



A new interpretation of the integer and real WZ factorization using block scaled ABS algorithms

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Abstract The WZ factorization suitable for parallel computing was introduced by Evans. Here, a block generalization of the ABS class of methods for the solution of linear system of equations is given and it is shown that it covers the whole class of conjugate direction methods defined by Stewart. The methods produce a factorization of the coefficient matrix implicitly, generating well-known matrix factorizations. We show how to set the parameters of a block ABS algorithm to compute the WZ and ZW factorizations of a nonsingular matrix as well as the $W^T W$ and $Z^T Z$ factorizations of a symmetric positive definite matrix. We also provide a new interpretation of the necessary and sufficient conditions for the existence of the WZ and ZW factorizations of a nonsingular matrix. We show how to appropriate the parameters to construct algorithms for computing the QZ and the QW factorizations, where $Q^T Q$ is an X-matrix. We also provide the conditions for the existence of the integer WZ and the integer ZW factorizations of an integer matrix and compute the factorizations by the integer ABS algorithm.

Keywords ABS algorithm, WZ factorization, ZW factorization, $W^T W$ factorization, $Z^T Z$ factorization, X factorization, Cross orthogonal matrix.

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1. Introduction

The central problem of linear algebra is the solution of linear system of equations $Ax = b$. Direct methods used to solve a linear system of equations are based on a factorization of the coefficient matrix into factors that are easy to be used in solving equations. The Gaussian elimination process (i.e., the LU factorization) is optimal among the methods involving only linear combinations of rows and columns [4].

Implicit matrix elimination schemes for solving linear systems were introduced by Evans [8] and Evans and Hatzopoulos [7]. These schemes propose the elimination of two matrix elements simultaneously (as opposed to a single element in Gaussian elimination) and is eminently suitable for parallel implementation [9]. ABS class of algorithms was constructed for the solution of linear systems utilizing some basic ideas such as vector projection and rank one update techniques [1, 3]. The ABS class was later extended to solve optimization problems [3] and systems of linear Diaphantine equations (see [5, 6, 17, 18]). Reviews of ABS methods and their extensions can be found in [21, 22].

Consider the linear system

$$Ax = b, \quad x \in R^n, \quad A \in R^{n \times n}, \quad b \in R^n. \quad (1)$$

For simplicity, assume that A is a nonsingular matrix. A scaled version of the linear ABS class was introduced in [3]. Let $V \in R^{n \times n}$ be a nonsingular matrix. Obviously, the system (1) is equivalent to the following scaled system:

$$V^T Ax = V^T b. \quad (2)$$

Algorithm 1. Basic ABS algorithm.

(1) Let $H_1 \in R^{n \times n}$ be an arbitrary nonsingular matrix. Let $i=1$ and $r = 0$.

(2) Compute $s_i = H_i a_i$.

(3) **If** $s_i = 0$ (the i th row of A is dependent on its first $i-1$ rows) **then** let $H_{i+1} = H_i$ and **go to** (6).

(4) ($s_i \neq 0$ and hence the i th row of A is independent of its first $i-1$ rows) Compute the search vectors p_i by

$$p_i = H_i^T f_i, \quad (3)$$

where $f_i \in R^n$ is arbitrary, save for the condition $s_i^T f_i \neq 0$.

(5) (Updating the null space generator) Update H_i by

$$H_{i+1} = H_i - \frac{H_i a_i q_i^T H_i}{q_i^T H_i a_i}, \tag{4}$$

where $q_i \in R^n$ is an arbitrary vector satisfying $s_i^T q_i \neq 0$, and let $r = r + 1$.

(6) **If** $i = m$ **then Stop** (H_{m+1}^T generates the null space of A and r is its rank) **else** let $i = i + 1$ and **go to** (2).

The matrices H_i are generalizations of projections matrices. The updating formula (4) for the H_i is a rank reduction formula first appeared in a book by Wedderburn [23]. They have been named *Abaffians* since the First International Conference on ABS Methods (Luyoyang, China, 1991).

Considering (2), it is easily verified that the scaled ABS class is obtained by replacing a_i with $A^T v_i$ [3] in Algorithm 1. The algorithms produce a matrix factorization $V^T AP = L$, where L is a lower triangular matrix (see [3]). Choices of the parameters H_1, v_i, f_i and q_i determine particular methods within the class so that various matrix factorizations such as LU, QR and Cholesky factorizations are derived [3]. Recently, Golpar-Raboky and Mahdavi-Amiri showed how to compute the Smith normal form of an integer matrix using the scaled integer ABS algorithm [14, 15]. Later Mahdavi-Amiri and Golpar-Raboky [19] extended the rank reducing process and developed a new class of algorithms containing the scaled extended ABS class of algorithms and the Wedderburn rank reducing process.

It has been proved in [11, 13] that all full rank factorizations can be produced only by a subset of the ABS methods.

The integer ABS (IABS) class of algorithms has been developed by Esmaeili et al. [5, 6] to compute the general integer solution of linear Diophantine equations. There, conditions for the existence of an integer solution and determination of all integer solutions of a linear Diophantine system have been established.

The block ABS algorithm, due to Abaffy and Galantai [2] for the scaled ABS class, further developed in several papers by Galantai, is a block form of the ABS algorithm [3]. Let n_1, \dots, n_s be positive integer numbers so that $n_1 + \dots + n_s = n$. Let V be partitioned in the form $V = [V_1, \dots, V_s]$, where $V_i \in R^{n \times n_i}$. The block scaled ABS algorithm is as follows.

- Let $H_1 \in R^{n \times n}$ be arbitrary and nonsingular.

- For $i = 1$ to s do
 - Compute $S_i = H_i A^T V_i$. Compute the matrix $P_i = H_i^T F_i$, where $F_i \in R^{n \times n_i}$ is an arbitrary matrix so that $S_i^T F_i$ is nonsingular, and update H_i by

$$H_{i+1} = H_i - H_i A^T V_i (Q_i^T H_i A^T V_i)^{-1} Q_i^T H_i, \quad (5)$$
 where $Q_i \in R^{n \times n_i}$ is an arbitrary matrix so that $S_i^T Q_i$ is nonsingular.
 - Define $V_{i+1} \in R^{n \times n_{i+1}}$, an arbitrary full column rank matrix with column being linearly independent of columns of V_1, \dots, V_i .

One main result of the ABS algorithms has been the derivation of the ABS class of algorithms for linear Diophantine equations given by Esmaeili, Mahdavi-Amiri and Spedicato [6] and its extension using the scaled ABS algorithms [21]. Each integer ABS algorithm decides whether the Diophantine system has an integer solution, and, if so, obtains a particular solution along with an integer matrix with possibly dependent rows generating the integer null space of the equations. Consider the linear Diophantine system:

$$Ax = b, \quad A \in Z^{m \times n}, x \in Z^n, b \in Z^m, m \leq n. \quad (6)$$

The system (6) is equivalent to the scaled system,

$$V^T Ax = V^T b, \quad (7)$$

where V is an arbitrary m by m unimodular matrix, an integer matrix with its determinant being equal to $+1$ or -1 .

First, we recall some results from number theory and then present the steps of an integer ABS (IABS) algorithm.

Note that if A is unimodular, then A^{-1} is also unimodular.

Definition 1

A matrix A is called totally unimodular if each square submatrix of A has determinant equal to 0 , $+1$, or -1 . In particular, each entry of a totally unimodular matrix is 0 , $+1$, or -1 .

Theorem 1

(Fundamental theorem of the single linear Diophantine equation) Let a_1, \dots, a_n and b be integer numbers. The Diophantine linear equation $a_1 x_1 + \dots + a_n x_n = b$ has an integer solution if and only if $\gcd(a_1, \dots, a_n) \mid b$ (if $n > 1$, then there are an infinite number of integer solutions).

The scaled integer ABS algorithm (SIABS) has the following structure, with $\gcd(u)$ being the greatest common divisor of a vector u .

- Let $H_1 \in Z^{n \times n}$ be arbitrary and unimodular.
- For $i = 1$ to n do
 - Compute $s_i = H_i A^T v_i$.
 - Compute $\delta_i = \gcd(s_i)$ and $p_i = H_i^T f_i$, where, $f_i \in Z^n$ is an arbitrary integer vector satisfying $s_i^T f_i = \delta_i$. Update H_i by

$$H_{i+1} = H_i - \frac{H_i A^T v_i q_i^T H_i}{q_i^T H_i A^T v_i}, \tag{8}$$

where $q_i \in Z^n$ is an arbitrary integer vector satisfying $s_i^T q_i = \delta_i$.

- Define $v_{i+1} \in Z^n$ so that v_1, \dots, v_{i+1} make the columns of a unimodular matrix.

The WZ factorization corresponds to a version of the block LU algorithm, with block sizes equal to two. The W-matrix and Z-matrix are constructed by applying appropriate permutation matrices on rows and columns of a lower triangular matrix.

Here, making use of permutations in the block ABS method we compute the WZ and WZ factorizations of a nonsingular matrix as well as the $W^T W$ and the $Z^T Z$ factorizations of a symmetric positive definite matrix. We also show how to compute the QZ and the QW factorizations, where $Q^T Q$ is an X-matrix, using the ABS algorithms.

The remainder of our work is organized as follows. In Section 3, we explain the WZ factorization. In Section 2, we present a new formulation of the necessary and sufficient conditions for existence of the real and integer WZ factorizations. Then, we present an algorithm for computing the WZ factorization as well as the $W^T W$ factorization of a positive definite matrix using the block scaled ABS algorithm. An algorithm for computing the QZ factorization where $Q^T Q$ is an X-matrix is also given. In Section 3, we compute the real and integer ZW factorizations by appropriate setting of the parameters of the block scaled ABS algorithm. We also compute the $Z^T Z$ factorization of a positive definite matrix and the QW factorization, where $Q^T Q$ is an X-matrix, using the block scaled ABS algorithms. Necessary and sufficient conditions for existence of ZW factorization based on the block ABS algorithm are also given. We conclude in Section 4.

2. WZ factorization using the block scaled ABS algorithm

To solve a system of linear equations, the WZ factorization splitting procedure proposed in [8, 9] is convenient for parallel computing. The WZ factorization offers a parallel method for solving dense linear systems, where A is a square $n \times n$ matrix, and b is an n -vector.

Definition 2

Let s be a real number, and denote by $\lfloor s \rfloor$ ($\lceil s \rceil$), the greatest (least) integer less (bigger) than or equal to s .

Definition 3

We say that a matrix A is factorized in the form WZ if

$$A = WZ, \quad (9)$$

where the matrices W and Z have the following structures:

$$W = \begin{pmatrix} 1 & \circ & \circ & \circ & \bullet \\ \bullet & 1 & \circ & \bullet & \bullet \\ \bullet & \bullet & 1 & \bullet & \bullet \\ \bullet & \bullet & \circ & 1 & \bullet \\ \bullet & \circ & \circ & \circ & 1 \end{pmatrix}, Z = \begin{pmatrix} \bullet & \bullet & \bullet & \bullet & \bullet \\ \circ & \bullet & \bullet & \bullet & \circ \\ \circ & \circ & \bullet & \circ & \circ \\ \circ & \bullet & \bullet & \bullet & \circ \\ \bullet & \bullet & \bullet & \bullet & \bullet \end{pmatrix}, \quad (10)$$

with the empty bullets standing for zero and the other bullets standing for possible nonzeros.

Definition 4

We define an X -matrix as follows:

$$X = \begin{pmatrix} \bullet & \circ & \circ & \circ & \bullet \\ \circ & \bullet & \circ & \bullet & \circ \\ \circ & \circ & \bullet & \circ & \circ \\ \circ & \bullet & \circ & \bullet & \circ \\ \bullet & \circ & \circ & \circ & \bullet \end{pmatrix}. \quad (11)$$

The following theorems express the conditions for the existence of a WZ factorization of a nonsingular matrix (see [20]). Later, we give a new set of conditions useful for our purposes.

The well-known LU factorization is one of the most commonly used algorithms to solve linear systems and the WZ factorization is an interesting variation of the block LU factorization with block sizes equal to two. Assume that $P = [p_{\delta_1}, \dots, p_{\delta_n}]$ is a permutation matrix so that $p_{\delta_i} = i$, if i is odd, and $p_{\delta_i} = n - i + 1$, if i is even, $i = 1, \dots, n/2$, with n being even. Let L and U be the block lower and upper triangular matrices with block sizes equal to two,

respectively. Then , $P^{-T}LP^T$ is a W -matrix and $P^{-T}UP^T$ is a Z -matrix.

We know that the matrix A has an LU decomposition if and only if the leading principal block submatrices of A are nonsingular; see [12]. Therefore, we have the following result.

Theorem 2
(Factorization Theorem). Let $A \in R^{n \times n}$ be nonsingular. The matrix A has a WZ factorization if and only if for every k , $1 \leq k \leq s$, with $s = \lfloor n/2 \rfloor$, if n is even, and $s = \lceil n/2 \rceil$, if n is odd, the submatrix

$$\Delta_k = \begin{pmatrix} a_{1,1} & \cdots & a_{1,k} & a_{1,n-k+1} & \cdots & a_{1,n} \\ \vdots & \cdots & \vdots & \vdots & \cdots & \vdots \\ a_{k,1} & \cdots & a_{k,k} & a_{k,n-k+1} & \cdots & a_{k,n} \\ a_{n-k+1,1} & \cdots & a_{n-k+1,k} & a_{n-k+1,n-k+1} & \cdots & a_{n-k+1,n} \\ \vdots & \cdots & \vdots & \vdots & \cdots & \vdots \\ a_{n,1} & \cdots & a_{n,k} & a_{n,n-k+1} & \cdots & a_{n,n} \end{pmatrix}_{2k,2k} \quad (12)$$

of A is invertible. Moreover, the factorization is unique.

Proof

See Theorem 2 in [20]. □

The following results have also been established in [20].

- If $A \in R^{n \times n}$ is nonsingular, then a WZ factorization can always be obtained by pivoting. That is, there exists a row permutation matrix Π and the factors W and Z such that

$$\Pi A = WZ. \quad (13)$$

- Every symmetric positive definite matrix has a WZ factorization.

When A is a symmetric positive definite matrix, it is possible to factor A in the Cholesky factorization form $A = LL^T$, for some lower triangular matrix L . A variant of the classical Cholesky factorization, called Cholesky QIF, is given by Evans [10]. Existence and stability of this factorization are proved by Khazal [16].

Here, we propose a new formulation of Theorem 2 using the block scaled ABS algorithm.

Theorem 3

Let $A \in Z^{n \times n}$ be nonsingular. If the Δ_k , $k = 1, \dots, n/2$, are unimodular, then

the block ABS algorithm with parameter choices $H_1 = I$, $Q_i = V_i = [e_i, e_{n-i+1}]$ is well defined and the implicit factorization $V^T AP$ with $p_i = H_i^T e_i$, $p_{n-i+1} = H_i^T e_{n-i+1}$, $i = 1, \dots, n/2$, and $V = [V_1, \dots, V_{n/2}]$ leads to a WZ factorization.

Proof

Let $H_1 = I$, $V_i = [v_{2i-1}, v_{2i}] = [e_i, e_{n-i+1}]$, $Q_i = [q_{2i-1}, q_{2i}] = [e_i, e_{n-i+1}]$ and H_{i+1} be defined by (5). Then, according to Theorem 5.5 in [3], we have

$$H_{i+1} = \begin{bmatrix} 0 & 0 & 0 \\ K_i & I_{n-2i} & L_i \\ 0 & 0 & 0 \end{bmatrix}, \quad (14)$$

with $K_i, L_i \in Z^{n-2i, i}$. Let $P_i = [p_{2i-1}, p_{2i}] = H_i^T [e_i, e_{n-i+1}]$, $P = [P_1, \dots, P_{n/2}]$ and $V = [V_1, \dots, V_{n/2}]$. Then, the block ABS algorithm produces $V^T AP = L$, where P is an upper triangular matrix and L is a lower triangular matrix. Now, we have

$$V^T AP = L \Rightarrow APV^T = V^{-T} LV^T \Rightarrow AZ = W \quad (15)$$

where $Z = PV^T$ is an Z -matrix, with 1 as diagonal and 0 as off diagonal entries, and $W = V^{-T} LV^T$ is an W -matrix. \square

Next, we present an algorithm for computing the WZ factorization.

Algorithm 2. The WZ factorization.

- (1) Let $H_1 = I_n$ and $i=1$.
- (2) Let $A_i = [a_i, a_{n-i+1}]^T$ and $[p_i, p_{n-i+1}] = H_i^T [e_i, e_{n-i+1}]$.
- (4) Let $Q_i = [e_i, e_{n-i+1}]$. Update H_i by

$$H_{i+1} = H_i - H_i A_i (Q_i^T H_i A_i)^{-1} Q_i^T H_i. \quad (16)$$

- (5) Let $i = i + 1$. If $i \leq n/2$ then **go to** (2).
- (6) Compute $AP = W$, where $P = [p_1, \dots, p_n]$. **Stop**.

We now have the following results.

Theorem 4

Let A be a symmetric positive definite matrix. Then, there exists a $Z^T D Z$ factorization for A , obtained by the ABS algorithm.

Proof

Consider the assumptions of Theorem 3 and let $V_i = P_i$, for $i = 1, \dots, s$. Then, $V^T AP$ is an X -matrix, V and P are Z -matrices and we have

$$V^T AP = X \Rightarrow A = V^{-T} X P^{-1} = Z^T D Z, \tag{17}$$

where D is an X -matrix, $Z^T = V^{-T}$ and $Z = P^{-1}$. □

Theorem 5

Let A be symmetric positive definite. Then, there exists a $Z^T Z$ factorization for A , obtained by the ABS algorithm.

Proof

Consider the assumptions of Theorem 3 and let $V_i = P_i$, for $i = 1, \dots, s$. Then, $V^T AP$ is an X -matrix, V and P are Z -matrices and we have

$$V^T AP = X \Rightarrow A = V^{-T} X P^{-1} = Z^T Z, \tag{18}$$

where X is an X -matrix, $Z^T = V^{-T} X^{1/2}$ and $Z = X^{1/2} P^{-1}$. □

Remark 1

Now, let $A \in Z^{n \times n}$, $H_1 = I$ and Δ_k , $k = 1, \dots, n/2$, are unimodular. Then, the sequence of H_{i+1} , $i = 1, \dots, s$ generated by the Algorithm 2 are integer matrices [5] and the Algorithm 2, leads to a WZ factorization, where the W and the Z are integer W -matrix and Z -matrix, respectively.

2.1. The QZ Algorithm

Definition 5

A matrix Q is said a cross orthogonal matrix if $Q^T Q$ is an X -matrix.

Definition 6

We say that a matrix A is factorized in the form QZ if

$$A = QZ, \tag{19}$$

where the matrix Z is a Z -matrix and Q is a cross orthogonal matrix.

Theorem 6

Let $A \in R^{n \times n}$, $H_1 = I$, $V_i = [e_i, e_{n-i+1}]$, $Q_i = F_i = A^T V_i$ and $PV_i = H_i^T A^T V_i$, $i = 1, \dots, s$, with $s = \lfloor n/2 \rfloor$, if n is even, and $s = \lceil n/2 \rceil$, if n is odd. Then, there exists a QZ factorization, obtained by the block scaled ABS algorithm.

Proof

Making use of the proof of Theorem 5.1 in [3], P is a cross orthogonal matrix and we have

$$AP = W \Rightarrow A^T = QZ, \tag{20}$$

where $Z = W^T$ is a Z-matrix, $Q = P^{-T}$ and Q is a cross orthogonal matrix. Thus, we admit a QZ factorization for A^T . Of course, a QZ factorization for A is easily found by applying the above process to A^T . \square

3. ZW factorization using the block scaled ABS algorithm

Here, we present conditions for the existence and compute the ZW factorization of a nonsingular matrix using the ABS algorithm.

Definition 7

We say that a matrix A is factorized in the form ZW if

$$A = ZW, \tag{21}$$

where the matrices W and Z have the following structures:

$$Z = \begin{pmatrix} 1 & \bullet & \bullet & \bullet & \bullet \\ \circ & 1 & \bullet & \bullet & \circ \\ \circ & \circ & 1 & \circ & \circ \\ \circ & \bullet & \bullet & 1 & \circ \\ \bullet & \bullet & \bullet & \bullet & 1 \end{pmatrix}, W = \begin{pmatrix} \bullet & \circ & \circ & \circ & \bullet \\ \bullet & \bullet & \circ & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \circ & \bullet & \bullet \\ \bullet & \circ & \circ & \circ & \bullet \end{pmatrix}. \tag{22}$$

Now, we give a new characterization of the necessary and sufficient conditions for the ZW factorization.

Theorem 7

Let $A \in R^{n,n}$ be a nonsingular matrix. A has a ZW factorization if and only if for every $k, 1 \leq k \leq s$, with $s = \lfloor n/2 \rfloor$, if n is even, and $s = \lceil n/2 \rceil$, if n is odd, the submatrix

$$\Lambda_k = \begin{pmatrix} a_{s-k+1, s-k+1} & \cdots & a_{s-k+1, s+k} \\ \vdots & \cdots & \vdots \\ a_{s+k, s-k+1} & \cdots & a_{s+k, s+k} \end{pmatrix}_{2k, 2k} \tag{23}$$

is invertible. Moreover, the factorization is unique.

Proof

The proof follows the lines of the proof for Theorem 5.5 in [3] replacing Δ_i by Λ_i . \square

Next, we propose a new formulation of Theorem 7 using the block scaled ABS algorithm.

Theorem 8

Let $A \in Z^{n \times n}$ be nonsingular. If Δ_k , $k = 1, \dots, n/2$, is nonsingular, then the block ABS algorithm with parameter choices $H_1 = I$, $Q_i = V_i = [e_{\frac{n}{2}-i+1}, e_{\frac{n}{2}+i}]$ is well defined and the implicit factorization $V^T AP$ with $p_{\frac{n}{2}-i+1} = H_i^T e_{\frac{n}{2}-i+1}$, $p_{\frac{n}{2}+i} = H_i^T e_{\frac{n}{2}+i}$, $i = 1, \dots, n/2$, and $V = [V_1, \dots, V_{n/2}]$ leads to a ZW factorization.

Proof

Let $H_1 = I$, $V_i = [v_{2i-1}, v_{2i}] = [e_{\frac{n}{2}-i+1}, e_{\frac{n}{2}+i}]$, $Q_i = [q_{2i-1}, q_{2i}] = [e_{\frac{n}{2}-i+1}, e_{\frac{n}{2}+i}]$ and H_{i+1} be defined by (5). Then, according to Theorem 5.5 in [3], we have

$$H_{i+1} = \begin{bmatrix} I_i & K_{2i} & 0 \\ 0 & 0 & 0 \\ 0 & L_{2i} & I_i \end{bmatrix}, \tag{24}$$

with $K_i, L_i \in R^{n-2i, 2i}$. Let $P_i = [p_{2i-1}, p_{2i}] = H_i^T [e_{\frac{n}{2}-i+1}, e_{\frac{n}{2}+i}]$, $P = [P_1, \dots, P_{n/2}]$ and $V = [V_1, \dots, V_{n/2}]$. Then, the block ABS algorithm produces $V^T AP = L$, where P is upper triangular and L is lower triangular. Now, we have

$$V^T AP = L \Rightarrow APV^T = V^{-T} LV^T \Rightarrow AZ = W, \tag{25}$$

where $W = PV^T$ is a W -matrix with 1 as diagonal and 0 as off diagonal entries, and $Z = V^{-T} LV^T$ is a Z -matrix. □

Now, we are ready to present an algorithm for computing the WZ factorization.

Algorithm 3. The ZW factorization.

- (1) Let $H_1 = I_n$ and $i=1$.
- (2) Let $A_i = [a_{\frac{n}{2}-i+1}, a_{\frac{n}{2}+i}]^T$ and $[p_{\frac{n}{2}-i+1}, p_{\frac{n}{2}+i}] = H_i^T [e_{\frac{n}{2}-i+1}, e_{\frac{n}{2}+i}]$.
- (4) Let $Q_i = [e_{\frac{n}{2}-i+1}, e_{\frac{n}{2}+i}]$. Update H_i by

$$H_{i+1} = H_i - H_i A_i (Q_i^T H_i A_i)^{-1} Q_i^T H_i, \tag{26}$$

- (5) Let $i = i + 1$. If $i \leq n/2$ go to (2).
- (6) Compute $AP = Z$, where $P = [p_1, \dots, p_n]$. **Stop.**

Theorem 9

Let A be symmetric positive definite. Then, there exists a $W^T W$ factorization for A , obtained by the ABS algorithm.

Proof

Consider the assumption of Theorem 8 and let $V_i = P_i$, for $i = 1, \dots, s$. Then, $V^T AP$ is an X -matrix, V and P are W -matrices and we have

$$V^T AP = X \Rightarrow A = V^{-T} X P^{-1} = W^T W, \quad (27)$$

where X is an X -matrix, $W^T = V^{-T} X^{1/2}$ and $W = X^{1/2} P^{-1}$. \square

Remark 2

Now, let $A \in Z^{n \times n}$, $H_1 = I$ and Δ_k , $k = 1, \dots, n/2$, are unimodular. Then, the sequence of H_{i+1} , $i = 1, \dots, s$ generated by the Algorithm 3 are integer matrices [5] and the algorithm leads to a ZW factorization, where the W and the Z are integer W -matrix and Z -matrix, respectively.

3.1. The QW Algorithm*Definition 8*

We say that a matrix A is factorized in the form QW if

$$A = QW, \quad (28)$$

where the matrix W is a W -matrix and Q is a cross orthogonal matrix.

Theorem 10

Let $A \in R^{n \times n}$, $H_1 = I$, $V_i = [e_{s-i+1}, e_{s+i}]$, $Q_i = F_i = A^T V_i$ and $P V_i = H_i^T A^T V_i$, $i = 1, \dots, s$, with $s = \lfloor n/2 \rfloor$, if n is even, and $s = \lceil n/2 \rceil$, if n is odd. Then, there exists a QZ factorization, obtained by the block scaled ABS algorithm.

Proof

Making use of the proof of Theorem 5.1 in [3], P is a cross orthogonal matrix and we have

$$AP = Z \Rightarrow A^T = QW, \quad (29)$$

where $W = Z^T$ is a W -matrix and $Q = P^{-T}$ is a cross orthogonal matrix. Thus, we admit a QW factorization for A^T . Of course, a QW factorization for A is easily found by applying the above process to A^T . \square

4. Conclusions

We showed how to appropriate the parameters of the block scaled ABS algorithms to construct algorithms for computing the WZ and ZW factorizations of a nonsingular matrix and the $W^T W$ and $Z^T Z$ factorizations of a symmetric positives definite matrix. New formulation of the necessary and sufficient

conditions for the existence of the WZ and the ZW factorizations of a nonsingular matrix were given. We also derived two new factorizations, the QZ and QW, with $Q^T Q$ being an X-matrix, and showed how to compute the factorizations using the block scaled ABS algorithms. We also provided the conditions for the existence of the integer WZ and ZW factorizations of an integer matrix and computed the factorizations by the integer ABS algorithms.

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