STATISTICS, OPTIMIZATION AND INFORMATION COMPUTING
Stat., Optim. Inf. Comput., Vol. **12**, November 2024, pp 1745–1761.
Published online in International Academic Press (www.IAPress.org)

Complexity analysis of primal-dual interior-point methods for semidefinite optimization based on a new type of kernel functions

Bachir Bounibane [1,*], Randa Chalekh [2]

[1] *Department of Statistics and Data Sciences , University of Batna 2, Fesdis, Batna, 05078, Algeria*
[2] *Department of Mathematics, University of Batna 2, Fesdis, Batna, 05078, Algeria*

Abstract Kernel functions are essential for designing and analyzing interior-point methods (IPMs). They are used to determine search directions and reduce the computational complexity of the interior point method. Currently, IPM based on kernel functions is one of the most effective methods for solving LO [1, 20], second-order cone optimization (SOCO) [2], and symmetric optimization (SO) and is a very active research area in mathematical programming. This paper presents a large-update primal-dual IPM for SDO based on a new bi-parameterized hyperbolic kernel function. Then we proved that the proposed large-update IPM has the same complexity bound as the best-known IPMs for solving these problems. Taking advantage of the favorable characteristics of the kernel function, we can deduce that the iteration bound for the large update method is $\mathcal{O}\left(\sqrt{n}\log n \log \frac{n}{\varepsilon}\right)$ when a takes a special value utilizing the favorable properties of the kernel function. These theoretical results play an essential role in the design and analysis of IPMs for CQSCO [8] and the Cartesian $P_*(\kappa)$-SCLCP [7]. The proximity function has never been used. To validate our algorithm's efficacy and effectiveness, examples illustrate the applicability of our main results, and we compare our numerical results with some alternatives presented in the literature.

Keywords Linear semidefinite programming, Kernel functions, Complexity analysis, Primal-dual interior point methods, Large-update methods.

AMS 2010 subject classifications 90C22, 90C51

DOI: 10.19139/soic-2310-5070-1927

1. Introduction

We consider the semidefinite optimization problem in its primal format.

$$(P) \qquad \begin{cases} \min \operatorname{tr}(CX) \\ \text{subject to tr } (A_i X) = b_i,\ 1 \le i \le m,\ X \succeq 0, \end{cases}$$

and its dual problems.

$$(D) \qquad \begin{cases} \max b^T y \\ \text{subject to } \sum_{i=1}^{m} y_i A_i + S = C, \quad S \succeq 0, \end{cases}$$

where each $A_i \in \mathbf{S}^n$, $b = (b_1, b_2, ..., b_m)^T \in \mathbb{R}^m$, $y \in \mathbb{R}^m$ and $C \in \mathbf{S}^n$. Moreover, the matrices A_i are linearly independent, with $y \in \mathbb{R}^m$ and $S \in \mathbf{S}^n$. In addition, $X \succeq 0$ indicates that X is a symmetric positive semidefinite matrix. Moreover, the matrices A_i are linearly independent.

*Correspondence to: Bachir Bounibane (Email: b.bounibane@univ-batna2.dz). Department of Statistics and Data Sciences, University of Batna 2, Fesdis, Batna, 05078, Algeria.

ISSN 2310-5070 (online) ISSN 2311-004X (print)

Semidefinite programming is an essential numerical analysis tool in systems and control theory. (SDP) covers many scientific fields, including engineering, control theory, electronic structure problems, and statistics, see [23, 26].

In (2018), Fathi-Hafshejani et al [13] in contrast, hand demonstrated that kernel functions with trigonometric barrier terms produce good results.

The majority of the kernel functions that are utilized in IPMs may be categorized as either logarithmic, simple algebraic, exponential, or trigonometric, given the precedence that has been established. The remaining kernel functions are a binary combination of these different kinds. For more research on primal-dual IPMs that are based on a kernel function, see various authors, including Bouafia and Yassine [4]; Boudjellal et al. [5]; Fathi-Hafshejani et al. [13]; Li et al., [16]; Fathi-Hafshejani and Moaberfard, [12]. Inspired by their work, this research addresses primal-dual IPMs for SDO based on the novel bi-parameterized hyperbolic kernel function.

$$\psi_r(z) = \frac{z^2-1}{2} - \int_1^z r^{\left(\frac{1}{w}-1\right)} dw \quad r \geq e. \tag{1}$$

We deduce that the iteration bounds are $\mathcal{O}\left(\sqrt{n}(\log n)\log\frac{n}{\varepsilon}\right)$ for large-update methods, currently the best-known bounds.

Additionally, based on numerical results, our newly proposed kernel function performs favorably in practice compared to certain existing kernel functions in the literature.

This paper is structured as follows. Section 2 starts by reviewing the basics of IPMs for SDO, such as the central path. Section 3 presents details concerning the parametric kernel function and barrier function. We show that the kernel function meets the eligibility conditions. In Section 4, we derive the algorithm's inner iteration bound and total iteration bound. The results of the experimental tests are presented in Section 5. Section 6 is the concluding part of the paper. It offers some conclusions and remarks.

The following notational conventions are utilized throughout the paper. The sets of real, nonnegative real, and positive real vectors with n components are denoted by $\mathbb{R}^n$, $\mathbb{R}^n_+$, and $\mathbb{R}^n_{++}$, respectively. **E** represente an $n\times n$ identity matrix. If A is a $n\times n$ matrix, then its trace is written as $\mathrm{tr}(A) = \sum_{i=1}^{n} A_{ii}$. We use the matrix inner product, i.e., $\mathrm{tr}(MN) := \sum_{i,j=1}^{n} m_{ij}n_{ij}$. For any $Q \in S^n_{++}$, the expression $Q^{\frac{1}{2}}$ denotes its symmetric square root. For any $V \in S^n$, we denote by $\lambda(V)$ the vector of eigenvalues of V arranged in non-increasing order, that is, $\lambda_1(V) \leq \lambda_2(V) \leq, ..., \lambda_n(V)$. If $h(x) \geq 0$ is a real-valued function of the real nonnegative variable, the notation $h(x) = \mathcal{O}(x)$ means that $h(x) \leq kx$ for some positive constant k and $h(x) = \Theta(x)$ that $k_1x \leq h(x) \leq k_2x$ for two positive constants k_1 and k_2.

2. Preliminaries

2.1. The central path and search direction for SDO

We assume that both (P) and (D) satisfy the interior-point condition (IPC), i.e., there exists an $\left(X^0 \succ 0, y^0,\ S^0 \succ 0\right)$ such that

$$\mathrm{tr}\left(A_iX^0\right) = b_i,\ 1 \leq i \leq m,\ \sum_{i=1}^{m} y_i^0 A_i + S^0 = C,\ X_0 \succ 0,\ S^0 \succ 0.$$

We can immediately confirm that a pair of optimal solutions for (P) and (D) corresponds to solving the following Newton system:

$$\begin{cases} \mathrm{tr}\left(A_iX\right) = b_i, \quad i = 1, ..., m, \quad X \succeq 0, \\ \sum_{i=1}^{m} y_iA_i + S = C, \qquad\qquad S \succeq 0, \\ XS = 0. \end{cases} \tag{2}$$

The basic idea of primal-dual IPMs is to replace the complementarity condition in (2) with the parameterized equation $XS = \mu\mathbf{E}$ $(\mu > 0)$. This provides the next system.

$$\begin{cases} \text{tr}\,(A_iX) = b_i, \quad 1 \le i \le m, \;\; X \succeq 0, \\ \sum\limits_{i=1}^{m} y_iA_i + S = C, \quad S \succeq 0, \\ XS = \mu\mathbf{E}. \end{cases} \tag{3}$$

This system (3) possesses a unique solution, indicated by $(X(\mu), y(\mu), S(\mu))$ for any $\mu > 0$.

The set of μ-centers (with $\mu > 0$) defines a homotopy path, which is called the central path of (P) and (D) (De Klerk.,[9]). Newton's method is a well-known procedure to solve a system of nonlinear equations. Suppose the point $(X, y,\ S)$ is strictly feasible. Applying Newton's method to the system (3), thus yielding the following system:

$$\begin{cases} \text{tr}\,(A_i\Delta X) = 0,\ 1 \le i \le m,\ X \succeq 0, \\ \sum\limits_{i=1}^{m} \Delta y_iA_i + \Delta S = 0,\ S \succeq 0, \\ X\Delta S + \Delta XS = \mu\mathbf{E} - XS. \end{cases} \tag{4}$$

Note that, ΔS is symmetric due to the second equation in (4). Important observation ΔX is not always symmetric.

There are various ways for symmetrizing the third equation of (4). In this paper, we examine the symmetrization approach that produces NT-direction. (Nesterov and Todd, [17]), which uses the positive definite matrix. Define the matrix

$$P := X^{\frac{1}{2}}(X^{\frac{1}{2}}SX^{\frac{1}{2}})^{-\frac{1}{2}}X^{\frac{1}{2}} = S^{-\frac{1}{2}}\left(S^{\frac{1}{2}}XS^{\frac{1}{2}})\right)^{\frac{1}{2}}S^{-\frac{1}{2}}.$$

Moreover, also define $D = P^{\frac{1}{2}}$, where $P^{\frac{1}{2}}$ denotes the symmetric square root of P. Then the matrix D can be used to scale X and S to the same matrix V, defined by.

$$V = \frac{1}{\sqrt{\mu}}D^{-1}XD^{-1} = \frac{1}{\sqrt{\mu}}DSD = \frac{1}{\sqrt{\mu}}(D^{-1}XSD)^{\frac{1}{2}}. \tag{5}$$

The matrices D and V are symmetric and positive definite.

Applying on (4) some fundamental reductions of (5) we have

$$\begin{cases} \text{tr}\left(\bar{A}_iD_X\right) = 0,\ \ 1 \le i \le m, \\ \sum\limits_{i=1}^{m} \Delta y_i\bar{A}_i + D_S = 0, \\ D_X + D_S = V^{-1} - V, \end{cases} \tag{6}$$

with

$$\bar{A}_i = \frac{1}{\sqrt{\mu}}DA_iD,\ \ 1 \le i \le m;$$

and

$$D_X = \frac{1}{\sqrt{\mu}}D^{-1}\Delta XD^{-1},\ D_S = \frac{1}{\sqrt{\mu}}D\Delta SD. \tag{7}$$

We can say that $\text{tr}(D_XD_S) = 0$, which is coming from the first and second equations of (6) or from the orthogonality of ΔX and ΔS.

We introduce the new search direction in this section. However, we begin by defining the concept of a matrix function. (Horn and Johnson, [15]; Roos et al. [21])

Definition 1

Let $V \in \mathbf{S}^n_{++}$ and $V = Q_V^T diag\,(\lambda\,(V))\,Q_V$ where Q_V is any orthonormal matrix that diagonalizes V. Let $\psi\,(t)$ be defined in (1). Then the matrix valued-function $\psi(V) : \mathbf{S}^n_{++} \longrightarrow \mathbf{S}^n$ is defined by

$$\psi(V) = Q^T diag\,(\psi(\lambda_1(V)), \psi\,(\lambda_2(V))\,, ..., \psi\,(\lambda_n(V)))\,Q. \tag{8}$$

If the function $\psi(t)$ is differentiable on the interval $]0, +\infty[$ such that $\psi'(t) > 0, \forall t > 0$, the matrix function $\psi'(V)$ may be obtained by substituting $\psi(\lambda_i(V))$ in (8) with $\psi'(\lambda_i(V))$ for each i.

Definition 2
$\Psi(V) : \mathbf{S}^n_{++} \to \mathbb{R}_+$ such that

$$\Psi(V) = tr(\psi(V)) = \sum_{i=1}^{n} \psi(\lambda_i(V)), \tag{9}$$

where $\psi(V)$ is given by (8).

Addressing (Peng et al., [18, 19]), the second equation in the system (6) can be rewritten as $D_X + D_S = -\nabla\Psi(V)$. As a result, this system might be constructed as follows:

$$\begin{cases} \text{tr}\left(\bar{A}_i D_X\right) = 0, \ \ 1 \le i \le m, \\ \sum\limits_{i=1}^{m} \Delta y_i \bar{A}_i + D_S = 0, \\ D_X + D_S = -\nabla\Psi(V). \end{cases} \tag{10}$$

where $\nabla\Psi(V)$ denotes the gradient of $\Psi(V)$, i.e., $\psi'(V)$. This system has a unique solution D_X, D_S, and Δy, which can be used to compute ΔX and ΔS from (7) (Wang et al. [25])

Choosing an appropriate step size α, we will use $(\Delta X, \Delta y, \Delta S)$ as the new search direction, the new iterate (X_+, y_+, S_+) is given by

$$X_+ = X + \alpha\Delta X, \ y_+ = y + \alpha\Delta y, \ S_+ = S + \alpha\Delta S. \tag{11}$$

Due to the first two equations of the system (10), D_X and D_S are orthogonal i.e., $\text{tr}(D_X D_S) = tr(D_X D_S) = 0$. Then we have

$$\Psi(V) = 0 \Leftrightarrow V = \mathbf{E} \Leftrightarrow D_X = D_S = 0_{n\times n} \Leftrightarrow X = X(\mu), S = S(\mu).$$

The algorithm is presented in its generic form.

Algorithm 1
Generic primal-dual algorithm for SDO

Input
a threshold parameter $\tau \ge 1$;
an accuracy parameter $\varepsilon > 0$;
a fixed barrier update parameter θ, $0 < \theta < 1$;
$X^0 \succ 0$, $S^0 \succ 0$ and $\mu^0 = 1$ such that $\Psi\left(X^0, S^0, \mu^0\right) \le \tau$.
begin
$X := X^0$; $S = S^0$; $\mu = \mu^0$
while $n\mu \ge \varepsilon$ do
  begin
  $\mu := (1-\theta)\mu$
  while $\Psi(X, S, \mu) \ge \tau$ do
      begin
      Solve system (10) and use (7) for $(\Delta X, \ \Delta y, \ \Delta S)$
    Choose a suitable step size α
    $(X, y, S) := (X, y, S) + \alpha(\Delta X, \Delta y, \Delta S)$.
   end
  end
end

3. Properties of the new parametric kernel function

This section introduces a novel parametric kernel function. Subsequently, several advantageous characteristics of this function are presented.

3.1. *Kernel function properties*

In the sequel, we derive the three first derivatives of $\psi_r(z)$ with respect to z as follows:

$$\psi_r'(z) = z - r^{\left(\frac{1}{z}-1\right)}, \text{ for all } z > 0, \quad (12)$$

$$\psi_r''(z) = 1 + \frac{\log r}{z^2} r^{\left(\frac{1}{z}-1\right)}, \text{ for all } z > 0, \quad (13)$$

$$\psi_r'''(z) = -\frac{\log r\,(2z + \log r)}{z^4} r^{\left(\frac{1}{z}-1\right)}, \text{ for all } z > 0, \quad (14)$$

We can deduce from (13) that $\psi_r''(z) > 1$ for $z > 0$, implying that $\psi_r(z)$ is strongly convex over $\mathbb{R}_{++}$. There is also $\psi_r(1) = \psi_r'(1) = 0$. Thus, $\psi_r(z)$ is indeed a kernel function.

Because of the conditions $\psi_r(1) = \psi_r'(1) = 0$, we can completely describe $\psi_r(z)$ by its second derivative:

$$\psi_r(z) = \int_1^z \int_1^\xi \psi_r''(\zeta)\, d\zeta d\xi.$$

Next, the lemma proves the qualification of our new kernel function (1).

3.2. *The new kernel function's eligibility*

Lemma 1

Let the function $\psi_r(z)$ be defined as in (1). Then, we have

$$z\psi_r''(z) + \psi_r'(z) > 0, \quad z < 1, \quad (15)$$

$$z\psi_r''(z) - \psi_r'(z) > 0, \quad z > 1, \quad (16)$$

$$\psi_r'''(z) < 0 \quad z > 0, \quad (17)$$

$$2\left(\psi_r''(z)\right)^2 - \psi_r'(z)\psi_r'''(z) > 0, \quad z < 1, \quad (18)$$

$$\psi_r''(z)\psi_r'(\beta z) - \beta\psi_r'(z)\psi_r''(\beta z) > 0, \; z > 1, \; \beta > 1. \quad (19)$$

Proof

For (15) and all $z > 0$, we get the following:

$$\begin{aligned} z\psi_r''(z) + \psi_r'(z) &= 2z + \left(\frac{\log r}{z} - 1\right) r^{\left(\frac{1}{z}-1\right)} \\ &\geq \left[2z + \left(\frac{\log r}{z} - 1\right)\left(1 + \left(\frac{1-z}{z}\right)\log r\right)\right]. \end{aligned}$$

$$z\psi_r''(z) + \psi_r'(z) \geq 0 \Leftrightarrow \log r - z \geq 0$$

This last inequality is due to the fact that $r \geq e$, and $z \in (0,1)$, this proves the condition is satisfied. For (16) By substituting $\psi_r'(z)$ and $\psi_r''(z)$, we obtain,

$$z\psi_r''(z) - \psi_r'(z) = \left(\frac{\log r}{z} + 1\right) r^{\left(\frac{1}{z}-1\right)} > 0, \; z > 0.$$

The proof (17). It is simple to observe $\psi_r'''(z) < 0$ from (14).

For (18) , we have

$$\begin{aligned} 2\left(\psi_r''(z)\right)^2 - \psi_r'(z)\psi_r'''(z) &= 2\left[1 + \frac{\log r}{z^2} r^{\left(\frac{1}{z}-1\right)}\right]^2 \\ &\quad + \left[\frac{(2z + \log r)\log r}{z^4} r^{\left(\frac{1}{z}-1\right)}\right] \times \left[z - r^{\left(\frac{1}{z}-1\right)}\right] \\ &= \left[2 + \frac{z\log r\,(\log r + 6z)\, r^{\left(\frac{1}{z}-1\right)} + \log r\,(\log r - 2z)\, r^{2\left(\frac{1}{z}-1\right)}}{z^4}\right], \end{aligned}$$

If $r \geq e^2$, the condition (18) is unquestionably satisfied for $0 < z < 1$. So it's still clear that:

$$\begin{aligned} & \left[2\left(\psi_r''(z)\right)^2 - \psi_r'(z)\psi_r'''(z) > 0\right] \\ \Leftrightarrow \quad & \left[z \log r\left(\log r + 6z\right) r^{\left(\frac{1}{z}-1\right)} + \log r\left(\log r - 2z\right) r^{2\left(\frac{1}{z}-1\right)} > 0\right] \end{aligned} \tag{20}$$

Let's examine the case $0 < r < e^2$ and $z \in \left(0, \frac{\log r}{2}\right)$. The relationship (20) is obviously satisfied. It's sufficient to prove that (18) holds for

$$\begin{cases} z \in \left(\frac{\log r}{2}, 1\right) \\ r \in \left[e, e^2\right[. \end{cases}$$

Then

$$\begin{aligned} \left[2\left(\psi_r''(z)\right)^2 - \psi_r'(z)\psi_r'''(z) > 0\right] \quad & \Leftrightarrow \quad \left[r^{\left(\frac{1}{z}-1\right)} \leq \frac{\left(\log r + 6z\right) z}{2z - \log r}\right] \\ & \Leftrightarrow \quad \left[r^{\left(\frac{1}{z}-1\right)} \leq \left(\begin{array}{c} \frac{z \log r}{2z - \log r} \\ + \frac{6z^2}{2z - \log r} \end{array}\right)\right], \end{aligned}$$

and this is true if

$$r^{\left(\frac{1}{z}-1\right)} < \frac{\log r}{2 - \frac{\log r}{z}} \tag{21}$$

Let $u = \frac{1}{z}$. The relation (21) can then be expressed as follows:

$$r^{u-1} < \frac{\log r}{2 - u \log r}, \; u \in \left(\frac{1}{\log r}, \frac{2}{\log r}\right),$$

which to

$$1 > \left(\frac{2}{\log r} - u\right) r^{u-1}. \tag{22}$$

For (22) , let $h\left(u\right) = 1 - \left(\frac{2}{\log r} - u\right) r^{u-1}$, then

$$\begin{cases} h'\left(u\right) = r^{u-1}\left(-1 + u \log r\right) \\ h''\left(u\right) = r^{u-1}. \left(\log r\right)^2 u > 0 \text{ for } t > 0. \end{cases}$$

If we set $h'\left(u\right) = 0$, we obtain $u = \frac{1}{\log r}$. Since $h(u)$ is strictly convex and has a global minimum,

$$h\left(\frac{1}{\log r}\right) = 1 - \left(\frac{1}{\log r}\right) r^{\frac{1-\log r}{\log r}} = 1 - \left(\frac{1}{\log r}\right) r^{-1+\frac{1}{\log r}} > 0.$$

We have the result □

From the exponential convexity property of the kernel function ψ, we can deduce the following result for the matrix barrier function $\Psi\left(V\right)$.

Lemma 2 (Proposition 3 in Peng et al., [20])
For any $V_1, V_2 \succ 0$,

$$\Psi\left(\left[V_1^{\frac{1}{2}} V_2 V_1^{\frac{1}{2}}\right]^{\frac{1}{2}}\right) \leq \frac{1}{2}\left(\Psi\left(V_1\right) + \Psi\left(V_2\right)\right).$$

Lemma 3
Given $\psi_r\left(z\right)$, we have the following results confirmed.

$$\frac{1}{2}\left(z-1\right)^2 \leq \quad \psi_r\left(z\right) \quad \leq \frac{1}{2}\psi_r'\left(z\right)^2, \; z > 0, \tag{23}$$

$$\psi_r\left(z\right) \quad \leq \frac{1}{2}\psi_r''\left(1\right)\left(z-1\right)^2, \; z \geq 1, \tag{24}$$

$$\|V\| \quad \leq \sqrt{n} + \sqrt{2 \;\; \Psi\left(V\right)}. \; \forall V \succ 0. \tag{25}$$

Proof
For (23) , according to the definition of $\psi_r(z)$, we have:

$$\psi_r(z) \geq \frac{1}{2}(z-1)^2,$$

which proves the first inequality. The second inequality is obtained as follows:

$$\begin{aligned}\psi_r(z) = \int_1^z \int^{\xi} \psi_r''(\zeta)\, d\zeta d\xi &\leq \int_1^z \int^{\xi} \psi_r''(\xi)\, \psi_r''(\zeta)\, d\zeta d\xi \\ &= \int_1^z \psi_r''(\xi)\, \psi_r'(\xi)\, d\xi \\ &= \int_1^z \psi_r'(\xi)\, d\psi_r'(\xi) \\ &= \frac{1}{2}(\psi_r'(z))^2.\end{aligned}$$

For (24), since $\psi_r(1) = \psi_r'(1) = 0$, $\psi_r'''(z) < 0$, $\psi_r''(1) = 1 + \log r$, and by using Taylor's expansion we have for some ξ, such that $1 \leq \xi \leq z$.

$$\begin{aligned}\psi_r(z) &= \psi_r(1) + \psi_r'(1)(z-1) + \frac{1}{2}\psi_r''(1)(z-1)^2 + \frac{1}{6}\psi_r'''(\xi)(\xi-1)^3 \\ &= \frac{1}{2}\psi_r''(1)(z-1)^2 + \frac{1}{6}\psi_r'''(\xi)(\xi-1)^3 \\ &< \frac{1}{2}\psi_r''(1)(z-1)^2,\end{aligned}$$

which completes the proof.
For (25) , using, the left-hand side of (23) , and the Cauchy-Schwarz inequality, one can obtain

$$\begin{aligned}2\Psi(V) &= 2\sum_{i=1}^{n} \psi_a(\lambda_i(V_i)) \geq \sum_{i=1}^{n} (\lambda_i(V_i) - 1)^2 \\ &= \left[\sum_{i=1}^{n} \lambda_i(V_i)^2 - 2\sum_{i=1}^{n} \lambda_i(V_i) + n\right] \\ &= \|V\|^2 - 2.\mathbf{E}^T V + \|\mathbf{E}\|^2 \geq \left(\|V\|^2 - 2\|V\|\,\|\mathbf{E}\| + n\right) \\ &= \left(\|V\| - \sqrt{n}\right)^2,\end{aligned}$$

that is to say

$$\|V\| \leq \sqrt{n} + \sqrt{2\Psi(V)} = \sqrt{n} + \sqrt{2\Psi(V)}.$$

where $\mathbf{E}$ denotes the all one vector. This completes the proof. □

Lemma 4
Let $\beta \geq 1$. Then

$$\psi_r(\beta z) \leq \psi_r(z) + \frac{1}{2}\left(\beta^2 - 1\right) z^2.$$

Proof
Let us define $\psi_r(z)$ as $\psi_r(z) = \frac{z^2-1}{2} + \varphi_r(z)$, where $\varphi_r(z) = -\int\limits_1^z r^{\left(\frac{1}{z}-1\right)}$. Then we have:

$$\varphi_r'(z) = -r^{\left(\frac{1}{z}-1\right)} < 0$$

i.e., $\varphi_r(z)$ is thus a decreasing function when $z > 0$. Thus $\varphi_r(z)(\beta z) \leq \varphi_r(z)$ for $\beta \geq 1$. So

$$\psi_r(\beta z) - \psi_r(z) = \frac{1}{2}\left(\beta^2 - 1\right) z^2 + \varphi_r(\beta z) - \varphi_r(z) \leq \frac{1}{2}\left(\beta^2 - 1\right) z^2.$$

That implies the lemma. □

Lemma 5
Let $\varrho : [0, +\infty) \longrightarrow [1, +\infty)$ *be the inverse function of* $\psi_r(z)$ for $z \geq 1$ and $\rho : [0, +\infty) \longrightarrow (0, 1]$ *the inverse function of* $\frac{-1}{2}\psi_r'(z)$ for $z \in (0, 1]$, we have:

$$\sqrt{2u+1} \leq \varrho(u) \leq \sqrt{2u} + 1 \quad u \geq 0, \tag{26}$$

$$\rho(u) \geq \frac{1}{1 + \frac{\log(1+2u)}{\log r}} \quad u \geq 0. \tag{27}$$

Proof
For (26), let $u = \psi_r(z)$ for $z \geq 1$. Then $\varrho(u) = z, z \geq 1$, using (23) of Lemma 3, we have $u = \psi_r(z) \geq \frac{1}{2}(z-1)^2$, so $z = \varrho(u) \leq \sqrt{2u} + 1$. By the definition of $\psi_r(z)$ we have

$$u = \psi_r(z) = \psi_b(z) + \frac{z^2 - 1}{2} \leq \frac{z^2 - 1}{2} \quad \Leftrightarrow \quad 2u \leq z^2 - 1$$
$$\Leftrightarrow \quad z = \varrho(u) \geq \sqrt{1 + 2u}.$$

Thus

$$z = \varrho(u) \geq \sqrt{1 + 2u}.$$

For (27). To find the inverse function of the restriction of $\frac{-1}{2}\psi_r'(z)$ in the interval $(0, 1]$, we need to solve the equation $\frac{-1}{2}\psi_r'(z) = u$ for $z \in (0, 1]$. To do so, we have

$$2u = -\psi_r'(z) \Leftrightarrow -\left(z - r^{\left(\frac{1}{z} - 1\right)}\right) = 2u.$$

This implies that

$$r^{\left(\frac{1}{z} - 1\right)} = z + 2u \leq 1 + 2u \quad \Leftrightarrow \quad \frac{1}{z} \leq 1 + \frac{\log(1+2u)}{\log r}$$
$$\Leftrightarrow \quad z = \rho(u) \geq \frac{1}{1 + \frac{\log(1+2u)}{\log r}},$$

where the last inequality is obtained from the fact that $z \leq 1$. This completes the proof. □

We now present a norm-based proximity measure

$$\delta(V) := \frac{1}{2}\|\nabla \Psi(V)\| = \frac{1}{2}\sqrt{\operatorname{tr}(\psi'(V)^2)}. \tag{28}$$

This lemma establishes a lower bound for the function $\delta(V)$ using the proximity function $\Psi(V)$.

Lemma 6
Let $\delta(V)$ be defined as in (28).

$$\delta(V) \geq \sqrt{\frac{\Psi(V)}{2}}. \; V \in \mathbf{S}^n_{++}, \; r \geq e \tag{29}$$

Proof
Using (23)

$$\begin{aligned}\Psi(V) &= \sum_{i=1}^{n} \psi_r(\lambda_i(V)) \\ &\leq \frac{1}{2}\sum_{i=1}^{n} \psi_r'(\lambda_i(V))^2 \\ &= \frac{1}{2}\|\nabla\Psi\|^2 = 2\delta(V)^2.\end{aligned}$$

So that $\delta(V) \geq \sqrt{\frac{\Psi(V)}{2}}$. This finishes the proof. □

Remark 1
We always assume that $\tau \geq 1$. During this work, we use Lemma 6 and the assumption that $\Psi(v) \geq \tau$ we have

$$\delta(V) \geq \sqrt{\frac{1}{2}}.$$

Theorem 1 (Theorem 3.2, [1])
Assume that ϱ it is defined as in Lemma 5. and $V \succ 0$, $\beta \geq 1$, then

$$\Psi(\beta V) \leq n\psi\left(\beta\varrho\left(\frac{\Psi(V)}{n}\right)\right).$$

Lemma 7
Let $0 \leq \theta < 1$ and $V_+ = \frac{V}{\sqrt{1-\theta}}$. If $\Psi(V) \leq \tau$ then we have:

$$\Psi(V_+) \leq \Psi(V) + \frac{1}{2}\left(\frac{\theta}{1-\theta}\right)\left[n + 2\,\Psi(V) + 2\sqrt{2n\,\Psi(V)}\right], \tag{30}$$

Proof
For (30) , using Lemma 4 with $\beta = \frac{1}{\sqrt{1-\theta}}$, Lemma 3 and (25) we obtain

$$\begin{aligned}\Psi(V_+) &= \Psi(\beta V) = \sum_{i=1}^{n} \psi_a(\beta V_i) \leq \sum_{i=1}^{n}\left[\psi_a(V_i) + \frac{1}{2}\left(\beta^2 - 1\right)V_i^2\right] \\ &= \Psi(V) + \frac{1}{2}\left(\beta^2 - 1\right)\sum\nolimits_{i=1}^{n} V_i^2 \\ &= \Psi(V) + \frac{1}{2}\left(\frac{\theta}{1-\theta}\right)\|V\|^2 \\ &\leq \Psi(V) + \frac{1}{2}\left(\frac{\theta}{1-\theta}\right)\left(\sqrt{n} + \sqrt{2\Psi(V)}\right)^2 \\ &= \Psi(V) + \frac{1}{2}\left(\frac{\theta}{1-\theta}\right)\left(n + 2\Psi(V) + 2\sqrt{2n\Psi(V)}\right).\end{aligned}$$

We obtain

$$\Psi(V_+) \leq \tau + \frac{\theta}{2(1-\theta)}\left(n + 2\tau + 2\sqrt{2n\tau}\right).$$

This completes the proof. □

Denote

$$\bar{\Psi}_0 = \frac{2\tau + n\theta + 2\theta\sqrt{2n\tau}}{1-\theta}, \tag{31}$$

We'll utilize $\bar{\Psi}_0$ for the upper bounds of $\Psi(V)$ for large-update methods throughout the algorithm.

Remark 2
For the large-update method, by taking $\tau = \mathcal{O}(n)$, $\theta = \Theta(1)$ we have $\bar{\Psi}_0 = \mathcal{O}(n)$

Now we determine a default step size and obtain an upper bound for the decrease of the barrier function $\Psi(V)$ during an inner iteration.

4. Analysis of the interior-point algorithm for SDO

4.1. Default value for the step size

We are utilizing (11) and (7). After a step of size α, the next iteration is determined by

$$X_+ := X + \alpha\Delta X = X + \alpha\sqrt{\mu}DD_XD = \sqrt{\mu}D(V+\alpha D_X)D,$$
$$S_+ := S + \alpha\Delta S = S + \alpha\sqrt{\mu}D^{-1}D_SD^{-1} = \sqrt{\mu}D^{-1}(V+\alpha D_S)D^{-1}.$$

We obtain the result from (5)by defining the matrix V after the step as V_+.

$$V_+ = \frac{1}{\sqrt{\mu}}(D^{-1}X_+S_+D)^{\frac{1}{2}}$$

We can verify that V_+^2 is unitarily similar to the matrix $X_+^{\frac{1}{2}}S_+X_+^{\frac{1}{2}}$ and thus to $(V+\alpha D_X)^{\frac{1}{2}}(V+\alpha D_S)(V+\alpha D_X)^{\frac{1}{2}}$. This implies that the eigenvalues of V_+ are precisely the same as those of the matrix

Consequently, the eigenvalues of the matrix V_+ are the same as those of

$$\left[(V+\alpha D_X)^{\frac{1}{2}}(V+\alpha D_S)(V+\alpha D_X)^{\frac{1}{2}}\right]^{\frac{1}{2}}.$$

Since the proximity after one step is defined by $\Psi(V_+)$, and then we have

$$\Psi(V_+) = \Psi\left(\left[(V+\alpha D_X)^{\frac{1}{2}}(V+\alpha D_S)(V+\alpha D_X)^{\frac{1}{2}}\right]^{\frac{1}{2}}\right).$$

By Lemma 2, we obtain

$$\Psi(V_+) \leq \frac{1}{2}\left[\Psi(V+\alpha D_X) + \Psi(V+\alpha D_S)\right].$$

Defining

$$f(\alpha) := \Psi(V_+) - \Psi(V).$$

Due to Lemma 2 and the definition of $f(\alpha)$, it follows that $f(\alpha) \leq f_1(\alpha)$ where

$$f_1(\alpha) := \frac{1}{2}\left(\Psi(V+\alpha D_X) + \Psi(V+\alpha D_S)\right) - \Psi(V).$$

Obviously

$$f(0) = f_1(0) = 0.$$

When we take the first two derivatives of $f_1(\alpha)$ with respect to α, we get the following:

$$f_1'(\alpha) = \frac{1}{2}\text{tr}\left(\psi'(V+\alpha D_X)D_X + \psi'(V+\alpha D_S)D_S\right),$$

and

$$\begin{aligned} f_1''(\alpha) &= \frac{1}{2}\frac{d^2}{d\alpha^2}\text{tr}\left(\psi(V+\alpha D_X)+\psi(V+\alpha D_S)\right) \\ &= \frac{1}{2}\text{tr}\left(\psi''(V+\alpha D_X)D_X^2+\psi''(V+\alpha D_S)D_S^2\right). \end{aligned}$$

It is clear that $f_1(\alpha) > 0$ unless $D_X = D_S = 0$.

Using the third equation of system (10) and (28) , we get

$$\begin{aligned} f_1'(0) &= \frac{1}{2}\text{tr}[\psi(V)'D_X+\psi(V)'D_S] = \frac{1}{2}\text{tr}\left(\psi'(V)(D_X+D_S)\right) \\ &= \frac{1}{2}\text{tr}\left[\psi(V)'\left(-\psi'(V)\right)\right] = \frac{1}{2}\text{tr}\left(-\psi'(V)^2\right) = -2\delta^2\left(V\right). \end{aligned} \tag{32}$$

In the following, we will utilize the abbreviated notation: $\delta := \delta(V)$ and $\Psi := \Psi(V)$.

Similar to the LO case, the following lemma holds for all kernel functions that satisfy $\psi'''(z) < 0$. (see Wang and Bai., [24]).

Lemma 8
Let δ be defined as in (28). Then we have

$$f_1''(\alpha) \leq 2\delta^2\psi''(\lambda_n(V) - 2\alpha\delta),$$

where $\lambda_n(V)$ is the smallest eigenvalue of V.

Our objective in introducing a suitable step size is for it to be chosen so that X_+ and S_+ are realizable and $f(\alpha)$ decreases adequately.

Without proof, from Lemmas 4.2-4.5 in (Bai et al.,[1]). We have the following Lemmas 9, 10, 11 and 13.

Lemma 9
If the step size α satisfies

$$\psi'(\lambda_n(V)) - \psi'(\lambda_n(V) - 2\alpha\delta) \leq 2\delta, \tag{33}$$

then

$$f'(\alpha) \leq 0.$$

Lemma 10
Let $\rho : [0, \infty) \to (0, 1]$ denote the inverse function of the restriction of -$\frac{1}{2}\psi'(z)$ on the interval $(0, 1]$, then the largest possible value of the step size of α satisfying (33) is given by

$$\bar{\alpha} := \frac{1}{2\delta}(\rho(\delta) - \rho\left(2\delta\right)).$$

Lemma 11
Let ρ and $\bar{\alpha}$ as defined in Lemma 10. Then

$$\bar{\alpha} \geq \frac{1}{\psi''\left(\rho\left(2\delta\right)\right)}.$$

As we did with LO, we use

$$\bar{\alpha} = \frac{1}{\psi''\left(\rho\left(2\delta\right)\right)} \tag{34}$$

The necessary step size α is determined for the algorithm.

Lemma 12
Let ρ and $\bar{\alpha}$ be as defined in Lemma 11. If $\Psi(v) \geq \tau \geq 1$, then we have

$$\bar{\alpha} \geq \frac{1}{8\delta\log r\left[1+\frac{\log(1+4\delta)}{\log r}\right]^2}.$$

Proof

From Lemma 11 using $z = 2\delta$, (27) and (29), we get

$$\begin{aligned} \bar{\alpha} &\geq \frac{1}{\psi_r''\left(\rho\left(2\delta\right)\right)} \\ &= \frac{1}{1+\frac{\log r}{(\rho(2\delta))^2}r^{\left(\frac{1}{\rho(2\delta)}-1\right)}} \\ &\geq \frac{1}{1+\log r\left(4\delta+1\right)\left[\frac{\log(4\delta+1)}{\log r}+1\right]^2}. \end{aligned}$$

Using Remark 1, one has

$$\begin{aligned} \bar{\alpha} &\geq \frac{1}{\sqrt{2}\delta\log r+\left(1+\frac{\log(1+4\delta)}{\log r}\right)^2\left(4\delta+\sqrt{2}\delta\right)\log r} \\ &\geq \frac{1}{2\delta\log r+\left(1+\frac{\log(1+4\delta)}{\log r}\right)^2\left(4\delta+2\delta\right)\log r}. \end{aligned}$$

This implies that

$$\bar{\alpha} \geq \frac{1}{8\delta\left(1+\frac{\log(1+4\delta)}{\log r}\right)^2\log r}.$$

This completes the proof. □

Denoting

$$\tilde{\alpha} = \frac{1}{8\delta\left[1+\frac{\log(1+4\delta)}{\log r}\right]^2\log r}, \tag{35}$$

4.2. Decrease the Value of $\Psi\left(V\right)$

Lemma 13

If the step size α is such that $\alpha \leq \bar{\alpha}$, then

$$f(\alpha) \leq -\alpha\delta^2.$$

Lemma 14

If the step size $\tilde{\alpha}$ in as (34) Then we have

$$f\left(\tilde{\alpha}\right) \leq -\frac{\delta^2}{\psi''\left(\rho\left(2\delta\right)\right)}. \tag{36}$$

Indeed, the upper bound for the decreasing value of proximity in the inner iteration may be obtained through the following lemma

Lemma 15

Let $\tilde{\alpha}$ be as defined in (35) and $\Psi\left(v\right) \geq 1$. Then we have the following upper bound for $f\left(\tilde{\alpha}\right)$:

$$f\left(\tilde{\alpha}\right) \leq -\frac{\sqrt{\Psi}}{16\log r\left[1+\frac{\log\left(1+2\sqrt{\Psi_0}\right)}{\log r}\right]^2}. \tag{37}$$

Proof
According to Lemma 13, with $\alpha = \tilde{\alpha}$ and (35), we have

$$\begin{aligned} f(\tilde{\alpha}) &\leq -\tilde{\alpha}\delta^2 \\ &= -\frac{\delta^2}{8\delta \log r \left[1 + \frac{\log(1+4\delta)}{\log r}\right]^2} \\ &\leq -\frac{\sqrt{\Psi}}{16 \log r \left[1 + \frac{\log\left(1+2\sqrt{\Psi_0}\right)}{\log r}\right]^2}. \end{aligned}$$

This proves the theorem. □

4.3. Iteration complexity

We first offer the following technical result to determine an upper bound K for the number of inner iterations.

Lemma 16
Suppose that a sequence $\left\{t^k > 0,\ k = 0, 1, 2, ..., K\right\}$ is satisfying the following inequality:

$$t_{k+1} \leq t_k - \eta t_k^{1-\gamma}, \quad k = 0, 1, 2, ..., K-1,$$

where $\eta > 0$ and $\gamma \in (0, 1]$. Then $K \leq \left\lceil \frac{t_0^\gamma}{\eta\gamma} \right\rceil$.

(35) shows the diminution of every inner iteration. In [18] we may obtain the proper values of η and $\gamma \in (0, 1]$.

$$\eta = \frac{1}{16 \log r \left(1 + \frac{\log\left(1+2\sqrt{\Psi_0}\right)}{\log r}\right)^2}, \quad \gamma = \frac{1}{2}.$$

Theorem 2
Let $\bar{\Psi}_0$ be defined as in (31) and let L is the total number of inner iterations in the outer iteration for large-update methods. We have

$$L \leq 32 \log r \left(1 + \frac{\log\left(1 + 2\sqrt{\bar{\Psi}_0}\right)}{\log r}\right)^2 \bar{\Psi}_0^{\frac{1}{2}},$$

Proof
By Lemma 16 and Theorem 1, we have

$$L \leq \frac{\bar{\Psi}_0^\gamma}{\eta\gamma} = 32 \log r \left(1 + \frac{\log\left(1 + 2\sqrt{\bar{\Psi}_0}\right)}{\log r}\right)^2 \bar{\Psi}_0^{\frac{1}{2}}.$$

This proves the lemma □

The number of outer iterations is bounded above by $\frac{\log \frac{n}{\epsilon}}{\theta}$ (see [21] Lemma $II.17$, page 116). By multiplying the number of outer iterations by the number of inner iterations, we get an upper bound for the total number of iterations, which is

$$\left\lceil 32 \log r \left(\frac{\log r + \log\left(1 + 2\sqrt{\bar{\Psi}_0}\right)}{\log r}\right)^2 \bar{\boldsymbol{\Psi}}_0^{\frac{1}{2}} \frac{1}{\theta} \log \frac{n}{\epsilon} \right\rceil, \text{ for large -update methods.}$$

For large-update methods, set $\tau = \mathcal{O}(n)$ and $\theta = \Theta(1)$. In special cases, by choosing

$$r = 1 + 2\left(\frac{n\theta + 2\tau + 2\sqrt{2n\tau}}{2(1-\theta)}\right)^{\frac{1}{2}}, \tag{38}$$

we get the so far best known LO case complexity, i.e. $\mathcal{O}\left(\sqrt{n}\log n \log \frac{n}{\varepsilon}\right)$, for large update primal dual interior point methods for SDO also.

5. Numerical results

In this section, we present some numerical results on some problems to confirm the effectiveness of our proposed function where the experiments were manipulated in **MATLAB (R2017a)** and run it on a PC. We take, the accuracy parameter $\epsilon = 10^{-8}$, a threshold parameter $\tau = 3$, barrier update $\theta \in \{0.15, 0.3, 0.5, 0.75, 0.9\}$, for each parametrized function, we choose the barrier parameter p, which satisfies the best complexity for large updates and the practical value for step size α_{pra} are given by $\alpha_{pra} = \rho \min(\alpha_X, \alpha_S)$ with $\rho \in (0,1)$ where

$$\alpha_X = \begin{cases} \dfrac{-1}{\lambda_{min}(X^{-1}\Delta X)} & \text{if } \lambda_{min}(X^{-1}\Delta X) < 0 \\ 1 & \text{else} \end{cases}$$

and

$$\alpha_S = \begin{cases} \dfrac{-1}{\lambda_{min}(S^{-1}\Delta S)} & \text{if } \lambda_{min}(S^{-1}\Delta S) < 0 \\ 1 & \text{else} \end{cases}$$

We assume that Iter and Cpu are used to represent the number of iterations and the time (seconds) produced by our algorithm, respectively. Our main goal is to compare iteration numbers and the calculation time of the algorithm for the following kernel functions.

i	The kernel function $\psi_i(z)$	Ref
r	$\frac{z^2-1}{2} - \int_1^z r^{(\frac{1}{w}-1)}dw, \quad r \geq e$	new
cl	$\frac{z^2-1}{2} - \log(z)$	[14]
1	$z^2 - 1 - \log(z) + \frac{z^{-p}-1}{p}, \quad p \geq 1$	[6]
2	$z^2 - z + \frac{z^{-p+1}-1}{p+1}, \quad p > 1$	[5]
3	$z^2 - 1 - \frac{z^{-2p+1}-1}{-2p+1} - \frac{z^{-p+1}-1}{-p+1}, \quad p > 1$	[3]
4	$(p+1)z^2 - \frac{1}{z^p} - (p+2)z, \quad p > 4$	[11]
5	$\frac{z^2-1-\log(z)}{2} + \frac{e^{\frac{1}{z^p}-1}-1}{2p}, \quad p \geq 1$	[10]

Table 1. Some kernel functions.

Problem 1
(Example 1, [22]) For this problem, we have
$b = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, C = \begin{pmatrix} -1 & -1 \\ -1 & -1 \end{pmatrix}, \quad A_1 = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$ and $A_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.
The procedure begins by generating initial strictly feasible primal and dual point solutions to this test problem.

$\psi(z) \setminus \theta$	0.15		0.3		0.5		0.75		0.9	
	Iter	Cpu	Iter	Cpu	Iter	Cpu	Iter	Cpu	Iter	Cpu
$\psi_r(z)$	118	0.312	56	0.321	29	0.256	16	0.222	11	0.199
$\psi_{cl}(z)$	124	0.450	67	0.440	36	0.405	24	0.401	18	0.391
$\psi_1(z)$	124	0.435	79	0.379	36	0.375	23	0.340	13	0.333
$\psi_2(z)$	124	0.434	79	0.417	45	0.377	29	0.338	17	0.310
$\psi_3(z)$	122	0.335	58	0.316	31	0.279	18	0.268	18	0.211
$\psi_4(z)$	120	0.239	56	0.337	34	0.312	19	0.294	18	0.292
$\psi_5(z)$	119	0.401	62	0.353	34	0.341	22	0.361	18	0.252

Table 2. NUMERICAL RESULTS FOR SOME KERNEL FUNCTIONS.

$$X^0 = \begin{pmatrix} 0.5 & 0 \\ 0 & 0.5 \end{pmatrix}, \quad y^0 = \begin{pmatrix} 0 \\ -3 \end{pmatrix} \text{ and } S^0 = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$$

The numerical outcomes are as follows

Problem 2
(problem, [25]) The primal-dual pair (SDO) and (SDD) are treated as the following data.

$$b = \begin{pmatrix} -2 & 2 & -2 \end{pmatrix}^T,$$

$$C = \begin{pmatrix} 3 & 3 & -3 & 1 & 1 \\ 3 & 5 & 3 & 1 & 2 \\ -3 & 3 & -1 & 1 & 2 \\ 1 & 1 & 1 & -3 & -1 \\ 1 & 2 & 2 & -1 & -1 \end{pmatrix}, \quad A_1 = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -2 & -1 \\ 0 & -1 & 1 & -1 & -2 \end{pmatrix},$$

$$A_2 = \begin{pmatrix} 0 & 0 & -2 & 2 & 0 \\ 0 & 2 & 1 & 0 & 2 \\ -2 & 1 & -2 & 0 & 1 \\ 2 & 0 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 & 2 \end{pmatrix} \quad \text{and} \quad A_3 = \begin{pmatrix} 2 & 2 & -1 & -1 & 1 \\ 2 & 0 & 2 & 1 & 1 \\ -1 & 2 & 0 & 1 & 0 \\ -1 & 1 & 1 & -2 & 0 \\ 1 & 1 & 0 & 0 & 2 \end{pmatrix}.$$

To get the optimal solution (X^*, y^*, S^*) for our problem, we use the feasible point for the primal and dual problems, which are

$$X^0 = I, \quad S^0 = I, \quad y^0 = \begin{pmatrix} 1 & 1 & 1 \end{pmatrix}^T,$$

respectively. The results are presented in the tables below.

$\psi(z) \setminus \theta$	0.15		0.3		0.5		0.75		0.9	
	Iter	Cpu	Iter	Cpu	Iter	Cpu	Iter	Cpu	Iter	Cpu
$\psi_r(z)$	121	0.308	57	0.272	31	0.266	16	0.229	11	0.188
$\psi_{cl}(z)$	126	0.400	60	0.389	35	0.366	19	0.355	15	0.278
$\psi_1(z)$	128	0.375	60	0.359	32	0.347	17	0.323	12	0.262
$\psi_2(z)$	127	0.414	60	0.407	37	0.401	19	0.378	13	0.372
$\psi_3(z)$	126	0.428	59	0.391	32	0.390	20	0.339	15	0.327
$\psi_4(z)$	128	0.354	61	0.352	32	0.298	17	0.268	13	0.259
$\psi_5(z)$	127	0.331	59	0.326	32	0.299	22	0.282	15	0.220

Table 3. NUMERICAL RESULTS FOR SOME KERNEL FUNCTIONS.

Problem 3
(Example 4, [22]) The information in this problem is as follows

$$n = 2m, m \in \{25, 50\}, \quad b = \begin{pmatrix} 2 & \dots & 2 \end{pmatrix}^T, \quad C = -I,$$

$$A_k(i,j) = \begin{cases} 1 & \text{if} \quad i = j = k \text{ or } i = j = k + m \\ 0 & \text{else} \end{cases},$$

$$X^0 = \begin{cases} 1.5 & \text{if} \quad i \leq j \\ 0.5 & \text{if} \quad i > j \end{cases}, \quad y^0 = \begin{pmatrix} -2 & \dots & -2 \end{pmatrix}^T \text{ and } S^0 = I$$

The numerical results are as follows

$\psi(z) \setminus \theta$	0.15		0.3		0.5		0.75		0.9	
	Iter	Cpu	Iter	Cpu	Iter	Cpu	Iter	Cpu	Iter	Cpu
					$m = 25$					
$\psi_r(z)$	147	4.086	61	2.976	32	2.391	18	1.692	11	1.382
$\psi_{cl}(z)$	162	4.176	75	3.249	39	2.593	20	2.403	13	2.074
$\psi_1(z)$	148	4.004	62	2.901	33	2.281	18	1.849	11	1.605
$\psi_2(z)$	150	4.289	64	3.169	34	2.507	19	2.104	11	2.052
$\psi_3(z)$	150	4.419	64	3.224	34	2.113	20	1.868	11	1.731
$\psi_4(z)$	150	4.486	64	3.024	34	2.640	19	2.114	11	2.054
$\psi_5(z)$	153	6.149	65	4.303	36	3.270	20	2.540	12	2.334
					$m = 50$					
$\psi_r(z)$	167	16.790	82	12.005	47	8.312	23	6.739	16	6.643
$\psi_{cl}(z)$	171	17.673	85	13.862	51	8.928	25	6.892	18	6.712
$\psi_1(z)$	168	16.711	83	12.209	48	8.495	24	6.803	16	6.670
$\psi_2(z)$	170	17.581	84	12.461	50	8.511	24	6.765	17	6.713
$\psi_3(z)$	170	16.956	84	12.671	50	8.898	24	7.129	17	6.743
$\psi_4(z)$	170	17.046	84	12.937	50	8.974	24	7.181	17	6.818
$\psi_5(z)$	170	21.070	83	12.701	49	8.695	23	8.356	16	6.446

Table 4. NUMERICAL RESULTS FOR $\psi_r(z)$ AND SOME KERNEL FUNCTIONS.

The numerical results shown in tables 3,2,4 demonstrate that the approach based on our novel kernel function $\psi_r(z)$ outperforms $\psi_{cl}(z)$ and $\psi_i(z)$ for $i = 1, \dots, 5$ us in terms of iterations and time. A few observations may be drawn from the above tables :

- With the values of r selected in (38) and the step size α_{pra}, our kernel function $\psi_r(z)$ gives the best results in terms of iterations and time taken in all circumstances, regardless of θ.
- The algorithm's iteration numbers are determined by the values of the parameter θ. For each θ to be close to 1, we achieve the fewest possible number of iterations in the shortest possible time.
- Even if the number of iterations is equal (see table 3 when $\theta = 0.3$ and table 4 when $\theta = 0.75, 0.9$ for $m = 25, 50$) our new function $\psi_r(z)$ always contributes to solving in the shortest time.

6. CONCLUSIONS AND VARIOUS PROPOSALS FOR FURTHER RESEARCH

This paper proposes the first bi-parameterized hyperbolic kernel function for semidefinite programming (1). We proved that the new kernel function is eligible by examining several properties. Based on the empirical findings, the kernel function being considered has significant potential in practical applications compared to other evaluated kernel functions. Finally, with a special value given to the parameter a, we obtain the complexity bound of the algorithm as $\mathcal{O}\left(\sqrt{n}\log n \log \frac{n}{\varepsilon}\right)$, which matches the best-known iteration bound for large-update IPMs so far.

The numerical results indicate that the new proposed kernel function exhibits promising performance in practice compared to other considered kernel functions. The results of the implemented numerical trials validate the use of our new kernel function.

Expanding this study to include linear and convex quadratic optimization problems, complementarity, and conic problems would be interesting.

Acknowledgement

The authors thank anonymous reviewers whose constructive comments and guidance significantly improved the quality of the presentation of this article.

REFERENCES

1. Y.Q. Bai, M. EL Ghami, C. Roos, A comparative study of kernel functions for primal-dual interior-point algorithms in linear optimization. SIAM J. Optim., **15** (**2**) : 101–128, 2004.
2. Y.Q. Bai, G.Q. Wang, C. Roos, Primal-dual interior-point algorithms for second-order cone optimization based on kernel functions. Nonlinear Analysis: Theory, Methods and Applications., **70** (**10**) : 3584–3602, 2009.
3. A. Benhadid, F. Merahi, Complexity analysis of an interior-point algorithm for linear optimization based on a new kernel function with a double barrier term. Numerical Algebra, Control and Optimization., **13** (**2**) : 224–238, 2023.
4. M. Bouafia, A. Yassine, An efficient twice parameterized trigonometric kernel function for linear optimization. Optim. Eng., **21** (**2**) : 651–672, 2020.
5. N. Boudjellal, H. Roumili, D. Benterki, Complexity Analysis of Interior Point Methods for Convex Quadratic Programming Based on a Parameterized Kernel Function, Bol. Soc. Paran. Mat, **4** : 1–16, 2022.
6. B. Bounibane, El.A. Djeffal, Kernel function-based interior-point algorithms for linear optimization, Int. J. Mathematical Modelling and Numerical Optimisation., **9** (**2**): 158–177, 2019.
7. Cai, X.Z., Li, L., El Ghami, M., T. Steihaug and G.Q. Wang. A new parametric kernel function yielding the best-known iteration bounds of interior-point methods for the Cartesian $P_*(\kappa)$-LCP over symmetric cones. Pacific Journal of Optimization, 13(4): 547-570, 2017.
8. X.Z. Cai, L. Wu, Y.J. Yue, M.M. Li, G.Q. Wang. Kernel-function based primaldual interior-point methods for convex quadratic optimization over symmetric cone. Journal of inequalities and applications, 308, 1-22, 2014.
9. E. De Klerk, Aspects of Semidefinite Programming. Applied Optimization, Kluwer Academic, Dordrecht. **65**, 2002.
10. L. Derbal, Z. Kebbiche, An Efficient Parameterized Logarithmic Kernel Function for Semidefinite Optimization, Acta Mathematicae Applicatae Sinica, English Series., **36** (**3**): 753–770, 2020.
11. El.A. Djeffal, M. Laouar, A primal-dual interior-point method based on a new kernel function for linear complementarity problem, Asian-European., **12** (**07**): 2050001, 2019.
12. S. Fathi-Hafshejani, Z. Moaberfard, A generic kernel function for interior point methods. Optim. Eng., **22** : 261–291, 2021.
13. S. Fathi-Hafshejani, H. Mansouri, M.R. Peyghami, and S. Chen, Primal-dual interior-point method for linear optimization based on a kernel function with trigonometric growth term. Optimization., **67** (**10**) : 1605–1630, 2018.
14. M. El Ghami, I.D. Ivanov, C. Roos, T. Steihaug, A polynomial-time algorithm for LO based on generalized logarithmic barrier functions. Int. J. Appl. Math., **21** : 99–115, 2008.
15. R.A. Horn, C.R. Johnson, Topics in matrix analysis. Cambridge University Press, Cambridge, (1991).
16. M. Li, M. Zhang, K. Huang, Z. Huang, A new primal-dual interior-point method for semidefinite optimization based on a parameterized kernel function. Optim. Eng., **22** : 293–319, 2021.
17. Yu. E. Nesterov, and M. J. Todd, Self-scaled barriers and interior-point methods for convex programming, Math. Oper. Res., **22** (**1**) : 1–42, 1997.
18. J. Peng, C. Roos, T. Terlaky, A new class of polynomial primal-dual methods for linear and semidefinite optimization. Eur. J. Oper. Res., **143** : 234–256, 2002.
19. J. Peng, C. Roos, T. Terlaky, Self-regular functions and new search directions for linear and semi-definite optimization. Mathematical Programming., **93** (**1**) : 129–171, 2002.
20. J. Peng, C. Roos, T. Terlaky, Self-Regularity. A New Paradigm for Primal-Dual Interior Point Algorithms, Princeton University Press, Princeton 2002.
21. C. Roos, T. Terlaky, J.Ph. Vial, Theory and Algorithms for Linear Optimization, An interior point approach. Wiley, Chichester, 1997.
22. I. Touil, D. Benterki, A. Yassine, A feasible primal-dual interior point method for linear semidefinite programming. J. Comput. Appl. Math., **312** : 216–230, 2017.
23. L. Vandenberghe, S. Boyd, Semidefinite programming. SIAM Review., **38** (**1**) : 49–95, 1996.
24. G.Q. Wang, Y.Q. Bai, A class of polynomial primal-dual interior-point algorithm for semidefinite optimization. J Shanghai Univ., **10** (**3**) : 198–207, 2006.
25. G.Q. Wang, Y.Q, Bai, C. Roos, Primal-dual interior point algorithms for semidefinite optimization based on a simple kernel function. Journal of Mathematical Modelling and Algorithms., **4** (**4**) : 409-433, 2005.
26. H. Wolkowicz, R. Saigal, L. Vadenberghe, Handbook of Semidefinite Programming, Theory, Algorithm, and Applications. Kluwer Academic Publishers, Dordrecht 2000.