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On Quantile Credibility Estimators Under An Equal Correlation Structure Over Risks

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Abstract In traditional quantile credibility models, it is typically assumed that claims are independent across different risks. Nevertheless, there are numerous scenarios where dependencies among insured individuals can emerge, thereby breaching the independence assumption. This study focuses on examining the quantile credibility model and extending some established results within the context of an equal correlation structure among risks. Specifically, we compute the credibility premiums for both homogeneous and inhomogeneous cases utilizing the orthogonal projection method.

Keywords Credibility Premiums, Equal Correlation, Quantile Model, Orthogonal Projection, Heterogeneity, Insurance Risks

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1. Introduction

Credibility theory is one of the most useful tools used by the actuaries to make a pricing which can be expressed as a balance between the individual experience of a policyholder and the portfolio's collective experience. In this context, the theory of credibility mainly aims to determine to a policyholder the weight which should be assigned to his own information to make a fair pricing to be charged. For more historical references on this topic, we can refer to Whitney (1918) [29], Mowbray (1914) [16], Bailey (1950) [1], Bühlmann (1967) [3], Kahn (1975) [14], Heilmann (1989) [11], Goovaerts et al. (1990) [10] and Herzog (1996) [13]. Recent detailed introduction to the credibility theory can be found in Landsman and Makov (1999, 2000) [15], Promislow and Young (2000) [23], Young (2000) [31] and Gómez et al. (2006) [9]. Also, the latest findings of Oscar et al (2024) [17] show how the credibility theory is utilized in calculating short-term insurance premiums, by taking advantage of real data from general insurance contracts that extend back nearly a decade.
In the framework of classical credibility theory, we consider a portfolio of K risks, we presume that the individual risk denoted as ,X_i (i=1,2,. . . ,K), is characterized by a density $f(X_i|\theta)$ depending on an unknown risk parameter $\theta \in \Theta$ following a prior distribution (structure distribution) with density $\pi(\theta)$. The Probability density $f(X_i|\theta)$ contributes a sequence of a total past claims X_i, (i=1,2,. . . ,K) over n_i time periods. The classical credibility models state that to estimate a possible future loss X_{i,n_i+1} , the sequence of historical claims amounts is to be observed under the two following fundamental assumptions:
i) Independence over risks, i.e. the random vectors (X_i, Θ_i) are supposed to be independent across individuals.

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ii) Conditional Independence across time: given the risk parameter θ_i, the total claims X_{i1},X_{i2},...,X_{i,n_i} for an individual risk are also conditionally independent.
However, to reflect more the reality, it is necessary to focus on a credibility models including the assumption of dependence between risks under a general dependence structure. This assumption recognizes that events are often correlated or interconnected. In reality, insurance risks and claims can be influenced by several factors that are not independent of each other, such as weather patterns, societal trends and economic conditions. Hence, insurance companies can better assess and manage risks by incorporating dependencies into their models, leading to more accurate pricing and coverage decisions.
In the current actuarial and insurance literature, there are many papers which are dealing with the notion of claims dependence in different aspects. The papers of Yeo and Valdez (2006) [30], Wen et al. (2009) [27], and Ebrahimzadeh et al. (2013) [6], propose the use of common effects to take into account claim dependence and provide the credibility premiums. The paper of Wen and Deng (2011) [28] generalizes the Bühlmann's credibility model by considering that there is an equal correlation structure over risks and gives an estimation for the structural parameters. In addition, Zhang et al. (2018) [32] gives an extended version of the idea of [28] with respect to a balanced loss function, he assumes the existing of constant interest rate and a structure of equal correlation over risks and time to obtain new premiums for the Bühlmann credibility model. Other models have utilized the works of Bolancé et al. (2003) [2], Purcaru and Denuit (2002, 2003) [24] [25], Frees et al. (1999, 2001) [7] [8] and references therein where conditional dependence of claims on time have been studied.
The concept of quantiles is progressively becoming an important part of the credibility framework, for many reasons including that quantiles play a fundamental role in credibility theory, which aims to improve risk prediction by combining individual experience data with collective data. Quantiles help to determine the proper weight given to individual and collective data when estimating future risks, allowing insurers to make more precise predictions. Insurers can also make educated decisions about risk management and pricing by understanding the range of potential losses at different confidence levels through the analysis of quantiles.
Application of quantiles for providing credibility premiums has been suggested for the first time in Pitselis (2013) [19], The works of Pitselis (2016) [20] and Pitselis (2017) [21] investigate the use of quantile credibility in the measuring of risk. Other generalizations and alternative credibility quantile models have since followed including, Bozikas and Pitselis (2020) [5] and Pitselis (2020) [22], addressing their impact on the regression framework. Furthermore, Reference Wang et al.(2020) [26] modeling the quantile credibility model under common effects.
Our study provides additional perspectives on modelling claim dependencies, in order to develop credibility premiums. We focus on modelling the possible dependence present across the insured individuals by using quantiles and equal correlation assumption, we incorporate quantiles into Wen et al.(2011) [28] credibility models with equal correlation risks, with some addition from Zhang et al. (2018) [32]. Nothing similar has been studied in the insurance and actuarial literature, and we trust this makes a valuable addition to the already advancing literature on the issue of claim dependence.
Using an equal correlation assumption in insurance risk modeling is more realistic in certain situations because of its simplicity, ease of calibration, analytical tractability, it saves time and reduces the complexity involved in estimating a large number of pairwise correlations between hazards, also by positing that all risks are equally correlated with each other, this facilitates the analysis of data and simplifies the dependency structure and mathematical models used in insurance pricing and risk management. These factors make it a practical choice for insurers who need to balance manageability with accuracy in their risk assessments. On the other hand, while the common effects assumption can provide a more detailed and potentially accurate representation of dependencies, it introduces significant complexity, requiring more data and computational resources.
In the insurance industry, equal correlation between risks can arise in diverse scenarios for example is when a portfolio of similar properties is insured against a certain type of risk, such as fire damage. To clarify, consider an insurance company that offers home insurance for a group of residential houses in the same neighborhood. these homes might be at equal risk of fire if they are built similarly, have comparable fire protection measures, and are situated in the same geographical area. In this instance, the correlation between the risks associated with insuring these homes against fire damage could be considered relatively equal. So here, the insurance company assesses and guarantees the risks linked with insuring multiple homes based on their shared characteristics and susceptibility to

similar hazards. This approach enables insurers to effectively manage their portfolios, establish suitable premium rates and maintain sufficient coverage for policyholders.
The motivation of our paper differs from that of [19] where he discusses only the assumption of independence between risks. Also Furthermore, this work develops a credibility premium that can be represented in a linear form. We compare our model's numerical results with those of Pitselis (2013) [19] and Wang et al (2020) [26].
Our paper is presented as follows. Section 2 reviews some necessary preliminaries and assumptions which will be used in the other sections of the paper. Sections 3 and 4 provide the principal contribution of this article by the determination of credibility estimators based on quantiles with equal correlation structure between risks, in the homogeneous and inhomogeneous cases. Applications to claims data are given in Section 5. Section 6 draws concluding remarks, discussing the limitations and disadvantages of our model and providing suggestions for future research.

2. Model Formulation and Preliminaries

Definition 1
Let X_j be a random variable with cumulative distribution function $F_j(x)$, ($F_j(x)$ is differentiable and continuous everywhere), and p be a real number comprised between 0 and 1, i.e. $0 < p < 1$. We can write the p-quantile function ξ_p as follows:

$$\xi_{pj} = F_j^{-1}(p) = inf\,\{x : F_j(x), \geq p\}$$

Given a sample $X_1, X_2, \ldots, X_n$ of a continuous random variable X, with the corresponding orders statistics $X^{(1)}, X^{(2)}, \ldots, X^{(n)}$, let $\widehat{\xi}_{pj}$ denote the sample p-quantile which has the property that if

$$-\infty < X < +\infty \quad \text{and } X \in [0,1[: F_j(x) \geq p \quad \text{if and only if} \quad \xi_{pj} \leq p,$$

Now, suppose that the vector $\xi_{pj} = (\xi_{p1}, \xi_{p2}, \ldots, \xi_{pk})'$ contains the pth quantiles for the corresponding K types of risks. If the distribution function of X is unknown, the empirical pth quantile, ξ_{pj} , becomes the natural distribution free estimator of p^{th} quantile, ξ_{pj}. Consequently, the empirical pth quantiles are grouped now in the vector $\widehat{\xi}_{pj} = (\widehat{\xi}_{p1}, \widehat{\xi}_{p2}, \ldots, \widehat{\xi}_{pk})'$ Thus, the empirical distribution function can be obtained as follows

$$\widehat{\xi}_{pj} = n\left(\frac{i}{n} - p\right) X_{j(i-1)} + n\left(p - \frac{i-1}{n}\right) X_{ji} \quad \text{for} \quad \frac{i-1}{n} \leq p \leq \frac{i}{n}, \quad i = 1, 2, \ldots, n$$

For more details on quantiles, we can refer to Parzen (1979) [18] or Herbert and Nagaraja (2003) [12].

In the following we shall consider a portfolio which contains k insured individuals. Each individual in this portfolio has a claim experience X_{ij}, which are random variables where $j = 1, 2, \ldots, n_i$ represents the time period. Here, K represents the number of portfolio contracts, and n_i denotes the time period where past claims of the insured are available.
Having observed the risks for n_i periods of experience, a sequence of observed claim experiences $X_i = (X_{i1}, X_{i2}, \ldots, X_{in_i})'$, $i = 1, 2, \ldots, k$ is contributed. For each insured individual, we aim to estimate $\Xi_p(\Theta_i)$ which can be represented as the risk premium associated with p^{th} quantile. Using the framework mentioned in Pitselis (2013) [19], we can construct a quantile credibility model characterized by an equal correlation structure between risks. Our model's assumptions are given as follows:

Assumption 1
Conditionally on θ_i, the observations $X_{ij}, \ldots, X_{in_i}$ are uniformly dependent having the same distribution function $F(X|\theta)$. Then given θ, the corresponding conditional distribution function at p-quantile is defined as $F(\xi_{pj}|\theta)$.

Assumption 2
ϕ_i is a random variable, where the parameters $\theta_1, \theta_2, \ldots, \theta_k$ are equally correlated, and its density function represented by $\Pi(\theta)$ which is called prior distribution in Actuariat . Equivalently, we assume that there is an

equal correlation between the individual premiums $\Xi_p(\theta_1), \Xi_p(\theta_2), \ldots, \Xi_p(\theta_k)$ explained by the same correlation coefficient ρ. i.e, $Corr(\Xi_p(\theta_i), \Xi_p(\theta_s)) = \rho$ with $i \neq s$.

Assumption 3
we denote

$$E\left(\widehat{\xi}_i^p|\theta_i\right) = \Xi_p(\theta_i),$$

$$Var\left(\widehat{\xi}_i^p|\theta_i\right) = V_p(\theta_i) = \frac{\omega_p(\theta_i)}{n_i},$$

$$Cov\left(\widehat{\xi}_{is}^p, \widehat{\xi}_{it}^p|\theta_i\right) = \eta(\theta_i) \quad \text{with} \quad i \neq s.$$

Remark 1
The term $\omega_p(\theta_i)$ can be estimated from data by several methods, (see section 3.2.2 in Pitselis (2013) [19])

The structural parameters are defined as follows

The collective premium is

$$E\left[\Xi_p(\theta_i)\right] = \Xi_p$$

The mean portfolio variability is

$$E\left(V_p(\theta_i)\right) = \sigma_p^2$$

The variance between contract means is

$$Var\left(\Xi_p(\theta_i)\right) = \Psi_p$$

The average covariance between quantiles is

$$E\left[\eta(\theta_i)\right] = \eta$$

To simplify the exposition, we introduce the following notation:

$$d_i = \frac{n_i}{\sigma_p^2 + (n_i - 1)\eta + n_i(1-\rho)\Psi_p} \quad \text{and} \quad D = \sum_{i=1}^{k} d_i$$

The individual mean of quantiles is

$$\overline{\widehat{\xi}^p} = \frac{\sum_{i=1}^{K} \widehat{\xi}_i^p}{K}$$

The weighted mean of quantiles is

$$\overline{\widehat{\xi}^p}^d = \frac{\sum_{i=1}^{K} d_i \widehat{\xi}_i^p}{D}$$

The objective now is to estimate the future premium to be charged for the contract i. In order to obtain new credibility estimators based on quantiles of the risk premium $\Xi_p(\theta_i)$, in homogeneous and inhomogeneous cases. As proposed by Bühlmann (1967) [3], the following minimizing problems with a linear form and a constant (denoted by α in our model), must be solved:

$$\min_{\alpha_0, \alpha_i} E\left[\Xi_{i,n_i+1}^P - \alpha_0 - \sum_{i=1}^{K} \alpha_i^{'} \widehat{\xi}_i^p\right]^2, \tag{1}$$

$$\text{with} \quad E(\Xi_{i,n_i+1}^P) = E\left(\alpha_0 + \sum_{i=1}^{K} \alpha_i^{'} \widehat{\xi}_i^p\right) \quad where \quad \alpha_0 \in \mathbb{R} \quad \text{and} \quad \alpha_i \in \mathbb{R}^{n_i}$$

The solutions of (1) is defined as the inhomogeneous quantile credibility premium of Ξ_{i,n_i+1}, which is denoted by $\widehat{\Xi^P_{i,n_i+1}}^{inhom}$.
and

$$\min_{\alpha_i} E\left[\Xi^P_{i,n_i+1} - \sum_{i=1}^{K}\alpha_i^{'}\widehat{\xi}_i^p\right]^2 \quad , \tag{2}$$

$$\text{with}\quad E(\Xi^P_{i,n_i+1}) = E\left(\sum_{i=1}^{K}\alpha_i^{'}\widehat{\xi}_i^p\right),\quad \text{where}\quad \alpha_i\in\mathbb{R}^{n_i}$$

The solutions of (2) is defined as the homogeneous quantile credibility premium of Ξ_{i,n_i+1}, which is denoted by $\widehat{\Xi^P_{i,n_i+1}}^{hom}$.

We denote

$$L(\widehat{\xi}^p,1) := \left\{\alpha_0^j + \sum_{i=1}^{K}\alpha_i^{j'}\widehat{\xi}_i^p,\quad \alpha_0\in\mathbb{R}\quad \alpha_i\in\mathbb{R}^{n_i}\quad \text{with}\quad E\left(\alpha_0 + \sum_{i=1}^{K}\alpha_i^{'}\widehat{\xi}_i^p\right) = E(\Xi_{i,n_i+1})\right\} \tag{3}$$

a closed subspace of all square integrable functions $\mathbb{L}^2$.
The class $L(\widehat{\xi}^p,1)$ refers to the set of solutions to an inhomogeneous system of linear equations, in which the constant term is not necessarily equal to 0.

and

$$L_E(\widehat{\xi}^p) := \left\{\sum_{i=1}^{K}\alpha_i^{j'}\widehat{\xi}_i^p,\quad \alpha_i\in\mathbb{R}^{n_i}\quad \text{with}\quad E\left(\sum_{i=1}^{K}\alpha_i^{'}\widehat{\xi}_i^p\right) = E(\Xi_{i,n_i+1})\right\} \tag{4}$$

the class of all collectively unbiased estimators for $\Xi_p(\Theta_i)$ which is a closed affine subspace of $\mathbb{L}^2$. Where $\widehat{\xi}^p = (\widehat{\xi}_1^{p'},\widehat{\xi}_2^{p'},\ldots,\widehat{\xi}_k^{p'})'$.
The class $L_E(\widehat{\xi}^p)$ refers to the set of solutions to a homogeneous system of linear equations, in which the constant term is equal to 0.

According to Bühlmann and Gisler (2005) [4], the homogeneous estimator $\widehat{\Xi^P_{i,n_i+1}}^{hom}$ does not contain a constant term contrary to $\widehat{\Xi^P_{i,n_i+1}}^{inhom}$.
But while it is necessary for $\widehat{\Xi^P_{i,n_i+1}}^{hom}$ to be unbiased across the collective, there is no requirement for the estimator to be unbiased for any individual value of θ, this condition is satisfied for the inhomogeneous credibility estimator.

Furthermore, the orthogonal projection (represented by "PROJ" in mathematics) of Ξ_{i,n_i+1} on the closed subspaces $L(\widehat{\xi}^p,1)$ and $L_E(\widehat{\xi}^p)$, gives us the following premiums $\widehat{\Xi^P_{i,n_i+1}}^{inhom}$ and $\widehat{\Xi^P_{i,n_i+1}}^{hom}$, respectively, i.e.,

$$\widehat{\Xi^P_{i,n_i+1}}^{inhom} = PROJ\left(\Xi_{i,n_i+1}|L(\widehat{\xi}^p,1)\right)$$

$$\widehat{\Xi^P_{i,n_i+1}}^{hom} = PROJ\left(\Xi_{i,n_i+1}|L_E(\widehat{\xi}^p)\right)$$

To determine the quantile credibility estimators easily, we start by citing a preliminary lemma from Wen et al.(2009) [27] on projections of random variables.

lemma 1
The inhomogeneous and homogeneous credibility estimators are actually the orthogonal projection of a random variable Y on L(X,1) and $L_E(X)$. The following formulae hold true:

$$PROJ(Y|L(X,1)) = E(Y) + Cov(Y,X)Var(X)^{-1}(X - E(X)), \tag{5}$$

and

$$PROJ(Y|L_E(X)) = \left(Cov(Y,X) + \frac{\left(E(Y) - Cov(Y,X)Var(X)^{-1}E(X)\right) E(X^{'})}{E(X^{'})Var(X)^{-1}E(X)} \right) Var(X)^{-1}X \tag{6}$$

where $\quad X = (X_1^{'}, X_2^{'}, \ldots, X_k^{'})^{'}$ and $X_i = (X_{i1}, X_{i2}, \ldots, X_{i,n_i})^{'}$

Now, we give another lemma that also deals with orthogonal projection. The proof can be found in Bühlmann and Gisler(2005) [4].

lemma 2
For the two closed subspace $M^{'} \subset M \subset \mathbb{L}^2$, and $Y \in \mathbb{L}^2$, the following equality holds true

$$PROJ\left(Y|M^{'}\right) = PROJ\left(PROJ(Y|M)|M^{'}\right)$$

Consequently, since $L_E(X) \subset L(X,1)$, we have

$$PROJ(Y|L_E(X)) = PROJ(PROJ(Y|L(X,1))|L_E(X)) \tag{7}$$

The above formula will be used to calculate conveniently the homogeneous quantile credibility premium of Ξ_{i,n_i+1} in the next section.

lemma 3
Under the three Assumptions mentioned above and the previous notations, we have:
1) The expectation of $\widehat{\xi}_i^p$ is

$$E(\widehat{\xi}_i^p) = \Xi_p 1_{n_i}, \quad E(\Xi_{i,n_i+1}^p) = \Xi_p, \quad \text{and} \quad E(\widehat{\xi}^p) = \Xi_p 1_N; \quad \text{where} \quad N = \sum_{i=1}^{K} n_i \tag{8}$$

where $1_{n_i} = (1_{n_1}, 1_{n_2}, \ldots, 1_{n_k})^{'}$ is a n_i dimensional column vector with 1 in all of the n_i entries.

2) The covariance of $\widehat{\xi}^p$ is

$$\sum_{X,X} = \sum_{\widehat{\xi}^p, \widehat{\xi}^p} = Var(\widehat{\xi}^p) = Diag(\wedge_1, \ldots, \wedge_k) + \rho\Psi_p 1_N 1_N^{'} \tag{9}$$

$Diag(\ldots)$ represents a matrix where the elements inside the bracket form a block structure along the diagonal. and

$$\wedge_i = (\sigma_p^2 - \eta)I_{n_i} + (\eta + (1-\rho)\Psi_p)1_{n_i}1_{n_i}^{'}$$

where I_{n_i} is the identity matrix of dimension n_i.

3) The covariance matrix between Ξ_{i,n_i+1}^p and $\widehat{\xi}^p$ is given by

$$\sum_{YX} = \sum_{\Xi_{i,n_i+1}^p \widehat{\xi}^p} = Cov(\Xi_{i,n_i+1}^p, \widehat{\xi}^p) = (\eta + (1-\rho)\Psi_p)v_i^{'} \otimes 1_{n_i}^{'} + \rho\Psi_p 1_N^{'} \tag{10}$$

where v_i is a vector characterized by a value of 1 in the ith entry, and 0 in all other entries. Additionally, $\otimes$ represents the Kronecker product of matrices.
4) The covariance matrix's inverse of $\widehat{\xi}^p$ is given by

$$\sum_{\widehat{\xi}^p\widehat{\xi}^p}^{-1} = Diag(\wedge_1^{-1},\dots,\wedge_k^{-1}) - \frac{1}{\frac{1}{\rho\Psi_p}+\sum_{i=1}^{k}\frac{n_i}{\sigma_p^2+(n_i-1)\eta+n_i(1-\rho)\Psi_p}}\begin{bmatrix}\wedge_1^{-1}1_{n_1}\\ \wedge_2^{-1}1_{n_2}\\ \vdots\\ \wedge_k^{-1}1_{n_k}\end{bmatrix}(1_{n_1}^{'}\wedge_1^{-1},\dots,1_{n_k}^{'}\wedge_k^{-1}) \quad (11)$$

where

$$\wedge_i^{-1} = \frac{1}{\sigma_p^2-\eta}\left(I_{n_i} - \frac{\eta+(1-\rho)\Psi_p}{\sigma_p^2+(n_i-1)\eta+n_i(1-\rho)\Psi_p}1_{n_i}1_{n_1}^{'}\right)$$

As we have the notations $\frac{n_i}{\sigma_p^2+(n_i-1)\eta+n_i(1-\rho)\Psi_p} = d_i$ and $D=\sum_{i=1}^{k} d_i$ so the formula (11) becomes

$$\sum_{\widehat{\xi}^p\widehat{\xi}^p}^{-1} = Diag(\wedge_1^{-1},\dots,\wedge_k^{-1}) - \frac{\rho\Psi_p}{1+\rho D\Psi_p}\begin{bmatrix}\wedge_1^{-1}1_{n_1}\\ \wedge_2^{-1}1_{n_2}\\ \vdots\\ \wedge_k^{-1}1_{n_k}\end{bmatrix}(1_{n_1}^{'}\wedge_1^{-1},\dots,1_{n_k}^{'}\wedge_k^{-1})$$

And

$$\wedge_i^{-1} = \frac{1}{\sigma_p^2-\eta}\left(I_{n_i} - \frac{(\eta+(1-\rho)\Psi_p)d_i}{n_i}1_{n_i}1_{n_1}^{'}\right)$$

Proof
1) Let $\Theta=(\Theta_1,\Theta_2,\dots,\Theta_k)$ By using the theorem of dual expectation, we can write

$$E\left(\Xi_{i,n_i+1}^p\right) = E\left[E(\Xi_{i,n_i+1}^p|\Theta)\right] = E\left[\Xi_p(\theta_i)\right] = \Xi_p$$

And

$$E\left(\widehat{\xi}_i^p\right) = E\left[E(\widehat{\xi}_i^p|\theta_i)\right] = E\left[\Xi_p(\theta_i)1_{n_i}\right] = \Xi_p 1_{n_i}$$

Then

$$E(\widehat{\xi}^p) = E\left(\widehat{\xi}_1^{p'},\widehat{\xi}_2^{p'},\dots,\widehat{\xi}_k^{p'}\right)^{'} = \Xi_p 1_N$$

Write $\Theta=(\Theta_1,\dots,\Theta_k)^{'}$. Then

$$E\left(\widehat{\xi}_i^p\right) = E\left[E(\widehat{\xi}_i^p|\theta_i)\right] = E\left[\Xi_p(\theta_i)1_{n_i}\right] = E\left[E((\widehat{\xi}_1^p,\widehat{\xi}_2^p,\dots,\widehat{\xi}_k^p)^{'}|\theta_i)\right] = \Xi_p\left[1_1^{'},\dots,1_k^{'}\right]^{'}$$

2) According to Assumptions 2 and 3, it is evident that

$$Cov\left(\widehat{\xi}_i^p,\widehat{\xi}_j^p|\Theta_i\right) = \begin{cases}(V_p(\theta_i)-\eta(\theta_i))I_{n_i}+\eta(\theta_i)1_{n_i}1_{n_j}^{'}, & i=j\\ 0, & i\neq j\end{cases} \quad (12)$$

And

$$Cov\left(E(\widehat{\xi}_i^p|\Theta_i),E(\widehat{\xi}_j^p|\Theta_j)\right) = \begin{cases}\Psi_p 1_{n_i}1_{n_j}^{'}, & i=j\\ \rho\Psi_p 1_{n_i}1_{n_j}^{'}, & i\neq j\end{cases} \quad (13)$$

Using the conditional variance formula, we can write

$$\begin{aligned}Cov\left(\widehat{\xi}_i^p,\widehat{\xi}_j^p\right) &= E\left[Cov(\widehat{\xi}_i^p,\widehat{\xi}_j^p|\Theta)\right] + Cov\left[E(\widehat{\xi}_i^p|\Theta_i),E(\widehat{\xi}_j^p|\Theta_j)\right]\\ &= \begin{cases}(\sigma_p^2-\eta)I_{n_i}+(\eta+\Psi_p)1_{n_i}1_{n_j}^{'}, & i=j\\ \rho\Psi_p 1_{n_i}1_{n_j}^{'}, & i\neq j\end{cases}\end{aligned} \quad (14)$$

Now, it is easy to check that

$$Cov(\widehat{\xi}_i^p, \widehat{\xi}_j^p) = Var(\widehat{\xi}^p) = Diag(\wedge_1, \ldots, \wedge_k) + \rho\Psi_p 1_N 1_N^{'}$$

3) Since $Cov\left(\Xi_{i,n_i+1}^p, \widehat{\xi}^p|\Theta\right) = \eta(\theta_i)v_i^{'} \otimes 1_{n_i}^{'}$ Then we can write

$$\begin{aligned} Cov(\Xi_{i,n_i+1}^p, \widehat{\xi}^p) &= E\left[Cov(\Xi_{i,n_i+1}^p, \widehat{\xi}^p|\Theta)\right] + Cov\left[E(\Xi_{i,n_i+1}^p|\Theta_i), E(\widehat{\xi}^p|\Theta_j)\right] \\ &= E\left[\eta(\theta_i)v_i^{'} \otimes 1_{n_i}^{'}\right] + Cov\left[\Xi_p(\theta_i), \Xi_p(\theta_i)1_{n_i}^{'}\right] \\ &= (\eta + (1-\rho)\Psi_p)v_i^{'} \otimes 1_{n_i}^{'} + \rho\Psi_p 1_N^{'} \end{aligned}$$

Thus, formula (10) has been proven.

4) Using the result of Rao and Toutenburg (1995), the matrix inverse formula can be written as follows

$$(A + BCD)^{-1} = A^{-1} - A^{-1}B(C^{-1} + DA^{-1}B)^{-1}DA^{-1} \tag{15}$$

it follows that

$$\begin{aligned} \sum_{\widehat{\xi}^p\widehat{\xi}^p}^{-1} &= [Diag(\wedge_1, \ldots, \wedge_k) + \rho\Psi_p 1_{n_i} 1_{n_1}^{'}]^{-1} \\ &= Diag(\wedge_1^{-1}, \ldots, \wedge_k^{-1}) - \begin{bmatrix} \wedge_1^{-1}1_{n_1} \\ \wedge_2^{-1}1_{n_2} \\ \vdots \\ \wedge_k^{-1}1_{n_k} \end{bmatrix} \left(\frac{1}{\rho\Psi_p} + \sum_{i=1}^{k} 1_{n_i}^{'} \wedge_i^{-1} 1_{n_i}\right)^{-1}(1_{n_1}^{'}\wedge_1^{-1}, \ldots, 1_{n_k}^{'}\wedge_k^{-1}) \end{aligned}$$

Because

$$\wedge_i = (\sigma_p^2 - \eta)I_{n_i} + (\eta + (1-\rho)\Psi_p)1_{n_i}1_{n_i}^{'}$$

Using equation (15) we derive

$$\begin{aligned} \wedge_i^{-1} &= \left[(\sigma_p^2 - \eta)I_{ni} + (\eta + (1-\rho)\Psi_p)1_{n_i}1_{n_i}^{'}\right]^{-1} \\ &= \frac{1}{\sigma_p^2 - \eta}\left[I_{n_i} - I_{n_i} \wedge_i \left(\frac{1}{\eta + (1-\rho)\Psi_p} + \frac{n_i}{\sigma_p^2 - \eta}\right)^{-1}\left(1_{n_i}^{'}\frac{1}{\sigma_p^2 - \eta}I_{n_i}\right)\right] \\ &= \frac{1}{\sigma_p^2 - \eta}\left[I_{n_i} - \frac{(\eta + (1-\rho)\Psi_i)d_i}{n_i}1_{n_i}1_{n_i}^{'}\right] \end{aligned}$$

Using the notation of d_i mentioned in assumption 3, we obtain

$$\wedge_i^{-1} = \frac{1}{\sigma_p^2 - \eta}\left(I_{n_i} - \frac{\eta + (1-\rho)\Psi_p}{\sigma_p^2 + (n_i - 1)\eta + n_i(1-\rho)\Psi_p}1_{n_i}1_{n_1}^{'}\right)$$

Here we also note that $\wedge_i^{-1}1_{n_i} = \frac{1_{n_i}}{\sigma_p^2 + (n_i-1)\eta + n_i(1-\rho)\Psi_p}$, $1_{n_i}^{'}\wedge_i^{-1} = \frac{1_{n_i}^{'}}{\sigma_p^2 + (n_i-1)\eta + n_i(1-\rho)\Psi_p}$, and $1_{n_i}^{'} \wedge_i^{-1} 1_{n_i} = \frac{n_i}{\sigma_p^2 + (n_i-1)\eta + n_i(1-\rho)\Psi_p} = d_i$

Consequently, we can derive easily the formula of $\sum_{\widehat{\xi}^p\widehat{\xi}^p}^{-1}$ as (11). □

3. Derivation of Inhomogeneous Quantile Credibility Estimator with an equal Correlation between Risks

This section is dedicated to find the inhomogeneous quantile credibility estimator for authentic premium Ξ^p_{i,n_i+1} for $i = 1, 2, \ldots, k$ with a dependence structure between risks. The theorem below derives the quantile credibility premium in the inhomogeneous case.

Theorem 1
From Assumptions 1 and 2, the inhomogeneous credibility estimator of future claim Ξ^p_{i,n_i+1}, $i = 1, \ldots, K$ are given by

$$\widehat{\Xi^P_{i,n_i+1}}^{inhom} = Z_{i1}\widehat{\xi}^P_i + Z_{i2}\overline{\widehat{\xi}^p}^d + (1 - Z_{i1} - Z_{i2})\Xi_p \tag{16}$$

where Z_{i1} and Z_{i2} are credibility factors satisfying,

$$Z_{i1} = (\eta + (1-\rho)\Psi_p)d_i \quad \text{and} \quad Z_{i2} = \frac{\rho D\Psi_p[1 - (\eta + (1-\rho)\Psi_p)d_i]}{\rho D\Psi_p + 1}$$

Proof
The optimization of problem (1) can be simplified as follows

$$\min_{\alpha_0 \in \mathbb{R}, \alpha_i \in \mathbb{R}^{n_i}} E\left[\left(\Xi_{i,n_i+1} - \alpha_0 - \sum_{i=1}^{K} \alpha_i^{'} \widehat{\xi}^p_i\right)^2\right]$$

Using Lemma 2, we can write the credibility estimator of Ξ^p_{i,n_i+1} in inhomogeneous case as

$$\widehat{\Xi^P_{i,n_i+1}}^{inhom} = PROJ\left(\Xi_{i,n_i+1} | L(\widehat{\xi}^P, 1)\right) = E\left(\Xi_{i,n_i+1}\right) + \sum_{\Xi^P_{i,n_i+1}\widehat{\xi}^P} \sum^{-1}_{\widehat{\xi}^P\widehat{\xi}^P} (\widehat{\xi}^P - \Xi_p) \tag{17}$$

Note that $1^{'}_{n_i} \wedge_i^{-1} 1_{n_i} = d_i$ and $E(\Xi^p_{i,n_i+1}) = \Xi_p$, then replacing Equations (8), (10) and (11) into Equation (17) and after some straighforward calculations, we can obtain the quantile credibility estimator

$$\begin{aligned}
\widehat{\Xi^P_{i,n_i+1}}^{inhom} &= \Xi_p + (\eta + (1-\rho)\Psi_p)1^{'}_{n_i} \wedge_i^{-1} (\widehat{\xi}^p_i - \Xi_p 1_{n_i}) + \\
&\left(\rho\Psi_p - \frac{\rho\Psi_p \sum_{i=1}^k d_i + (\eta + (1-\rho)\Psi_p)d_i}{\frac{1}{\rho\Psi_p} + \sum_{i=1}^k d_i}\right)\left(\sum_{i=1}^k 1^{'}_{n_i} \wedge_i^{-1} (\widehat{\xi}^p_i - \Xi_p 1_{n_i})\right) \\
&= \Xi_p + (\eta + (1-\rho)\Psi_p)\frac{n_i}{\sigma_p^2 + (n_i - 1)\eta + n_i(1-\rho)\Psi_p}(\widehat{\xi}^p_i - \Xi_p) + \\
&\frac{\rho\Psi_p[1 - (\eta + (1-\rho)\Psi_p)d_i]}{\rho D\Psi_p + 1}\sum_{i=1}^k \frac{n_i(\widehat{\xi}^p_i - \Xi_p)}{\sigma_p^2 + (n_i - 1)\eta + n_i(1-\rho)\Psi_p} \\
&= \Xi_p + (\eta + (1-\rho)\Psi_p)d_i(\widehat{\xi}^p_i - \Xi_p) + \frac{\rho D\Psi_p[1 - (\eta + (1-\rho)\Psi_p)d_i]}{\rho D\Psi_p + 1}(\overline{\widehat{\xi}^p}^d - \Xi_p) \\
&= ((\eta + (1-\rho)\Psi_p)d_i)\,\widehat{\xi}^p_i + \left(\frac{\rho D\Psi_p[1 - (\eta + (1-\rho)\Psi_p)d_i]}{\rho D\Psi_p + 1}\right)\overline{\widehat{\xi}^p}^d + \\
&\left(1 - (\eta + (1-\rho)\Psi_p)d_i - \frac{\rho D\Psi_p[1 - (\eta + (1-\rho)\Psi_p)d_i]}{\rho D\Psi_p + 1}\right)\Xi_p \\
&= Z_{i1}\widehat{\xi}^P_i + Z_{i2}\overline{\widehat{\xi}^p}^d + (1 - Z_{i1} - Z_{i2})\Xi_p
\end{aligned}$$

□

Remark 2
If we assume in our model that all the time periods are equal, $n_1 = n_2 = \ldots = n_k = n$, $\rho = 0$ and $\eta = 0$ then

$$d_i = \frac{n_i}{\sigma_p^2 + (n_i - 1)\eta + n_i(1-\rho)\Psi_p} = \frac{n}{\sigma_p^2 + n\Psi_p},$$

$$D = \sum_{i=1}^{k} d_i = \frac{kn}{\sigma_p^2 + n\Psi_p},$$

$$\overline{\widehat{\xi^p}}^d = \frac{\sum_{i=1}^{K} d_i \widehat{\xi}_i^p}{D} = \frac{\sum_{i=1}^{K} \widehat{\xi}_i^p}{k}$$

In this situation, the credibility factors and the credibility estimators can be written as

$$Z_{i1} = \frac{n\Psi_p}{n\Psi_p + \sigma_p^2} = \frac{\Psi_p}{\Psi_p + \frac{\sigma_p^2}{n}} \quad and \quad Z_{i2} = 0 \tag{18}$$

$$\begin{aligned}\widehat{\Xi_{i,n_i+1}^P}^{inhom} &= Z_{i1}\widehat{\xi}_i^P + (1 - Z_{i1})\Xi_p \\ &= \frac{\Psi_p}{E(V_p(\theta_i)) + \Psi_p}\widehat{\xi}_i^P + \left(1 - \frac{\Psi_p}{E(V_p(\theta_i)) + \Psi_p}\right)\Xi_p = \widehat{\Xi_{i,n_i+1}^P}^{Pitselis}\end{aligned}$$

which is termed as Quantile credibility estimator (i.e., Pitselis (2013) [19]). Therefore, our result is reduced to the premium of Pitselis (2013).

Remark 3
If we take $n_1 = n_2 \ldots = n_k = n$ with $\eta = 0$, $\sigma_{0p}^2 = n\rho\Psi_p$, $\sigma_{1p}^2 = \sigma_p^2$ and $\sigma_{2p}^2 = n(1-\rho)\Psi_p$
Hence, the inhomogeneous credibility estimator $\widehat{\Xi_{i,n_i+1}^P}^{inhom}$ for $i = 1, \ldots, k$ in equation (theorem 1) becomes

$$\widehat{\Xi_{i,n_i+1}^P}^{inhom} = Z_1\widehat{\xi}_i^P + Z_2\overline{\widehat{\xi^p}} + (1 - Z_1 - Z_2)\Xi_p$$

Where $Z_1 = \frac{\sigma_{2p}^2}{\sigma_{1p}^2 + \sigma_{2p}^2}$, $Z_2 = \frac{k\sigma_{0p}^2\sigma_{1p}^2}{(\sigma_{1p}^2+\sigma_{2p}^2)(\sigma_{1p}^2+\sigma_{2p}^2+k\sigma_{0p}^2)}$, $Z_3 = 1 - Z_1 - Z_2$, and $\overline{\widehat{\xi^p}} = \frac{\sum_{i=1}^{K}\widehat{\xi}_i^p}{k}$

which are just formula in Theorem 1 in Wang et al. (2020) [26]. Obviously, the Quantile credibility estimators with random common effects will also yield equal correlations and they represent a particular case of our model. For the inhomogeneous quantile credibility estimators mentioned in this section, the value of the collective premium Ξ_p must be calculated using the prior distribution. In the case when collective premium Ξ_p is unknown, we can overcome this problem by solving minimization problem 2 under the class $L_E(\widehat{\xi^p})$ and establishing the homogeneous quantile credibility estimators of Ξ_{i,n_i+1}^P for $i = 1, \ldots, k$..

4. Derivation of Homogeneous Quantile Credibility Estimator with an equal Correlation between Risks

Theorem 2
Under the above assumptions, the homogeneous credibility estimator of Ξ_{i,n_i+1} for $i = 1, \ldots, k$ can be expressed as

$$\widehat{\Xi_{i,n_i+1}^P}^{hom} = Z_{i1}\widehat{\xi}_i^P + (1 - Z_{i1})\overline{\widehat{\xi^p}}^d \tag{19}$$

Where $Z_{i1} = (\eta + (1-\rho)\Psi_p)d_i$.

Proof
From equation (7), it is apparent that

$$PROJ(\Xi_{i,n_i+1}^P | L(\widehat{\xi^p}, 1)) = PROJ(PROJ(\Xi_{i,n_i+1}^P | L(\widehat{\xi^p}, 1)) | L_E(\widehat{\xi^p}))$$

Theorem 1 indicates that

$$PROJ(\Xi^P_{i,n_i+1}|L(\widehat{\xi}^p,1)) = Z_{i1}\widehat{\xi}^P_i + Z_{i2}\overline{\widehat{\xi}^p}^d + (1-Z_{i1}-Z_{i2})\Xi_p$$

Since $\widehat{\xi}^P_i, \overline{\widehat{\xi}^p}^d \in L_E(\widehat{\xi}^p)$ we can write

$$\widehat{\Xi^P_{i,n_i+1}}^{hom} = Z_{i1}\widehat{\xi}^P_i + Z_{i2}\overline{\widehat{\xi}^p}^d + (1-Z_{i1}-Z_{i2})PROJ(\Xi_p|L_E(\widehat{\xi}^p)) \tag{20}$$

In addition, $Cov(Y,X)=0$, let $Y=\Xi_p(\theta)$ and $X=\widehat{\xi}^p$ in equation (6), then it is apparent that

$$PROJ(\Xi_p|L_E(\widehat{\xi}^p)) = \frac{\Xi_p E(\widehat{\xi}^{P\prime})\sum^{-1}_{\widehat{\xi}^P\widehat{\xi}^P}\widehat{\xi}^P}{E(\widehat{\xi}^{P\prime})\sum^{-1}_{\widehat{\xi}^P\widehat{\xi}^P}E(\widehat{\xi}^P)} \tag{21}$$

Substituting equations (8) and (11) into (21), we obtain

$$PROJ(\Xi_p|L_E(\widehat{\xi}^p)) = \frac{\Xi_p^2 1'_N\left(Diag(\wedge_1^{-1},\dots,\wedge_k^{-1}) - \frac{\rho\Psi_p}{1+\rho D\Psi_p}\begin{bmatrix}\wedge_1^{-1}1_{n_1}\\ \vdots \\ \wedge_k^{-1}1_{n_k}\end{bmatrix}(1'_{n_1}\wedge_1^{-1},\dots,1'_{n_k}\wedge_k^{-1})\right)\begin{bmatrix}\widehat{\xi}^P_1\\ \vdots \\ \widehat{\xi}^P_k\end{bmatrix}}{\Xi_p 1'_N\left(Diag(\wedge_1^{-1},\dots,\wedge_k^{-1}) - \frac{\rho\Psi_p}{1+\rho D\Psi_p}\begin{bmatrix}\wedge_1^{-1}1_{n_1}\\ \vdots \\ \wedge_k^{-1}1_{n_k}\end{bmatrix}(1'_{n_1}\wedge_1^{-1},\dots,1'_{n_k}\wedge_k^{-1})\right)\Xi_p 1_N} = \overline{\widehat{\xi}^p}^d$$

Then, the homogeneous credibility estimator $\widehat{\Xi^P_{i,n_i+1}}^{hom}$ becomes

$$\begin{aligned}\widehat{\Xi^P_{i,n_i+1}}^{hom} &= Z_{i1}\widehat{\xi}^P_i + Z_{i2}\overline{\widehat{\xi}^p}^d + (1-Z_{i1}-Z_{i2})\overline{\widehat{\xi}^p}^d \\ &= Z_{i1}\widehat{\xi}^P_i + (Z_{i2}+1-Z_{i1}-Z_{i2})\overline{\widehat{\xi}^p}^d \\ &= Z_{i1}\widehat{\xi}^P_i + (1-Z_{i1})\overline{\widehat{\xi}^p}^d\end{aligned}$$

which gives the result. □

The two above theorems provide the estimators of the inhomogeneous and homogeneous credibility premiums for each contract. But, the structural parameters $\Xi_p, \sigma^2_p, \Psi_p, \eta$, and ρ in real applications are generally unknown in formula and that we leads us to estimate them based on quantiles $\widehat{\xi}^p = (\widehat{\xi}^{p\prime}_1, \widehat{\xi}^{p\prime}_2, \dots, \widehat{\xi}^{p\prime}_k)'$ Concerning the estimator of Ξ_p, it does not represent a subject to much controversy because it is already available into the formula for $\widehat{\Xi^P_{i,n_i+1}}^{hom}$ and well established in the actuarial literature (see [19]). So we will investigate only an estimation for the three structural parameters σ^2_p, Ψ_p and η.
The estimators of σ^2_p, Ψ_p and η are given in the following proposition, we will also presume that these estimators are unbiased.

Proposition 1
The estimators of σ^2_p, η and Ψ_p are given by

$$(i)\quad \widehat{\sigma}^2_p = E(\widehat{V}_p(\theta_i)) = \frac{1}{K}\sum_{i=1}^{K} Var(\widehat{\xi}^p_i|\theta_i) = \frac{1}{K}\sum_{i=1}^{K}\frac{\widehat{\omega}_p(\theta_i)}{n_i} \tag{22}$$

$$(ii)\quad \widehat{\eta} = \widehat{\sigma^2_p} - \frac{1}{K}\sum_{j=1}^{K}\frac{1}{n_j-1}\sum_{i=1}^{n_i}(\widehat{\xi}^p_i - \overline{\widehat{\xi}^p})^2 \tag{23}$$

$$(iii) \quad \widehat{\Psi}_p = Var(\Xi_p(\theta_i)) = \frac{1}{1-\rho}\left(\frac{1}{K-1}\sum_{i=1}^{K}(\widehat{\xi}_i^p - \overline{\widehat{\xi}^p})^2 - \frac{1}{K}\sum_{i=1}^{K}\frac{\widehat{\sigma_p^2}+(n_i-1)\widehat{\eta}}{n_i}\right) \tag{24}$$

Proof

(i) Let $S_i^2 = \frac{1}{n_i-1}\sum_{i=1}^{n_i}(\widehat{\xi}_i^p - \overline{\widehat{\xi}^p})^2$, where $\overline{\widehat{\xi}^p} = \frac{1}{K}\sum_{i=1}^{K}\widehat{\xi}_i^p$

We have

$$E(S_i^2) = E\left[\frac{1}{n_i-1}\sum_{i=1}^{n_i}(\widehat{\xi}_i^p - \overline{\widehat{\xi}^p})^2\right] = \frac{1}{n_i-1}\sum_{i=1}^{n_i}E(\widehat{\xi}_i^p - \overline{\widehat{\xi}^p})^2$$

$$\begin{aligned}
E\left[(\widehat{\xi}_i^p - \overline{\widehat{\xi}^p})^2|\theta_i\right] &= Var\left[(\widehat{\xi}_i^p - \overline{\widehat{\xi}^p})|\theta_i\right] \\
&= Var(\widehat{\xi}_i^p|\theta_i) + Var(\overline{\widehat{\xi}^p}|\theta_i) - 2Cov(\widehat{\xi}_i^p, \overline{\widehat{\xi}^p}|\theta_i) \\
&= V_p(\theta_i) + \frac{1}{n_i}V_p(\theta_i) - 2\frac{1}{n_i}V_p(\theta_i) = \frac{n_i-1}{n_i}V_p(\theta_i)
\end{aligned}$$

We know that

$$\begin{aligned}
E\left[(\widehat{\xi}_i^p - \overline{\widehat{\xi}^p})^2\right] &= E\left[E(\widehat{\xi}_i^p - \overline{\widehat{\xi}^p}|\theta_i)\right] = E\left(\frac{n_i-1}{n_i}V_p(\theta_i)\right) \\
&= \frac{n_i-1}{n_i}E\left(V_p(\theta_i)\right) = \frac{n_i-1}{n_i}\sigma_p^2(\theta_i)
\end{aligned}$$

Then

$$E(S_i^2) = \frac{1}{n_i-1}\sum_{i=1}^{n_i}\frac{n_i-1}{n_i}\sigma_p^2(\theta_i) = \sigma_p^2$$

$$E(\widehat{\sigma}_p^2) = \frac{1}{K}\sum_{i=1}^{K}E(S_i^2) = \sigma_p^2$$

$$E(\widehat{\sigma}_p^2) = \frac{1}{K}\sum_{i=1}^{K}E(V_p(\theta_i)) = E\left(\frac{1}{K}\sum_{i=1}^{K}V_p(\theta_i)\right)$$

So we obtain immediately that

$$\widehat{\sigma}_p^2 = \frac{1}{K}\sum_{i=1}^{K}V_p(\theta_i)$$

Furthermore, since $Var(\widehat{\xi}_i^p|\theta_i) = V_p(\theta_i) = \frac{\widehat{\omega}_p(\theta_i)}{n_i}$ the estimator of σ_p^2 can be formulated as follows

$$\widehat{\sigma}_p^2 = \frac{1}{K}\sum_{i=1}^{K}Var(\widehat{\xi}_i^p|\theta_i) = \frac{1}{K}\sum_{i=1}^{K}\frac{\widehat{\omega}_p(\theta_i)}{n_i}$$

(ii) For $\widehat{\eta}$ Note that

$$\sum_{i=1}^{n_i}(\widehat{\xi}_i^p - \overline{\widehat{\xi}^p})^2 = \sum_{i=1}^{n_i}(\widehat{\xi}_i^p - \Xi_p(\theta_i))^2 - n_i(\overline{\widehat{\xi}^p} - \Xi_p(\theta_i))^2$$

and then from the iterated expectation law , we obtain

$$E\left(\frac{1}{n_j-1}\sum_{i=1}^{n_i}(\widehat{\xi}_i^p - \overline{\widehat{\xi}^p})^2\right) = E\left(E\left(\frac{1}{n_j-1}\sum_{i=1}^{n_i}(\widehat{\xi}_i^p - \overline{\widehat{\xi}^p})^2|\theta_i\right)\right) = \sigma^2 - \eta$$

then

$$\widehat{\eta} = \widehat{\sigma^2} - \frac{1}{K}\sum_{j=1}^{K}\frac{1}{n_j - 1}\sum_{i=1}^{n_i}(\widehat{\xi}_i^p - \overline{\widehat{\xi}^p})^2$$

(iii) For $\widehat{\Psi}_p$ we have to calculate these quantities firstly

$$Var(\widehat{\xi}_i^p) = \frac{\sigma_p^2 + (n_i - 1)\eta}{n_i} + \Psi_p$$

$$Cov(\widehat{\xi}_i^p, \overline{\widehat{\xi}^p}) = \rho\Psi_p + \frac{(1-\rho)\Psi_p}{K} + \frac{\sigma_p^2 + (n_i - 1)\eta}{Kn_i}$$

and

$$Var(\overline{\widehat{\xi}^p}) = \frac{1}{K^2}\sum_{i=1}^{K}\frac{\sigma_p^2 + (n_i - 1)\eta}{n_i} + \rho\Psi_p + \frac{(1-\rho)\Psi_p}{K}$$

Then

$$\frac{1}{K-1}\sum_{i=1}^{K}E(\widehat{\xi}^p - \overline{\widehat{\xi}^p})^2 = \frac{1}{K-1}\sum_{i=1}^{K}\left[Var(\widehat{\xi}^p) + Var(\overline{\widehat{\xi}^p}) - 2Cov(\widehat{\xi}^p, \overline{\widehat{\xi}^p})\right]$$

$$= (1-\rho)\Psi_p + \frac{1}{K}\sum_{i=1}^{K}\frac{\sigma_p^2 + (n_i - 1)\eta}{n_i}$$

The structural parameter $\widehat{\Psi}_p$ is not biased because

$$E(\widehat{\Psi}_p) = \frac{1}{1-\rho}E\left[\frac{1}{K-1}\sum_{i=1}^{K}(\widehat{\xi}_i^p - \overline{\widehat{\xi}^p})^2 - \frac{1}{K}\sum_{i=1}^{K}\frac{\widehat{\sigma_p^2} + (n_i - 1)\widehat{\eta}}{n_i}\right] = \Psi_p$$

□

5. Numerical experiment

After demonstrating how to obtain both credibility estimators $\widehat{\Xi^P_{i,n_i+1}}^{inhom}$ and $\widehat{\Xi^P_{i,n_i+1}}^{hom}$ by theorems 1 and 2, in this section, we want to make a comparison between the results of quantile credibility estimators under equal correlation assumption in homogeneous and inhomogeneous cases with those of the quantile credibility premium with common effects of wang et al.(2020) [26] and the quantile credibility premium without dependence of Pitselis(2013) [19]. For this reason, we used the same data of wang et al.(2020), we generated n=1000 groups of 10 years paths of claims and 10 different individuals and we present the experience data for the first group in Table 1. Then, descriptive statistics of the Quantile credibility premiums with p =0.5 for 10 individuals in this group is given in Table 2.

Table 1. Portfolio data

Group	Contract	$X_{1,1}$	$X_{1,2}$	$X_{1,3}$	...	$X_{1,10}$
	1	268.71	182.69	394.92	...	283.68
	2	296.85	347.79	370.99	...	374.21
	3	367.35	213.87	215.27	...	387.67
	4	337.74	443.31	332.42	...	371.85
1	5	309.75	338.14	221.49	...	244.74
	6	314.40	317.58	374.51	...	265.73
	7	265.50	409.18	394.40	...	340.09
	8	181.41	215.33	370.74	...	332.11
	9	322.73	388.63	306.62	...	384.12
	10	329.49	326.14	272.04	...	323.35

Table 2. Descriptive statistics

Contract	Mean	Median	Minimum	Maximum	Standard Deviation
1	313.23	309.06	182.69	396.30	67.05
2	318.74	332.38	132.85	416.51	81.35
3	292.43	317.26	184.44	387.66	75.83
4	350.54	338.39	247.74	443.31	66.23
5	277.10	278.14	194.31	339.05	49.66
6	340.68	339.77	265.73	401.97	39.56
7	280.31	302.79	84.68	451.73	129.96
8	290.28	271.71	181.41	458.11	81.69
9	305.30	319.45	106.62	388.63	84.48
10	295.46	306.24	210.88	348.69	45.80

Table 3, present a comparison between the results of our model in homogeneous and inhomogenous cases and those of Pitselis model (2013) [19] where the assumption of dependence is not considered, and Wang et al model with common effects (2020) [26] where they assumed that Θ_i varies among 10 individuals but the common effect random variable Λ stay the same. The quantile we use is median (the order p =0.5).
In order to show how the correlation coefficient ρ (which indicates dependence among risks) affects the estimators or our premiums, we consider that it takes two known values: ρ=0.4 with η=0.32 and ρ=0.65 with η=0.12. Then, by straightforward calculations, we can obtain the credibility estimators for the next period $(11^{th} year)$.

Table 3. Quantile credibility premiums comparison

Contracts	1	2	3	.....	9	10
$\widehat{\xi_i^{0.5}}$	309.06	332.26	317.26	.....	319.45	306.24
$\widehat{\xi^{0.5}}$	311.52	311.52	311.52	.....	311.52	311.52
Quantile credibility model with common effects (2020) as Λ=132,12						
Θ_i	98.80	175.68	144.48	.....	69.69	111.81
Z_1	0.43	0.43	0.43	.....	0.43	0.43
Z_2	0.18	0.18	0.18	.....	0.18	0.18
Z_3	0.39	0.39	0.39	.....	0.39	0.39
$\widehat{\Xi_{i,n_i+1}^P}^C$	305.96	315.99	309.43	.....	310.43	304.75
Ξ_p=300	$\sigma_{2,p}^2$=836.01	$\sigma_{1,p}^2$=1126.77	$\sigma_{2,p}^2$=89.10			
Quantile credibility model without dependence (Pitselis 2013)						
Z	0.75	0.75	0.75	.....	0.75	0.75
$\widehat{\Xi_{i,n_i+1}^P}^{Pitselis}$	309.67	327.22	315.84	.....	317.49	307.54
Quantile credibility model with equal correlation as ρ=0.4 and η=0.32						
Z_1	0.52	0.52	0.52	.....	0.52	0.52
Z_2	0.12	0.12	0.12	.....	0.12	0.12
Z_3	0.36	0.36	0.36	.....	0.36	0.36
$\widehat{\Xi_{i,n_i+1}^P}^{inhom}$	310.23	322.45	314.53	.....	315.68	308.75
$\widehat{\Xi_{i,n_i+1}^P}^{hom}$	310.23	322.45	314.53	.....	315.68	308.75
Quantile credibility model with equal correlation as ρ=0.65 and η=0.12						
Z_1	0.27	0.27	0.27	.....	0.27	0.27
Z_2	0.24	0.24	0.24	.....	0.24	0.24
Z_3	0.49	0.49	0.49	.....	0.49	0.49
$\widehat{\Xi_{i,n_i+1}^P}^{inhom}$	310.84	317.20	313.08	.....	313.68	310.08
$\widehat{\Xi_{i,n_i+1}^P}^{hom}$	310.84	317.20	313.08	.....	313.68	310.08

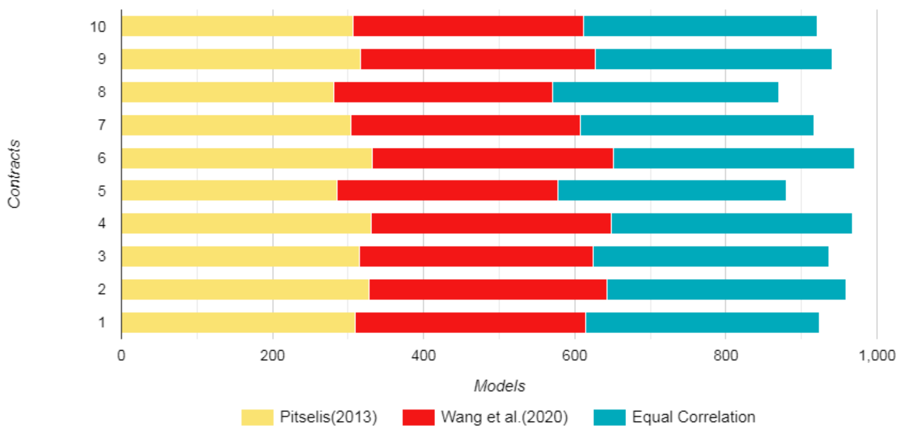

Figure 1. Credibility premiums comparisons as $\rho = 0.65$

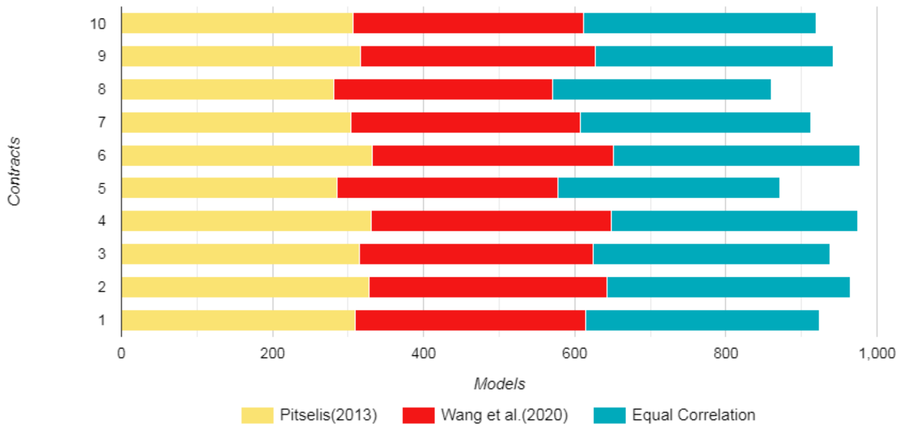

Figure 2. Credibility premiums comparisons as $\rho = 0.4$

From table 3 and as Figures 1 and 2 show:
(i) Most weight is put on the individual's experience in our model compared to quantile credibility premiums in Wang et al (2020) and Pitselis (2013), which explains the fact that the values of the pitselis model are closer to the average loss of the portfolio (the collective premium is 311.52).
(ii) As the value of ρ increases, the premiums of the three models become closer to each other. This approximation between the three premiums, is justified by the convergence of the parameter towards 1, this latter value means a total independence between risks, which leads us to return to the Pitselis model.

6. Conclusion

To summarize, the fundamental goal of this work has been to define a new model for the quantile credibility model with a general dependence structure over individual risks. The credibility estimator of individual premium is derived. A numerical comparison of our credibility premium and those of Pitselis (2013) [19] and Wang et al (2020) [26] is carried out to illustrate the performance of the three models using a real data study. According to comparison results, the newly defined credibility premium should be given a higher priority in practical use because the numerical illustration on real data shows the feasibility of the model. the fact that this assumption is more realistic in insurance market gives it more and more opportunity to be investigated and used by actuaries. This assumption simplifies the analysis of the insurance portfolio by assuming a uniform level of correlation between risks. It may be used in certain mathematical models or calculations within insurance credibility theory to make calculations more tractable or to provide a baseline scenario for analysis.
However, it's essential to recognize that in practice, the correlation between risks in an insurance portfolio may not be equal across all pairs of risks. Real-world correlations can vary based on factors such as the nature of the risks, geographical location, economic conditions, and other external factors. While the equal correlation structure assumption provides a convenient simplification, it's important for insurance professionals and actuaries to critically evaluate whether it accurately reflects the correlation patterns in the specific insurance portfolio being analyzed. In many cases, more sophisticated models or techniques may be needed to capture the true correlation structure effectively, our contribution in this work is limited in this case and where certain structure of dependence over risks is considered and loss function is used. It would be interesting in future research topic to add certain identical conditional dependence for each individual risk (over time) and extend it to a credibility model with respect to balanced loss functions.

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