STATISTICS, OPTIMIZATION AND INFORMATION COMPUTING
Stat., Optim. Inf. Comput., Vol. **3**, December 2015, pp 336–347.
Published online in International Academic Press (www.IAPress.org)

Construction of exact solutions to the modified forms of DP and CH equations by analytical methods

Jalil Manafian [1,*], Reza Shahabi [2], Mohammad Asadpour [3], Isa Zamanpour [4], Jalal Jalali[5]

[1] *Department of Applied Mathematics, Faculty of Mathematical Sciences, University of Tabriz, Tabriz, Iran,*
[2] *University of Applied Science and Technology of IDEM , Tabriz, Iran,*
[3] *Department of Engineering, Faculty of engineering, Marand, Iran,*
[4] *Department of Mathematics, Karaj Branch, Islamic Azad university, Karaj, Iran,*
[5] *Department of Mathematics, Ahar Branch, Islamic Azad university, Ahar, Iran*

(Received: 23 February 2015; Accepted: 13 September 2015)

Abstract In this work, we establish the exact solutions to the modified forms of Degasperis–Procesi (DP) and Camassa–Holm (CH) equations. The generalized (G'/G)-expansion and generalized tanh-coth methods were used to construct solitary wave solutions of nonlinear evolution equations. The generalized (G'/G)-expansion method presents a wider applicability for handling nonlinear wave equations. It is shown that the (G'/G)-expansion method, with the help of symbolic computation, provides a straightforward and powerful mathematical tool for solving nonlinear evolution equations in mathematical physics.

Keywords The generalized (G'/G)-expansion method; tanh-coth method; Modified forms of Degasperis–Procesi; Camassa–Holm equations; Solitary wave solutions; Solitons

AMS 2010 subject classifications 35Dxx

DOI: 10.19139/soic.v3i4.131

1. Introduction

In the recent years, the investigation of the traveling wave solutions for nonlinear partial differential equations plays an important role in the study of nonlinear physical phenomena. During the past decades, both mathematicians and physicists have devoted considerable effort to the study of exact and numerical solutions of the nonlinear ordinary or partial differential equations corresponding to the nonlinear problems. Many powerful methods have been presented. For instance, Hirota's bilinear method [1], the inverse scattering transform [2], F-expansion method [3], sine-cosine method [4], homotopy perturbation method [5], homotopy analysis method [6, 7], variational iteration method [9], tanh-coth method [10, 11], Exp-function method [12, 13, 14, 15], central difference and Newton iteration method [16], septic B-spline collocation method [17] and so on. Here, we use of an effective method, (G'/G)-expansion method, for constructing a range of exact solutions for the nonlinear partial differential equations, first proposed by Wang et al [19]. Zhang et al. [20] examined the generalized (G'/G)-expansion method and its applications. Authors of [21] obtained the exact solutions for the symmetric regularized long wave equation using the (G'/G)-expansion method. Fazli and Manafian [22] applied the (G'/G)-expansion method for solving the couple Boiti-Leon-Pempinelli system. Also, Bekir [23] used to application of the (G'/G)-expansion method for nonlinear evolution equations. In [24], solitary wave and periodic wave solutions via (G'/G)-expansion method

*Correspondence to: Jalil Manafian (Email: j_ manafianheris@tabrizu.ac.ir). Department of Applied Mathematics, Faculty of Mathematical Sciences, University of Tabriz, Tabriz, Iran.

ISSN 2310-5070 (online) ISSN 2311-004X (print)

have been obtained. The CH equation

$$u_t - u_{xxt} + 3uu_x = 2u_x u_{xx} + uu_{xxx}, \tag{1}$$

has been investigated in the literature in [25, 40] and the references therein. However, changing the coefficients 3 and 2 in Eq. (1) to 4 and 3, respectively, gives the DP equation

$$u_t - u_{xxt} + 4uu_x = 3u_x u_{xx} + uu_{xxx}. \tag{2}$$

The CH equation has peaked solitary wave solutions of the form [40]

$$u(x,t) = ce^{-|x-ct|}, \tag{3}$$

where c is the wave speed. The name 'peakon', that is, solitary wave with slope discontinuities, was used to single them from general solitary wave solutions since they have a corner at the peak of height c [40]. In this article an application of the proposed method to the modified forms of Degasperis–Procesi and Camassa–Holm equations is illustrated. we will investigate modified forms of the DP and the CH equations given by

$$u_t - u_{xxt} + 4u^2 u_x = 3u_x u_{xx} + uu_{xxx}, \tag{4}$$

and

$$u_t - u_{xxt} + 3u^2 u_x = 2u_x u_{xx} + uu_{xxx}, \tag{5}$$

respectively. In [25] Mustafa studied DP equation. Multi-peakon solutions of DP equation was investigated by Lundmark [26]. Shen have obtained the new integrable equation with peakon and compactons [27]. Chen have acquired the new type of bounded waves for DP equation [28]. In [29, 30] have obtained the integrable shallow water equation and completely integrable shallow water equation respectively. In [31, 32] Liu and et al. have studied the CH equation and have obtained peaked wave solutions [31] and peakons [32]. Peakons and periodic cusp waves of the generalized CH equation have been studied by [33]. Also in [34] Tian and et al. have studied the generalized CH equation and obtained new peaked solitary wave solutions. Boyd [35] derived a perturbation series which converges even at the peakon limit, and gave three analytical representation for the spatially periodic generalization of the peakon, called "coshoidal wave". Cooper and Shepard [36] derived approximate solitary wave solution by using some variational functions. Constantin [37] gave a mathematical description of the existence of interacting solitary waves. Wazwaz [38] derived a solitary wave solutions for modified forms of Degasperis-Procesi and Camassa-Holm equations. A class of nonlinear fourth order variant of a generalized Camassa-Holm equation was investigated by [39] where have obtained compact and noncompact solutions. Recently, CH equation and some its generalized forms have been studied by many authors, for instance, Wazwaz [40] have obtained new solitary wave solutions to the modified forms of DP and CH equations. He [41] derived exact travelling wave solutions of a generalized CH equation using the integral bifurcation method. In [42] an integrable shallow water equation has been discussed with linear and nonlinear dispersion. Also, new integrable equation with peakon solutions has been studied by Degasperis [43]. Explicit solutions of the Camassa-Holm equation have been obtained in [44]. Guo [45] obtained periodic cusp wave solutions and single-solitons for the b-equation. Bäcklund transformation has been applied for the modified DGH equation by [46]. In [47] bifurcations of travelling wave solutions for a variant of CH equation investigated by He. In [48] Rui et. al, applied the integral bifurcation method and its application for solving a family of third-order dispersive PDEs. Finally, Liu and Qian [49] have derived peakons and their bifurcation for the generalized CH equation. The article is organized as follows: In Section 2, first we briefly give the steps of the methods and apply these methods to solve the nonlinear partial differential equations. In Section 3, the application of the (G'/G)-expansion method to the modified DP equation will be introduced briefly. Also, Section 4 by using the results obtained in Section 2, the corresponding solutions of the modified CH equation can be obtained. Also a conclusion is given in Section 5. Finally some references are given at the end of this paper.

2. Basic ideas of two methods

2.1. The $(\frac{G'}{G})$-expansion method

Step 1. For a given NLPDE with independent variables $X = (x, t)$ and dependent variable u:

$$\mathcal{P}(u, u_t, u_x, u_{xx}, u_{xt}, ...) = 0, \tag{6}$$

can be converted to an ODE

$$\mathcal{M}(u, -cu', u', u'', -cu'', ...) = 0, \tag{7}$$

by transformation $\xi = x - ct$ is wave variable. Also, c is constant to be determined later.
Step 2. We seek its solutions in the more general polynomial form as following

$$u(\xi) = a_0 + \sum_{k=1}^{m} a_k \left(\frac{G'(\xi)}{G(\xi)}\right)^k, \tag{8}$$

where $G(\xi)$ satisfies the second order LODE in the form

$$G''(\xi) + \lambda G'(\xi) + \mu G(\xi) = 0, \tag{9}$$

where a_0, $a_k(k = 1, 2, ..., m)$, λ and μ are constants to be determined later, $a_m = 0$, but the degree of which is generally equal to or less than $m - 1$, the positive integer m can be determined by considering the homogeneous balance between the highest order derivatives and nonlinear terms appearing in Eq. (7).
Step 3. Substituting Eq. (8) and Eq. (9) into Eq. (7) with the value of n obtained in Step 1. Collecting the coefficients of $\left(\frac{G'(\xi)}{G(\xi)}\right)^k$ $(k = 0, 1, 2, ...)$, then setting each coefficient to zero, we can get a set of over-determined partial differential equations for $a_0, a_i(i = 1, 2, ..., n), \lambda, c$ and μ with the aid of symbolic computation Maple.
Step 4. Solving the algebraic equations in Step 3, then substituting $a_i, ..., a_m, c$ and general solutions of Eq. (9) into Eq. (8) we can obtain a series of fundamental solutions of Eq. (6) depending of the solution $G(\xi)$ of Eq. (9).

2.2. The generalized tanh–coth method

Step 1. For a given NLPDE with independent variables $X = (x, t)$ and dependent variable u:

$$\mathcal{P}(u, u_t, u_x, u_{xx}, u_{xt}, ...) = 0, \tag{10}$$

can be converted to an ODE

$$\mathcal{M}(u, -cu', u', u'', -cu'', ...) = 0, \tag{11}$$

which transformation $\xi = x - ct$ is wave variable. Also, c is constant to be determined later.
Step 2. We introduce the Riccati equation as following

$$\Phi' = r + p\Phi + q\Phi^2, \qquad \Phi = \Phi(\xi), \qquad \xi = x - ct, \tag{12}$$

leads to the change of derivatives

$$\frac{d}{d\xi} = \left(r + p\Phi + q\Phi^2\right) \frac{d}{d\Phi}, \tag{13}$$

$$\frac{d^2}{d\xi^2} = \left(r + p\Phi + q\Phi^2\right) \left[(p + 2q\Phi)\frac{d}{d\Phi} + (r + p\Phi + q\Phi^2)\frac{d^2}{d\Phi^2}\right], \tag{14}$$

$$\frac{d^3}{d\xi^3} = \left(r + p\Phi + q\Phi^2\right)\left[\left(6q^2\Phi^2 + 6pq\Phi + 2rq + p^2\right)\frac{d}{d\Phi} + \left(6q^2\Phi^3 + 9pq\Phi^2 + 3(p^2+2rq)\Phi + 3rp\right)\frac{d^2}{d\Phi^2} + \left(r + p\Phi + q\Phi^2\right)^2\frac{d^3}{d\Phi^3}\right], \tag{15}$$

which admits the use of a finite series of functions of the form

$$u(\xi) = S(\Phi) = \sum_{k=0}^{m} a_k\Phi^k + \sum_{k=1}^{m} b_k\Phi^{-k}, \tag{16}$$

where $a_k(k = 0, 2, ..., m)$, $b_k(k = 1, 2, ..., m)$, p, r and q are constants to be determined later. But, the positive integer m can be determined by considering the homogeneous balance between the highest order derivatives and nonlinear terms appearing in Eq. (11). If m is not an integer, then a transformation formula should be used to overcome this difficulty. For aforementioned method, expansion (16) reduces to the standard tanh method for $b_k = 0, 1 \leq k \leq m$.

Step 3. Substituting Eqs. (12) –(15) into Eq. (11) with the value of m obtained in Step 2. Collecting the coefficients of Φ^k $(k = 0, 1, 2, ...)$, then setting each coefficient to zero, we can get a set of over-determined partial differential equations for $a_0, a_i(i = 1, 2, ..., m), b_i(i = 1, 2, ..., m)$ p, q and r with the aid of symbolic computation Maple.

Step 4. Solving the algebraic equations in Step 3, then substituting $a_0, a_1, b_1, ..., a_m, b_m, c$ in Eq. (16).

Step 5. We will consider the following twenty seven solutions of generalized Riccati differential equation (12) are given in [10, 11].

Case 1: **For each** $pq \neq 0$ **or** $qr \neq 0$ **and** $\Delta = p^2 - 4qr > 0$**, Eq. (12) has the following solutions**

$$\Phi_1(\xi) = \frac{-1}{2q}\left[p + \sqrt{\Delta}\tanh\left(\sqrt{\Delta}\xi/2\right)\right], \tag{17}$$

$$\Phi_2(\xi) = \frac{-1}{2q}\left[p + \sqrt{\Delta}\coth\left(\sqrt{\Delta}\xi/2\right)\right], \tag{18}$$

$$\Phi_3(\xi) = \frac{-1}{2q}\left[p + \sqrt{\Delta}\left[\tanh\left(\sqrt{\Delta}\xi\right) \pm i\operatorname{sech}\left(\sqrt{\Delta}\xi\right)\right]\right], \quad i = \sqrt{-1}, \tag{19}$$

$$\Phi_4(\xi) = \frac{-1}{2q}\left[p + \sqrt{\Delta}\left[\coth\left(\sqrt{\Delta}\xi\right) \pm \operatorname{csch}\left(\sqrt{\Delta}\xi\right)\right]\right], \tag{20}$$

$$\Phi_5(\xi) = \frac{-1}{4q}\left[2p + \sqrt{\Delta}\left[\tanh\left(\sqrt{\Delta}\xi/4\right) \pm \coth\left(\sqrt{\Delta}\xi/4\right)\right]\right], \tag{21}$$

$$\Phi_6(\xi) = \frac{1}{2q}\left[-p + \frac{\sqrt{(A^2+B^2)\Delta} - A\sqrt{\Delta}\cosh\left(\sqrt{\Delta}\xi\right)}{A\sinh\left(\sqrt{\Delta}\xi\right) + B}\right], \tag{22}$$

$$\Phi_7(\xi) = \frac{1}{2q}\left[-p - \frac{\sqrt{(B^2-A^2)\Delta} + A\sqrt{\Delta}\cosh\left(\sqrt{\Delta}\xi\right)}{A\sinh\left(\sqrt{\Delta}\xi\right) + B}\right], \tag{23}$$

where A and B are two nonzero real constants and satisfy $B^2 - A^2 > 0$

$$\Phi_8(\xi) = \frac{2r\cosh\left(\sqrt{\Delta}\xi/2\right)}{\sqrt{\Delta}\sinh\left(\sqrt{\Delta}\xi/2\right) - p\cosh\left(\sqrt{\Delta}\xi/2\right)}, \tag{24}$$

$$\Phi_9(\xi) = \frac{-2r\sinh\left(\sqrt{\Delta}\xi/2\right)}{p\sinh\left(\sqrt{\Delta}\xi/2\right) - \sqrt{\Delta}\cosh\left(\sqrt{\Delta}\xi/2\right)}, \tag{25}$$

$$\Phi_{10}(\xi) = \frac{2r\cosh\left(\sqrt{\Delta}\xi/2\right)}{\sqrt{\Delta}\sinh\left(\sqrt{\Delta}\xi\right) - p\cosh\left(\sqrt{\Delta}\xi\right) \pm i\sqrt{\Delta}}, \tag{26}$$

$$\Phi_{11}(\xi) = \frac{2r\sinh\left(\sqrt{\Delta}\xi/2\right)}{-p\sinh\left(\sqrt{\Delta}\xi\right) + \sqrt{\Delta}\cosh\left(\sqrt{\Delta}\xi\right) \pm \sqrt{\Delta}}, \tag{27}$$

$$\Phi_{12}(\xi) = \frac{4r\sinh\left(\sqrt{\Delta}\xi/4\right)\cosh\left(\sqrt{\Delta}\xi/4\right)}{-2p\sinh\left(\sqrt{\Delta}\xi/4\right)\cosh\left(\sqrt{\Delta}\xi/4\right) + 2\sqrt{\Delta}\cosh^2\left(\sqrt{\Delta}\xi/4\right) - \sqrt{\Delta}}. \tag{28}$$

Case 2: **For each $pq \neq 0$ or $qr \neq 0$ and $\Delta = p^2 - 4qr < 0$, Eq. (12) has the following solutions**

$$\Phi_{13}(\xi) = \frac{-1}{2q}\left[p - \sqrt{-\Delta}\tan\left(\sqrt{-\Delta}\xi/2\right)\right], \tag{29}$$

$$\Phi_{14}(\xi) = \frac{-1}{2q}\left[p + \sqrt{-\Delta}\cot\left(\sqrt{-\Delta}\xi/2\right)\right], \tag{30}$$

$$\Phi_{15}(\xi) = \frac{-1}{2q}\left[p - \sqrt{-\Delta}\left[\tan\left(\sqrt{-\Delta}\xi\right) \pm \sec\left(\sqrt{-\Delta}\xi\right)\right]\right], \tag{31}$$

$$\Phi_{16}(\xi) = \frac{-1}{2q}\left[p + \sqrt{-\Delta}\left[\cot\left(\sqrt{-\Delta}\xi\right) \pm \csc\left(\sqrt{-\Delta}\xi\right)\right]\right], \tag{32}$$

$$\Phi_{17}(\xi) = \frac{-1}{4q}\left[2p - \sqrt{-\Delta}\left[\tan\left(\sqrt{-\Delta}\xi/4\right) - \cot\left(\sqrt{-\Delta}\xi/4\right)\right]\right], \tag{33}$$

$$\Phi_{18}(\xi) = \frac{1}{2q}\left[-p + \frac{\pm\sqrt{(A^2+B^2)\Delta} - A\sqrt{-\Delta}\cos\left(\sqrt{-\Delta}\xi\right)}{A\sin\left(\sqrt{-\Delta}\xi\right) + B}\right], \tag{34}$$

$$\Phi_{19}(\xi) = \frac{1}{2q}\left[-p - \frac{\pm\sqrt{(B^2-A^2)\Delta} + A\sqrt{-\Delta}\cos\left(\sqrt{-\Delta}\xi\right)}{A\sin\left(\sqrt{-\Delta}\xi\right) + B}\right], \tag{35}$$

where A and B are two nonzero real constants and satisfy $B^2 - A^2 > 0$

$$\Phi_{20}(\xi) = \frac{-2r\cos\left(\sqrt{-\Delta}\xi/2\right)}{\sqrt{-\Delta}\sin\left(\sqrt{-\Delta}\xi/2\right) + p\cos\left(\sqrt{-\Delta}\xi/2\right)}, \tag{36}$$

$$\Phi_{21}(\xi) = \frac{2r\sin\left(\sqrt{-\Delta}\xi/2\right)}{-p\sin\left(\sqrt{-\Delta}\xi/2\right) + \sqrt{-\Delta}\cos\left(\sqrt{-\Delta}\xi/2\right)}, \tag{37}$$

$$\Phi_{22}(\xi) = \frac{-2r\cos\left(\sqrt{-\Delta}\xi/2\right)}{\sqrt{-\Delta}\sin\left(\sqrt{-\Delta}\xi\right) + p\cos\left(\sqrt{-\Delta}\xi\right) \pm \sqrt{-\Delta}}, \tag{38}$$

$$\Phi_{23}(\xi) = \frac{2r\sin\left(\sqrt{-\Delta}\xi/2\right)}{-p\sin\left(\sqrt{-\Delta}\xi\right) + \sqrt{-\Delta}\cos\left(\sqrt{-\Delta}\xi\right) \pm \sqrt{-\Delta}}, \tag{39}$$

$$\Phi_{24}(\xi) = \frac{4r\sin\left(\sqrt{-\Delta}\xi/4\right)\cosh\left(\sqrt{-\Delta}\xi/4\right)}{-2p\sin\left(\sqrt{-\Delta}\xi/4\right)\cos\left(\sqrt{-\Delta}\xi/4\right) + 2\sqrt{-\Delta}\cos^2\left(\sqrt{-\Delta}\xi/4\right) - \sqrt{-\Delta}}. \tag{40}$$

Case 3: **For $r = 0$ and $pq \neq 0$ Eq. (12) has the following solutions**

$$\Phi_{25}(\xi) = \frac{-pd}{q[d + \cosh(p\xi) - \sinh(p\xi)]}, \tag{41}$$

$$\Phi_{26}(\xi) = \frac{p[\cosh(p\xi) + \sinh(p\xi)]}{q[d + \cosh(p\xi) + \sinh(p\xi)]}, \tag{42}$$

where d is an arbitrary constant.
Case 4: For $r = p = 0$ and $q \neq 0$ Eq. (12) has the following solution

$$\Phi_{27}(\xi) = \frac{-1}{q\xi + c}, \tag{43}$$

where c is an arbitrary constant.
But from (G'/G)-expansion method, we have

$$\left(\frac{G'}{G}\right)' = \frac{G''G - G'^2}{G^2} = \frac{(-\lambda G' - \mu G)G - G'^2}{G^2} = -\mu - \lambda\left(\frac{G'}{G}\right) - \left(\frac{G'}{G}\right)^2, \tag{44}$$

we set $\mathrm{F} = \frac{\mathrm{G}'}{\mathrm{G}}$, then

$$F' = -\mu - \lambda F - F^2. \tag{45}$$

Thus we obtain that the exact solutions derived by (G'/G)-expansion are same as ones by the generalized tanh-coth methods. Hence we use only the generalized (G'/G)-expansion method.

3. Applications of the (G'/G)-expansion method

3.1. The modified Degasperis–Procesi equation

In this section we employ the (G'/G)-expansion method to the modified DP equation as

$$\mathrm{u_t - u_{xxt} + 4u^2u_x = 3u_xu_{xx} + uu_{xxx}}. \tag{46}$$

The wave variable $\xi = \mathrm{x - ct}$ PDE transforms to an ODE

$$\mathrm{-cu' + cu''' + 4u^2u' = 3u'u'' + uu'''}. \tag{47}$$

Integrating Eq. (47) with respect to ξ and considering the zero constants for integration, we obtain

$$\mathrm{c(u'' - u) + \frac{4}{3}u^3 - uu'' - (u')^2 = 0}. \tag{48}$$

To determine the index, m, we balance the linear term of the highest order with the highest order nonlinear terms. Therefore, in Eq. (48) we balance $\mathrm{uu''}$ with u^3, so that $2m + 2 = 3m$, and this gives us $m = 2$. Hence, the expression for $u(\xi)$ now simplifies to:

$$\mathrm{u(\xi) = a_0 + a_1\left(\frac{G'(\xi)}{G(\xi)}\right) + a_2\left(\frac{G'(\xi)}{G(\xi)}\right)^2, \qquad a_2 \neq 0,} \tag{49}$$

and therefore

$$\mathrm{u^2(\xi) = a_2^2\left(\frac{G'(\xi)}{G(\xi)}\right)^4 + 2a_1a_2\left(\frac{G'(\xi)}{G(\xi)}\right)^3 + (a_1^2 + 2a_0a_2)\left(\frac{G'(\xi)}{G(\xi)}\right)^2 + 2a_0a_1\left(\frac{G'(\xi)}{G(\xi)}\right) + a_0^2,} \tag{50}$$

and

$$\mathrm{u_{\xi\xi}(\xi) = 6a_2\left(\frac{G'(\xi)}{G(\xi)}\right)^4} \tag{51}$$

$$+(2\mathrm{a}_1+10\mathrm{a}_2\lambda)\left(\frac{\mathrm{G}'(\xi)}{\mathrm{G}(\xi)}\right)^3+(8\mathrm{a}_2\mu+3\mathrm{a}_1\lambda+4\mathrm{a}_2\lambda^2)\left(\frac{\mathrm{G}'(\xi)}{\mathrm{G}(\xi)}\right)^2$$
$$+(6\mathrm{a}_2\lambda\mu+2\mathrm{a}_1\mu+\mathrm{a}_1\lambda^2)\left(\frac{\mathrm{G}'(\xi)}{\mathrm{G}(\xi)}\right)+2\mathrm{a}_2\mu^2+\mathrm{a}_1\lambda\mu.$$

Substituting (49)-(51) into Eq. (48) and equating the coefficients of $(G'/G)^i$; $i = 0, ..., 6$ to zero, leads to the following nonlinear system of algebraic equations:

$$(G'/G)^0 : 2a_2\mu^2+\frac{4}{3}a_0^3-a_0a_1\mu\lambda+ca_1\mu\lambda-a_1^2\mu^2-ca_0-2a_0a_2\mu^2=0, \tag{52}$$
$$(G'/G)^1 : ca_1\lambda^2+4a_0^2a_1-3a_1^2\mu\lambda+6ca_2\lambda\mu-ca_1-a_0a_1\lambda^2-2a_0a_1\mu-6a_0a_2\lambda\mu-6a_1a_2\mu^2+2ca_1\mu=0,$$
$$(G'/G)^2 : -2(\lambda^2+2\mu)a_1^2+3a_1\lambda(c-a_0)+4a_0^2a_2-6a_2^2\mu^2+4(2c\mu+\lambda^2)a_2(c-a_0)+4a_0a_1^2-15a_1a_2\lambda\mu-ca_2=0,$$
$$(G'/G)^3 : -2a_0a_1-14a_2^2\lambda\mu-18a_1a_2\mu+10ca_2\lambda-9a_1a_2\lambda^2+\frac{4}{3}a_1^3+8a_0a_1a_2-10a_0a_2\lambda+2ca_1-5a_1^2\lambda=0,$$
$$(G'/G)^4 : -6a_0a_2+6ca_2+4a_1^2a_2-21a_1a_2\lambda-8a_2^2\lambda^2+4a_0a_2^2-16a_2^2\mu-3a_1^2=0,$$
$$(G'/G)^5 : 4a_1a_2^2-12a_1a_2-18a_2^2\lambda=0,$$
$$(G'/G)^6 : -10a_2^2+\frac{4}{3}a_2^3=0.$$

Solving system (52) gives:

$$\mathrm{a}_0=\frac{15\mu}{2},\qquad \mathrm{a}_1=\frac{15\lambda}{2},\qquad \mathrm{a}_2=\frac{15}{2},\qquad \mathrm{c}=\frac{5}{2}, \tag{53}$$

or

$$\mathrm{a}_0=\frac{-9\pm\sqrt{15}\mathrm{i}}{8}+\frac{15\mu}{2},\qquad \mathrm{a}_1=\frac{15\lambda}{2},\qquad \mathrm{a}_2=\frac{15}{2},\qquad \mathrm{c}=\frac{11\mp\sqrt{15}\mathrm{i}}{8}, \tag{54}$$

where λ and μ are arbitrary constants. Substituting (53) and (54) into expression Eq. (49), can be written as

$$\mathrm{u}(\xi)=\frac{15\mu}{2}+\frac{15\lambda}{2}\left(\frac{\mathrm{G}'(\xi)}{\mathrm{G}(\xi)}\right)+\frac{15}{2}\left(\frac{\mathrm{G}'(\xi)}{\mathrm{G}(\xi)}\right)^2,\qquad \xi=\mathrm{x}-\frac{15}{2}\mathrm{t}, \tag{55}$$

or

$$\mathrm{u}(\xi)=\frac{-9\pm\sqrt{15}\mathrm{i}}{8}+\frac{15\mu}{2}+\frac{15\lambda}{2}\left(\frac{\mathrm{G}'(\xi)}{\mathrm{G}(\xi)}\right)+\frac{15}{2}\left(\frac{\mathrm{G}'(\xi)}{\mathrm{G}(\xi)}\right)^2,\qquad \xi=\mathrm{x}-\frac{11\mp\sqrt{15}\mathrm{i}}{8}\mathrm{t}. \tag{56}$$

Substituting the general solutions of Eq. (9) into (55) and (56) we have three types of exact solutions of Eq. (46) as follows:

1. When $\lambda^2-4\mu>0$, we obtain hyperbolic function solution

$$\mathrm{u}_1(\xi)=\frac{15}{8}(\lambda^2-4\mu)\left(\frac{\mathrm{C}_1\sinh\left(\frac{\sqrt{\lambda^2-4\mu}\xi}{2}\right)+\mathrm{C}_2\cosh\left(\frac{\sqrt{\lambda^2-4\mu}\xi}{2}\right)}{\mathrm{C}_1\cosh\left(\frac{\sqrt{\lambda^2-4\mu}\xi}{2}\right)+\mathrm{C}_2\sinh\left(\frac{\sqrt{\lambda^2-4\mu}\xi}{2}\right)}\right)^2 -\frac{15}{8}\lambda^2+\frac{15\mu}{2}, \tag{57}$$

where $\xi=\mathrm{x}-\frac{15}{2}\mathrm{t}$ and

$$\mathrm{u}_2(\xi)=\frac{15}{8}(\lambda^2-4\mu)\left(\frac{\mathrm{C}_1\sinh\left(\frac{\sqrt{\lambda^2-4\mu}\xi}{2}\right)+\mathrm{C}_2\cosh\left(\frac{\sqrt{\lambda^2-4\mu}\xi}{2}\right)}{\mathrm{C}_1\cosh\left(\frac{\sqrt{\lambda^2-4\mu}\xi}{2}\right)+\mathrm{C}_2\sinh\left(\frac{\sqrt{\lambda^2-4\mu}\xi}{2}\right)}\right)^2 \tag{58}$$

$$-\frac{15}{8}\lambda^2 + \frac{-9 \pm \sqrt{15}\mathrm{i}}{8} + \frac{15\mu}{2},$$

where $\xi = \mathrm{x} - \frac{11 \mp \sqrt{15}\mathrm{i}}{8}\mathrm{t}$.

2. When $\lambda^2 - 4\mu < 0$, we have trigonometric function solution

$$\mathrm{u}_3(\xi) = \frac{15}{8}(4\mu - \lambda^2)\left(\frac{-\mathrm{C}_1 \sin\left(\frac{\sqrt{4\mu-\lambda^2}\xi}{2}\right) + \mathrm{C}_2 \cos\left(\frac{\sqrt{4\mu-\lambda^2}\xi}{2}\right)}{\mathrm{C}_1 \cos\left(\frac{\sqrt{4\mu-\lambda^2}\xi}{2}\right) + \mathrm{C}_2 \sin\left(\frac{\sqrt{4\mu-\lambda^2}\xi}{2}\right)}\right)^2 \tag{59}$$

$$-\frac{15}{8}\lambda^2 + \frac{15\mu}{2},$$

where $\xi = x - \frac{15}{2}t$ and

$$\mathrm{u}_4(\xi) = \frac{15}{8}(4\mu - \lambda^2)\left(\frac{-\mathrm{C}_1 \sin\left(\frac{\sqrt{4\mu-\lambda^2}\xi}{2}\right) + \mathrm{C}_2 \cos\left(\frac{\sqrt{4\mu-\lambda^2}\xi}{2}\right)}{\mathrm{C}_1 \cos\left(\frac{\sqrt{4\mu-\lambda^2}\xi}{2}\right) + \mathrm{C}_2 \sin\left(\frac{\sqrt{4\mu-\lambda^2}\xi}{2}\right)}\right)^2 \tag{60}$$

$$-\frac{15}{8}\lambda^2 + \frac{-9 \pm \sqrt{15}\mathrm{i}}{8} + \frac{15\mu}{2},$$

where $\xi = \mathrm{x} - \frac{11 \mp \sqrt{15}\mathrm{i}}{8}\mathrm{t}$.

3. When $\lambda^2 - 4\mu = 0$, we get rational solution

$$\mathrm{u}_5(\xi) = \frac{15}{2}\frac{\mathrm{C}_2^2}{(\mathrm{C}_1 + \mathrm{C}_2\xi)^2} - \frac{15}{8}\lambda^2 + \frac{15\mu}{2}, \qquad \xi = \mathrm{x} - \frac{15}{2}\mathrm{t}, \tag{61}$$

$$\mathrm{u}_6(\xi) = \frac{15}{2}\frac{\mathrm{C}_2^2}{(\mathrm{C}_1 + \mathrm{C}_2\xi)^2} - \frac{15}{8}\lambda^2 + \frac{-9 \pm \sqrt{15}\mathrm{i}}{8} + \frac{15\mu}{2}, \qquad \xi = \mathrm{x} - \frac{11 \mp \sqrt{15}\mathrm{i}}{8}\mathrm{t}.$$

If $C_1 \neq 0, \mathrm{C}_2 = 0, \lambda > 0, \mu = 0$, then (57) and (58) give respectively

$$\mathrm{u}_7(\xi) = -\frac{15}{8}\lambda^2 \operatorname{sech}^2\left[\frac{\lambda}{2}\left(\mathrm{x} - \frac{15}{2}\mathrm{t}\right)\right], \tag{62}$$

$$\mathrm{u}_8(\xi) = \frac{-9 \pm \sqrt{15}\mathrm{i}}{8} - \frac{15}{8}\lambda^2 \operatorname{sech}^2\left[\frac{\lambda}{2}\left(\mathrm{x} - \frac{11 \mp \sqrt{15}\mathrm{i}}{8}\mathrm{t}\right)\right].$$

In particular, If $\mathrm{C}_1 \neq 0, \mathrm{C}_2 = 0, \lambda = 0, \mu > 0$, then (59) and (60) can be written as respectively

$$\mathrm{u}_9(\xi) = \frac{15}{2}\mu \sec^2\left[\sqrt{\mu}\left(\mathrm{x} - \frac{15}{2}\mathrm{t}\right)\right], \tag{63}$$

$$\mathrm{u}_{10}(\xi) = \frac{-9 \pm \sqrt{15}\mathrm{i}}{8} + \frac{15}{2}\mu \sec^2\left[\sqrt{\mu}\left(\mathrm{x} - \frac{15}{2}\mathrm{t}\right)\right].$$

But, if $\mathrm{C}_1 \neq 0, \mathrm{C}_2 = 0, \lambda = 0, \mu < 0$, then (57) and (58) get respectively

$$\mathrm{u}_{11}(\xi) = \frac{15}{2}\mu \operatorname{sech}^2\left[\sqrt{-\mu}\left(\mathrm{x} - \frac{15}{2}\mathrm{t}\right)\right], \tag{64}$$

$$\mathrm{u}_{12}(\xi) = \frac{-9 \pm \sqrt{15}\mathrm{i}}{8} + \frac{15}{2}\mu \operatorname{sech}^2\left[\sqrt{\mu}\left(\mathrm{x} - \frac{11 \mp \sqrt{15}\mathrm{i}}{8}\mathrm{t}\right)\right],$$

which are the exact solutions of the DP equation. It can be seen that some results are similar to the results in [40].

3.2. The modified Camassa–Holm equation

We next consider the modified CH equation [40] as

$$u_t - u_{xxt} + 3u^2u_x = 2u_xu_{xx} + uu_{xxx}. \tag{65}$$

The wave variable $\xi = x - ct$ PDE transforms to an ODE

$$-cu' + cu''' + 43u^2u' = 2u'u'' + uu''', \tag{66}$$

where by integrating Eq. (66) with respect to ξ and considering the zero constants for integration, we get

$$c(u'' - u) + u^3 - uu'' - \frac{1}{2}(u')^2 = 0. \tag{67}$$

Applying the procedure given in the previous sections and balancing uu'' and u^3 in Eq. (67), we obtain $m = 2$. Proceeding as before we get

$$(G'/G)^0 : a_0^3 + 2ca_2\mu^2 - ca_0 + ca_1\mu\lambda - 2a_0a_2\mu^2 - a_0a_1\mu\lambda - \frac{1}{2}a_1^2\mu^2 = 0, \tag{68}$$
$$(G'/G)^1 : 6ca_2\lambda\mu - 4a_1a_2\mu^2 - a_0a_1\lambda^2 - 2a_1^2\mu\lambda - 6a_0a_2\lambda\mu - ca_1 + 3a_0^2a_1 + ca_1\lambda^2 + 2ca_1\mu - 2a_0a_1\mu = 0,$$
$$(G'/G)^2 : (3a_1\lambda + 8a_2\mu)(c - a_0) + 3a_1^2(a_0 - \mu) + (4ca_2 - 4a_0a_2 - \frac{3}{2}a_1^2)\lambda^2 + a_2(3a_0^2 - 4a_2\mu^2 - 11a_1\lambda\mu - c) = 0,$$
$$(G'/G)^3 : -10a_0a_2\lambda + 2ca_1 - 14a_1a_2\mu - 2a_0a_1 + a_1^3 - 10a_2^2\lambda\mu + 10ca_2\lambda - 7a_1a_2\lambda^2 + 6a_0a_1a_2 - 4a_1^2\lambda = 0,$$
$$(G'/G)^4 : 3a_0a_2^2 - 12a_2^2\mu - 6a_0a_2 - 17a_1a_2\lambda - 6a_2^2\lambda^2 + 3a_1^2a_2 + 6ca_2 - \frac{5}{2}a_1^2 = 0,$$
$$(G'/G)^5 : -14a_2^2\lambda - 10a_1a_2 + 3a_1a_2^2 = 0,$$
$$(G'/G)^6 : -8a_2^2 + a_2^3 = 0.$$

Solving system (68) gives:

$$a_0 = 8\mu, \qquad a_1 = 8\lambda, \qquad a_2 = 8, \qquad c = 2, \tag{69}$$

or

$$a_0 = 8\mu - 1, \qquad a_1 = 8\lambda, \qquad a_2 = 8, \qquad c = 1, \tag{70}$$

where λ and μ are arbitrary constants. Substituting (69) and (70) into expression (49), can be written as

$$u(\xi) = 8\mu + 8\lambda\left(\frac{G'(\xi)}{G(\xi)}\right) + 8\left(\frac{G'(\xi)}{G(\xi)}\right)^2, \qquad \xi = x - 2t, \tag{71}$$

or

$$u(\xi) = 8\mu - 1 + 8\lambda\left(\frac{G'(\xi)}{G(\xi)}\right) + 8\left(\frac{G'(\xi)}{G(\xi)}\right)^2, \qquad \xi = x - t. \tag{72}$$

1. When $\lambda^2 - 4\mu > 0$, we obtain hyperbolic function solution

$$u_1(\xi) = 2(\lambda^2 - 4\mu)\left(\frac{C_1\sinh\left(\frac{\sqrt{\lambda^2-4\mu}\xi}{2}\right) + C_2\cosh\left(\frac{\sqrt{\lambda^2-4\mu}\xi}{2}\right)}{C_1\cosh\left(\frac{\sqrt{\lambda^2-4\mu}\xi}{2}\right) + C_2\sinh\left(\frac{\sqrt{\lambda^2-4\mu}\xi}{2}\right)}\right)^2 \tag{73}$$
$$-2\lambda^2 + 8\mu, \qquad \xi = x - 2t,$$

and

$$u_2(\xi) = 2(\lambda^2 - 4\mu)\left(\frac{C_1 \sinh\left(\frac{\sqrt{\lambda^2-4\mu}\xi}{2}\right) + C_2 \cosh\left(\frac{\sqrt{\lambda^2-4\mu}\xi}{2}\right)}{C_1 \cosh\left(\frac{\sqrt{\lambda^2-4\mu}\xi}{2}\right) + C_2 \sinh\left(\frac{\sqrt{\lambda^2-4\mu}\xi}{2}\right)}\right)^2 \tag{74}$$

$$-2\lambda^2 + 8\mu - 1, \qquad \xi = x - t.$$

2. When $\lambda^2 - 4\mu < 0$, we have trigonometric function solution

$$u_3(\xi) = 2(4\mu - \lambda^2)\left(\frac{-C_1 \sin\left(\frac{\sqrt{4\mu-\lambda^2}\xi}{2}\right) + C_2 \cos\left(\frac{\sqrt{4\mu-\lambda^2}\xi}{2}\right)}{C_1 \cos\left(\frac{\sqrt{4\mu-\lambda^2}\xi}{2}\right) + C_2 \sin\left(\frac{\sqrt{4\mu-\lambda^2}\xi}{2}\right)}\right)^2 \tag{75}$$

$$-2\lambda^2 + 8\mu, \qquad \xi = x - 2t,$$

and

$$u_4(\xi) = 2(4\mu - \lambda^2)\left(\frac{-C_1 \sin\left(\frac{\sqrt{4\mu-\lambda^2}\xi}{2}\right) + C_2 \cos\left(\frac{\sqrt{4\mu-\lambda^2}\xi}{2}\right)}{C_1 \cos\left(\frac{\sqrt{4\mu-\lambda^2}\xi}{2}\right) + C_2 \sin\left(\frac{\sqrt{4\mu-\lambda^2}\xi}{2}\right)}\right)^2 \tag{76}$$

$$-2\lambda^2 + 8\mu - 1, \qquad \xi = x - t.$$

3. When $\lambda^2 - 4\mu = 0$, we get rational solution

$$u_5(\xi) = 2\frac{C_2^2}{(C_1 + C_2\xi)^2} - 2\lambda^2 + 8\mu, \qquad \xi = x - 2t, \tag{77}$$

$$u_6(\xi) = 2\frac{C_2^2}{(C_1 + C_2\xi)^2} - 2\lambda^2 + 8\mu - 1, \qquad \xi = x - t.$$

If $C_1 \neq 0, C_2 = 0, \lambda > 0, \mu = 0$, then (73) and (74) can be written respectively as

$$u_7(\xi) = -2\lambda^2 \operatorname{sech}^2\left[\frac{\lambda}{2}(x - 2t)\right], \tag{78}$$

$$u_8(\xi) = -1 - 2\lambda^2 \operatorname{sech}^2\left[\frac{\lambda}{2}(x - 2t)\right].$$

In particular, If $C_1 \neq 0, C_2 = 0, \lambda = 0, \mu > 0$, then (75) and (76) give respectively

$$u_9(\xi) = 8\mu \sec^2[\sqrt{\mu}(x - 2t)], \tag{79}$$

$$u_{10}(\xi) = -1 + 8\mu \sec^2[\sqrt{\mu}(x - t)].$$

But, if $C_1 \neq 0, C_2 = 0, \lambda = 0, \mu < 0$, then (73) and (74) get respectively

$$u_{11}(\xi) = 8\mu \operatorname{sech}^2[\sqrt{-\mu}(x - 2t)], \tag{80}$$

$$u_{12}(\xi) = -1 + 8\mu \operatorname{sech}^2[\sqrt{-\mu}(x - t)],$$

which are the exact solutions of the CH equation. It can be seen that the results are similar to the results in [40].

4. Conclusion

The generalized (G'/G)-expansion method was successfully used to establish periodic wave and solitary wave solutions. The obtained results complement the useful works of others for this important equations. Generalized (G'/G)-expansion method is a useful method for finding travelling wave solutions of nonlinear evolution equations. These exact solutions include three types hyperbolic function solution, trigonometric function solution and rational solution. The generalized (G'/G)-expansion method is more powerful in searching for exact solutions of NLPDEs. Some of these results are in agreement with the results reported specially by [40]. Also, new results are formally developed in this article. It can be concluded that the this method is a very powerful and efficient technique in finding exact solutions for wide classes of problems.

Acknowledgement

The authors are very grateful to the reviewers for carefully reading the paper and for their constructive comments and suggestions which have improved the paper.

REFERENCES

1. R. Hirota, *The Direct Method in Soliton Theory*, Cambridge University Press, 2004.
2. M.J. Ablowitz, P.A. Clarkson, *Solitons, nonlinear evolution equations and inverse scattering*, Cambridge: Cambridge University Press, 1991.
3. M.A. Abdou, *The extended F-expansion method and its application for a class of nonlinear evolution equations*, Chaos Solitons Fractals, vol. 31, pp. 95-104, 2007.
4. A. M. Wazwaz, *Travelling wave solutions for combined and double combined sine-cosine-Gordon equations by the variable separated ODE method*, Applied Mathematics and Computation, vo. 177, pp. 755-760 2006.
5. M. Dehghan, J. Manafian, The solution of the variable coefficients fourth–order parabolic partial differential equations by homotopy perturbation method, Z. Naturforsch, vol. 64a, pp. 420-430, 2009.
6. M. Dehghan, J. Manafian, A. Saadatmandi, *Solving nonlinear fractional partial differential equations using the homotopy analysis method*, Numer Methods Partial Differential Eq, vol. 26, pp. 448-479, 2010.
7. M. Dehghan, J. Manafian, A. Saadatmandi, *The solution of the linear fractional partial differential equations using the homotopy analysis method*, Z. Naturforsch, vol. 65a, pp. 935-949, 2010.
8. J. H. He, *Variational iteration method a kind of non-linear analytical technique: some examples*, International Journal Nonlinear Mechanics, vol. 34, pp. 699-708, 1999.
9. M. Dehghan, J. Manafian, A. Saadatmandi, *Application of semi–analytic methods for the Fitzhugh–Nagumo equation, which models the transmission of nerve impulses*, Mathematical Methods in the Applied Sciences, vol. 33, pp. 1384-1398, 2010.
10. J. Manafian Heris, M. Lakestani, *Solitary wave and periodic wave solutions for variants of the KdV-Burger and the K(n, n)-Burger equations by the generalized tanh-coth method*, Communications in Numerical Analysis, vol. 2013, pp. 1-18, 2013.
11. J. Manafian Heris, M. Lakestani, *Exact solutions for the integrable sixth-order Drinfeld-Sokolov-Satsuma-Hirota system by the analytical methods*, International Scholarly Research Notices, vol. 2014, pp. 1-8, 2014.
12. M. Dehghan, J. Manafian Heris, A. Saadatmandi, *Application of the Exp-function method for solving a partial differential equation arising in biology and population genetics*, International Journal of Numerical Methods for Heat and Fluid Flow, vol. 21, pp. 736-753, 2011.
13. M. Dehghan, J. Manafian Heris, A. Saadatmandi, *Analytical treatment of some partial differential equations arising in mathematical physics by using the Exp-function method*, International Journal Modern Physics B, vol. 25, pp. 2965-2981, 2011.
14. J. Manafian Heris, M. Bagheri, *Exact solutions for the modified KdV and the generalized KdV equations via Exp-function method*, Journal of Mathematical Extension, vol. 24, pp. 77-98, 2010.
15. J. Manafian Heris, I. Zamanpour, *Analytical treatment of the coupled Higgs equation and the Maccari system via Exp-Function method*, Acta Universitatis Apulensis, vol. 33, pp. 203-216, 2013.
16. X. Wang, T. Zeng, *Solve thermal explosion model by central difference and Newton iteration method*, Statistics, Optimization and Information Computing, vol. 3, pp. 197-205, 2015.
17. B. G. Karakoc, H. Zeybek, T. Ak, *Numerical solutions of the Kawahara equation by the septic B-spline collocation method*, Statistics, Optimization and Information Computing, vol. 2, pp. 2011-221, 2015.
18. J. Manafian, M. Lakestani, *Optical solitons with Biswas-Milovic equation for Kerr law nonlinearity*, Europen Physics Journal-Plus vol. 130, pp. 1-13, 2015.
19. M. Wang, X. Li, J. Zhang, *The* $(\frac{G'}{G})$*-expansion method and travelling wave solutions of nonlinear evolution equations in mathematical physics*, Physics Letter A, vol. 372, pp. 417-423, 2008.
20. J. Zhang, X. Wei, Y. Lu, *A generalized* $(\frac{G'}{G})$*-expansion method and its applications evolution equations in mathematical physics*, Physics Letter A, vol. 372, pp. 3653-3658, 2008.

21. J. Manafian, I. Zamanpour, *Exact travelling wave solutions of the symmetric regularized long wave (SRLW) using analytical methods*, Statistics, Optimization and Information Computing, vol. 2, pp. 47-55, 2014.
22. M. Fazli Aghdaei, J. Manafianheris, *Exact solutions of the couple Boiti-Leon-Pempinelli system by the generalized* $(\frac{G'}{G})$*-expansion method*, Journal of Mathematical Extension, vol. 5, pp. 91-104, 2011.
23. A. Bekir, *Application of the* $(\frac{G'}{G})$*-expansion method for nonlinear evolution equations*, Physics Letter A, vol. 372, pp. 3400-3406, 2008.
24. J. Manafian, M. Lakestani, *Solitary wave and periodic wave solutions for Burgers, Fisher, Huxley and combined forms of these equations by the (G'/G)-expansion method*, Pramana Journal of Physics, vol. 2, pp. 1-22, 2015.
25. OG. Mustafa, *A note on the Degasperis-Procesi equation*, Journal Nonlinear Mathematical Physics, vol. 12, pp. 10-14, 2005.
26. H. Lundmark, J. Szmigielski, *Multi–peakon solutions of Degasperis–Procesi equation*, Inverse Problems, vol. 19, pp. 1241-1245, 2003.
27. J. Shen, W. Xu, W. Lei, *Bifurcations of travelling wave solutions in a new integrable equation with peakon and compactons*, Chaos Solitons Fractals, vol. 27, pp. 413-425, 2006.
28. C. Chen, M. Tang, *A new type of bounded waves for Degasperis–Procesi equation*, Chaos Solitons Fractals, vol. 27, pp. 698-704, 2006.
29. R. Camassa, D. Holm, *An integrable shallow water equation with peaked solitons*, Physical Review Letters, vol. 71, pp. 1661-1664, 1993.
30. R. Camassa, *Characteristics and the initial value problem of a completely integrable shallow water equation*, Discrete and Continuous Dynamical System-Series B, vol. 3, pp. 115-139, 2003.
31. Z. Liu, R. Wang, Z. Jing, *Peaked wave solutions of Camassa–Holm equation*, Chaos Solitons Fractals, vol. 19, pp. 77-92, 2004.
32. Z. Liu, T. Qian, *Peakons of the Camassa–Holm equation*, Applied Mathematics and Modeling, vol. 26, pp. 473-480, 2002.
33. T. Qian, M. Tang, *Peakons and periodic cusp waves in a generalized Camassa–Holm equation*, Chaos Solitons Fractals, vol. 12, pp. 1347-1360, 2001.
34. L. Tian, X. Song, *New peaked solitary wave solutions of the generalized Camassa-Holm equation*, Chaos Solitons Fractals, vol. 19, pp. 621-637, 2004.
35. J. P. Boyd, *Peakons and coshoidal waves: traveling wave solutions of the Camassa-Holm equation*, Applied Mathematics and Computation, vol. 81, pp. 173-187, 1997.
36. F. Cooper, H. Shepard, *Solitons in the Camassa-Holm shallow water equation*, Physics Letter A, vol. 194, pp. 246-250, 1994.
37. A. Constantin, *Soliton interactions for the CamassaHolm equation*, Exposition Mathematics, vol. 15, pp. 251-264, 1997.
38. A. M. Wazwaz, *Solitary wave solutions for modified forms of Degasperis-Procesi and Camassa-Holm equations*, Physics Letter A, vol. 352, pp. 500-504, 2006.
39. A. M. Wazwaz, *A class of nonlinear fourth order variant of a generalized Camassa–Holm equation with compact and noncompact solutions*, Applied Mathematics and Computation, vol. 165, pp. 485-501 , 2005.
40. A. M. Wazwaz, *New solitary wave solutions to the modified forms of Degasperis–Procesi and Camassa–Holm equations*, Applied Mathematics and Computation, vol. 186, pp. 130-141, 2007.
41. B. He, W. Rui, C. Chen, S. Li, *Exact travelling wave solutions of a generalized Camassa-Holm equation using the integral bifurcation method*, Applied Mathematics and Computation, vol. 206, pp. 141-149, 2008.
42. H.R. Dullin, G.A. Gottwald, D.D. Holm, *An integrable shallow water equation with linear and nonlinear dispersion*, Physics Review Letter, vol. 87, pp. 194501-194504, 2001.
43. A. Degasperis, D.D. Holm, A.N.W. Hone, *A new integrable equation with peakon solutions*, Theor. Math. Phys, vol. 33, pp. 1463-1474, 2002.
44. E.J. Parkes, V.O. Vakhnenko, *Explicit solutions of the Camassa-Holm equation*, Chaos Solitons Fractal, vol. 26, pp. 1309-1316, 2005.
45. B.L. Guo, Z.R. Liu, *Periodic cusp wave solutions and single-solitons for the b-equation*, Chaos Solitons and Fractal, vol. 23, pp. 1451-1463, 2005.
46. J.L. Yin, *Painleve integrability, Backlund transformation and solitary solutions stability of modified DGH equation*, Int. J. Nonlinear Sciences, vol. 2, pp. 183-187, 2006.
47. B. He, J.B. Li, Y. Long, W.G. Rui, *Bifurcations of travelling wave solutions for a variant of Camassa-Holm equation*, Nonlinear Analysis: Real World Applied, vol. 9, pp. 222-232, 2008.
48. W.G. Rui, B. He, Y. Long, C. Chen, *The integral bifurcation method and its application for solving a family of third-order dispersive PDEs*, Nonlinear Analysis, vol. 69, pp. 1256-1267, 2008.
49. Z.R. Liu, T.F. Qian, *Peakons and their bifurcation in a generalized Camassa-Holm equation*, International Journal Bifurcation Chaos, vol. 11, pp. 781-792, 2001.