STATISTICS, OPTIMIZATION AND INFORMATION COMPUTING
Stat., Optim. Inf. Comput., Vol. **3**, September 2015, pp 249–258.
Published online in International Academic Press (www.IAPress.org)

Estimation Approaches for Mean Response Time of a Two Stage Open Queueing Network Model

V. K. Gedam [1], S. B. Pathare [2,*]

[1] *Department of Statistics, Savitribai Phule Pune University, Pune-411007, Maharashtra, India.*
[2] *Indira College of Commerce and Science,Pune-411033, Maharashtra, India.*

Received: 16 April 2014; Accepted: 25 March 2015
Editor: Paulo Rodrigues

Abstract In the analysis of queueing network models, the response time plays an important role in studying the various characteristics. In this paper data based recurrence relation is used to compute a sequence of response time. The sample means from those response times, denoted by $\hat{r_1}$ and $\hat{r_2}$ are used to estimate true mean response time r_1 and r_2. Further we construct some confidence intervals for mean response time r_1 and r_2 of a two stage open queueing network model. A numerical simulation study is conducted in order to demonstrate performance of the proposed estimator $\hat{r_1}$ and $\hat{r_2}$ and bootstrap confidence intervals of r_1 and r_2. Also we investigate the accuracy of the different confidence intervals by calculating the coverage percentage, average length, relative coverage and relative average length.

Keywords Coverage percentage, Response Time, Relative coverage, Relative average length.

AMS 2010 subject classifications: 60K20, 60K25, 68M20, 90B22.

DOI: 10.19139/soic.v3i3.79

1. Introduction

The response time is defined as the time spent by a customer from arrival until it departs. It can also be viewed as the time elapsed from the instant of job arrival until its completion. The statistical inference in queueing networks are rarely found in the literature and the work of related problems in the past mainly concentrates on only parametric statistical inference, in which the distribution of population is with a known form. Burke [1] has shown that the output of an M/M/1 queue is also Poisson with rate λ. Jackson [17]showed that the product form solution also applies to open network of Markovian queues with feedback, also Jackson theorem states that each node behaves like an independent queue. Disney [5] introduces basic properties of queueing networks. Thiruvaiyaru, Basawa and Bhat [26] established maximum likelihood estimators of the parameters of an open Jackson network. Thiruvaiyaru and Basawa [25] considered the problem of estimation for the parameters in a Jacksons type queueing network. Ke and Chu [18] constructed various confidence intervals for intensity parameter of a queueing system.

So far very few authors have studied the nonparametric statistical inferences. Efron, the greatest statistician in the field of nonparametric resampling approach, originally developed and proposed the bootstrap[6, 7, 8], which is a resampling technique that can be effectively applied to estimate the sampling distribution of any statistic. For necessary background on bootstrap technique, we refer to Efron and Gong [9], Efron and Tibshirani [10, 11],

*Correspondence to: S. B. Pathare (Email: sureshpathare23@gmail.com). Indira College of Commerce and Science, Pune-411033, Maharashtra, India.

ISSN 2310-5070 (online) ISSN 2311-004X (print)

Guntur [15], Mooney and Duval [21], Young [27], Rubin [24], Miller [20]. Gedam and Pathare [12, 13, 14, 22] have studied the nonparametric statistical estimation approaches of various queueing network models.

Chu and Ke [3] examined the statistical behavior of the mean response time for the M/G/1 queueing system using bootstrapping simulation. Chu and Ke [2] studied the interval estimation of mean response time for the G/M/1 queueing system using empirical Laplace function approach. Chu and Ke [4] developed a data based recurrence relation to compute a sequence mean response times and constructed confidence intervals of mean response times for the G/G/1 queueing system using simulation. This motivates us to develop the nonparametric statistical inferences of mean response time for a queueing network model. Consider the two-stage open queueing network shown in Figure 1.

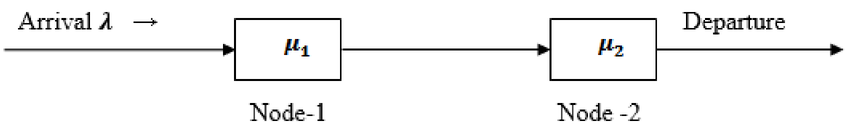

Figure 1. Two-stage open queueing network.

The system consists of two nodes with respective service rates μ_1 and μ_2. The external arrival rate is λ.

In section 2, we described the method of estimation of mean response time and nonparametric estimation approach for mean response time. In section 3 to 9 we proposed CAN, Normal, exact-t, Standard Bootstrap, Bootstrap-t, Variance-stabilized Bootstrap-t, Percentile Bootstrap and Bias-corrected and accelerated bootstrap confidence intervals for response time $r_i, i = 1, 2$. In Section 10, Numerical simulation study is conducted. All simulation results are shown by appropriate tables for illustrating performances of all estimation approaches. In section 11, Conclusions are provided.

2. Mean Response Time and Nonparametric Estimation Approach for Mean Response Time

Let $(X_i, Y_i, i = 1, 2)$ be nonnegative continuous random variables representing respectively inter-arrival times and service times at node-1 and node-2 of a queueing network model. The random variables $(Y_i, i = 1, 2)$ and $(X_i, i = 1, 2)$ are independent.

Let $(X_{ij}, Y_{ij}, i = 1, 2, j = 1, 2, \cdots, n)$ be a random sample drawn from $(X_i, Y_i, i = 1, 2)$ represents inter-arrival times and service times for j^{th} customer at i^{th} node of a queueing network.

Let $R_{ij}, i = 1, 2, j = 1, 2, \cdots, n$ represents the response time of j^{th} customer at i^{th} node of a queueing network and are determined from $(X_{ij}, Y_{ij}, i = 1, 2, j = 1, 2, \cdots, n)$.

Let $W_{ij}, i = 1, 2, j = 1, 2, \cdots, n$ represents the waiting time of j^{th} customer at i^{th} node of a queueing network. Then

$$R_{ij} = W_{ij} + Y_{ij}, i = 1, 2, j = 1, 2, \cdots, n, \tag{1}$$

With the help of analysis by Kleinrock [19] we can evaluate $W_{ij}, i = 1, 2, j = 1, 2, \cdots, n$ using recurrence relation given by

$$W_{ij} = (R_{i,j-1} - X_{ij}) I(R_{i,j-1} > X_{ij}) \tag{2}$$

for $i = 1, 2, j = 1, 2, \cdots, n$ and $W_{i1} = 0, i = 1, 2$ and $I(.)$ denote the indicator function. Using (1) we get

$$R_{ij} = (R_{i,j-1} - X_{ij}) I(R_{i,j-1} > X_{ij}) + Y_{ij} \tag{3}$$

for $i = 1, 2, j = 2, 3, \cdots, n$ and $R_{i1} = Y_{i1}, i = 1, 2$. Equation (3)is the exact data based recurrence relation for calculating response times $R_{ij}, i = 1, 2, j = 1, 2, \cdots, n$ that are exactly as a sequence of customers response times for queueing network. Hence

$$\hat{r}_i = \frac{1}{n}\sum_{j=1}^{n} R_{ij}, \quad i = 1,2 \tag{4}$$

the arithmetic mean of these response times is a natural estimator of the mean response times $r_i, i = 1,2$ for queueing network. According to the Strong Law of Large Numbers [23], we know that $\hat{r}_i, i = 1,2$ is strongly consistent estimator of $r_i, i = 1,2$. The true distributions of $(X_i, Y_i, i = 1,2)$ are not often known in practice so the exact distributions of $\hat{r}_i, i = 1,2$ cannot be derived. But under the assumption that X_i and Y_i being independent, the asymptotical distributions of $\hat{r}_i, i = 1,2$ can be developed. By Slutskys theorem [16], we have

$$\sqrt{n}(\hat{r}_i - r_i) \xrightarrow{D} N(0,\sigma_i^2), i = 1,2$$

where $\hat{\sigma}_i^2 = \frac{1}{n}\sum_{j=1}^{n}(R_{ij} - \hat{r}_i)^2, \quad i = 1,2$ is the variance of $R_{ij}, i = 1,2, j = 1,2,\cdots,n$ and $\xrightarrow{D}$ denotes convergence in distribution. Then $\hat{\sigma}_i^2, \quad i = 1,2$ is a strongly consistent estimator of $\sigma_i^2 \quad i = 1,2$. Again applying the Slutskys theorem we have,

$$\frac{\sqrt{n}(\hat{r}_i - r_i)}{\hat{\sigma}_i} \xrightarrow{D} N(0,1), \quad i = 1,2.$$

Thus $\hat{r}_i, i = 1,2$ is a strongly consistent and asymptotically normal (CAN) estimator with approximate variances $\frac{\hat{\sigma}_i^2}{n}, \quad i = 1,2.$

3. Consistent and Asymptotical Normal (CAN) and Normal Confidence Intervals(Normal) for mean response times

Using CAN estimators $\hat{r}_i, i = 1,2$ and its associated approximate variances $\hat{\sigma}_i^2/n, i = 1,2,$ we construct confidence intervals for mean response times $r_i, i = 1,2$ of a distribution-free two-stage open queueing network. Let z_α be the upper α^{th} quantile of the standard normal distribution. Then approximate $100(1-\alpha)\%$ CAN confidence intervals for mean response times $r_i, i = 1,2$ are

$$\left(\hat{r}_i \pm z_{\alpha/2}\hat{\sigma}_i/\sqrt{n}\right), \quad i = 1,2. \tag{5}$$

Also for sufficiently large value of n the CAN confidence intervals approaches to normal confidence intervals.

4. Exact Confidence Intervals for mean response times based on Student-t distribution(Exact-t)

As sample size is small, using the student t-distribution we construct confidence intervals for mean response times $r_i, i = 1,2$ of a two-stage open queueing network. Let t_α be the upper α^{th} quantile of the student t-distribution. Then approximate $100(1-\alpha)\%$ confidence intervals for mean response times $r_i, i = 1,2$ are

$$\left(\hat{r}_i \pm t_{(n-1),\alpha/2}\frac{\hat{\sigma}_i}{\sqrt{n}}\right), \quad i = 1,2. \tag{6}$$

5. Standard Bootstrap (SB)Confidence Intervals for mean response times

According to the bootstrap procedure, a simple random sample $(X_{ij}^*, Y_{ij}^*, i = 1,2, j = 1,2,\cdots,n)$ called a bootstrap sample can be taken from the empirical distribution function of $(X_{ij}, Y_{ij}, i = 1,2; j = 1,2,\cdots,n)$. Using (3) we can obtain $r_{ij}, i = 1,2, j = 1,2,\cdots,n$ as a sequence of customer' response time. Similarly we can obtain $r_{ij}^*, i = 1,2, j = 1,2,\cdots,n$. It follows that $\hat{r}_i = \frac{1}{n}\sum_{j=1}^{n} r_{ij}, \quad i = 1,2$ is natural estimate of the mean response time $r_i, i = 1, 2$ for a queuing network. And $\hat{r}_i^* = \frac{1}{n}\sum_{j=1}^{n} r_{ij}^*,\ \ i = 1, 2$ is called bootstrap estimate of $\hat{r}_i, i = 1, 2$. The above re-sampling process can be repeated N times. The N bootstrap estimates $\hat{r}_{i1}^*, \hat{r}_{i2}^*, \cdots, \hat{r}_{iN}^*, i = 1, 2$ can be computed from the bootstrap resample. Averaging the N bootstrap estimates we get

$$\hat{r}_N(i) = \frac{1}{N}\sum_{j=1}^{N} \hat{r}_{ij}^*,\ \ i = 1, 2$$

is the bootstrap estimate of $r_i, i = 1, 2$ and standard deviation of $\hat{r}_i, i = 1, 2$,can be estimated by

$$sd(\hat{r}_N(i)) = \left[\frac{1}{N-1}\sum_{j=1}^{N}\left[r_{ij}^* - \hat{r}_N(i)\right]^2\right]^{\frac{1}{2}},\ \ i = 1, 2$$

By central limit theorem, the distribution of $\hat{r}_i, i = 1, 2$ is approximately normal. Therefore $100(1-\alpha)\%$ SB confidence intervals for mean response times $r_i, i = 1, 2$ are

$$\left(\hat{r}_i \pm z_{\alpha/2} sd(\hat{r}_N(i))\right),\ \ i = 1, 2. \tag{7}$$

6. Bootstrap-t Confidence Intervals for mean response times (Boot-t)

Consider N bootstrap estimates $\hat{r}_{i1}^*, \hat{r}_{i2}^*, \cdots, \hat{r}_{iN}^*, i = 1, 2$ computed from the bootstrap resample. We obtain $Z_{ij}^* = \frac{(\hat{r}_{ij}^* - \hat{r}_N(i))}{sd(\hat{r}_N(i))}\ \ i = 1, 2, j = 1, 2, \cdots, N$ and sample $Z_{i1}^*, Z_{i2}^*, \cdots, Z_{iN}^*, i = 1, 2$ considered as an approximate t distribution. Thus we have $100(1-\alpha)\%$ Bootstrap-t confidence intervals for mean response times $r_i, i = 1, 2$ are

$$\left(\hat{r}_i \pm \hat{t}_{a/2} sd(\hat{r}_N(i))\right),\ \ i = 1, 2. \tag{8}$$

where $\hat{t}_{a/2}$ equals the $\alpha/2$ percentile of the random sample $Z_{i1}^*, Z_{i2}^*, \cdots, Z_{iN}^*, i = 1, 2.$

7. Variance-stabilized Bootstrap-t (VST) Confidence Intervals for mean response times

Let $\hat{r}_i, i = 1, 2$ is a strongly consistent and asymptotically normal estimator with approximate variances $\hat{\sigma}_i^2/n, i = 1, 2$ and consider $\hat{\sigma}_i = \phi(\hat{r}_i)$. Now to find a transformation $f(\hat{r}_i)$ such that $Var(f(\hat{r}_i)) \approx$ constant,by the first order Taylor series expansion:

$$f(\hat{r}_i) \approx f(r_i) + (\hat{r}_i - r_i)f'(r_i) \Rightarrow [f(\hat{r}_i) - f(r_i)]^2 \approx (\hat{r}_i - r_i)^2(f'(r_i))^2,\ \ i = 1, 2.$$

Taking expectations on both sides, we get:

$$Var[f(\hat{r}_i)] \approx Var(\hat{r}_i)(f'(r_i))^2 = (\phi(r_i))^2(f'(r_i))^2,\ \ i = 1, 2.$$

Now consider $f(\hat{r}_i) = \sqrt{n}\log(\phi(\hat{r}_i)), i = 1, 2$ is the variance-stabilizing transformation. Then we have,

$$V[f(\hat{r}_i)] \approx \left(\frac{\sqrt{n}}{\phi(\hat{r}_i)}\right)^2 Var[\hat{r}_i] = \left(\frac{\sqrt{n}}{\hat{\sigma}_i}\right)^2 Var[\hat{r}_i] = \frac{n}{\hat{\sigma}_i^2}\frac{\hat{\sigma}_i^2}{n} = 1, i = 1, 2.$$

Here we consider N bootstrap estimates $\hat{r}_{i1}^*, \hat{r}_{i2}^*, \cdots, \hat{r}_{iN}^*, i = 1, 2$ computed from the bootstrap resample. We obtain

$$\theta_{ij}^* = \sqrt{n}\log(\hat{r}_{ij}^*) - \sqrt{n}\log(\hat{r}_i), i = 1, 2, j = 1, 2, \cdots, N.$$

Thus we have $100(1-\alpha)\%$ VST confidence intervals for mean response times $r_i, i = 1, 2$ are

$$\left(e^{\log(\hat{r}_i) - \frac{1}{\sqrt{n}}\hat{v}_i t_{1-\alpha/2}},\ \ e^{\log(\hat{r}_i) - \frac{1}{\sqrt{n}}\hat{v}_i t_{\alpha/2}}\right) \tag{9}$$

where $\hat{v}_i t_{\alpha/2}$ and $\hat{v}_i t_{1-\alpha/2}$ are $(\alpha/2)^{th}$ and $(1-\alpha/2)^{th}$ percentile of the random sample $\theta_{i1}^*, \theta_{i2}^*, \cdots, \theta_{iN}^*, i = 1, 2.$

8. Percentile Bootstrap (PB) Confidence Intervals for mean response times

Now call $\hat{r}^*_{i1}, \hat{r}^*_{i2}, \cdots, \hat{r}^*_{iN}, i = 1,2$ the bootstrap distribution of $\hat{r}_i, i = 1,2$. Let $\hat{r}^*_i(1), \hat{r}^*_i(2), \cdots, \hat{r}^*_i(N), i = 1,2$ be the order statistics of $\hat{r}^*_{i1}, \hat{r}^*_{i2}, \cdots, \hat{r}^*_{iN}, i = 1,2$. Then utilizing the $100(\alpha/2)^{th}$ and $100(1-\alpha/2)^{th}$ percentage points of the bootstrap distribution, $100(1-\alpha)\%$ PB confidence intervals for mean response times $r_i, i = 1,2$ are

$$\left(\hat{r}^*_i\left(\left[N\left(\frac{\alpha}{2}\right)\right]\right),\ \hat{r}^*_i\left(\left[N\left(1-\frac{\alpha}{2}\right)\right]\right)\right),\ \ i = 1,2. \tag{10}$$

where $[x]$ denotes the greatest integer less than or equal to x.

9. Bias-corrected and accelerated bootstrap(BCaB) Confidence Intervals for mean response times

The bootstrap distribution $\hat{r}^*_{i1}, \hat{r}^*_{i2}, \cdots, \hat{r}^*_{iN}, i = 1,2$ may be biased, consequently the PB confidence intervals of mean response times is designed to correct this potential bias of the bootstrap designed. Set $p_i = \frac{1}{N}\sum_{j=1}^{N} I(\hat{r}^*_{ij} < \hat{r}_i),\quad i = 1,2$ where $I(.)$ is the indicator function. Define $\hat{z}_i = \phi^{-1}(p_i), i = 1,2$ where ϕ^{-1} denotes the inverse function of the standard normal distribution ϕ. Except for correcting the potential bias of the bootstrap distribution, we can accelerate convergence of bootstrap distribution. Let $(\widetilde{X}_i(k), \widetilde{Y}_i(k), i = 1,2, k = 1,2\cdots,n)$ denote the original samples with the k^{th} observation $(X_{ik}, Y_{ik}, i = 1,2)$ deleted, also $\hat{r}_{ik}, i = 1,2$ be the estimator of $r_i, i = 1,2$ calculated by using $(\widetilde{X}_i(k), \widetilde{Y}_i(k), i = 1,2)$.

Define

$$\widetilde{r}_i = \frac{1}{n}\sum_{k=1}^{n} \hat{r}_{ik}, i = 1,2$$

and

$$\hat{a}_i = \frac{\sum_{k=1}^{n}(\widetilde{r}_i - \hat{r}_{ik})^3}{6\left(\sum_{k=1}^{n}(\widetilde{r}_i - \hat{r}_{ik})^2\right)^{\frac{3}{2}}}, i = 1,2$$

where $\hat{z}_i$ and $\hat{a}_i, i = 1,2$ are named bias-correction and acceleration respectively. Thus $100(1-\alpha)\%$ BCaB confidence intervals for mean response times $r_i, i = 1,2$ are

$$\left(\hat{r}^*_i\left([N\alpha_{i1}]\right),\ \hat{r}^*_i\left([N\alpha_{i2}]\right)\right),\ \ i = 1,2. \tag{11}$$

where $a_{i1} = \phi\left[\hat{z}_i + \frac{\hat{z}_i - z_{\alpha/2}}{1-\hat{a}_i\left(\hat{z}_i - z_{\alpha/2}\right)}\right]$ and $a_{i2} = \phi\left[\hat{z}_i + \frac{\hat{z}_i + z_{\alpha/2}}{1-\hat{a}_i\left(\hat{z}_i + z_{\alpha/2}\right)}\right], i = 1,2$

10. Simulation Study

A numerical simulation study was undertaken to evaluate performance of the various interval estimation approaches mentioned above for a two-stage open queueing network. It is observed that most statisticians assess performances of interval estimations in terms of coverage percentages or average lengths of confidence intervals. However, through simulation study in the research work, we find that larger coverage percentages of confidence intervals may often be due to wider standard deviation of interval estimation methods. Moreover, narrower confidence intervals may often lead to smaller coverage percentages. Hence, both coverage percentage and average length are not efficient for appraising interval estimation methods. In order to overcome above two shortcomings, we consider two measures namely relative coverage and relative average length to evaluate performances of interval estimation methods.

Table 1. Different queueing network models simulated for study.

Models simulated	Distribution of $X_i, i = 1, 2$ $x_1 \geq 0, x_2 \geq 0$	Distribution of $Y_i, i = 1, 2$ $y_1 \geq 0, y_2 \geq 0$
$E_4/H_4^{Pe}/1$ to $H_4^{Pe}/E_4/1$	$f(x_1) = \frac{128}{3} x_1^3 e^{-4x_1}$ $f(x_2) = 0.1e^{-x_2} + 0.4e^{-2x_2} + 0.8e^{-8x_2/3} + 3.2e^{-8x_2}$	$f(y_1) = 0.1e^{-y_1} + 0.4e^{-2y_1} + 0.8e^{-8y_1/3} + 3.2e^{-8y_1}$ $f(y_2) = \frac{1}{1536} y_2^3 e^{-y_2/4}$
$E_4/H_4^{Po}/1$ to $H_4^{Po}/E_4/1$	$f(x_1) = \frac{128}{3} x_1^3 e^{-4x_1}$ $f(x_2) = 2e^{-2x_2} + 4e^{-4x_2} + \frac{16}{3} e^{-16x_2/3} + 16e^{-16x_2}$	$f(y_1) = 2e^{-2y_1} + 4e^{-4y_1} + \frac{16}{3} e^{-16y_1/3} + 16e^{-16y_1}$ $f(y_2) = \frac{1}{96} y_2^3 e^{-y_2/2}$
$H_4^{Pe}/H_4^{Po}/1$ to $H_4^{Po}/H_4^{Pe}/1$	$f(x_1) = \frac{3}{8} e^{-x_1} + \frac{1}{4} e^{-2x_1} + \frac{2}{3} e^{-8x_1/3} + 2e^{-8x_1}$ $f(x_2) = 4e^{-4x_2} + 8e^{-8x_2} + \frac{32}{3} e^{-32x_2/3} + 32e^{-32x_2}$	$f(y_1) = 4e^{-4y_1} + 8e^{-8y_1} + \frac{32}{3} e^{-32y_1/3} + 32e^{-32y_1}$ $f(y_2) = \frac{1}{4} e^{-y_2} + \frac{1}{2} e^{-2y_2} + \frac{2}{3} e^{-8y_2/3} + 2e^{-8y_2}$

Table 2. Simulation analysis for consistency of $\hat{r}_i, i = 1, 2$ based on large sample size.

Models simulated	The true value of $r_i, i = 1, 2$	The mean of 1000 simulated $\hat{r}_i, i = 1, 2$		
		$n = 100$	$n = 150$	$n = 200$
$E_4/H_4^{Pe}/1$ to $H_4^{Pe}/E_4/1$	r_1=0.3627& r_2=0.3525	$\hat{r}_1$=0.36276& $\hat{r}_2$=0.35212	$\hat{r}_1$=0.36285& $\hat{r}_2$=0.35364	$\hat{r}_1$=0.36257& $\hat{r}_2$=0.35432
$E_4/H_4^{Po}/1$ to $H_4^{Po}/E_4/1$	r_1=1.0206& r_2=0.5869	$\hat{r}_1$=1.02429& $\hat{r}_2$=0.58495	$\hat{r}_1$=1.02122& $\hat{r}_2$=0.58556	$\hat{r}_1$=1.02481& $\hat{r}_2$=0.58601
$H_4^{Pe}/H_4^{Po}/1$ to $H_4^{Po}/H_4^{Pe}/1$	r_1=1.3329& r_2=0.3886	$\hat{r}_1$=1.31528& $\hat{r}_2$=0.38800	$\hat{r}_1$=1.32267& $\hat{r}_2$=0.38918	$\hat{r}_1$=1.32720& $\hat{r}_2$=0.38855

Table 3. Simulation analysis for consistency of $\hat{r}_i, i = 1, 2$ based on small sample size.

Models simulated	The true value of $r_i, i = 1, 2$	The mean of 1000 simulated $\hat{r}_i, i = 1, 2$	
		$n = 15$	$n = 25$
$E_4/H_4^{Pe}/1$ to $H_4^{Pe}/E_4/1$	r_1=0.3627& r_2=0.3525	$\hat{r}_1$=0.36384& $\hat{r}_2$=0.33187	$\hat{r}_1$=0.36212& $\hat{r}_2$=0.33867
$E_4/H_4^{Po}/1$ to $H_4^{Po}/E_4/1$	r_1=1.0206& r_2=0.5869	$\hat{r}_1$=1.02218& $\hat{r}_2$=0.58453	$\hat{r}_1$=1.02221& $\hat{r}_2$=0.58325
$H_4^{Pe}/H_4^{Po}/1$ to $H_4^{Po}/H_4^{Pe}/1$	r_1=1.3329& r_2=0.3886	$\hat{r}_1$=1.29045& $\hat{r}_2$=0.38586	$\hat{r}_1$=1.28124& $\hat{r}_2$=0.38578

Table 4. Simulation results of queueing network model $E_4/H_4^{Pe}/1$ to $H_4^{Pe}/E_4/1$ based on large sample size.

Estimation	Coverage Percentages			Average Lengths			Relative Coverage			Relative Average Length		
Approches	$n =$ 100	$n =$ 150	$n =$ 200	$n =$ 100	$n =$ 150	$n =$ 200	$n =$ 100	$n =$ 150	$n =$ 200	$n =$ 100	$n =$ 150	$n =$ 200
Normal1	0.891	0.882	0.886	0.062	0.050	0.044	14.476	17.551	20.358	0.170	0.139	0.120
Normal2	0.478	0.466	0.468	0.067	0.056	0.049	**7.135**	**8.310**	**9.492**	**0.189**	**0.160**	**0.139**
SB1	0.888	0.881	0.887	0.062	0.050	0.044	14.448	17.531	20.392	0.169	0.139	0.120
SB2	0.892	0.894	0.893	0.206	0.165	0.143	4.334	5.406	6.239	0.582	0.471	0.402
PB1	0.890	0.881	0.886	0.061	0.050	0.043	**14.532**	17.602	**20.444**	0.169	0.138	0.120
PB2	**0.925**	**0.940**	**0.951**	0.188	0.155	0.135	4.913	6.078	7.029	0.532	0.440	0.380
BCaB1	0.885	0.878	0.884	0.061	0.050	0.043	14.480	17.574	20.422	**0.169**	**0.138**	**0.119**
BCaB2	0.780	0.763	0.766	0.205	0.165	0.142	3.800	4.620	5.414	0.580	0.470	0.397
Boot-t1	0.890	0.884	**0.888**	0.061	0.050	0.044	14.493	17.610	20.431	0.169	0.138	0.120
Boot-t2	0.843	0.860	0.853	0.189	0.155	0.136	4.470	5.551	6.294	0.533	0.441	0.381
VST1	**0.893**	**0.887**	0.885	0.062	0.050	0.044	14.478	**17.623**	20.319	0.170	0.139	0.120
VST2	0.744	0.718	0.735	0.176	0.148	0.131	4.236	4.864	5.631	0.496	0.420	0.367

Table 5. Simulation results of queueing network model $E_4/H_4^{Po}/1$ to $H_4^{Po}/E_4/1$ based on large sample size.

Estimation	Coverage Percentages			Average Lengths			Relative Coverage			Relative Average Length		
Approches	$n=$ 100	$n=$ 150	$n=$ 200	$n=$ 100	$n=$ 150	$n=$ 200	$n=$ 100	$n=$ 150	$n=$ 200	$n=$ 100	$n=$ 150	$n=$ 200
Normal1	0.852	0.862	0.875	0.200	0.163	0.142	**4.261**	**5.284**	**6.167**	**0.195**	**0.160**	**0.139**
Normal2	0.680	0.709	0.691	0.104	0.086	0.075	**6.535**	**8.225**	**9.280**	**0.178**	**0.147**	**0.127**
SB1	0.883	0.886	0.912	0.218	0.179	0.156	4.054	4.958	5.858	0.213	0.175	0.152
SB2	0.882	0.913	0.886	0.177	0.146	0.125	4.997	6.249	7.071	0.302	0.249	0.213
PB1	**0.888**	**0.897**	**0.915**	0.217	0.178	0.155	4.097	5.043	5.904	0.212	0.174	0.151
PB2	**0.908**	**0.935**	**0.928**	0.172	0.143	0.123	5.292	6.547	7.538	0.293	0.244	0.210
BCaB1	0.856	0.873	0.880	0.215	0.177	0.154	3.991	4.940	5.710	0.209	0.173	0.150
BCaB2	0.785	0.802	0.777	0.174	0.146	0.125	4.506	5.495	6.239	0.298	0.249	0.212
Boot-t1	0.881	0.888	0.911	0.217	0.178	0.155	4.055	4.979	5.862	0.212	0.175	0.152
Boot-t2	0.869	0.902	0.886	0.172	0.143	0.123	5.055	6.303	7.181	0.294	0.244	0.210
VST1	0.861	0.865	0.884	0.218	0.179	0.156	3.944	4.841	5.681	0.213	0.175	0.152
VST2	0.788	0.816	0.797	0.170	0.141	0.122	4.649	5.789	6.536	0.290	0.240	0.208

Table 6. Simulation results of queueing network model $H_4^{Pe}/H_4^{Po}/1$ to $H_4^{Po}/H_4^{Pe}/1$ based on large sample size.

Estimation	Coverage Percentages			Average Lengths			Relative Coverage			Relative Average Length		
Approches	$n=$ 100	$n=$ 150	$n=$ 200	$n=$ 100	$n=$ 150	$n=$ 200	$n=$ 100	$n=$ 150	$n=$ 200	$n=$ 100	$n=$ 150	$n=$ 200
Normal1	0.601	0.622	0.604	0.282	0.234	0.204	**2.134**	**2.657**	**2.957**	**0.213**	**0.177**	**0.154**
Normal2	0.782	0.806	0.781	0.068	0.056	0.049	**11.495**	**14.366**	**16.069**	**0.175**	**0.144**	**0.125**
SB1	0.895	0.892	0.899	0.585	0.484	0.416	1.531	1.844	2.161	0.442	0.366	0.313
SB2	0.879	0.896	0.879	0.087	0.072	0.062	10.119	12.529	14.218	0.223	0.184	0.159
PB1	**0.925**	**0.942**	**0.946**	0.559	0.467	0.405	1.655	2.018	2.335	0.422	0.353	0.305
PB2	**0.904**	**0.915**	**0.910**	0.086	0.071	0.061	10.520	12.908	14.820	0.221	0.182	0.158
BCaB1	0.744	0.766	0.778	0.565	0.473	0.406	1.317	1.620	1.915	0.427	0.358	0.306
BCaB2	0.832	0.818	0.825	0.086	0.071	0.061	9.700	11.573	13.525	0.221	0.182	0.157
Boot-t1	0.860	0.875	0.879	0.560	0.468	0.406	1.536	1.871	2.166	0.423	0.354	0.305
Boot-t2	0.872	0.904	0.873	0.086	0.071	0.062	10.122	12.721	14.184	0.222	0.183	0.158
VST1	0.760	0.773	0.761	0.542	0.455	0.397	1.402	1.699	1.916	0.410	0.344	0.299
VST2	0.818	0.840	0.829	0.086	0.071	0.062	9.502	11.833	13.473	0.221	0.183	0.158

Table 7. Simulation results of queueing network model $E_4/H_4^{Pe}/1$ to $H_4^{Pe}/E_4/1$ based on small sample size.

Estimation	Coverage Percentages		Average Lengths		Relative Coverage		Relative Average Length	
Approches	$n=15$	$n=25$	$n=15$	$n=25$	$n=15$	$n=25$	$n=15$	$n=25$
CAN1	0.853	0.886	0.155	0.122	5.506	7.284	0.427	0.336
CAN2	0.512	0.487	0.137	0.113	**3.742**	**4.299**	**0.405**	**0.337**
Exact-t1	0.867	0.891	0.160	0.124	5.406	7.174	0.442	0.343
Exact-t2	0.834	0.871	0.377	0.350	2.212	2.489	1.114	1.040
SB1	0.840	0.877	0.150	0.119	5.609	7.345	0.413	0.329
SB2	0.816	0.867	0.352	0.336	2.318	2.577	1.041	1.000
PB1	0.845	0.884	0.149	0.119	5.665	**7.440**	0.411	0.328
PB2	0.844	0.885	0.327	0.306	2.582	2.890	0.966	0.910
BCaB1	0.842	0.866	0.148	0.118	**5.693**	7.327	**0.408**	**0.326**
BCaB2	0.726	0.744	0.339	0.335	2.144	2.224	1.001	0.995
Boot-t1	0.842	0.875	0.150	0.119	5.631	7.345	0.413	0.329
Boot-t2	0.748	0.807	0.327	0.307	2.285	2.632	0.968	0.912
VST1	0.845	0.886	0.155	0.121	5.470	7.302	0.426	0.335
VST2	0.721	0.740	0.301	0.272	2.396	2.719	0.890	0.809

Table 8. Simulation results of queueing network model $E_4/H_4^{Po}/1$ to $H_4^{Po}/E_4/1$ based on small sample size.

Estimation Approches	Coverage Percentages		Average Lengths		Relative Coverage		Relative Average Length	
	$n=15$	$n=25$	$n=15$	$n=25$	$n=15$	$n=25$	$n=15$	$n=25$
CAN1	0.846	0.842	0.496	0.389	**1.704**	**2.166**	**0.485**	**0.380**
CAN2	0.732	0.668	0.251	0.198	**2.922**	**3.382**	**0.433**	**0.337**
Exact-t1	0.874	0.883	0.564	0.435	1.550	2.028	0.551	0.426
Exact-t2	0.892	0.873	0.468	0.363	1.907	2.407	0.808	0.619
SB1	0.861	0.869	0.527	0.419	1.635	2.076	0.514	0.409
SB2	0.875	0.859	0.437	0.349	2.003	2.464	0.754	0.596
PB1	0.858	0.875	0.520	0.415	1.652	2.111	0.507	0.405
PB2	0.891	0.878	0.405	0.326	2.198	2.694	0.700	0.557
BCaB1	0.828	0.826	0.507	0.405	1.634	2.037	0.495	0.396
BCaB2	0.800	0.790	0.424	0.341	1.888	2.317	0.732	0.582
Boot-t1	0.852	0.860	0.521	0.416	1.636	2.070	0.508	0.406
Boot-t2	0.854	0.828	0.406	0.327	2.103	2.536	0.701	0.558
VST1	0.852	0.857	0.540	0.424	1.579	2.024	0.527	0.414
VST2	0.836	0.784	0.392	0.315	2.134	2.488	0.677	0.538

Table 9. Simulation results of queueing network model $H_4^{Pe}/H_4^{Po}/1$ to $H_4^{Po}/H_4^{Pe}/1$ based on small sample size.

Estimation Approches	Coverage Percentages		Average Lengths		Relative Coverage		Relative Average Length	
	$n=15$	$n=25$	$n=15$	$n=25$	$n=15$	$n=25$	$n=15$	$n=25$
CAN1	0.591	0.617	0.624	0.503	**0.947**	**1.226**	**0.483**	**0.387**
CAN2	0.795	0.788	0.166	0.132	**4.782**	**5.992**	**0.427**	**0.338**
Exact-t1	0.860	0.855	1.391	1.131	0.618	0.756	1.077	0.870
Exact-t2	0.873	0.880	0.225	0.177	3.874	4.968	0.579	0.456
SB1	0.844	0.847	1.299	1.087	0.650	0.779	1.006	0.836
SB2	0.857	0.867	0.210	0.170	4.072	5.091	0.541	0.438
PB1	0.864	0.876	1.195	0.996	0.723	0.880	0.925	0.766
PB2	0.860	0.876	0.203	0.165	4.238	5.305	0.522	0.425
BCaB1	0.729	0.768	1.126	0.967	0.648	0.794	0.872	0.744
BCaB2	0.806	0.810	0.207	0.167	3.898	4.840	0.531	0.431
Boot-t1	0.797	0.804	1.197	0.998	0.666	0.806	0.927	0.767
Boot-t2	0.852	0.853	0.203	0.166	4.189	5.154	0.523	0.426
VST1	0.763	0.766	1.154	0.944	0.661	0.811	0.893	0.726
VST2	0.841	0.834	0.204	0.165	4.117	5.043	0.525	0.426

Note that, boldface denotes the greatest relative coverage and shortest relative average length among all estimation approaches.

Table 10. Performances of the estimation approaches to response time under various queueing networks based on large sample size.

Queueing Network simulated	Estimation approach with Greatest Relative coverage			Estimation approach with Shortest Relative Average Length		
	$n=100$	$n=150$	$n=200$	$n=100$	$n=150$	$n=200$
$E_4/H_4^{Pe}/1$ to $H_4^{Pe}/E_4/1$	PB1 Normal2	VST1 Normal2	PB1 Normal2	BCaB1 Normal2	BCaB1 Normal2	BCaB1 Normal2
$E_4/H_4^{Po}/1$ to $H_4^{Po}/E_4/1$	Normal1 Normal2	Normal1 Normal2	Normal1 Normal2	Normal1 Normal2	Normal1 Normal2	Normal1 Normal2
$H_4^{Pe}/H_4^{Po}/1$ to $H_4^{Po}/H_4^{Pe}/1$	Normal1 Normal2	Normal1 Normal2	Normal1 Normal2	Normal1 Normal2	Normal1 Normal2	Normal1 Normal2

Table 11. Performances of the estimation approaches to response time under various queueing networks based on small sample size.

Queueing Network simulated	Estimation approach with Greatest Relative coverage		Estimation approach with Shortest Relative Average Length	
	$n = 15$	$n = 25$	$n = 15$	$n = 25$
$E_4/H_4^{Pe}/1$ to $H_4^{Pe}/E_4/1$	BCaB1 CAN2	PB1 CAN2	BCaB1 CAN2	BCaB1 CAN2
$E_4/H_4^{Po}/1$ to $H_4^{Po}/E_4/1$	CAN1 CAN2	CAN1 CAN2	CAN1 CAN2	CAN1 CAN2
$H_4^{Pe}/H_4^{Po}/1$ to $H_4^{Po}/H_4^{Pe}/1$	CAN1 CAN2	CAN1 CAN2	CAN1 CAN2	CAN1 CAN2

Relative coverage is defined as the ratio of coverage percentage to average length of confidence interval. Larger relative coverage implies the better performances of the corresponding confidence interval. Also another approach named Relative Average Length is defined as the ratio of average length to the true value of $r_i, i = 1, 2$. For a given confidence level, the shorter the interval is, the more informative it is. Hence shorter relative average length implies the better performances of the corresponding confidence interval.

The consistency of $r_i, i = 1, 2$ is examined by comparing true value of $r_i, i = 1, 2$ with the average simulated estimates of $\hat{r}_i, i = 1, 2$ whereas the different confidence intervals are assessed in terms of their coverage accuracy, relative coverage and relative average length. In order to achieve these goals, in simulation study we select various queueing network modes as shown in Table 1. With regard to queueing network models shown in Table 1 there is no theoretical formula for the true value of $r_i, i = 1, 2$. Using strong law of large numbers [23], we have estimated the true value of $r_i, i = 1, 2$ by the simulated sample values of $\hat{r}_i, i = 1, 2$ for sufficiently large sample size n.

Thus for queueing network model $E_4/H_4^{Pe}/1$ to $H_4^{Pe}/E_4/1$ in Table 2 and Table 3, the approximated value of $r_i, i = 1, 2$ is 0.3627 and 0.3525 respectively which is obtained from the simulated sample value of $\hat{r}_i, i = 1, 2$ with sample size $n \geq 10^6$. Similarly, the approximated mean response time r_1=1.0206, r_2 =0.5869 for queueing network model $E_4/H_4^{Po}/1$ to $H_4^{Po}/E_4/1$ is produced with $n \geq 10^6$. And for the model $H_4^{Pe}/H_4^{Po}/1$ to $H_4^{Po}/H_4^{Pe}/1$ the approximated value of $r_i, i = 1, 2$ is 1.3329 and 0.3886 respectively. We find that the approximated mean response time approaches to the true value of $r_i, i = 1, 2$ with four decimal places when $n \geq 10^6$. Also we get the same interpretation for small samples where the approximate values of $r_i, i = 1, 2$ are given in Table-3. Here E_4 represents a 4-stage Erlang distribution,H_4^{Pe} a 4-stage hyper-exponential distribution and H_4^{Po} a 4-stage hypo-exponential distribution.

Thus for each specified queueing network in Table 1, a random sample of sample size n (= 15, 25, 100, 150, 200) is drawn from the original samples. Further N =1000 bootstrap resample's are drawn from the original samples. According to (5) to (11) we obtain CAN, Normal,Exact-t, SB, Boot-t, VST,PB and BCaB confidence intervals for mean response times with confidence level 90%. The above simulation process is replicated N =1000 times and computed coverage percentages, average lengths, relative coverage and relative average lengths. We utilize a PC Dual Core and apply Matlab 7.0.1 to accomplish all simulations. The coverage percentages, average lengths, relative coverage's and relative average lengths of mean response time $r_i, i = 1, 2$ based on simulation study for queuing network models for different interval estimation approaches are shown in Tables 4 to 6 for large sample and Tables 7 to 9 for small sample.

According to the simulation results based on small and large sample sizes we observe that, average lengths and relative average lengths are decreases but both coverage percentages and relative coverage are increases with sample size n. The PB method has the largest coverage percentage among almost all confidence intervals. The coverage percentage can approaches to 90% when n increases. All methods have decreasing relative average lengths with sample size n but the normal estimation method has the shortest relative average length. Normal method has greatest relative coverage among all estimation methods and has shortest relative average lengths among all estimation methods.

Finally among all estimation methods, normal confidence intervals method is the best for mean response time of queueing network for large sample and for small samples CAN method is the best among all confidence intervals.

11. Conclusions

This paper provides the interval estimation of mean response time $r_i, i = 1, 2$ for two-stage open queueing network. Using a recurrence relation we obtain a sequence of response time for the two-stage open queueing network. Different estimation approaches CAN, Normal, Exact-t, SB, Boot-t, VST, PB and BCaB are applied to produce confidence intervals for mean response times $r_i, i = 1, 2$. The relative coverage and relative average lengths are adopted to understand compare and assess performance of the resulted confidence intervals. The simulation results imply that the normal estimation method has the best performance for G/G/1 to G/G/1 queueing network among almost all estimation methods for large sample and for small sample the CAN method has the best among all estimation methods. The above mentioned approaches is easily applied to practical queueing network such as all types of open, closed, mixed queueing networks as well as cyclic, retrial queueing models.

REFERENCES

1. Burke P.J., *Output of a Queueing System*, Operations Research, Vol.4, pp.175-178, 1956.
2. Chu Y.K .and Ke J.C., *Interval Estimation of Mean Response Time for a G/M/1 Queueing System: Empirical Laplace Function Approach*, Mathematical Methods in the Applied Sciences, Vol.30, pp. 707-715, 2006.
3. Chu Y.K .and Ke J.C., *Confidence intervals of mean response time for an M/G/1 queueing system: Bootstrap simulation*, Applied Mathematics and Computation, Vol.180, pp. 255-263, 2006.
4. Chu Y.K .and Ke J.C., *Mean response time for a G/G/1 queueing system: Simulated computation*, Applied Mathematics and Computation, Vol.196, pp.772-779, 2007.
5. Disney R. L., *Random flow in queueing networks: a review and a critique*, Trans. A.I.E.E., Vol. 7, pp. 268-288, 1975.
6. Efron B., *Bootstrap methods; another look at the jackknife*, Annals of Statistics, Vol. 7, pp.1-26, 1979.
7. Efron B., *The jackknife, the bootstrap, and other resampling plans*, SIAM Monograph No. 38, 1982.
8. Efron B., *Better bootstrap confidence intervals*, Journal of the American Statistical Association, Vol. 82, pp.171-200, 1987.
9. Efron B. and Gong G., *A leisurely look at the boot strap, the jackknife and cross validation*, The American Statistician, Vol.37, pp. 36-48, 1983.
10. Efron B. and Tibshirani R.J., *Bootstrap Method for standard errors, confidence intervals and other measures of statistical accuracy*, Statistical Science, Vol. 1, pp. 54-77, 1986.
11. Efron B. and Tibshirani R.J., *An Introduction to the bootstrap*, Chapman and Hall, New York, 1993.
12. Gedam V.K. and Pathare S.B., *Calibrated Confidence Intervals for Intensities of a Two Stage Open Queueing Network*, Journal of Statistics Applications and Probability, Vol.3, No. 1, page 33-44, 2014.
13. Gedam V.K. and Pathare S.B., *Calibrated Confidence Intervals for Intensities of a Two Stage Open Queueing Network with Feedback*, Journal of Statistics and Mathematics, Vol. 4, No. 1, pp. 151-161, 2013.
14. Gedam V.K. and Pathare S.B., *Comparison of Different Confidence Intervals of Intensities for an Open Queueing Network with Feedback*, American Journal of Operations Research, Vol. 3 No. 2, pp. 307-327, 2013.
15. Guntur B., *Bootstrapping: how to make something from almost nothing and get statistically valid answers*, Quality Progress Part-I, pp. 97-103, 1991.
16. Hogg R.V. and Craig A.T, *Introduction to Mathematical Statistics*, Prentice- Hall, Inc, 1995.
17. Jackson J. R., *Networks of Waiting Lines*, Operations Research, Vol. 5, pp. 518-521, 1957.
18. Ke J. C. and Chu Y. K., *Comparison on five estimation approaches of intensity for a queueing system with short run*, Computational Statistics, Vol. 24 , No. 4, pp. 567-582, 2009.
19. Kleinrock L., *Queueing Systems Vol. II*, Computer Applications, John Wiley and Sons, New York, 1976.
20. Miller R.G., *The Jackknife -a review*, Biometrika, Vol. 61, pp.1-15, 1974.
21. Mooney C. Z. and Duval R.D., *Bootstrapping: a nonparametric approach to Statistical inference*, SAGE Newbury Park, 1993.
22. Pathare S.B. and Gedam V.K., *Some Estimation Approaches of Intensities for a Two Stage Open Queueing Network*, Statistics Optimization and Information Computing, Vol. 2, No.1. pp 33-46, 2014.
23. Rousses G.G. , *A course in mathematical statistics, 2nd edn*, Academic Press, New York, 1997.
24. Rubin D.B. , *The Bayesian bootstrap*, The Annals of Statistics, Vol.9, pp. 130-134, 1981.
25. Thiruvaiyaru D. and Basava I.V. , *Maximum likelihood estimation for queueing networks*, Stochastic Processes and Statistical Inference (Eds., B.L.S. Prakasa Rao and B. R. Bhat), pp.132-149,1996. New Age International Publications, New Delhi.
26. Thiruvaiyaru D., Basava I.V. and Bhat U. N., *Estimation for a class of simple queueing network*, Queueing Systems, Vol.9, pp. 301-312, 1991.
27. Young G.A., *Bootstrap: More than a stab in the dark?* , Statistical Science, Vol. 9, pp. 382-415, 1994.