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Modified random errors S-iterative process for stochastic fixed point theorems in a generalized convex metric space

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Abstract In this paper, we suggest the modified random S-iterative process and prove some stochastic fixed point theorems of a finite family of random uniformly quasi-Lipschitzian operators in a generalized convex metric space. Our results improves and extends various results in the literature.

Keywords Modified random S-iteration, Common random fixed point theorems, Generalized convex metric spaces, Convex structure

AMS 2010 subject classifications 47B80, 47H40

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1. Introduction

Fixed point theorems are a very performance tool of the present mathematical applications. Also, random fixed point theorems are stochastic generalizations of Banach's fixed point theorem and Banach's type fixed point theorems in complete metric spaces. Random nonlinear analysis is an important mathematical methodology which is mainly concerned with the study of random nonlinear operators and their properties and its development is required for study of wide classes of random operator equations. Random Techniques have been crucial in various areas from pure mathematics to applied sciences. The study of random fixed point theorem was first introduced by Prague school of probability in the 1950s. Later, Spacek [1] and Hans [2, 3] first proved random fixed point theorems for random contraction mappings in separable complete metric spaces. Moreover, there were many authors who have studied about random fixed point theorems and its application, for instance, in [4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21]. In the sense of another fixed point theorems, there are many mathematicians who studied common fixed point theorems in various mappings and spaces (see in [22, 23, 24, 25, 26])

On the other hand, in 1970, Takahashi [27] first suggested the knowledge of convex metric spaces and first studied the fixed point theorems for non-expansive mappings in this spaces. Later, the iterative processes for non-expansive mappings in the hyperbolic type space was studied by Kirk and Goebel, see in [28, 29]. Later, many paper of Liu [30, 31, 32] showed some sufficient and necessary conditions for two schemes of Ishikawa iterative process of asymptotically quasi-non-expansive mappings to converge to fixed point in a convex Banach space.

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Next, Tian [33] give some sufficient and necessary conditions for Ishikawa iterative process of two asymptotically quasi-non-expansive mappings which introduced by Das and Debats [34] to converge to fixed points in convex metric spaces.

Recently, in a convex metric spaces, Khan [35] suggested the iterative process for common fixed points of asymptotically quasi-non-expansive mappings $\{T_i : i \in J\}$ where $J = \{1, 2, 3, ..., k\}$ as follows:

$$\begin{cases} x_{n+1} = W(T_k^n y_{(k-1)n}, x_n; \alpha_{kn}), \\ y_{(k-1)n} = W(T_{k-1}^n y_{(k-2)n}, x_n; \alpha_{(k-1)n}), \\ y_{(k-2)n} = W(T_{k-2}^n y_{(k-3)n}, x_n; \alpha_{(k-2)n}), \\ \qquad \qquad ... \\ y_{2n} = W(T_1^n y_{1n}, x_n; \alpha_{2n}), \\ y_{1n} = W(T_1^n y_{0n}, x_n; \alpha_{1n}), \end{cases} \tag{1}$$

where $y_{0n} = x_n$ and $\{\alpha_{in}\}$ are real sequences in $[0, 1]$.

Also, Wang and Liu [36] introduced an Ishikawa type iterative process with errors to estimate the fixed point mappings of T and S which were uniformly quasi-Lipschitzian mappings in generalized convex metric spaces as follows:

$$\begin{cases} x_{n+1} = W(x_n, Sy_n, u_n, a_n, b_n, c_n), \\ y_n = W(x_n, Tx_n, v_n, a'_n, b'_n, c'_n), \end{cases} \tag{2}$$

where $\{a_n\}, \{b_n\}, \{c_n\}, \{a'_n\}, \{b'_n\}$ and $\{c'_n\}$ are real sequences in [0, 1] such that $a_n + b_n + c_n = a'_n + b'_n + c'_n = 1$ and $\{u_n\}, \{v_n\}$ are two bounded sequence.

Recently, the S-iterative process was suggested by Agarwal, O'Regan and Sahu [37] in a Banach space. They proved that this iterative process converges faster than Mann iterative process and Ishikawa iterative process.

$$\begin{cases} x_1 \in K, \\ x_{n+1} = (1 - \alpha_n)Tx_n + \alpha_n T(y_n), \\ y_n = (1 - \beta_n)x_n + \beta_n T(x_n), n \in \mathbb{N}, \end{cases} \tag{3}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are the sequences in (0, 1). Also, there were many authors who have studied about this iterative process for estimate a fixed point in various spaces and them results showed that the rate of convergence for this iterative process is much quicker than another iterative process (see e.g.,[38, 39, 40, 41, 42]).

Motivated and inspired by (1), (2) and (3), we modified the following random S-iterative process in a generalized convex metric space. Let $\{T_i : i \in J\}$ where $J = \{1, 2, 3, ..., k\}$ be a finite family of random uniformly quasi-Lipschitzian operators such that $T_i : \Omega \times K \to K$, where K is a non-empty closed convex subset of a separable generalized convex metric space (X, d) with a convex structure W (see in definition 3 and 4). Let $\zeta_1 : \Omega \to K$ be a measurable mapping, the sequence $\{\zeta_n(\omega)\}$ is generated by

$$\begin{cases} \zeta_{n+1} = W(T_k^n(\omega, \zeta_n(\omega)), T_k^n(\omega, \chi_{(k-1)n}(\omega)), v_{kn}(\omega); \alpha_{kn}, \beta_{kn}, \gamma_{kn}), \\ \chi_{(k-1)n} = W(\zeta_n(\omega), T_{k-1}^n(\omega, \chi_{(k-2)n}(\omega)), v_{(k-1)n}(\omega); \alpha_{(k-1)n}, \beta_{(k-1)n}, \gamma_{(k-1)n}), \\ \chi_{(k-2)n} = W(\zeta_n(\omega), T_{k-2}^n(\omega, \chi_{(k-3)n}(\omega)), v_{(k-2)n}(\omega); \alpha_{(k-2)n}, \beta_{(k-2)n}, \gamma_{(k-2)n}), \\ \qquad \qquad ... \\ \chi_{2n} = W(\zeta_n(\omega), T_2^n(\omega, \chi_{1n}(\omega)), v_{2n}(\omega); \alpha_{2n}, \beta_{2n}, \gamma_{2n}), \\ \chi_{1n} = W(\zeta_n(\omega), T_1^n(\omega, \chi_{0n}(\omega)), v_{1n}(\omega); \alpha_{1n}, \beta_{1n}, \gamma_{1n}), \end{cases} \tag{4}$$

where $\chi_{0n}(\omega) = \zeta_n(\omega)$, for any given $i \in J$, $\{\alpha_{in}\}, \{\beta_{in}\}, \{\gamma_{in}\}$ are real sequences in $[0,1]$ with $\alpha_{in} + \beta_{in} + \gamma_{in} = 1$ and the bounded sequence $\{v_{in}(\omega)\} : \Omega \to K$ is a sequence of measurable mappings which $\{v_{in}(\omega)\} \in K$, $\forall \omega \in \Omega$ and $\forall n \in \mathbb{N}$.

The purposes of this paper are to suggest the modified random S-iterative process and prove some stochastic fixed point theorems of a finite family of random uniformly quasi-Lipschitzian operators in a generalized convex metric space. In this paper was organized as follows. In section 2 and 3, we present preliminaries and main results, respectively.

2. Preliminaries

Overview on this paper, we denote the notation that $T^n(\omega, x)$ is the n th iteration $T(\omega, T(\omega, ...T(\omega, x)...))$ of T and the letter I denotes the random mapping $T : \Omega \times K \to K$ defined by $I(\omega, x) = x$ and $T^0 = I$. First, we present about the following definition of a random fixed point operator T.

Definition 1
[43] Let (Ω, Σ) be a measurable space with Σ be a σ-algebra of subsets of Ω, and let K be a non-empty subset of a metric space (X, d).

i) A mapping $\xi : \Omega \to X$ is measurable if $\xi^{-1}(U) \in \Sigma$ for any open subset U of X;
ii) the operator $T : \Omega \times K \to K$ is a random mapping iff for any fixed $x \in K$, $T(\cdot, x) : \Omega \to K$ is measurable and continuous if $\forall \omega \in \Omega$, $T(\omega, x) : K \to X$ is continuous;
iii) a measurable mapping $\xi : \Omega \to X$ is a random fixed point of the random operator $T : \Omega \times X \to X$ iff $T(\omega, \xi(\omega)) = \xi(\omega)$, $\forall \omega \in \Omega$.

By above definition 1, we denote the set of all random fixed points of a random operator T by $RF(T)$. Next, we present about the following definitions of operator T that using in our main results.

Definition 2
Let K be a non-empty subset of a separable metric space (X, d) and $T : \Omega \times K \to K$ be a random operator. The operator T is called

i) an asymptotically nonexpansive random operator if there exists a sequence of measurable mappings $\{k_n(\omega)\} : \Omega \to [1, \infty)$ with $\lim_{n\to\infty} k_n(\omega) = 0$ such that
$$d(T^n(\omega, x), T^n(\omega, y)) \le (1 + k_n(\omega))d(x, y)$$
for all $\omega \in \Omega$ and $x, y \in K$;
ii) a uniformly L-Lipschitzian random operator if
$$d(T^n(\omega, x), T^n(\omega, y)) \le Ld(x, y)$$
for all $\omega \in \Omega$, $x, y \in K$ and L is positive constant;
iii) an asymptotically nonexpansive random operator if there exists a sequence of measurable mappings $\{k_n(\omega)\} : \Omega \to [1, \infty)$ with $\lim_{n\to\infty} k_n(\omega) = 0$ such that
$$d(T^n(\omega, \eta(\omega)), \xi(\omega)) \le (1 + k_n(\omega))d(\eta(\omega), \xi(\omega))$$
for all $\omega \in \Omega$, where $\xi : \Omega \to K$ is a random fixed point of operator T and $\eta : \Omega \to K$ is any measurable mapping;
iv) a uniformly quasi-Lipschitzian random operator if
$$d(T^n(\omega, \eta(\omega)), \xi(\omega)) \le Ld(\eta(\omega), \xi(\omega))$$
for all $\omega \in \Omega$, where $\xi : \Omega \to K$ is a random fixed point of operator T, $\eta : \Omega \to K$ is any measurable mapping and L is positive constant;

v) an semi-compact random mapping if for any sequence of measurable mappings $\{\xi_n(\omega)\} : \Omega \to K$, with $\lim_{n\to\infty} d(T(\omega, \xi_n(\omega)), \xi_n(\omega)) = 0$, for all $\omega \in \Omega$ there exists a subsequence $\{\xi_{n_j}\}$ of $\{\xi_n\}$ which converges pointwise to ξ, where $\xi : \Omega \to K$ is a measurable mapping.

Now, we present about the following definition of a convex structure in a metric space and a generalized convex metric space.

Definition 3
[27] A convex structure in a metric space (X, d) is a mapping $W : X \times X \times [0, 1] \to X$ satisfying, for any $x, y, u \in X$ and any $\lambda \in [0, 1]$

$$d(u, W(x, y, \lambda)) \leq \lambda d(u, x) + (1 - \lambda)d(u, y).$$

A metric space together with a convex structure is said to be a convex metric space. A non-empty subset K of X is called convex if $W(x, y, \lambda) \in K$ for any $(x, y, \lambda) \in K \times K \times [0; 1]$.

Definition 4
[33, 36] Let X be a metric space, $I = [0, 1]$, $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ be real sequence in $[0, 1]$ with $\alpha_n + \beta_n + \gamma_n = 1$. A mapping $W : X^3 \times I^3 \times \to X$ is called a convex structure on X, if it satisfies the following conditions: for each $(x, y, z; \alpha_n, \beta_n, \gamma_n) \in X^3 \times I^3$ and $u \in X$,

$$d(u, W(x, y, z; \alpha_n, \beta_n, \gamma_n)) \leq \alpha_n d(u, x) + \beta_n d(u, y) + \gamma_n d(u, z).$$

A metric space together with a convex structure is called a generalized convex metric space.

In the sense of random fixed point, a non-empty subset K of X is called convex if $W(x, y, z; \alpha_n, \beta_n, \gamma_n) \in K$ for any $(x, y, z; \alpha_n, \beta_n, \gamma_n) \in X^3 \times I^3$. The mapping $W : K^3 \times I^3 \to X$ is called a random convex structure if for any measurable mappings $\xi, \eta, \zeta : \Omega \to K$ and each fixed $\alpha_n, \beta_n, \gamma_n \in [0, 1]$ with $\alpha_n + \beta_n + \gamma_n = 1$, the mapping $W(\xi(\cdot), \eta(\cdot), \zeta(\cdot)) : \Omega \to K$ is measurable.

The last, we present the following lemmas for proving the main results as follow.

Lemma 1
[44] Let X be a separable metric space and Y a metric space. If $f : \Omega \times X \to Y$ is measurable in $\omega \in \Omega$ and continuous in $x \in X$, and if $x : \Omega \to X$ is measurable, then $f(\cdot, x(\cdot)) : \Omega \to Y$ is measurable.

Lemma 2
[31] Let $\{p_n\}, \{q_n\}, \{r_n\}$ be sequences of nonnegative real numbers satisfying the following conditions:

$$p_{n+1} \leq (1 + q_n)p_n + r_n, \sum_{n=0}^{\infty} q_n < \infty, \sum_{n=0}^{\infty} r_n < \infty$$

we have

i) $\lim_{n\to\infty} p_n$ exists;
ii) if $\liminf_{n\to\infty} p_n = 0$, then $\lim_{n\to\infty} p_n = 0$.

3. The main results

In this section, we state and prove some stochastic fixed point theorems of a finite family of random uniformly quasi-Lipschitzian operators in a generalized convex metric space as follow.

Lemma 3
Let K be a nonempty closed convex subset of a separable generalized convex metric space (X, d). Let $\{T_i : i \in J\}$ where $J = \{1, 2, 3, ..., k\}$ be a finite family of uniformly quasi- Lipschitzian random mappings $L_i > 0$. Suppose that the sequence $\{\zeta_n(\omega)\}$ is as in (4) and $\sum_{n=0}^{\infty}(\beta_{kn} + \gamma_{kn}) < \infty$. If $\mathcal{F} = \cap_{i=1}^{k} RF(T_i) \neq \emptyset$, then

1. there exist two positive constants $\mathcal{M}_0, \mathcal{M}_1$, such that;
$$d(\zeta_{n+1}(\omega, \zeta(\omega)) \leq (1+\Theta_n \mathcal{M}_0) d(\zeta_n(\omega), \zeta(\omega)) + \Theta_n \mathcal{M}_1 \tag{5}$$
where $\Theta_n = \beta_{kn} + \gamma_{kn}$ and $\alpha_{kn} \leq \gamma_{nk}$, for all $\zeta(\omega) \in \mathcal{F}$ and $n \in \mathbb{N}$;
2. there exist a positive constants $\mathcal{M}_2$, such that;
$$d(\zeta_{n+m}(\omega), \zeta(\omega)) \leq \mathcal{M}_2 d(\zeta_n(\omega), \zeta(\omega)) + \mathcal{M}_1 \mathcal{M}_2 \Sigma_{j=n}^{n+m-1} \Theta_j, \tag{6}$$
for all $\zeta(\omega) \in F$ and $n \in \mathbb{N}$.

Proof

1. Let $\zeta(\omega) \in \mathcal{F}$. Since $\{v_{in}(\omega)\}$ are bounded sequences in K for any $j \in J$, there exists $\mathcal{M} > 0$ such that
$$\mathcal{M} = \sup_{\omega \in \Omega} \max_{1 \leq i \leq k} d(v_{in}(\omega), \zeta(\omega)).$$
Let $L = \max_{1 \leq i \leq k}\{L_i\} > 0$. From the sequence in (4), we get
$$\begin{aligned} & d(\chi_{1n}(\omega), \zeta(\omega)) \\ = \;& d(W(\zeta_n(\omega), T_1^n(\omega, \chi_{0n}(\omega)), v_{1n}(\omega); \alpha_{1n}, \beta_{1n}, \gamma_{1n}), \zeta(\omega)) \\ \leq \;& \alpha_{1n} d(\zeta_n(\omega), \zeta(\omega)) + \beta_{1n} d(T_1^n(\omega, \chi_{0n}(\omega)), \zeta(\omega)) + \gamma_{1n} d(v_{1n}(\omega), \zeta(\omega)) \\ \leq \;& \alpha_{1n} d(\zeta_n(\omega), \zeta(\omega)) + \beta_{1n} L d(\chi_{0n}(\omega), \zeta(\omega)) + \gamma_{1n} \mathcal{M} \\ = \;& \alpha_{1n} d(\zeta_n(\omega), \zeta(\omega)) + \beta_{1n} L d(\zeta_{0n}(\omega), \zeta(\omega)) + \gamma_{1n} \mathcal{M} \\ \leq \;& d(\zeta_n(\omega), \zeta(\omega)) + L d(\zeta_{0n}(\omega), \zeta(\omega)) + \mathcal{M} \\ = \;& (1+L) d(\zeta_n(\omega), \zeta(\omega)) + \mathcal{M}. \end{aligned}$$
For $1 \leq i \leq k-1$, suppose that
$$d(\chi_{in}(\omega), \zeta(\omega)) = (1+L)^i d(\zeta_n(\omega), \zeta(\omega)) + \Sigma_{j=0}^{i-1} L^j \mathcal{M}$$
holds. Then
$$\begin{aligned} & d(\chi_{(i+1)n}(\omega), \zeta(\omega)) \\ = \;& d(W(\zeta_n(\omega), T_{i+1}^n(\omega, \chi_{in}(\omega)), v_{(i+1)n}(\omega); \alpha_{(i+1)n}, \beta_{(i+1)n}, \gamma_{(i+1)n}), \zeta(\omega)) \\ \leq \;& \alpha_{(i+1)n} d(\zeta_n(\omega), \zeta(\omega)) + \beta_{(i+1)n} d(T_{i+1}^n(\omega, \chi_{in}(\omega)), \zeta(\omega)) \\ & + \gamma_{(i+1)n} d(v_{(i+1)n}(\omega), \zeta(\omega)) \\ \leq \;& \alpha_{(i+1)n} d(\zeta_n(\omega), \zeta(\omega)) + \beta_{(i+1)n} L d(\chi_{in}(\omega), \zeta(\omega)) \\ & + \gamma_{(i+1)n} d(v_{(i+1)n}(\omega), \zeta(\omega)) \\ \leq \;& \alpha_{(i+1)n} d(\zeta_n(\omega), \zeta(\omega)) + \beta_{(i+1)n} L\{(1+L)^i d(\zeta_n(\omega), \zeta(\omega)) + \Sigma_{j=0}^{i-1} L^j \mathcal{M}\} \\ & + \gamma_{(i+1)n} d(v_{1n}(\omega), \zeta(\omega)) \\ \leq \;& \{\alpha_{(i+1)n} + \beta_{(i+1)n} L(1+L)^i\} d(\zeta_n(\omega), \zeta(\omega)) + \beta_{(i+1)n} L \Sigma_{j=0}^{i-1} L^j \mathcal{M} + \gamma_{(i+1)n} \mathcal{M} \\ \leq \;& \{1 + L(1+L)^i\} d(\zeta_n(\omega), \zeta(\omega)) + \Sigma_{j=1}^{i} L^j \mathcal{M} + \mathcal{M} \\ = \;& \{1 + L(1+L)^i\} d(\zeta_n(\omega), \zeta(\omega)) + \Sigma_{j=1}^{i} L^j \mathcal{M} + L^0 \mathcal{M} \\ \leq \;& \{(1+L)^i\} d(\zeta_n(\omega), \zeta(\omega)) + \Sigma_{j=1}^{i} L^j \mathcal{M}. \end{aligned}$$

Therefore, for any $1 \leq i \leq k$, by mathematical induction, we get

$$d(\chi_{in}(\omega), \zeta(\omega)) \leq (1+L)^i d(\zeta_n(\omega), \zeta(\omega)) + \Sigma_{j=0}^{i-1} L^j \mathcal{M}.$$

Then, it follows from (4) that

$$\begin{aligned}
& d(\zeta_{n+1}(\omega), \zeta(\omega)) \\
= \; & d(W(T_k^n(\omega, \zeta_n(\omega)), T_k^n(\omega, \chi_{(k-1)n}(\omega)), v_{kn}(\omega); \alpha_{kn}, \beta_{kn}, \gamma_{kn}), \zeta(\omega)) \\
\leq \; & \alpha_{kn} d(T_k^n(\omega, \zeta_n(\omega)), \zeta(\omega)) + \beta_{kn} d(T_k^n(\omega, \chi_{(k-1)n}(\omega)), \zeta(\omega)) \\
& + \gamma_{kn} d(v_{kn}(\omega), \zeta(\omega)) \\
\leq \; & \alpha_{kn} L d(\zeta_n(\omega), \zeta(\omega)) + \beta_{kn} L d(\chi_{(k-1)n}(\omega), \zeta(\omega)) \\
& + \gamma_{kn} d(v_{kn}(\omega), \zeta(\omega)) \\
\leq \; & \alpha_{kn} L d(\zeta_n(\omega), \zeta(\omega)) + \beta_{kn} L \{(1+L)^{k-1} d(\zeta_n(\omega), \zeta(\omega)) \\
& + \Sigma_{j=0}^{k-2} L^j \mathcal{M}\} + \gamma_{kn} \mathcal{M} \\
\leq \; & \{\alpha_{kn} L + \beta_{kn} L (1+L)^{k-1}\} d(\zeta_n(\omega), \zeta(\omega)) + \beta_{kn} L \Sigma_{j=0}^{k-2} L^j \mathcal{M} + \gamma_{kn} \mathcal{M} \\
\leq \; & \{\alpha_{kn} L + \frac{\beta_{kn} L (1+L)^k}{1+L}\} d(\zeta_n(\omega), \zeta(\omega)) + (\beta_{kn} + \gamma_{kn})(\Sigma_{j=0}^{k-2} L^j \mathcal{M} + \mathcal{M}) \\
\leq \; & \{\frac{\alpha_{kn} L (1+L)}{1+L} + \frac{\beta_{kn} L (1+L)^k}{1+L}\} d(\zeta_n(\omega), \zeta(\omega)) + \Theta_n \Sigma_{j=0}^{k-1} L^j \mathcal{M} \\
\leq \; & \{\alpha_{kn}(1+L) + \beta_{kn}(1+L)^k\} d(\zeta_n(\omega), \zeta(\omega)) + + \Theta_n \Sigma_{j=0}^{k-1} L^j \mathcal{M} \\
\leq \; & \{1 + \alpha_{kn}(1+L)^k + \beta_{kn}(1+L)^k\} d(\zeta_n(\omega), \zeta(\omega)) + \Theta_n \Sigma_{j=0}^{k-1} L^j \mathcal{M} \\
\leq \; & \{1 + (\beta_{kn} + \gamma_{kn})(1+L)^k\} d(\zeta_n(\omega), \zeta(\omega)) + \Theta_n \Sigma_{j=0}^{k-1} L^j \mathcal{M} \\
= \; & (1 + \Theta_n \mathcal{M}_0) d(\zeta_n(\omega), \zeta(\omega)) + \Theta_n \mathcal{M}_1
\end{aligned}$$

where $\Theta_n = \beta_{kn} + \gamma_{kn}$, $\mathcal{M}_0 = (1+L)^k$ and $\mathcal{M}_1 = \Sigma_{j=0}^{k-1} L^j \mathcal{M}$.

2. By $1 + x \leq e^x$ for any $x \geq 0$, we get

$$\begin{aligned}
& d(\zeta_{n+m}(\omega), \zeta(\omega)) \\
\leq \; & (1 + \Theta_{n+m-1} \mathcal{M}_0) d(\zeta_{n+m-1}(\omega), \zeta(\omega)) + \Theta_{n+m-1} \mathcal{M}_1 \\
\leq \; & e^{\Theta_{n+m-1} \mathcal{M}_0} \{(1 + \Theta_{n+m-2} \mathcal{M}_0) d(\zeta_{n+m-2}(\omega), \zeta(\omega)) + \Theta_{n+m-2} \mathcal{M}_1\} \\
& + \Theta_{n+m-1} \mathcal{M}_1 \\
\leq \; & e^{\Theta_{n+m-1} \mathcal{M}_0} e^{\Theta_{n+m-2} \mathcal{M}_0} d(\zeta_{n+m-2}(\omega), \zeta(\omega)) + \Theta_{n+m-2} \mathcal{M}_1 + \Theta_{n+m-1} \mathcal{M}_1 \\
\leq \; & e^{(\Theta_{n+m-1} + \Theta_{n+m-2}) \mathcal{M}_0} d(\zeta_{n+m-2}(\omega), \zeta(\omega)) + e^{(\Theta_{n+m-1} \mathcal{M}_0} (\Theta_{n+m-2} + \Theta_{n+m-1}) \mathcal{M}_1 \\
& \ldots \\
\leq \; & e^{\mathcal{M}_0 \Sigma_{j=1}^{n+m-1} \Theta_j} d(\zeta_{n+m-2}(\omega), \zeta(\omega)) + e^{\mathcal{M}_0 \Sigma_{j=1}^{n+m-1} \Theta_j} \mathcal{M}_1 \Sigma_{j=1}^{n+m-1} \Theta_j \\
\leq \; & e^{\mathcal{M}_0 \Sigma_{j=1}^{\infty} \Theta_j} d(\zeta_{n+m-2}(\omega), \zeta(\omega)) + e^{\mathcal{M}_0 \Sigma_{j=1}^{\infty} \Theta_j} \mathcal{M}_1 \Sigma_{j=1}^{n+m-1} \Theta_j \\
\leq \; & \mathcal{M}_2 d(\zeta_{n+m-2}(\omega), \zeta(\omega)) + \mathcal{M}_1 \mathcal{M}_2 \Sigma_{j=1}^{n+m-1} \Theta_j,
\end{aligned}$$

where $\mathcal{M}_2 = e^{\mathcal{M}_0 \Sigma_{j=1}^{\infty} \Theta_j}$.

□

Theorem 1
Let K be a non-empty closed convex subset of a separable complete generalized convex metric space (X, d) with a random convex structure W. Let $\{T_i : i \in J\} : \Omega \times K \to K$ be a finite family of continuous uniformly quasi-Lipchitzian random operators with $L_i > 0$. Suppose that the sequence $\{\zeta_n(\omega)\}$ is generated by (4) and $\Sigma_{n=1}^{\infty}(\beta_{kn} + \gamma_{kn}) < \infty$. If $\mathcal{F} = \bigcap_{i=1}^{k} RF(T_i) \neq \emptyset$, then $\{\zeta_n(\omega)\}$ converges to a common fixed point of $\{T_i : i \in J\}$ if and only if $\liminf_{n\to\infty} d(\zeta_n(\omega), \mathcal{F}) = 0$, where $d(x, \mathcal{F}) = \inf\{d(x, y) : \forall y \in \mathcal{F}\}$.

Proof
By lemma 3, we have

$$d(\zeta_{n+1}(\omega, \mathcal{F}) \leq (1 + \Theta_n \mathcal{M}_0) d(\zeta_n(\omega), \mathcal{F}) + \Theta_n \mathcal{M}_1.$$

Because $\Sigma_{n=1}^{\infty}(\beta_{kn} + \gamma_{kn}) < \infty$, by Lemma 2, thus $\lim_{n\to\infty} d(\zeta_n(\omega), \mathcal{F})$ exists. From hypothesis, $\liminf_{n\to\infty} d(\zeta_n(\omega), \mathcal{F}) = 0$, we get

$$\lim_{n\to\infty} d(\zeta_n(\omega), \mathcal{F}) = 0.$$

Now, we prove that $\{\zeta_n(\omega)\}$ is a Cauchy sequence. Actually, for each $\varepsilon > 0$, there exists a constant N_0 such that for all $n \geq N_0$, we get

$$d(\zeta_n(\omega), \mathcal{F}) \leq \tfrac{\varepsilon}{4\mathcal{M}_2} \text{ and } \Sigma_{n=N_0}^{\infty} \Theta_n \leq \tfrac{\varepsilon}{4\mathcal{M}_1\mathcal{M}_2}.$$

Especially, there exists $\varrho_1(\omega) \in \mathcal{F}$ and $N_1 > N_0$, where N_1 is constant such that

$$d(\zeta_{N_1}(\omega), \varrho_1(\omega)) \leq \tfrac{\varepsilon}{4\mathcal{M}_2}.$$

By Lemma 3, we get

$$\begin{aligned}
& d(\zeta_{n+m}(\omega), \zeta_n(\omega)) \\
= \;& d(\zeta_{n+m}(\omega), \varrho(\omega)) + d(\varrho(\omega), \zeta_n(\omega)) \\
\leq \;& \mathcal{M}_2 d(\zeta_{N_1}(\omega), \zeta(\omega)) + \mathcal{M}_1\mathcal{M}_2 \Sigma_{j=N_1}^{n+m-1} \Theta_j + \mathcal{M}_2 d(\zeta_{N_1}(\omega), \zeta(\omega)) \\
& + \mathcal{M}_1\mathcal{M}_2 \Sigma_{j=N_1}^{n-1} \Theta_j \\
\leq \;& 2\mathcal{M}_2 d(\zeta_{N_1}(\omega), \zeta(\omega)) + \mathcal{M}_1\mathcal{M}_2 (\Sigma_{j=N_1}^{n+m-1} \Theta_j + \Sigma_{j=N_1}^{n-1} \Theta_j) \\
\leq \;& 2\mathcal{M}_2 d(\zeta_{N_1}(\omega), \zeta(\omega)) + 2\mathcal{M}_1\mathcal{M}_2 \Sigma_{j=N_1}^{\infty} \Theta_j \\
\leq \;& 2\mathcal{M}_2 (\frac{\varepsilon}{4\mathcal{M}_2}) + 2\mathcal{M}_1\mathcal{M}_2 (\frac{\varepsilon}{4\mathcal{M}_1\mathcal{M}_2}) = \varepsilon.
\end{aligned}$$

That is $\{\zeta_n(\omega)\}$ is a Cauchy sequence in closed convex subset of complete generalized convex metric spaces. Therefore, $\{\zeta_n(\omega)\}$ converges to a point of K. Suppose $\lim_{n\to\infty} \zeta_n(\omega) = \varrho(\omega), \forall \omega \in \Omega$. Since T_i are continuous by Lemma 1, we know that for any measurable mapping $f : \Omega \to K, T_i^n(\omega, f(\omega)) : \Omega \to K$ are measurable mappings. So, $\{\zeta_n(\omega)\}$ is a sequence of measurable mappings. Hence, $\varrho : \Omega \to K$ is also measurable. Now, we show that $\varrho(\omega) \in \mathcal{F}$. From

$$d(\varrho(\omega), \mathcal{F}) \leq d(\zeta_n(\omega), \varrho(\omega)) + d(\zeta_n(\omega), \mathcal{F}).$$

Since $d(\zeta_n(\omega), \varrho(\omega)) = 0$ and $d(\zeta_n(\omega), \mathcal{F}) = 0$, so, we get $d(\varrho(\omega), \mathcal{F}) = 0$. Hence, $\varrho(\omega) \in \mathcal{F}$. □

From definition 2, if T is an asymptotically quasi-nonexpansive random operator, then T is a uniformly quasi-Lipschitzian random operator $((L = \sup_{n\geq 1}\{k_n\}))$. And if $RF(T) \neq \emptyset$, then every uniformly L-Lipschitzian random operator is a uniformly quasi-Lipschitzian random operator, we get the following corollary.

Corollary 1
Let K be a non-empty closed convex subset of a separable complete generalized convex metric space (X, d) with a random convex structure W. Let $\{T_i : i \in J\} : \Omega \times K \to K$ be a finite family of continuous asymptotically quasi-nonexpansive random operator with $L_i > 0$. Suppose that the sequence $\{\zeta_n(\omega)\}$ is generated by (4) and $\Sigma_{n=1}^{\infty}(\beta_{kn} + \gamma_{kn}) < \infty$. If $\mathcal{F} = \bigcap_{i=1}^{k} RF(T_i) \neq \emptyset$, then $\{\zeta_n(\omega)\}$ converges to a common fixed point of $\{T_i : i \in J\}$ if and only if $\liminf_{n\to\infty} d(\zeta_n(\omega), \mathcal{F}) = 0$, where $d(x, \mathcal{F}) = \inf\{d(x, y) : \forall y \in \mathcal{F}\}$.

Theorem 2
Let K be a non-empty closed convex subset of a separable complete generalized convex metric space (X, d) with a random convex structure W. Let $\{T_i : i \in J\} : \Omega \times K \to K$ be a finite family of continuous uniformly quasi-Lipchitzian random mappings with $L_i > 0$. Suppose that the sequence $\{\zeta_n(\omega)\}$ generated by (4), $\Sigma_{n=1}^{\infty}(\beta_{kn} + \gamma_{kn}) < \infty$ and $\mathcal{F} = \bigcap_{i=1}^{k} RF(T_i) \neq \emptyset$. If for some given $1 \leq l \leq k$,

(i) $\lim_{n\to\infty} d(T_l(\omega, \zeta_n(\omega)), \zeta_n(\omega)) = 0$;
(ii) there exists a positive constant $\mathcal{M}_3$ such that

$$d(T_l(\omega, \zeta_n(\omega)), \zeta_n(\omega)) \geq M_3 d(\zeta_n(\omega), \mathcal{F}).$$

Then $\{\zeta_n(\omega)\}$ converges to a common fixed point of $\{T_i : i \in J\}$.

Proof
By condition (i), we get

$$\lim_{n\to\infty} d(T_l(\omega, \zeta_n(\omega)), \zeta_n(\omega)) = 0,$$

that is,

$$d(T_l(\omega, \zeta_n(\omega)), \zeta_n(\omega)) = 0.$$

Also, by condition (ii), we get

$$\mathcal{M}_3 d(\zeta_n(\omega), \mathcal{F}) \leq d(T_l(\omega, \zeta_n(\omega)), \zeta_n(\omega)) = 0,$$

that is,

$$\mathcal{M}_3 d(\zeta_n(\omega), \mathcal{F}) = 0.$$

Thus, follow proof from Theorem 3, we get, $\{\zeta_n(\omega)\}$ converges to a common fixed point of $\{T_i : i \in J\}$. This completes the proof. □

Theorem 3
Let K be a non-empty closed convex subset of a separable complete generalized convex metric space (X, d) with a random convex structure W. Let $\{T_i : i \in J\} : \Omega \times K \to K$ be a finite family of continuous uniformly quasi-Lipchitzian random mappings with $L_i > 0$. Suppose that the sequence $\{\zeta_n(\omega)\}$ generated by (4), $\Sigma_{n=1}^{\infty}(\beta_{kn} + \gamma_{kn}) < \infty$ and $\mathcal{F} = \bigcap_{i=1}^{k} RF(T_i) \neq \emptyset$. If

(i) for all $1 \leq i \leq k$; $\lim_{n\to\infty} d(T_l(\omega, \zeta_n(\omega)), \zeta_n(\omega)) = 0$;
(ii) for some $1 \leq l \leq k$; T_l is semi-compact.

Then $\{\zeta_n(\omega)\}$ converges to a common fixed point of $\{T_i : i \in J\}$.

Proof
By conditions (i) and (ii), there exists a subsequence $\{\zeta_{n_j}(\omega)\} \subset \{\zeta_n(\omega)\}$ such that $\lim_{j\to\infty} \zeta_{n_j} = \zeta^*(\omega)$, $\forall \omega \in \Omega$, where $\zeta^*(\omega) \in K$. Since T_i are continuous for $i \in J$. So, $\{\zeta_n(\omega)\}$ is a sequence of measurable mappings. Hence, $\zeta^* : \Omega \to K$ is also measurable. Since $\lim_{j\to\infty} d(T_i(\omega, \zeta_{n_j}(\omega)), \zeta_{n_j}(\omega)) = d(T_i(\omega, \zeta^*(\omega)), \zeta^*(\omega)) = 0$, we get, $\zeta^*(\omega) \in \mathcal{F}$, $\forall \omega \in \Omega$. By Lemma 6, we get

$$d(\zeta_{n+1}(\omega), \zeta^*(\omega)) \leq (1 + \Theta_n \mathcal{M}_0) d(\zeta_n(\omega), \zeta^*(\omega)) + \Theta_n \mathcal{M}_1.$$

Since $\Sigma_{n=1}^{\infty} \Theta_n < \infty$, by Lemma 2, there exists $\rho \geq 0$ such that

$$\lim_{n\to\infty} d(\zeta_n(\omega), \zeta^*(\omega)) = \rho. \tag{7}$$

Since $\liminf_{n\to\infty} d(\zeta_n(\omega), \zeta^*(\omega)) = 0$, we get

$$\lim_{n\to\infty} \zeta_n(\omega) = \zeta^*(\omega),$$

that is

$$\lim_{n\to\infty} d(\zeta_n(\omega) = \zeta^*(\omega)) = 0. \tag{8}$$

By (7) and (8), we get $\rho = 0$.

Hence, $\{\zeta_n(\omega)\}$ converges to common fixed point of $\{T_i : i \in J\}$. This completes the proof. □

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