STATISTICS, OPTIMIZATION AND INFORMATION COMPUTING
Stat., Optim. Inf. Comput., Vol. **12**, Month 2024, pp 0–12.
Published online in International Academic Press (www.IAPress.org)

Generalized variables approach to generalized inverted exponential distribution reliability analyses with progressively type II censored data

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Abstract Reliability analysis plays a crucial role in various fields, including engineering, manufacturing, and quality control. It provides valuable insights into the failure behavior of systems and products. One commonly used distribution in reliability modeling is the generalized inverted exponential distribution. The generalized inverted exponential distribution is known for its flexibility and adaptability to a wide range of failure data. This paper presents a new method for estimating confidence intervals and testing hypotheses for generalized inverted exponential distribution reliability functions based on a generalized value approach. By transforming the reliability function into the generalized value domain and calculating the generalized lower confidence limit, the proposed method offers enhanced accuracy and precision. Furthermore, the generalized p-value approach for hypothesis testing provides a robust and computationally efficient method for analyzing reliability data. The results from a real data set and Monte Carlo simulations confirm the superiority of the proposed approach over classical methods. The proposed method offers improved accuracy and computational efficiency, making it a valuable tool for reliability analysis using GIED distributions.

Keywords fixed removals, generalized inverted exponential distribution, generalized p-value, generalized confidence interval, progressive type II censored sample.

AMS 2010 subject classifications 62N02, 62N05

DOI: 10.19139/soic-2310-5070-2013

1. Introduction

The one-parameter exponential distribution is the simplest and most widely used model in life testing and reliability analysis. However, it has certain limitations, such as a constant hazard rate. Researchers have proposed various modifications to overcome these limitations. These modifications increase the flexibility and applicability of the exponential model, making it a valuable tool in fields such as life testing and reliability analysis. In order to make the distribution function more flexible for use in a wide range of applications, its exponentiation has been extended extensively (see [1]). Furthermore, an inverted version of the exponential distribution was developed, known as the inverted exponential distribution (IED). A detailed discussion of IED as a lifetime model is presented in [2]. The authors developed maximum likelihood estimates (MLEs), confidence limits, and uniformly minimum variance unbiased estimators of the parameter and reliability function in IED using complete samples. Using maintenance data, the authors also compared this model with the inverted Gaussian distribution and the log-normal distribution. According to them, this model provided a better fit than the inverted Gaussian and log-normal distribution. Additionally, [3] has obtained Bayesian estimates for the parameter and risk functions under various loss functions. In [4], a generalized version of the IED is introduced, known as the Generalized Inverted Exponential Distribution (GIED). In addition to modeling various shapes of failure rates, this lifetime distribution can also model various

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ISSN 2310-5070 (online) ISSN 2311-004X (print)

shapes of aging criteria. The model can be considered as another useful two-parameter generalization of the IED. A discussion of the statistical and reliability properties as well as the estimation properties of the GIED is provided in [4].

In relation to the shape parameter of the distribution, the hazard function (HF) of GIED increases or decreases, but is not constant. For the shape parameter, $\beta > 4$, the GIED has an unimodal and right-skewed probability density function (PDF). Also, the GIED is observed to fit numerous situations better than the gamma distribution, the Weibull distribution, the generalized exponential distribution, and the inverted exponential distribution [17]. These characteristics enable the distribution to model a variety of failure rates based on aging criteria. [43] investigated the reliability estimation under progressive type II censoring in GIED.

In [15], generalized values and generalized test variables were introduced, which are useful for developing hypothesis tests in situations where traditional frequentist approaches are insufficient. Afterward, [16] introduced the concept of generalized pivotal quantities (GPQs) for scalar parameters, which can be used to construct interval estimators for parameters for which normal pivotal quantities may not be applicable. These intervals are referred to as generalized confidence intervals (GCIs). Generalized point estimators were introduced by [28]. In many practical applications, GPQs and GCIs have proven to be useful tools for making inferences. It is important to note that, while GCIs are not guaranteed to have exact frequentist coverage, several published and unpublished simulation studies have concluded that the coverage probabilities of such intervals are sufficiently close to their nominal values to make them useful in practice. The GPQ differs from the ordinary pivotal quantity in that it is based on observed statistics and random variables whose distributions are not affected by unknown parameters. When a GPQ for an individual parameter is available, a GPQ for a function of parameters can readily be obtained by substitution. Several complex problems have been solved using this generalized variable approach.

[27] introduced the concept of a generalized p-value about the comparison of two regression models and demonstrated that it is an exact probability of a well-defined subset of the sample space containing an unbiased extreme region based on sufficient statistics. Moreover, he demonstrated that the generalized p-value can be calculated numerically or by simulation. It should be noted, however, that the generalized p-value may not be uniformly distributed for some applications, resulting in non-exact solutions to some statistical problems. Weerahandi [40] further developed the concept of confidence intervals. Due to the failure of conventional methods to provide exact solutions to even simple problems involving nuisance parameters or complicated functions of parameters, practitioners tend to work with asymptotic results in order to obtain approximate solutions to such problems. There may be reasonable and exact solutions for such difficult and interesting problems using this newly developed promising approach, the generalized variable method. Through simulation studies conducted to date, several authors have shown that the generalized variable method outperforms approximate and exact tests used so far for finding solutions to more complex situations (cf. Ananda and Weerahandi [31], Weerahandi and Johnson /cite[weej92], Gamage and Weerahandi [38]. Various practical settings have been successfully applied to generalized variable methods, including ANOVA, regressions, mixed and growth curve models. However, this method does not produce satisfactory results for arbitrary covariance matrices, such as those found in the MANOVA method [45] for example. The reader is referred to Weerahandi [21, 20] for a complete explanation of these generalized tests and confidence intervals.

The generalized variable method is a statistical approach that has gained significant attention due to its time efficiency and easy computation compared to other methods. This method offers several advantages that make it a valuable tool for statistical research. One of the key advantages of the generalized variable method is its ability to handle various distribution-driven tests. By leveraging the generalized distribution, the generalized variable method enables researchers to conduct tests that are specific to their data distribution, providing more accurate and precise results. This flexibility enables researchers to tailor their analysis to the specific characteristics of their data, ensuring more accurate interpretations and conclusions. Another advantage of the generalized variable method is its ability to handle both small and large samples. One of the main challenges faced by researchers is determining the appropriate sample size for their study. The GVM eliminates the need for arbitrary assumptions regarding large sample sizes, making it suitable for studies with smaller sample sizes. This enables researchers to leverage available data more effectively, saving time and resources.

[18] identified a subclass of generalized pivotal quantities, called fiducial generalized pivotal quantities, and demonstrated that under some mild conditions, GCIs constructed using fiducial GPQs have correct frequentist coverage at least asymptotically. In [19], a heuristic method was proposed for estimating confidence intervals for the mean of a delta-lognormal distribution. Based on a generalized pivotal quantity asymptotic to the delta-lognormal distribution mean, this heuristic method constructed a generalized confidence interval. [26] showed that the Bayesian p-value has a numerical equivalent to the generalized p-value. In recent years, a number of practical problems have been developed using generalized methods. For example, see [37, 44, 32, 29, 39, 30, 33].

Data from lifetimes is usually censored due to time and cost constraints. When lifetimes are only partially known in fields such as statistics, engineering, and medicine, censoring occurs. It is expensive and time-consuming to conduct lifetime experiments. To reduce the cost and time of experimentation, various types of censoring schemes have been developed, such as type I, type II, Double type II, random, and progressive censoring schemes. Among the various censoring schemes, Type-I and Type-II are the most prevalent. The withdrawal of live items during an experiment is impossible under these censoring schemes. This article discusses a generalization of the classical Type-II censoring scheme, which allows live items to be withdrawn during an experiment, referred to as progressive Type-II censoring. As a result of its time and cost savings, progressive type II censoring has become very popular in life-testing experiments.

Several authors developed statistical inferences and predictions based on progressively Type-II censored data for failure times. Based on progressively censored Type-II data, maximum likelihood estimators of the distribution parameters are derived for the normal distribution as well as for the exponential distribution in [5]. [6] provides maximum likelihood estimators for the three-parameter gamma distribution (Pearson's Type III Distribution) when samples are progressively censored. An in-depth review of progressive censorship's theory, methods, and applications can be found in [7]. In addition, [8] offers an overview of the theory and methodology of progressive censorship, as well as practical applications to reliability and survival analysis.

A Bayesian approach to Pareto distribution prediction and inference is described in [10]. By applying different methods, bounds for expected values and variances of progressive type II censored order statistics are derived by [9]. These bounds provide useful approximations for the behavior of these order statistics. This enables researchers and practitioners to analyze the uncertainty associated with progressive Type II censoring. Moreover, [11] obtained Bayesian bounds for Rayleigh models for two samples with progressive Type-II censoring.

A statistical inference was presented in [12] using progressively type II censored data for Weibull parameters. In the study, maximum likelihood estimators were derived. The Bayesian approach was considered to incorporate previous information with the current data. Under squared error loss with a bivariate prior distribution, Bayes estimators were obtained, and credible intervals were derived for the parameters of the Weibull distribution. Bayes prediction intervals were calculated for future observations for both the one and two-sample cases.

A non-Bayesian two-sample prediction was developed by [14] using a progressive right censoring scheme based on Type-II. Maximum likelihood predictions were obtained for lifetime models including the Weibull distribution in a general form. An overview of progressive censoring can be found in [32]. Several recent studies have been published, including those by [24, 42, 43].

There are two common types of censoring schemes: Types I and II. However, none of these two allows units to be removed other than at the final point of termination. As a result, we use a more appropriate censoring scheme called the progressive Type-II censoring scheme, which allows the intermediate removals to be performed. In this case study, n units are tested, and after the first failure, R_1 units are randomly removed from the remaining $n-1$ surviving units. As a result of the second failure, R_2 surviving units are randomly removed from the remaining $n-R_1-2$ surviving units. The test continues till the mth failure time at which, all remaining $R_m = n - R_1 - R_2 - R_{m-1} - m + 1$ units are removed. It is assumed that R_i's are prefixed here. For $R_1 = R_2 = ... = R_{m-1} = 0$ and $R_m = n - m$, this scheme reduces to the conventional Type-II C.S. and for $R_1 = R_2 = ... = R_{m-1} = R_m = 0$ it reduces to complete the sample case. Early works on progressive censorship can be found in [5] and [23]. An algorithm has recently been developed by [24] for simulating general progressively Type-II censored samples from any distribution. There have also been studies on estimating parameters from different lifetime distributions using progressively Type-II censored data by [7, 9, 11, 14, 25]

The statistical and reliability properties of the GIED are presented in [4]. Further, maximum likelihood estimation and least squares estimation were used to evaluate the parameters and the reliability of the distribution. The reliability characteristics and distributional properties of the generalized inverted exponential distribution discussed in [43] provide valuable insights into its applicability to reliability analysis. The methods of maximum likelihood estimation and method of least squares estimation, along with their application to type II progressively right censored data, offer practical methods for estimating the parameters and calculating reliability and failure rate functions.

This paper presents a generalized variables approach to reliability estimation using progressively type-II censored data from a generalized inverted exponential model distribution. Section 2 presents the distributional properties of GIED as well as its reliability and failure rate functions. Assuming that the scaled parameter λ is known, in Section 3 MLEs of the shape parameter β and the reliability function are derived using progressively censored type II data. In Section 4, two methods are used to develop the method for testing the reliability function, namely the classical method and the generalized method. In Section 5, a real-data example and a simulation study are performed to examine the effects of sample sizes and censoring schemes on these estimates. Section 6 concludes with a conclusion.

2. The model

Now, we will examine some properties of the two-parameter Generalized Inverted Exponential distribution. The probability density function (pdf) of $GIED(\lambda, \beta)$ is given by

$$f(x;\lambda,\beta) = \left(\frac{\beta\lambda}{x^2}\right) exp\left(\frac{-\lambda}{x}\right)\left[1 - exp\left(\frac{-\lambda}{x}\right)\right]^{\beta-1} \qquad x \geq 0,\ \lambda,\ \beta > 0 \tag{1}$$

and the corresponding cumulative distribution function (cdf) is given by

$$F(x;\lambda,\beta) = 1 - \left[1 - exp\left(\frac{-\lambda}{x}\right)\right]^{\beta} \qquad x \geq 0,\ \lambda,\ \beta > 0 \tag{2}$$

where, λ and β are scaled and shape parameter, respectively.

So, the reliability and failure rate functions of the $GIED(\lambda, \beta)$, respectively, are given by

$$S(x) = \left[1 - exp\left(\frac{-\lambda}{x}\right)\right]^{\beta} \qquad x \geq 0,\ \lambda,\ \beta > 0 \tag{3}$$

$$h(x) = \frac{\beta\lambda}{x^2(exp(\frac{\lambda}{x}) - 1)} \qquad x \geq 0,\ \lambda,\ \beta > 0 \tag{4}$$

In Figure (1), the pdf and failure rate function of the GIED for $\lambda = 1$ and several representative values of β are shown. A non-monotone unimodal shape can be seen in the graph of GIED's failure rate function. In many practical situations, it is known that the data are derived from a distribution with a non-monotone failure rate function. Thus, if the empirical study indicates that the failure rate function for the underlying distribution is non-monotone and has an unimodal shape, then the GIED could be used to analyze such data sets.

By the transformation $Y = -ln\big[1 - exp(-\frac{\lambda}{x})\big]$, and when the distribution of Y is an exponential distribution, the pdf and cdf of Y are

$$f(x,\lambda) = \beta exp(-\beta y)\ ;\ \ y > \mu,\ \beta > 0, \tag{5}$$

and

$$F(x,\beta) = 1 - exp(-\beta y);\ \ y > \mu,\ \beta > 0, \tag{6}$$

respectively.

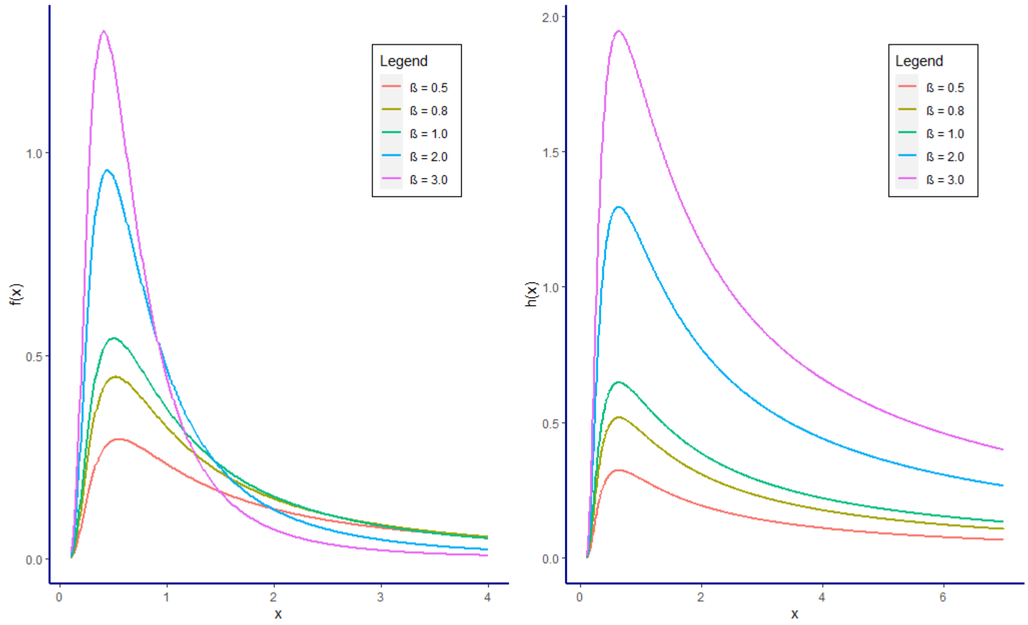

Figure 1. Plots of pdf and failure rate function of GIED for $\lambda = 1$ and different values of β

3. Maximum-likelihood estimation

Suppose $x_1, x_2, ..., x_m$ is a progressive type II right censored sample from $GIED(\lambda, \beta)$ with progressive censoring scheme $R_1, R_2, ..., R_m$. The likelihood function of this sample is given by [32]

$$L(x; \lambda, \beta) = C \prod_{i=1}^{m} f(x_i)\{1 - F(x_i)\}^{R_i}$$
$$= C \prod_{i=1}^{m} \left(\frac{\beta\lambda}{x_i^2}\right) exp\left(\frac{-\lambda}{x_i}\right)\left[1 - exp\left(\frac{-\lambda}{x_i}\right)\right]^{\beta-1} \left\{\left[1 - exp\left(\frac{-\lambda}{x_i}\right)\right]^{\beta}\right\}^{R_i} \quad (7)$$

where $C = n(n - R_1 - 1)(n - R_1 - R_2 - 2)...(n - R_1 - R_2 - R_3 - ... - R_{m-1} - m + 1)$.

3.1. Estimation of the reliability function

First, we find the MLE of β which is denoted by $\hat{\beta}$. The likelihood function can be given by [32];
Assuming λ is known, The MLE of β can be derived as follows:
Based on likelihood function (7), log-likelihood can be expressed as follows:

$$ln\, L(x_i; \lambda, \beta) = ln\, C + m\, ln\, \beta + m\, ln\, \lambda - 2\sum_{i=1}^{m} ln\, x_i - \sum_{i=1}^{m} \frac{\lambda}{x_i} + \sum_{i=1}^{m}\left[\left((1 + R_i)\,\beta - 1\right) ln\left(1 - exp\left(\frac{-\lambda}{x_i}\right)\right)\right]$$

MLE of β is a solution to the following log-likelihood equation:

$$\frac{\partial\, ln\, L(x; \lambda, \beta)}{\partial\beta} = \frac{m}{\hat{\beta}} + \sum_{i=1}^{m}(1 + R_i)\, ln\left(1 - exp\left(\frac{-\lambda}{x_i}\right)\right) = 0 \quad (8)$$

which gives

$$\implies \hat{\beta} = \frac{-m}{\sum_{i=1}^{m}(1 \ + \ \beta_i)\ ln\left(1 - exp\left(\frac{-\lambda}{x_i}\right)\right)} \tag{9}$$

By invariance property, MLE of $S(x)$ can be estimated as;

$$\widehat{S(x)} = \left[1 - exp\left(\frac{-\lambda}{x}\right)\right]^{\hat{\beta}} \ ; x \geq 0$$

4. Testing Procedure for the Reliability Function

Consider the following null and alternative hypothesis tests;

$$H_0 \ : \ S \leq s^* \ vs \ H_1 \ : \ S > s^*$$

where s^* denotes the target value. The classical testing procedure for the above hypothesis testing can be presented as follows;

4.1. Classical testing procedure

4.1.1. Critical Value The MLE of $S(x)$, $\widehat{S(x)})$ can be used as the test statistics. Then the rejection region is $\{\hat{s} > s_0^*\}$, where s_0^* is the critical value. Now, the critical value s_0^* can be calculated for a given the specified significance level α as follows:

$$\begin{aligned}
sup\ p(\hat{s} > s_0^*) &= \alpha \\
\implies p(\hat{S} > s_0^* \ \Big| \ s = s^*) &= \alpha \\
\implies p\left((1 - e^{\frac{-\lambda}{x}})^{\frac{m}{W}} > s_0^* \ \Big| \ (1 - e^{\frac{-\lambda}{x}})^{\beta} = s^*\right) &= \alpha \\
\implies p\left(\frac{m}{W}\ ln\ (1 - e^{\frac{-\lambda}{x}}) > ln\ s_0^* \ \Big| \ \beta\ ln\ (1 - e^{\frac{-\lambda}{x}}) = ln\ s^*\right) &= \alpha \\
\implies p\left(W < \frac{m\ ln(1 - e^{\frac{-\lambda}{x}})}{ln\ s_0^*} \ \Big| \ \beta = \frac{ln\ s^*}{ln\ (1 - e^{\frac{-\lambda}{x}})}\right) &= \alpha \\
\implies p\left(2\beta W < \frac{2m\ ln\ s^*}{ln\ s_0^*}\right) &= \alpha \qquad (10)
\end{aligned}$$

Since $U = 2\beta W$ where $U \sim \chi^2_{2m}$, from the above equation,

$$2m\ \frac{ln\ s^*}{ln\ s_0^*} = \chi^2_{2m,1-\alpha}$$

So, s^* can be represented as;

$$s_0^* = exp\left\{\frac{2m\ ln\ s^*}{\chi^2_{2m,1-\alpha}}\right\} \tag{11}$$

where s^*, α and m denote the target value, the specified significance level and the number of observed failures before termination, respectively.

4.1.2. The confidence interval Since the pivotal quantity, $2\beta W \sim \chi^2_{2m}$ and $\chi^2_{2m,1-\alpha}$ represents the lower $1-\alpha$ percentile of $\chi^2_{(2m)}$, we can express $p(2\theta W \geq \chi^2_{2m,1-\alpha}) = 1-\alpha$.

Since, $S = (1-e^{\frac{\lambda}{x}})^{\beta}$ and $\hat{S} = (1-e^{-\frac{\lambda}{x}})^{\frac{m}{w}}$,

$$\Longrightarrow p\left(\frac{2\,lnS}{ln(1-e^{\frac{\lambda}{x}})}\frac{ln(1-e^{\frac{\lambda}{x}})m}{ln\hat{S}} \geq \chi^2_{2m,1-\alpha}\right) = 1-\alpha$$
$$\Longrightarrow p\left(\frac{2\;m\;lns}{ln\hat{S}} \geq \chi^2_{2m,1-\alpha}\right) = 1-\alpha$$
$$\Longrightarrow p\left(S \geq exp\left(\frac{\chi^2_{2m,1-\alpha}\;ln\hat{S}}{2m}\right)\right) = 1-\alpha \tag{12}$$

From the above equation, the lower limit of the confidence interval for s at $100(1-\alpha)\%$ level can be presented as;

$$S \geq exp\left(\frac{\chi^2_{2m,1-\alpha}\;ln\;\hat{S}}{2m}\right)$$

Thus, the classical confidence bound for s at level $100(1-\alpha)\%$ can be written as

$$LB^c(x) = exp\left(\frac{\chi^2_{2m,1-\alpha}\;ln\;\hat{s}}{2m}\right) \tag{13}$$

where m, α and $\hat{S}$ denote the number of observed failures before termination, the specified significance level, and the MLE of S respectively.

4.1.3. Power Function The power $p(s_1^*)$ of the test at the point $S = s_1^*(> s)$ is derived as follows:

$$\begin{aligned}
p(s_1^*) &= p(\hat{S} > s_0^* | s = s_1^*)\\
&= p\left((1-e^{-\frac{\lambda}{x}})^{\frac{m}{w}} > s_0^* | (1-e^{-\frac{\lambda}{x}})^{\beta} = s_1^*\right)\\
&= p\left(\frac{m}{W}\;ln(1-e^{-\frac{\lambda}{x}}) > ln\;s_0^* | \beta = \frac{ln\;s_1^*}{ln(1-e^{\frac{\lambda}{x}})}\right)\\
&= p\left(2W\beta < \frac{2\;ln\;s_1^*}{ln(1-e^{-\frac{\lambda}{x}})}m\frac{ln(1-e^{-\frac{\lambda}{x}})}{ln\;s_0^*}\right)\\
&= p\left(U <\; 2m\;\frac{ln\;s_1^*}{ln\;s_0^*}\right)\\
&= p\left(U < \frac{ln\;s_1^*}{ln\;s_0^*}\;\chi^2_{2m,1-\alpha}\right)
\end{aligned} \tag{14}$$

where $U \sim \chi^2_{2m}$.

4.2. Generalized Inference

Since $y_i = -ln\;(1-e^{-\frac{\lambda}{x_i}}),\;\; i = 1,2,...,m$ has an exponential distribution, observed ordered statistics, $y_1 \leq y_2 \leq ... \leq y_m$ is a progressive type II censored sample from the exponential distribution with parameter β. Letting $U = 2\beta W$, where $W = \sum_{i=1}^{m}(1+\beta_i)\,Y_i$, we can easily show that U has a chi-square distribution with $2m$ degrees of freedom. The reader is referred to [32] for more details.

We now present the generalized method for testing the above hypothesis by calculating generalized CI and p-value. For more information see [40, 20, 21].

The generalized pivotal statistic for S, ($Q^S(\mathbf{X};\mathbf{x};\boldsymbol{\delta})$, or simply Q^S) is given by

$$Q^S = \exp\left[Q^\beta ln\big[1 - exp(-\frac{\lambda}{x})\big]\right] ; \;\; x > 0. \tag{15}$$

Using $U = 2\beta W \sim \chi^2_{(2m)}$, A choice of a generalized pivotal statistics of β, Q^β can be given by

$$Q^\beta = \frac{U\widehat{\beta}_{obs}}{2m} = \frac{U}{2W}, \tag{16}$$

where, $W = \sum_{i=1}^{m} -(1+R_i)ln\big[1 - exp(-\frac{\lambda}{x})\big]$, and $\widehat{\beta}_{obs} = \frac{-m}{\sum_{i=1}^{m}(1+\beta_i)\; ln\;(1-exp(\frac{-\lambda}{x_i}))}$.

Now, a generalized pivotal statistic for S can be given by,

$$Q^S = \exp\left[\frac{U\widehat{\beta}_{obs}}{2m} ln\big[1 - exp(-\frac{\lambda}{x})\big]\right]. \tag{17}$$

Let $Q^S_{1-\alpha}(\mathbf{x}_{m,n};\widehat{\beta}_{obs})$, where $\mathbf{x}_{m,n} = (x_{1,n}, x_{2,n}, ..., x_{m,n})$, satisfy

$$P[Q^S \leq Q^S_{1-\alpha}(\mathbf{x}_{m,n};\widehat{\beta}_{obs})] = 1 - \alpha$$

The $Q^S_{1-\alpha}(\mathbf{x}_{m,n};\widehat{\beta}_{obs})$ is a lower confidence limit for S at $100(1-\alpha)\%$ level. The lower confidence bound is

$$LB^G(x) = Q^S_{1-\alpha}(\mathbf{x}_{m,n};\widehat{\beta}_{obs}) \tag{18}$$

4.3. Generalized Testing Procedure

Considering the above null and alternative hypothesis tests for S, we can conduct one-sided hypothesis testing using the generalized test variable $T^S(\mathbf{X};\mathbf{x};\boldsymbol{\delta}) = Q^S(\mathbf{X};\mathbf{x};\boldsymbol{\delta}) - S$, (or $T^S = Q^S - S$) where $T = T^S(\mathbf{X};\mathbf{x};\boldsymbol{\delta})$.

Hence, the generalized *p*-value, p_g can be given as:

$$p_g = \Pr\left\{\exp\left[\frac{U\widehat{\beta}_{obs}}{2m}\; ln\big[1 - exp(-\frac{\lambda}{x})\big]\right] \geq s^*\right\} \tag{4.4}$$

5. Application and Simulation

This section aims to shed light on the methodologies discussed in the prior sections by showcasing their performance on a real data set and conducting a broad simulation study using R. The performance of the proposed methods will be evaluated using a real-world dataset, which allows us to assess the practicality and effectiveness of the methodologies in real-world scenarios. This dataset will represent a range of data types and characteristics, thereby providing a comprehensive evaluation. In addition to the real-world dataset, a broad simulation study will be conducted using R. The purpose of this simulation study is to replicate the experimental conditions of different sample sizes, distributions, and missing data patterns. By conducting these simulations, we can evaluate the robustness and reliability of the methodologies under a variety of conditions.

5.1. An Application

In this subsection, we will illustrate the proposed procedure using a data set. The data chosen are results taken from testing performed to measure the endurance of deep groove ball bearings and were originally discussed by Lieblien and Zelen [22].

Abougmmoh and Alshingiti fit this data set to GIED using a complete sample (see [4]).

The ordered data list is given as:

17.88	28.92	33.0	41.52	42.12	45.60	48.40	51.84
51.96	54.12	55.56	67.80	68.64	68.64	68.88	84.12
93.12	98.64	105.12	105.84	127.92	128.04	173.4	

Progressively type II censored two samples are selected with failure times m = 9 and m=12 as follows.

17.88	28.92	33.0	55.56	67.80	68.64	68.88	128.04	173.4

and

33.0	48.40	51.84	54.12	55.56	67.80	68.64	68.88	84.12	98.64	105.84	128.04

Since we assume λ is known and λ needs to be found. For various λ values that minimized SSE, the value λ was 47.7. Furthermore, the $\hat{\beta}$ value corresponding to $\lambda = 47.7$ is 0.5788567. Now the pdf of GIED can be written as;

$$f(x, 47.7, \beta) = \left(\frac{47.7\beta}{x^2}\right) exp\left(\frac{-47.7}{x}\right)\left[1 - exp\left(\frac{-47.7}{x}\right)\right]^{\beta-1}$$

The scale-free goodness-of-fit test is used to test the null hypothesis $H_0 : x \sim GIED(47.7, \beta)$ at $\alpha = 0.05$. The Gini statistic is found to be

$$G_9 = \frac{\sum_{i=1}^{8} w_{i+1}}{(9-1)\sum_{i=1}^{9} w_i} = 0.50392$$

Where $w_i = (m - i + 1)(z_{(i)} - z_{(i+1)}), z_{(0)} = 0, i = 1, 2, \ldots, m$ $z_0 = nY_1, z_i = [n - \sum_{j=1}^{i-1}(B_j + 1)](Y_i - Y_{i-1}),\ i = 1, 2, \ldots m$ and the data transformation $Y_i = ln(1 - e^{-\frac{\lambda}{x_i}})$.

At the 0.05 level of significance, we see that $\xi_{0.025} = 0.36696 < G_9 = 0.50392 < \xi_{0.9775} > 0.63304$. So, the observed failure times from the GIED with the pdf is $f(x, 47.7, \beta) = (\frac{47.7\beta}{x^2}) exp(\frac{-47.7}{x})[1 - exp(\frac{-47.7}{x})]^{\beta-1}$, at $\alpha = 0.05$. Then

$$\hat{\beta} = \frac{-m}{\sum_{i=1}^{m}(1\ +\ \beta_i)\ ln\ (1 - exp(\frac{-\lambda}{x_i}))} = 0.57857.$$

Now the classical and generalized 95% lower bound and $p - values$ for the reliability function of the GIE distribution for several x values and censoring schemes will be compared for both $m = 9$ and $m = 12$ cases. Lower bounds and $p - values$ are represented in the tables (1) and (2) respectively.

Table 1. 95% interval estimates of $S(x)$

(n,m)	x	Scheme	Pt Est.	Gen.	Clas.	(n,m)	x	Scheme	Pt Est.	Gen.	Clas.
(23,9)	20	(3*3,0*5,5)	0.9420	0.9060	0.9086	(23,12)	20	(1*3,0*7,4*2)	0.9387	0.9054	0.9086
		(0*8,14)	0.9667	0.9456	0.9472			((3,2)*2,1,0*7)	0.9163	0.8715	0.8758
	30	(3*3,0*5,5)	0.8684	0.7920	0.7976		30	(1*3,0*7,4*2)	0.8614	0.7909	0.7975
		(0*8,14)	0.9232	0.8764	0.8798			((3,2)*2,1,0*7)	0.8135	0.7228	0.7312
	65	(3*3,0*5,5)	0.6673	0.5124	0.5227		65	(1*3,0*7,4*2)	0.6519	0.5102	0.5225
		(0*8,14)	0.7954	0.6849	0.6927			((3,2)*2,1,0*7)	0.5533	0.3942	0.4074

5.2. Simulated data

In this subsection, we use the Monte Carlo simulation to illustrate the behavior of the proposed methods by calculating coverage probability of the confidence interval based on $LB^C(x)$ and $LB^G(x)$, and empirical type-I error for testing $H_0 : S \leq s^*$ vs $H_1 : S > s^*$ at both $\alpha = 0.05$ and $\alpha = 0.10$ levels. Coverage probability and

Table 2. $p - values$ for testing $H_0 : S \leq s^*$ vs $H_1 : S > s^*$ at $\alpha = 0.05$.

(n,m)	x	Scheme	s^*	Gen.	Clas.	(n,m)	x	Scheme	s^*	Gen.	Clas.
(23,9)	20	(3*3,0*5,5)	0.90	0.023	0.103	(23,12)	20	(1*3,0*7,4*2)	0.78	0.021	0.075
		(0*8,14)	0.94	0.016	0.157			((3,2)*2,1,0*7)	0.87	0.047	0.071
	30	(3*3,0*5,5)	0.78	0.023	0.072		30	(1*3,0*7,4*2)	0.79	0.034	0.077
		(0*8,14)	0.88	0.049	0.092			((3,2)*2,1,0*7)	0.73	0.047	0.071
	65	(3*3,0*5,5)	0.50	0.030	0.061		65	(1*3,0*7,4*2)	0.52	0.047	0.063
		(0*8,14)	0.68	0.033	0.066			((3,2)*2,1,0*7)	0.40	0.041	0.062

empirical type-I error rates are calculated for various sample sizes n, various failure times m, different parameters (λ, β), and various x values. The calculations are conducted using RStudio, Version 1.1.463 - 2009-2018 RStudio, Inc.

Table 3. Percentage of the Coverage probability of the confidence interval

			x					
			1.7		3.0		8.0	
(λ, β)	(n, m)	Scheme	Generalized	Classical	Generalized	Classical	Generalized	Classical
$(1, 0.5)$	(10,6)	$(1^*4, 0^*2)$	94.95(89.80)	95.00(89.89)	94.67(89.73)	94.72(89.96)	94.94(89.92)	95.00(90.02)
	(20,5)	$(0^*2, 5^*3)$	94.93(90.19)	95.13(90.23)	95.03(89.89)	95.16(89.95)	95.07(90.31)	95.22(90.49)
	(20,8)	$(0^*3, 2^*3, 3^*2)$	95.24(89.97)	95.31(90.25)	95.35(89.92)	95.52(89.86)	95.24(90.78)	95.41(90.60)
	(20,10)	$(2, 0)^*4, 1^*2)$	94.74(94.95)	94.98(89.60)	94.59(89.53)	94.82(89.71)	94.61(90.44)	94.82(90.48)
	(20,15)	$(1^*3, 0^*10, 1^*3)$	95.27(89.62)	95.28(89.76)	94.90(90.08)	95.09(89.99)	95.04(89.49)	94.88(89.71)
	(20,20)	(0^*20)	95.12(89.89)	95.17(90.31)	94.99(90.07)	95.11(89.94)	95.25(90.00)	95.44(90.26)
	(50,20)	$(1^*7, 0^*4, 2^*4, 3^*5)$	94.72(89.80)	94.88(90.19)	95.18(89.86)	95.01(89.93)	94.73(89.92)	94.98(89.97)
	(50,25)	$((0, 2)^*1, 1^*3)$	95.34(90.23)	95.37(90.03)	94.77(89.96)	94.85(90.09)	95.05(90.29)	95.17(90.30)
	(50,50)	(0^*50)	94.73(89.88)	94.92(89.98)	95.19(89.92)	95.13(90.03)	94.93(89.97)	94.95(89.89)
$(1, 1.5)$	(10,6)	$(1^*4, 0^*2)$	95.16(89.97)	95.12(90.04)	95.02(89.68)	95.09(89.68)	94.63(89.78)	94.54(89.64)
	(20,5)	$(0^*2, 5^*3)$	95.35(89.54)	95.41(89.32)	95.26(90.71)	95.43(90.63)	95.28(89.76)	95.26(89.96)
	(20,8)	$(0^*3, 2^*3, 3^*2)$	94.91(90.16)	94.98(90.24)	94.53(89.89)	94.50(90.01)	95.03(89.91)	95.04(89.99)
	(20,10)	$(2, 0)^*4, 1^*2)$	94.84(89.95)	94.76(90.32)	94.93(90.45)	95.07(90.37)	94.87(89.87)	94.92(89.71)
	(20,15)	$(1^*3, 0^*10, 1^*3)$	94.89(90.34)	95.15(90.20)	95.16(89.79)	94.86(90.09)	94.93(89.98)	95.12(90.10)
	(20,20)	(0^*20)	94.78(89.84)	94.75(89.85)	95.00(89.70)	94.99(89.75)	95.02(89.53)	94.92(89.77)
	(50,20)	$(1^*7, 0^*4, 2^*4, 3^*5)$	95.19(90.21)	95.20(90.18)	94.90(90.45)	95.05(90.56)	95.03(89.58)	94.95(89.73)
	(50,25)	$((0, 2)^*1, 1^*3)$	94.93(90.02)	95.02(90.33)	95.06(90.01)	94.82(90.31)	95.11(89.58)	95.17(89.68)
	(50,50)	(0^*50)	95.20(90.12)	95.15(90.28)	94.91(90.30)	95.03(90.45)	94.98(90.27)	94.92(90.35)
$(2.5, 0.5)$	(10,6)	$(1^*4, 0^*2)$	95.10(90.22)	94.94(90.03)	94.90(89.67)	95.05(89.64)	95.09(89.32)	95.02(89.54)
	(20,5)	$(0^*2, 5^*3)$	95.07(89.53)	95.13(89.72)	95.19(89.86)	95.15(90.11)	94.89(90.05)	95.05(90.09)
	(20,8)	$(0^*3, 2^*3, 3^*2)$	94.58(90.30)	94.70(90.34)	95.14(90.09)	95.33(90.21)	(94.9989.93)	95.05(90.07)
	(20,10)	$(2, 0)^*4, 1^*2)$	94.49(90.60)	94.56(90.61)	95.14(90.15)	95.11(90.23)	94.99(90.55)	95.05(90.77)
	(20,15)	$(1^*3, 0^*10, 1^*3)$	95.32(90.13)	95.43(90.32)	95.44(89.98)	95.36(90.13)	94.76(90.11)	94.97(90.34)
	(20,20)	(0^*20)	94.93(90.23)	95.09(90.41)	95.37(90.68)	95.56(90.70)	94.93(89.87)	94.85(90.03)
	(50,20)	$(1^*7, 0^*4, 2^*4, 3^*5)$	95.38(89.74)	95.32(89.74)	94.75(90.63)	94.77(90.51)	94.92(89.71)	94.76(89.94)
	(50,25)	$((0, 2)^*11, 1^*3)$	95.36(90.02)	95.43(90.11)	95.02(89.86)	95.18(90.17)	94.77(90.12)	94.70(90.30)
	(50,50)	(0^*50)	95.11(90.37)	95.23(90.33)	94.61(90.35)	94.59(90.51)	95.17(89.57)	95.32(89.79)
$(2.5, 1.5)$	(10,6)	$(1^*4, 0^*2)$	95.16(90.01)	95.21(90.05)	95.08(90.10)	95.09(90.23)	95.00(90.61)	95.00(90.95)
	(20,5)	$(0^*2, 5^*3)$	94.81(90.08)	94.75(90.19)	94.77(90.38)	94.87(90.37)	94.99(89.94)	95.19(90.92)
	(20,8)	$(0^*3, 2^*3, 3^*2)$	95.26(89.85)	95.32(89.83)	95.05(89.79)	95.22(89.85)	95.27(89.84)	95.36(90.08)
	(20,10)	$(2, 0)^*4, 1^*2)$	95.49(89.82)	95.68(90.06)	95.16(89.48)	95.17(89.75)	94.95(89.80)	94.90(89.79)
	(20,15)	$(1^*3, 0^*10, 1^*3)$	95.00(90.27)	94.73(90.38)	94.99(90.11)	95.13(90.32)	94.64(90.15)	94.78(90.18)
	(20,20)	(0^*20)	94.87(89.92)	95.03(89.93)	94.92(90.08)	94.70(89.99)	94.70(90.11)	94.82(90.24)
	(50,20)	$(1^*7, 0^*4, 2^*4, 3^*5)$	94.82(89.86)	94.95(90.20)	95.31(89.87)	95.16(90.11)	94.95(90.24)	95.02(90.18)
	(50,25)	$((0, 2)^*1, 1^*3)$	94.56(90.34)	94.44(90.34)	95.01(90.00)	94.89(90.09)	95.10(89.50)	95.14(89.58)
	(50,50)	(0^*50)	95.18(90.98)	95.19(90.82)	95.35(89.81)	95.09(90.17)	94.98(90.43)	95.15(90.44)

Values without parentheses: $\alpha = .05\%$. Values within parentheses: $\alpha = .1\%$.

We consider four sets of parameter values as $(\lambda, \beta) = (1, 0.5),\ (1, 1.5),\ (2.5, 0.5),$ $(2.5, 1.5)$ and various choices of both sample sizes and failure times as $(n, m) =$ $(10, 6),\ (20, 5),\ (20, 8),\ (20, 10),\ (20, 15),\ (20, 20),\ (50, 20),\ (50, 25),\ (50, 50)$ and different x values, $x = 1.7, 3.0, 8.0$. In addition, different censoring schemes $R = (R_1, R_2, ..., R_m)$ are considered for coverage probability of the confidence interval as shown in the table (3). We use $x = 2.5$ for empirical type 1 error. All results that are reported in tables (3) and (4) are based on 10,000 simulations.

Type I error rates for the test $H_0 : S \leq s^*$ vs $H_1 : S > s^*$ when the nominal type I error rate is at $\alpha = 0.1$ ($\alpha = 0.05$) are represented in the following table.

Table 4. Percentage of the Empirical type I error rates for testing $H_0 : S \leq s^*$ vs $H_1 : S > s^*$

(n,m)	(λ,β)	s^*	Classical	Generalize	(n,m)	(λ,β)	s^*	Classical	Generalize
(10,5)	(1,2)	0.75	5.14(09.86)	5.10(10.18)	(10,10)	(1,2)	0.75	4.96(09.94)	5.13(09.96)
	(1,4)	0.05	4.25(09.94)	5.00(10.10)		(1,4)	0.05	5.01(09.98)	5.21(10.00)
	(2,2)	0.50	5.10(10.20)	4.84(09.86)		(2,2)	0.50	4.93(09.78)	5.18(10.15)
	(2,4)	0.10	4.94(09.89)	4.62(10.38)		(2,4)	0.10	5.19(09.98)	5.29(10.12)
	(4,7)	0.30	5.08(09.93)	4.94(09.58)		(4,7)	0.30	4.96(09.55)	5.02(10.50)
(15,10)	(1,2)	0.75	4.96(09.96)	4.63(09.90)	(15,15)	(1,2)	0.75	4.88(10.13)	5.15(10.15)
	(1,4)	0.05	4.84(10.16)	4.91(10.33)		(1,4)	0.05	5.04(10.25)	5.05(10.44)
	(2,2)	0.50	5.31(09.68)	4.90(10.09)		(2,2)	0.50	5.04(09.97)	5.13(09.99)
	(2,4)	0.10	4.79(10.28)	5.05(09.96)		(2,4)	0.10	4.98(09.92)	5.07(10.42)
	(4,7)	0.30	4.76(09.00)	5.23(10.10)		(4,7)	0.30	4.82(09.95)	5.31(10.30)

Values without parentheses: $\alpha = .05\%$. Values within parentheses: $\alpha = .1\%$.

6. Conclusions

The purpose of this study was to propose a simple method for performing inferences regarding the reliability functionality of generalized inverted exponential distributions using a generalized variable method with a fixed removal scheme. In the simulation process, it is straightforward to calculate a generalized p-value and confidence interval. We first illustrate the proposed method with a real-world example. Using the scale-free goodness-of-fit test for exponential distributions based on the Gini statistic, we confirmed that the data set follows the generalized inverted exponential distribution. We then used the data set to calculate the reliability function's generalized and classical lower boundaries, as shown in table (1). Table (2) compares the classical and generalized p-values. While both lower boundaries yield similar results, classical p-values are larger than they should be. For example, for $(n, m) = (23, 9)$ and $x = 20$, the p-values are 0.103 and 0.157, respectively, even though the p-values should be lower than 0.05. Compared to the classical method, the proposed method provides more accurate results. In the simulation example, the coverage probabilities of the generalized and classical lower confidence levels are presented in table (3), and the type I error rates of both the generalized and classical lower confidence levels are presented in table (4). Based on these results, we can conclude that the generalized variable method is significantly more effective than classical methods. In addition to being very flexible, the proposed model is straightforward to implement. Furthermore, it is possible to obtain small sample results, which is a significant advantage over frequentist inference.

In the article, we assumed that the scale parameter λ is known. However, it is important to note that this assumption may not always be applicable in practical application. In cases where λ is not readily available, it is necessary to estimate it. One commonly used method for estimating λ is minimizing the sum of squared errors (SSE). This method has been discussed in the application example. Additionally, the maximum likelihood estimation method or other reasonable methods can also be employed to estimate the parameter. The objective of future research is to develop generalized methods that can accommodate unknown λ and β parameters in GIED. By addressing these limitations, researchers aim to provide a more robust and reliable framework for analyzing and interpreting data from GIED.

In conclusion, we have presented our initial comparison of our results with classical methods, which are widely recognized and widely used in the field of statistical analysis. While we have emphasized the comparison with classical methods, we acknowledge the need to develop our method further and compare it with additional techniques in the future. By doing so, we aim to enhance the accuracy and robustness of our methods and contribute to the scientific community's understanding of data analysis.

Acknowledgement

We are greatly indebted to the Editor-in-Chief, Coordinating Editor, and Reviewers for the very constructive suggestions and the useful comments that improved the earlier version of the manuscript.

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