 1, ..., 21, j = 0, 1, ..., 12$. Here, we consider the data up to inspection time $t_9 = 0.09$. It was observed that the initial crack length at time $t_0 = 0$ is equal to 0.9. Since the degradation have to be started from zero, we subtract the degradation data from 0.9. Hence, we get, $X_i(t_j) = D_i(t_j) - 0.9, i = 1, ..., 21, j = 0, 1, ..., 9$. Nezakati and Razmkhah [14] showed that each sample of these data follows a gamma process. It is obvious that by averaging all paths, the degradation process $X(t_j) = \frac{1}{21}\sum_{i=1}^{21} X_i(t_j)$ still follows a gamma process. The observed values of $X(t_j)$ at times $t_0 < t_2 < ... < t_9$ are

$$0, 0.0295, 0.0652, 0.1014, 0.1405, 0.1824, 0.2314, 0.2814, 0.3500, 0.4238, \tag{19}$$

respectively. Figure 2 shows the plot of $X(t_j)$ versus time. From this figure, one of the linear, exponential or power functions seems to be appropriate to model the average degradation. Therefore, we consider a Gamma process for $X(t_j)$ with scale parameter β and set the shape parameter $\alpha(t)$ as one of different forms of αt, $e^{\alpha t}$ and t^{α}.

To check the exact behavior of degradation data, in the sequel, the maximum likelihood estimates (MLEs) of the unknown parameters α and β are studied. Since, the gamma increments $\Delta(t_j) = X(t_j) - X(t_{j-1})$, $j \geq 1$, have gamma distributions, the likelihood function of the parameters α and β is

$$Lik(\alpha, \beta) = \prod_{i=1}^{n_i} \frac{d_i^{\alpha(t_j)-\alpha(t_{j-1})-1} e^{-\frac{d_i}{\beta}}}{\beta^{\alpha(t_j)-\alpha(t_{j-1})}\Gamma(\alpha(t_j) - \alpha(t_{j-1}))},$$

where d_i is the observed value of $\Delta(t_j)$, and $\alpha(t)$ may be one of the functions αt, t^{α} or $e^{\alpha t}$. The MLEs of the unknown parameters α and β as well as the AIC based on different forms of $\alpha(t)$ are presented in Table 7. The p-value for the null hypothesis that the data come from a $Ga(\alpha(t), \beta)$ process is also reported in this table.

According to the values of AIC and p-values presented in Table 7, a gamma process with shape parameter $\alpha(t) = \alpha t$ seems to be suitable for the data. So, we accept that the data in (19) follow a gamma process with mean $\alpha\beta t = 4.7075t$ at time t.

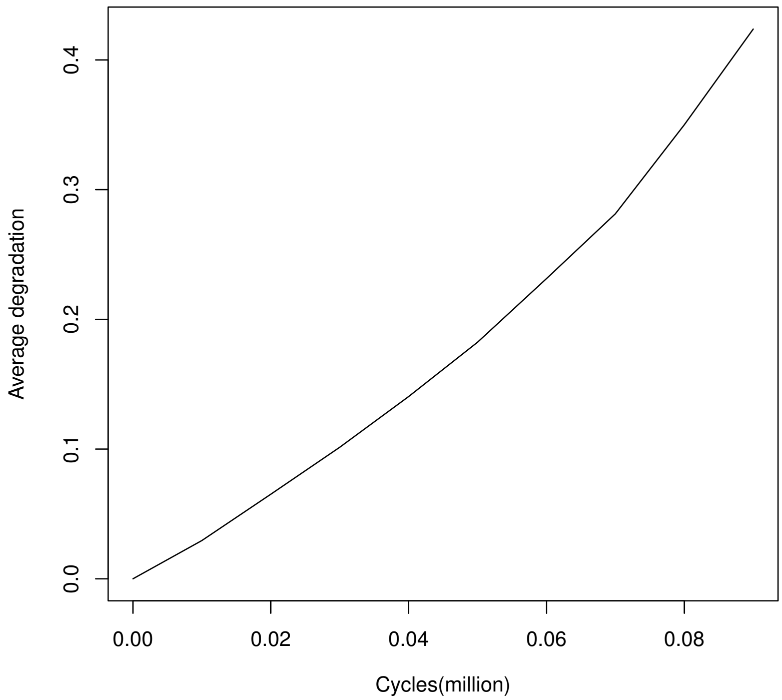

Figure 2. Average fatigue crack growth (inches).

Table 7. Values of MLE and AIC criterion.

	$\alpha(\tau)=\alpha\tau$	$\alpha(\tau)=\tau^{\alpha}$	$\alpha(\tau)=e^{\alpha t}$
$\widehat{\alpha}$	1185.77	0.3069	33.8041
$\widehat{\beta}$	0.00397	0.8875	0.0202
AIC	-322.7251	-39.0839	-12.8035
$p-value$	0.9134	1.22×10^{-5}	0.3387

To perform the maintenance policy proposed in the paper, let us assume the average fatigue crack growth data in (19) as degradation of a system. Moreover, assume that if it is realized that the crack growth reaches or exceeds $L=0.25$ in some inspection time t_j, the system will be considered as a failed system. Also, if the crack growth lies between M and L, a PM repair is done for the system. Note that the crack growth had been measured at inspection times $t_j=j\tau$, $j\geq 0$, where $\tau=0.01$. The question arises here is that what the optimal inspection period τ we would assign if we had the chance to apply the proposed maintenance policy. Toward this end, we would minimize the expected cost rate function in (10) for some choices of c_I and c_F. Moreover, the assumptions of Subsection 5.3 are considered to find the expected PM cost. The optimal inspection time and the corresponding minimum expected cost rate are reported in Table 8 for $L=0.25$, $M=0.15, 0.20$ and some choices of c_I and c_F. From this table, it is observed that:

- For fixed c_I (or c_F), when c_F (or c_I) increases, both of the optimal inspection time and the minimum expected cost rate increases for both $M=0.15$ and $M=0.20$.

Table 8. The optimal inspection time and the minimum cost for $M = 0.15, 0.2$, $L = 0.25$, $\alpha(\tau) = \alpha\tau$ and some choices of c_I and c_F.

		M=0.15		M=0.20	
c_I	c_F	τ^*	$ECR(\tau^*)$	τ^*	$ECR(\tau^*)$
5	70	0.0190	421.3892	0.0251	398.7245
	100	0.0198	426.9763	0.0263	421.7547
	120	0.0210	429.2351	0.0268	423.4387
	150	0.0243	433.2541	0.0279	430.5792
10	70	0.0192	425.8724	0.0259	410.9517
	100	0.0199	428.9341	0.0265	426.3409
	120	0.0221	430.3972	0.0271	427.7963
	150	0.0249	436.1823	0.0284	433.9743
15	70	0.0215	430.7210	0.0235	414.2863
	100	0.0223	435.3894	0.0238	431.6307
	120	0.0225	241.4571	0.0242	435.7862
	150	0.0231	453.1471	0.0256	439.8307
20	70	0.0223	430.9971	0.0237	421.1031
	100	0.0233	437.4973	0.0245	435.3947
	120	0.0235	448.7634	0.0253	441.6329
	150	0.0238	458.9132	0.0261	452.4218

- For given c_I and c_F, when the threshold level M increases, the optimal inspection interval τ^* increases, while the minimum cost rate decreases. This means that in addition to having smaller minimum expected cost rate, a larger inspection time period is required if M increases.

7. Discussion and conclusions

A maintenance policy was considered for a system subject to the gamma process. The PM cost of repairs was considered to be an uncertain variable, and its uncertainty distribution was determined based on the Delphi method. The expected cost rate criterion was applied to obtain the optimal maintenance policy. Since there were some random variables and an uncertain variable in the proposed cost function, the optimal policy was studied in a chance space. The maintenance costs of two different systems were compared in terms of chance ordering. Finally, the procedure was performed on a real data set following a gamma process. The model parameters were estimated using the maximum likelihood method. The optimal inspection interval was determined for some choices of costs and given degradation threshold. It was seen that when one of the parameters c_I, c_F or M increases while others are fixed, the value of optimal inspection interval increases. This means that the higher costs or larger PM threshold, the later inspection.

It is of worthwhile to note that during the use of the Delphi method, a series of challenges may arise in the proposed procedure in the paper. Some of these challenges which should be carefully implemented are described below:

- In practice, the right agreement from the consensus of experts' opinions is not defined.
- The right participating experts would not be chosen.
- There is no guidance and agreed standards on how to interpret and analyse the results.
- The number of rounds of Delphi method presented in Algorithm 1 may be prolonged due to distraction and fatigue of experts' opinions.

Despite existing the above-mentioned challenges in the process of determining the distribution of any uncertain variable, it is suggested to investigate any statistical or engineering problem in an uncertain or a chance space when there are no enough information or frequencies about the problem of interest. By the same reason, we preferred to study our maintenance policy in a chance space. The results of this paper may be extended to the following cases:

1. It was assumed that the degradation paths follow the gamma process. The behavior of other processes, such as inverse Gaussian process or Wiener processes may be also studied.
2. A maintenance model for a multi-component system may be extended in which the PM cost of the components are different uncertain variables.
3. A perfect repair action was done as a PM in the proposed maintenance policy in this paper. Other types of imperfect or minimal repairs may also be studied in different uncertain environments.
4. In this paper, we just assumed that the cost of PM is an uncertain variable depending on the experts' opinions, while the costs of CM and inspection were assumed to be constant. Though, one can imagine the situation that costs of PM, CM and inspections are all uncertain variables. This problem is the subject of our future research works.

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