



Relaxed resolvent operator for solving a variational inclusion problem

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Abstract In this paper, we introduce a new resolvent operator and we call it relaxed resolvent operator. We prove that relaxed resolvent operator is single-valued and Lipschitz continuous and finally we approximate the solution of a variational inclusion problem in Hilbert spaces by defining an iterative algorithm based on relaxed resolvent operator. A few concepts like Lipschitz continuity and strong monotonicity are used to prove the main result of this paper. Thus, no strong conditions are used. Some examples are constructed.

Keywords Relaxed; Inclusion; Space; Algorithm; Lipschitz

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1. Introduction and Preliminaries

Variational inequality is an inequality involving a functional, which has to be solved for all possible values of a given variable, belonging usually to a convex set. The theory of variational inequalities was initially developed to deal with equilibrium problems, precisely the Signorini problem. After that it has been extended and generalized to study a wide class of problems arising in mechanics, physics, optimization and control, non-linear programming, economics, finance, regional structural, transformation, elasticity, and applied sciences, etc., see e.g., [1, 2, 4, 12, 15, 16, 17, 18, 19, 20] and references therein.

A useful and important generalization of variational inequalities is called a variational inclusion which was introduced by *Hassouni* and *Moudafi* [10] and includes mixed variational inequalities as special cases. Many problems concerning variational inclusions are solved by using the concept of maximal monotonicity and its generalized concepts such as H -monotonicity [6], H -accretivity [5] etc., see e.g., [7, 8, 9, 13, 21] and references therein. Most of the splitting methods are based on the resolvent operator of the form $[I + \lambda M]^{-1}$, where M is a set-valued monotone mapping, λ is a positive constant and I is the identity mapping.

In this paper, we introduce a new resolvent operator of the form $[(I - H) + \lambda M]^{-1}$, where H is a relaxed Lipschitz continuous mapping, M is a set-valued monotone mapping, λ is a positive constant and I is an identity mapping. We call this new resolvent operator as relaxed resolvent operator and prove that it is single-valued and Lipschitz continuous. We define an iterative algorithm based on relaxed resolvent operator to solve a variational inclusion problem. Convergence of the iterative sequences generated by the iterative algorithm is also discussed. Some examples are constructed.

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Throughout this paper, we suppose that X is a real Hilbert space endowed with a norm $\|\cdot\|$ and an inner product $\langle \cdot, \cdot \rangle$, d is the metric induced by the norm $\|\cdot\|$, 2^X (respectively, $CB(X)$) is the family of all nonempty (respectively, closed and bounded) subsets of X , and $D(\cdot, \cdot)$ is the Hausdorff metric on $CB(X)$ defined by

$$D(P, Q) = \max \left\{ \sup_{x \in P} d(x, Q), \sup_{y \in Q} d(P, y) \right\},$$

where $d(x, Q) = \inf_{y \in Q} d(x, y)$ and $d(P, y) = \inf_{x \in P} d(x, y)$.

The following definitions are needed in the sequel.

Definition 1.1

A mapping $g : X \rightarrow X$ is said to be

(i) Lipschitz continuous if, there exist a constant $\lambda_g > 0$ such that

$$\|g(x) - g(y)\| \leq \lambda_g \|x - y\|, \forall x, y \in X;$$

(ii) monotone, if

$$\langle g(x) - g(y), x - y \rangle \geq 0, \forall x, y \in X;$$

(iii) strongly monotone if, there exists a constant $\xi > 0$ such that

$$\langle g(x) - g(y), x - y \rangle \geq \xi \|x - y\|^2, \forall x, y \in X;$$

(iv) relaxed Lipschitz continuous if, there exists a constant $r > 0$ such that

$$\langle g(x) - g(y), x - y \rangle \leq -r \|x - y\|^2, \forall x, y \in X.$$

Definition 1.2

A mapping $N : X \times X \times X \rightarrow X$ is said to be Lipschitz continuous with respect to first argument if, there exists a constant λ_{N_1} such that

$$\|N(x_1, x_2, x_3) - N(y_1, x_2, x_3)\| \leq \lambda_{N_1} \|x_1 - y_1\|, \forall x_1, y_1, x_2, x_3 \in X.$$

Similarly, we can define the Lipschitz continuity of N in rest of the arguments.

Definition 1.3

A set-valued mapping $A : X \rightarrow CB(X)$ is said to be D -Lipschitz continuous if, there exists a constant δ_A such that

$$D(A(x), A(y)) \leq \delta_A \|x - y\|, \forall x, y \in X.$$

2. Relaxed Resolvent operator

We begin this section with the introduction of relaxed resolvent operator and demonstrate some of its properties.

Definition 2.1

Let $H : X \rightarrow X$ be a mapping and $I : X \rightarrow X$ be an identity mapping. Then, a set-valued mapping $M : X \rightarrow 2^X$ is said to be $(I - H)$ -monotone if, M is monotone, H is relaxed Lipschitz continuous and

$$[(I - H) + \lambda M](X) = X,$$

where $\lambda > 0$ is a constant.

Definition 2.2

Let $H : X \rightarrow X$ be relaxed Lipschitz continuous mapping and $I : X \rightarrow X$ be an identity mapping. Suppose that $M : X \rightarrow 2^X$ is a set-valued, $(I - H)$ -monotone mapping. The relaxed resolvent operator $R_{\lambda, M}^{(I-H)} : X \rightarrow X$ associated with I, H and M is defined by

$$R_{\lambda, M}^{I-H}(x) = [(I - H) + \lambda M]^{-1}(x), \forall x \in X, \tag{1}$$

where $\lambda > 0$ is a constant.

Example 2.1

Let $X = S^2$, the space of all 2×2 real symmetric matrices equipped with inner product $\langle A, B \rangle = tr(A \cdot B)$, $\forall A, B \in X$, and let α, β be two positive real numbers such that $\beta \leq \alpha$. Let $H : X \rightarrow X$ be a mapping defined by

$$H \left(\begin{bmatrix} x_1 & a \\ a & x_2 \end{bmatrix} \right) = \begin{bmatrix} -\alpha x_1 & a \\ a & -\alpha x_2 \end{bmatrix}, \forall x_1, x_2, a \in \mathbb{R}.$$

For $x = \begin{bmatrix} x_1 & a \\ a & x_2 \end{bmatrix} \in X$, we calculate,

$$\begin{aligned} & \langle H(x) - H(y), x - y \rangle \\ &= \left\langle \begin{bmatrix} -\alpha(x_1 - y_1) & 0 \\ 0 & -\alpha(x_2 - y_2) \end{bmatrix}, \begin{bmatrix} (x_1 - y_1) & 0 \\ 0 & (x_2 - y_2) \end{bmatrix} \right\rangle \\ &= tr \left(\begin{bmatrix} -\alpha(x_1 - y_1) & 0 \\ 0 & -\alpha(x_2 - y_2) \end{bmatrix} \cdot \begin{bmatrix} (x_1 - y_1) & 0 \\ 0 & (x_2 - y_2) \end{bmatrix} \right) \\ &= -\alpha(x_1 - y_1)^2 - \alpha(x_2 - y_2)^2 \\ &= -\alpha[(x_1 - y_1)^2 + (x_2 - y_2)^2] \\ &\leq -\beta[(x_1 - y_1)^2 + (x_2 - y_2)^2]. \end{aligned}$$

Also,

$$\begin{aligned} \|x - y\|^2 &= \langle x - y, x - y \rangle \\ &= \left\langle \begin{bmatrix} x_1 - y_1 & 0 \\ 0 & x_2 - y_2 \end{bmatrix}, \begin{bmatrix} x_1 - y_1 & 0 \\ 0 & x_2 - y_2 \end{bmatrix} \right\rangle \\ &= tr \left(\begin{bmatrix} x_1 - y_1 & 0 \\ 0 & x_2 - y_2 \end{bmatrix} \cdot \begin{bmatrix} x_1 - y_1 & 0 \\ 0 & x_2 - y_2 \end{bmatrix} \right) \\ &= (x_1 - y_1)^2 + (x_2 - y_2)^2. \end{aligned}$$

From above it follows that

$$\langle H(x) - H(y), x - y \rangle \leq -\beta \|x - y\|^2, \forall x, y \in X.$$

i.e., H is β -relaxed Lipschitz continuous.

Suppose that $M : X \rightarrow 2^X$ is defined by

$$M \left(\begin{bmatrix} x_1 & a \\ a & x_2 \end{bmatrix} \right) = \begin{bmatrix} \alpha x_1 & a \\ a & \alpha x_2 \end{bmatrix}, \forall x_1, x_2, a \in \mathbb{R}.$$

We calculate

$$\begin{aligned}
 & \langle M(x) - M(y), x - y \rangle \\
 = & \left\langle M \left(\begin{bmatrix} x_1 & a \\ a & x_2 \end{bmatrix} \right) - M \left(\begin{bmatrix} y_1 & a \\ a & y_2 \end{bmatrix} \right), \begin{bmatrix} x_1 - y_1 & 0 \\ 0 & x_2 - y_2 \end{bmatrix} \right\rangle \\
 = & \left\langle \begin{bmatrix} \alpha(x_1 - y_1) & 0 \\ 0 & \alpha(x_2 - y_2) \end{bmatrix}, \begin{bmatrix} x_1 - y_1 & 0 \\ 0 & x_2 - y_2 \end{bmatrix} \right\rangle \\
 = & \operatorname{tr} \left(\begin{bmatrix} \alpha(x_1 - y_1) & 0 \\ 0 & \alpha(x_2 - y_2) \end{bmatrix} \cdot \begin{bmatrix} x_1 - y_1 & 0 \\ 0 & x_2 - y_2 \end{bmatrix} \right) \\
 = & \alpha(x_1 - y_1)^2 + \alpha(x_2 - y_2)^2,
 \end{aligned}$$

which implies that

$$\langle M(x) - M(y), x - y \rangle \geq 0, \forall x, y \in X.$$

i.e., M is monotone.

Now, we show that for $\lambda = 1$, every element $x = \begin{bmatrix} \alpha x_1 & a \\ a & \alpha x_2 \end{bmatrix} \in 2^X$ has a pre-image $y = \begin{bmatrix} \frac{\alpha x_1}{1+2\alpha} & a \\ a & \frac{\alpha x_2}{1+2\alpha} \end{bmatrix} \in X$.

$$\begin{aligned}
 [(I - H) + M](y) &= (I - H)(y) + M(y) \\
 &= y - H(y) + M(y) \\
 &= \begin{bmatrix} \frac{\alpha x_1}{1+2\alpha} & a \\ a & \frac{\alpha x_2}{1+2\alpha} \end{bmatrix} - \begin{bmatrix} \frac{-\alpha^2 x_1}{1+2\alpha} & a \\ a & \frac{-\alpha^2 x_2}{1+2\alpha} \end{bmatrix} + \begin{bmatrix} \frac{\alpha^2 x_1}{1+2\alpha} & a \\ a & \frac{\alpha^2 x_2}{1+2\alpha} \end{bmatrix} \\
 &= \begin{bmatrix} \frac{\alpha(1+2\alpha)x_1}{1+2\alpha} & a \\ a & \frac{\alpha(1+2\alpha)x_2}{1+2\alpha} \end{bmatrix} \\
 &= \begin{bmatrix} \alpha x_1 & a \\ a & \alpha x_2 \end{bmatrix} \in 2^X.
 \end{aligned}$$

It follows that

$$[(I - H) + M](X) = X,$$

i.e., M is $(I - H)$ -monotone mapping.

Now, we prove some of the properties of relaxed resolvent operator defined by (1).

Theorem 2.2

Let $H : X \rightarrow X$ be a r -relaxed Lipschitz continuous mapping, $I : X \rightarrow X$ be an identity mapping and $M : X \rightarrow 2^X$ be a set-valued $(I - H)$ -monotone mapping. Then the operator $[(I - H) + \lambda M]^{-1}$ is single-valued, where $\lambda > 0$ is a constant.

Proof

For any $z \in X$ and a constant $\lambda > 0$, let $x, y \in [(I - H) + \lambda M]^{-1}(z)$. Then,

$$\lambda^{-1}[z - (I - H)(x)] \in M(x);$$

$$\lambda^{-1}[z - (I - H)(y)] \in M(y).$$

Since M is monotone, we have

$$\begin{aligned} \langle -(I - H)(x) + z + (I - H)(y) - z, x - y \rangle &\geq 0; \\ -\langle (I - H)(x) - (I - H)(y), x - y \rangle &\geq 0; \\ -\langle x - H(x) - y + H(y), x - y \rangle &\geq 0; \\ \langle x - H(x) - y + H(y), x - y \rangle &\leq 0; \\ \langle x - H(x) - y + H(y), x - y \rangle &\leq 0; \\ \langle x - y, x - y \rangle - \langle H(x) - H(y), x - y \rangle &\leq 0. \end{aligned}$$

Since H is r -relaxed Lipschitz continuous, we have

$$0 \geq \langle x - y, x - y \rangle - \langle H(x) - H(y), x - y \rangle \geq \|x - y\|^2 + r\|x - y\|^2 \geq 0,$$

it follows that $(1 + r)\|x - y\|^2 = 0$, which implies that $x = y$. Thus $[(I - H) + \lambda M]^{-1}$ is single-valued. \square

Theorem 2.3

Let $H : X \rightarrow X$ be a r -relaxed Lipschitz continuous mapping, $I : X \rightarrow X$ be an identity mapping and $M : X \rightarrow 2^X$ be a set-valued, $(I - H)$ -monotone mapping. Then the resolvent operator $R_{\lambda, M}^{I-H} : X \rightarrow X$ is $\frac{1}{[1+r]}$ -Lipschitz continuous, i.e.,

$$\|R_{\lambda, M}^{I-H}(x) - R_{\lambda, M}^{I-H}(y)\| \leq \frac{1}{[1+r]}\|x - y\|, \forall x, y \in X.$$

Proof

Let x and y be any given point in X . It follows from (1) that

$$\begin{aligned} R_{\lambda, M}^{I-H}(x) &= [(I - H) + \lambda M]^{-1}(x), \\ R_{\lambda, M}^{I-H}(y) &= [(I - H) + \lambda M]^{-1}(y). \end{aligned} \tag{2}$$

It follows that

$$\begin{aligned} \frac{1}{\lambda} \left[x - (I - H)(R_{\lambda, M}^{I-H}(x)) \right] &\in M \left(R_{\lambda, M}^{I-H}(x) \right), \\ \frac{1}{\lambda} \left[y - (I - H)(R_{\lambda, M}^{I-H}(y)) \right] &\in M \left(R_{\lambda, M}^{I-H}(y) \right). \end{aligned} \tag{3}$$

Since M is $(I - H)$ -monotone i.e., M is monotone, we have

$$\begin{aligned} \frac{1}{\lambda} \left\langle x - (I - H)(R_{\lambda, M}^{I-H}(x)) - (y - (I - H)(R_{\lambda, M}^{I-H}(y))), R_{\lambda, M}^{I-H}(x) - R_{\lambda, M}^{I-H}(y) \right\rangle &\geq 0, \\ \frac{1}{\lambda} \left\langle x - y - \{(I - H)(R_{\lambda, M}^{I-H}(x)) - (I - H)(R_{\lambda, M}^{I-H}(y))\}, R_{\lambda, M}^{I-H}(x) - R_{\lambda, M}^{I-H}(y) \right\rangle &\geq 0. \end{aligned} \tag{4}$$

It follows that

$$\begin{aligned} &\left\langle x - y, R_{\lambda, M}^{I-H}(x) - R_{\lambda, M}^{I-H}(y) \right\rangle \\ &\geq \left\langle (I - H)(R_{\lambda, M}^{I-H}(x)) - (I - H)(R_{\lambda, M}^{I-H}(y)), R_{\lambda, M}^{I-H}(x) - R_{\lambda, M}^{I-H}(y) \right\rangle. \end{aligned} \tag{5}$$

By Cauchy-Schwartz inequality, (5) and r -relaxed Lipschitz continuity of H , we have

$$\begin{aligned}
 & \|x - y\| \left\| R_{\lambda, M}^{I-H}(x) - R_{\lambda, M}^{I-H}(y) \right\| \\
 \geq & \left\langle x - y, R_{\lambda, M}^{I-H}(x) - R_{\lambda, M}^{I-H}(y) \right\rangle \\
 \geq & \left\langle R_{\lambda, M}^{I-H}(x) - H(R_{\lambda, M}^{I-H}(x)) - R_{\lambda, M}^{I-H}(y) + H(R_{\lambda, M}^{I-H}(y)), R_{\lambda, M}^{I-H}(x) - R_{\lambda, M}^{I-H}(y) \right\rangle \\
 = & \left\langle R_{\lambda, M}^{I-H}(x) - R_{\lambda, M}^{I-H}(y), R_{\lambda, M}^{I-H}(x) - R_{\lambda, M}^{I-H}(y) \right\rangle - \\
 & \left\langle H(R_{\lambda, M}^{I-H}(x)) - H(R_{\lambda, M}^{I-H}(y)), R_{\lambda, M}^{I-H}(x) - R_{\lambda, M}^{I-H}(y) \right\rangle \\
 \geq & \left\| R_{\lambda, M}^{I-H}(x) - R_{\lambda, M}^{I-H}(y) \right\|^2 + r \left\| R_{\lambda, M}^{I-H}(x) - R_{\lambda, M}^{I-H}(y) \right\|^2 \\
 = & (1 + r) \left\| R_{\lambda, M}^{I-H}(x) - R_{\lambda, M}^{I-H}(y) \right\|^2.
 \end{aligned} \tag{6}$$

Thus, we have

$$\left\| R_{\lambda, M}^{I-H}(x) - R_{\lambda, M}^{I-H}(y) \right\| \leq \frac{1}{[1+r]} \|x - y\|,$$

i.e., the relaxed resolvent operator $R_{\lambda, M}^{I-H}$ is $\frac{1}{[1+r]}$ -Lipschitz continuous. \square

In support of Theorem 2.2, we have the following example.

Example 2.4

Let $X = \mathbb{R}^2$ with usual inner product. Let $H : X \rightarrow X$ be a mapping defined by

$$H(x) = (-2x_1, -4x_2), \forall x = (x_1, x_2) \in X,$$

and the mapping $M : X \rightarrow 2^X$ be defined by

$$M(x) = (4x_1, 2x_2), \forall x = (x_1, x_2) \in X.$$

Then, it easy to check that H is 2-relaxed Lipschitz continuous and M is monotone. In addition, it is easy to verify that for $\lambda = 1$, $[(I - H) + \lambda M](X) = X$, which shows that M is $(I - H)$ -monotone mapping. Hence, the relaxed resolvent operator $R_{\lambda, M}^{(I-H)} : X \rightarrow X$ associated with I , H and M is of the form:

$$R_{\lambda, M}^{I-H}(x) = \left(\frac{x_1}{7}, \frac{x_2}{7} \right), \forall x = (x_1, x_2) \in X. \tag{7}$$

It is easy to see that the relaxed resolvent operator defined by (2.7) is single-valued.

Now, we prove that $R_{\lambda, M}^{I-H}$ is Lipschitz continuous.

$$\begin{aligned}
 \left\| R_{\lambda, M}^{I-H}(x) - R_{\lambda, M}^{I-H}(y) \right\| &= \left\| \left(\frac{x_1}{7}, \frac{x_2}{7} \right) - \left(\frac{y_1}{7}, \frac{y_2}{7} \right) \right\| \\
 &= \left\| \left(\frac{x_1 - y_1}{7}, \frac{x_2 - y_2}{7} \right) \right\| \\
 &= \left[\frac{(x_1 - y_1)^2}{49} + \frac{(x_2 - y_2)^2}{49} \right]^{\frac{1}{2}} \\
 &= \frac{1}{7} [(x_1 - y_1)^2 + (x_2 - y_2)^2]^{\frac{1}{2}} \\
 &\leq \frac{1}{3} \|x - y\|.
 \end{aligned}$$

Hence, the resolvent operator $R_{\lambda, M}^{I-H}$ is $\frac{1}{3}$ -Lipschitz continuous.

3. Formulation of the problem and iterative algorithm

In this section, we formulate a variational inclusion problem and an iterative algorithm based on relaxed resolvent operator to approximate the solution of our problem.

Let X be a real Hilbert space and $H, g : X \rightarrow X$, $N : X \times X \times X \rightarrow X$ be the single-valued mappings, $I : X \rightarrow X$ be an identity mapping. Suppose that $A, B, C : X \rightarrow CB(X)$ and $M : X \times X \rightarrow 2^X$ are the set-valued mappings such that M is $(I - H)$ -monotone. We consider the problem of finding $x \in X$, $u \in A(x)$, $v \in B(x)$, $w \in C(x)$ and $g(x) \cap \text{dom}M(\cdot, x) \neq \emptyset$ such that

$$0 \in N(u, v, w) + M(g(x), x). \quad (8)$$

When $C \equiv 0$, $N(u, v, \cdot) = N(u, v)$, then problem (8) reduces to find $x \in X$, $u \in A(x)$, $v \in B(x)$ such that

$$0 \in N(u, v) + M(g(x), x). \quad (9)$$

Problem (9) was introduced and studied by *Kazmi and Khan* [11].

In addition if $M(g(x), x) = M(g(x))$, then a similar analogue of problem (9) was introduced and studied by *Chang, Cho, Lee and Jung* [3], *Chang* [2], *Chang, Jim and Kim* [4].

It is clear that for suitable choices of mappings involved in the formulation of problem (8), one can obtain many variational inclusion problems studied in recent past.

By applying the relaxed resolvent operator, we establish an equivalence result for variational inclusion problem (8) and a nonlinear equation.

Lemma 3.1

Let $x \in X$, $u \in A(x)$, $v \in B(x)$ and $w \in C(x)$ is a solution of variational inclusion problem (3.1) if and only if (x, u, v, w) satisfies the equation:

$$g(x) = R_{\lambda, M(\cdot, x)}^{I-H} [(I - H)g(x) - \lambda N(u, v, w)], \quad (10)$$

where

$$R_{\lambda, M(\cdot, x)}^{I-H} = [(I - H) + \lambda M(\cdot, x)]^{-1},$$

and $\lambda > 0$ is a constant.

Proof

The proof is a direct consequence of Definition 2.2. □

Theorem 3.2 (Nadler's Theorem [14])

Let (X, d) be a complete metric space. If $F : X \rightarrow CB(X)$ is a set-valued contraction mapping, then F has a fixed point.

Based on Lemma 3.1 and Theorem 3.2, we construct an iterative algorithm for finding approximate solutions of problem (8).

Iterative Algorithm 3.1

Let $g, H : X \rightarrow X$, $N : X \times X \times X \rightarrow X$ are the single-valued mappings, $I : X \rightarrow X$ be an identity mapping, and $A, B, C : X \rightarrow CB(X)$ are the set-valued mappings be such that for each $x \in X$, $Q(x_0) \subseteq g(x)$, where $Q : X \rightarrow 2^X$ be a set-valued mapping defined by

$$Q(x) = \bigcup_{u \in A(x)} \bigcup_{v \in B(x)} \bigcup_{w \in C(x)} [R_{\lambda, M(\cdot, x)}^{I-H} [(I - H)g(x) - \lambda N(u, v, w)]], \quad (11)$$

where $M : X \times X \rightarrow 2^X$ be a set-valued mapping such that a fixed $x \in X$, $M(\cdot, x)$ is $(I - H)$ -monotone.

For any given $x_0 \in X$, $u_0 \in A(x_0)$, $v_0 \in B(x_0)$, and $w_0 \in C(x_0)$, let

$$z_0 = R_{\lambda, M(\cdot, x_0)}^{I-H}[(I - H)g(x_0) - \lambda N(u_0, v_0, w_0)] \subseteq Q(x_0) \subseteq g(X).$$

Hence, there exist $x_1 \in X$ such that $z_0 = g(x_1)$. Since $u_0 \in A(x_0) \in CB(X)$, $v_0 \in B(x_0) \in CB(X)$ and $w_0 \in C(x_0) \in CB(X)$, by Theorem 3.2 there exist $u_1 \in A(x_1)$, $v_1 \in B(x_1)$ and $w_1 \in C(x_1)$ such that

$$\begin{aligned} \|u_1 - u_0\| &\leq D(A(x_1), A(x_0)), \\ \|v_1 - v_0\| &\leq D(B(x_1), B(x_0)), \\ \|w_1 - w_0\| &\leq D(C(x_1), C(x_0)). \end{aligned}$$

Let

$$z_1 = R_{\lambda, M(\cdot, x_1)}^{I-H}[(I - H)g(x_1) - \lambda N(u_1, v_1, w_1)] \subseteq Q(x_1) \subseteq g(X).$$

Hence, there exist $x_2 \in X$ such that $z_1 = g(x_2)$. Continuing the above process inductively, we can define the iterative sequences $\{x_n\}$, $\{u_n\}$, $\{v_n\}$ and $\{w_n\}$ by the following scheme:

$$g(x_{n+1}) = R_{\lambda, M(\cdot, x_n)}^{I-H}[(I - H)g(x_n) - \lambda N(u_n, v_n, w_n)], \quad (12)$$

$$u_{n+1} \in A(x_{n+1}), \|u_{n+1} - u_n\| \leq D(A(x_{n+1}), A(x_n)), \quad (13)$$

$$v_{n+1} \in B(x_{n+1}), \|v_{n+1} - v_n\| \leq D(B(x_{n+1}), B(x_n)), \quad (14)$$

$$w_{n+1} \in C(x_{n+1}), \|w_{n+1} - w_n\| \leq D(C(x_{n+1}), C(x_n)), \quad (15)$$

where $\lambda > 0$ is a constant and $n = 0, 1, 2, \dots$.

4. Existence and convergence result

In this section, we prove an existence and convergence result for variational inclusion problem (8).

Theorem 4.1

Let X be a real Hilbert space and $g, H : X \rightarrow X$ be the single-valued mappings such that g is strongly monotone with constant ξ , Lipschitz continuous with constant λ_g and H is relaxed Lipschitz continuous with constant r and Lipschitz continuous with constant λ_H . Suppose that $N : X \times X \times X \rightarrow X$ is a single-valued mapping such that N is Lipschitz continuous in all the three arguments with constants λ_{N_1} , λ_{N_2} and λ_{N_3} , respectively and $A, B, C : X \rightarrow CB(X)$ be the set-valued mappings such that A is D -Lipschitz continuous with constant λ_A , B is D -Lipschitz continuous with constant δ_B and C is D -Lipschitz continuous with constant δ_C . Suppose that set-valued mapping $M : X \times X \rightarrow 2^X$ is such that for a fixed $x \in X$, $M(\cdot, x)$ is $(I - H)$ -monotone with respect to the first arguments, where $I : X \rightarrow X$ is the identity mapping and for each $x \in X$, $Q(x) \subseteq g(x)$, where Q is defined by (11). Suppose that there exists constants $\lambda > 0$ and $h > 0$ such that the following conditions hold:

$$\left\| R_{\lambda, M(\cdot, x)}^{I-H}(z) - R_{\lambda, M(\cdot, y)}^{I-H}(z) \right\| \leq h \|x - y\|, \quad \forall x, y, z \in X, \quad (16)$$

and

$$\lambda_g + \lambda_H \lambda_g + \lambda \lambda_{N_1} \delta_A + \lambda \lambda_{N_2} \delta_B + \lambda \lambda_{N_3} \delta_C < [\xi(1 + r) - h]. \quad (17)$$

Then, there exist $x \in X$, $u \in A(x)$, $v \in B(x)$ and $w \in C(x)$ such that variational inclusion problem (8) is solvable. Moreover, $x_n \rightarrow x$, $u_n \rightarrow u$, $v_n \rightarrow v$ and $w_n \rightarrow w$, as $n \rightarrow \infty$, where $\{x_n\}$, $\{u_n\}$, $\{v_n\}$ and $\{w_n\}$ are the sequences defined in iterative Algorithm 3.1.

Proof

Using the strong monotonicity of g with constant ξ , we have

$$\begin{aligned} \|g(x_{n+1}) - g(x_n)\| \|x_{n+1} - x_n\| &\geq \langle g(x_{n+1}) - g(x_n), x_{n+1} - x_n \rangle \\ &\geq \xi \|x_{n+1} - x_n\|^2, \end{aligned}$$

which implies that

$$\|x_{n+1} - x_n\| \leq \frac{1}{\xi} \|g(x_{n+1}) - g(x_n)\|. \tag{18}$$

By iterative Algorithm 3.1, Theorem 2.2 and condition (16), we have

$$\begin{aligned} &\|g(x_{n+1}) - g(x_n)\| \\ = &\left\| R_{\lambda, M(\cdot, x_n)}^{I-H} [(I - H)g(x_n) - \lambda N(u_n, v_n, w_n)] - R_{\lambda, M(\cdot, x_{n-1})}^{I-H} [(I - H)g(x_{n-1}) \right. \\ &\quad \left. - \lambda N(u_{n-1}, v_{n-1}, w_{n-1})] \right\| \\ = &\left\| R_{\lambda, M(\cdot, x_n)}^{I-H} [(I - H)g(x_n) - \lambda N(u_n, v_n, w_n)] - R_{\lambda, M(\cdot, x_n)}^{I-H} [(I - H)g(x_{n-1}) \right. \\ &\quad \left. - \lambda N(u_{n-1}, v_{n-1}, w_{n-1})] + R_{\lambda, M(\cdot, x_n)}^{I-H} [(I - H)g(x_{n-1}) \right. \\ &\quad \left. - \lambda N(u_{n-1}, v_{n-1}, w_{n-1})] - R_{\lambda, M(\cdot, x_{n-1})}^{I-H} [(I - H)g(x_{n-1}) - \lambda N(u_{n-1}, v_{n-1}, w_{n-1})] \right\| \\ \leq &\left\| R_{\lambda, M(\cdot, x_n)}^{I-H} [(I - H)g(x_n) - \lambda N(u_n, v_n, w_n)] \right. \\ &\quad \left. - R_{\lambda, M(\cdot, x_n)}^{I-H} [(I - H)g(x_{n-1}) - \lambda N(u_{n-1}, v_{n-1}, w_{n-1})] \right\| + \left\| R_{\lambda, M(\cdot, x_n)}^{I-H} [(I - H)g(x_{n-1}) \right. \\ &\quad \left. - \lambda N(u_{n-1}, v_{n-1}, w_{n-1})] - R_{\lambda, M(\cdot, x_{n-1})}^{I-H} [(I - H)g(x_{n-1}) - \lambda N(u_{n-1}, v_{n-1}, w_{n-1})] \right\| \\ \leq &\frac{1}{[1 + r]} \|(I - H)g(x_n) - \lambda N(u_n, v_n, w_n) - ((I - H)g(x_{n-1}) - \lambda N(u_{n-1}, v_{n-1}, w_{n-1}))\| \\ &\quad + h \|x_n - x_{n-1}\|. \end{aligned} \tag{19}$$

Since g is Lipschitz continuous with constant λ_g , H is a Lipschitz continuous with constant λ_H , N is Lipschitz continuous in all three arguments with constants λ_{N_1} , λ_{N_2} , λ_{N_3} , respectively and A , B , C are D -Lipschitz continuous with constants δ_A , δ_B and δ_C , respectively, we have

$$\begin{aligned} &\|(I - H)g(x_n) - (I - H)g(x_{n-1}) - \lambda(N(u_n, v_n, w_n) - N(u_{n-1}, v_{n-1}, w_{n-1}))\| \\ \leq &\|(I - H)g(x_n) - (I - H)g(x_{n-1})\| + \lambda \|N(u_n, v_n, w_n) - N(u_{n-1}, v_{n-1}, w_{n-1})\| \\ \leq &\|g(x_n) - g(x_{n-1})\| + \|H(g(x_n)) - H(g(x_{n-1}))\| + \lambda \|N(u_n, v_n, w_n) - \\ &\quad N(u_{n-1}, v_n, w_n) + N(u_{n-1}, v_n, w_n) + N(u_{n-1}, v_{n-1}, w_n) - N(u_{n-1}, v_{n-1}, w_n) \\ &\quad - N(u_{n-1}, v_{n-1}, w_{n-1})\| \end{aligned}$$

$$\begin{aligned}
&\leq \|g(x_n) - g(x_{n-1})\| + \|H(g(x_n)) - H(g(x_{n-1}))\| + \lambda\|N(u_n, v_n, w_n) + \\
&\quad N(u_{n-1}, v_n, w_n)\| + \lambda\|N(u_{n-1}, v_n, w_n) - N(u_{n-1}, v_{n-1}, w_n)\| + \\
&\quad \lambda\|N(u_{n-1}, v_{n-1}, w_n) - N(u_{n-1}, v_{n-1}, w_{n-1})\| \\
&\leq \lambda_g\|x_n - x_{n-1}\| + \lambda_H\lambda_g\|x_n - x_{n-1}\| + \lambda\lambda_{N_1}\|u_n - u_{n-1}\| + \\
&\quad \lambda\lambda_{N_2}\|v_n - v_{n-1}\| + \lambda\lambda_{N_3}\|w_n - w_{n-1}\| \\
&\leq \lambda_g\|x_n - x_{n-1}\| + \lambda_H\lambda_g\|x_n - x_{n-1}\| + \lambda\lambda_{N_1}D(A(x_n), A(x_{n-1})) + \\
&\quad \lambda\lambda_{N_2}D(B(x_n), B(x_{n-1})) + \lambda\lambda_{N_3}D(C(x_n), C(x_{n-1})) \\
&\leq \lambda_g\|x_n - x_{n-1}\| + \lambda_H\lambda_g\|x_n - x_{n-1}\| + \lambda\lambda_{N_1}\delta_A\|x_n - x_{n-1}\| + \\
&\quad \lambda\lambda_{N_2}\delta_B\|x_n - x_{n-1}\| + \lambda\lambda_{N_3}\delta_C\|x_n - x_{n-1}\| \\
&\leq [\lambda_g + \lambda_H\lambda_g + \lambda\lambda_{N_1}\delta_A + \lambda\lambda_{N_2}\delta_B + \lambda\lambda_{N_3}\delta_C]\|x_n - x_{n-1}\|.
\end{aligned} \tag{20}$$

Using (20), (19) becomes

$$\|g(x_{n+1}) - g(x_n)\| \leq \left[\frac{[\lambda_g + \lambda_H\lambda_g + \lambda\lambda_{N_1}\delta_A + \lambda\lambda_{N_2}\delta_B + \lambda\lambda_{N_3}\delta_C]}{[1+r]} + h \right] \|x_n - x_{n-1}\|. \tag{21}$$

Using (21), (18) becomes

$$\|x_{n+1} - x_n\| \leq \theta \|x_n - x_{n-1}\|,$$

where

$$\theta = \frac{1}{\xi} \left[\frac{[\lambda_g + \lambda_H\lambda_g + \lambda\lambda_{N_1}\delta_A + \lambda\lambda_{N_2}\delta_B + \lambda\lambda_{N_3}\delta_C]}{[1+r]} + h \right]. \tag{22}$$

By condition (17), we have $0 \leq \theta < 1$, thus $\{x_n\}$ is a Cauchy sequence in X and as X is complete, there exists $x \in X$ such that $x_n \rightarrow x$ as $n \rightarrow \infty$. From (13), (14) and (15) of Algorithm 3.1 and D -Lipschitz continuity of A, B and C with constants δ_A, δ_B and δ_C , respectively, we have

$$\|u_{n+1} - u_n\| \leq D(A(x_{n+1}), A(x_n)) \leq \delta_A \|x_{n+1} - x_n\|; \tag{23}$$

$$\|v_{n+1} - v_n\| \leq D(B(x_{n+1}), B(x_n)) \leq \delta_B \|x_{n+1} - x_n\|; \tag{24}$$

$$\|w_{n+1} - w_n\| \leq D(C(x_{n+1}), C(x_n)) \leq \delta_C \|x_{n+1} - x_n\|. \tag{25}$$

It is clear from (23), (24) and (25) that $\{u_n\}$, $\{v_n\}$ and $\{w_n\}$ are also Cauchy sequences in X , so there exist u, v and w in X such that $u_n \rightarrow u, v_n \rightarrow v$ and $w_n \rightarrow w$. By using the continuity of the operators $I, N, M, g, A, B, C, H, R_{\lambda, M(\cdot, x)}^{I-H}$ and iterative Algorithm 3.1, we have

$$g(x) = R_{\lambda, M(\cdot, x)}^{I-H} [(I - H)g(x) - \lambda N(u, v, w)].$$

By Lemma 3.1, we conclude that (x, u, v) is a solution of problem (3.1). It remains to show that $u \in A(x), v \in B(x)$ and $w \in C(x)$. In fact

$$\begin{aligned}
d(u, A(u)) &\leq \|u - u_n\| + d(u_n, A(x)) \\
&\leq \|u - u_n\| + D(A(x_n), A(x)) \\
&\leq \|u - u_n\| + \delta_A \|x_n - x\| \rightarrow 0, \text{ as } n \rightarrow \infty.
\end{aligned}$$

Hence $u \in A(x)$. Similarly, we can show that $v \in B(x)$ and $w \in C(x)$. This completes the proof. \square

5. Conclusion

The resolvent operator techniques are applicable to solve several problem related to variational inequalities (inclusions), optimization problems, complementary problems etc..The aim of this work is to introduce a new type of resolvent operator based on relaxed Lipschitz continuity and monotonicity, and we call it as relaxed resolvent operator. We prove that relaxed resolvent operator is single valued and Lipschitz continuous. We define an iterative algorithm to approximate the solution of a variational inclusion problem. In our opinion, many other problems occurring in applied sciences may be solved by using relaxed resolvent operator in a different frame work.

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