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The Exponentiated-Gompertz-Marshall-Olkin-G Family of Distributions: Properties and Applications

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Abstract A new generalized family of distributions referred to as Exponentiated-Gompertz-Marshall-Olkin-G (EGom-MO-G) distribution is introduced. The distribution can be expressed as an infinite linear combination of the exponentiated-G family of distributions. Some mathematical properties are derived and studied. Several estimation techniques including maximum likelihood estimation, Cramér-von Mises, least squares estimation, weighted least squares, Anderson-Darling and right-tail Anderson-Darling methods are compared. A special case of the new family of distributions is adopted for application to two real data sets and compared to some existing models. Results revealed that the new family of distributions is superior than compared models.

Keywords Exponentiated-G, Gompertz-G, Marshall-Olkin-G, Maximum Likelihood Estimation, Simulations

Mathematics Subject Classifications 62E99; 60E05

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1. Introduction

Generalized distributions with additional parameters have proved to be attractive and more flexible than baseline distributions in the analysis of real life data from various disciplines such as reliability, engineering, medicine, economics, finance and insurance.. This flexibility allows for these distributions with additional parameters to handle data that exhibits a variety of levels of skewness and kurtosis. In addition, they are more flexible in fitting data that demonstrates varying shapes for the hazard rate functions. Some existing generalized families of distributions include among many; the beta-G distribution by Eugene et al. [26], Marshall-Olkin-G (MO-G) distribution by Marshall & Olkin [42], Exponentiated Generalized Marshall-Olkin distribution by Handique et al. [32], Kumaraswamy-G distribution by Nadarajah et al. [44], and Topp-Leone odd log-logistic-G distribution by Brito et al. [17].

Marshall and Olkin [42] defines a way of including a parameter to a baseline distribution as follows. Let $G(x;\xi)$ be a baseline cumulative distribution (cdf) with parameter vector ξ, then the Marshall-Olkin-G family of distributions has the cumulative distribution function (cdf) and probability density function (pdf) given by

$$F_{MO}(x;\delta,\xi) = 1 - \frac{\delta\bar{G}(x;\xi)}{1-\bar{\delta}\bar{G}(x;\xi)}, \tag{1}$$

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and

$$f_{MO}(x;\delta,\xi)=\frac{\delta g(x;\xi)}{[1-\bar{\delta}\bar{G}(x;\xi)]^2}, \tag{2}$$

respectively, for $\delta>0$, $\bar{\delta}=1-\delta$, and parameter vector ξ. An important property of the Marshall-Olkin-G (MO-G) family of distributions is their flexibility, which has rendered it a great substitute to the Weibull and gamma distributions (Marshall and Olkin[42]). In addition, the MO-G family of distributions has been said to model data that are characterized by heavy-tails, heavily-skewed, flat and peaked distributions, than classical distributions.

Some existing Marshall-Olkin generalized distributions comprise Marshall-Olkin Frechét distribution by Krisha et al. [40], Marshall-Olkin Odd Exponential Half Logistic-G family of distributions by Oluyede & Chipepa [49], Marshall-Olkin extended uniform distribution by Jose [37], Generalised Marshall-Olkin-Kumaraswamy-G family of distributions by Chakraborty & Handique [15], Marshall–Olkin extended Lomax distribution by Ghitany et al. [27], Marshall-Olkin extended Weibull distribution by Cordeiro & Lemonte [19], Marshall-Olkin extended Burr Type XII distribution by Al-Saiari et al. [5], Marshall-Olkin Log-logistic distribution by Gui [30], Marshall-Olkin alpha power family of distributions by Nasser et al. [46], Marshall-Olkin gamma Weibull distribution by Saboor & Pogány [52], Marshall-Olkin-Topp-Leone-Gompertz-G family of distributions by Oluyede et al. [48], Marshall-Olkin Chris-Jerry distribution by Obulezi et al. [47], Marshall-Olkin Pranav distribution by Alsultan [6], and generalized Marshall-Olkin inverse Lindley distribution by Bantan et al. [8].

Gompertz [29] developed the Gompertz distribution. This distribution is popular for fitting tumor gowth, model human mortality and fertility data, as well as fitting actuarial data. Notwithstanding these merits, Gompertz distribution has been reported to exhibit an increasing failure rate. However, the generalized Gompertz model has been reported to exhibit increasing or decreasing or constant failure rate subject to the shape parameter (El-Gohony et al. [24]). The cdf and pdf of the Gompertz-G (Gom-G) family of distributions (Alizadeh et al. [4]) are given as:

$$F_{Gom-G}(x;\lambda,\gamma,\xi)=1-exp\left(\frac{\lambda}{\gamma}\left[1-(1-G(x;\xi))^{-\gamma}\right]\right), \tag{3}$$

and

$$f_{Gom-G}(x;\lambda,\gamma,\xi)=\lambda g(x;\xi)\left[1-G(x;\xi)\right]^{-\gamma-1}exp\left(\frac{\lambda}{\gamma}\left[1-(1-G(x;\xi))^{-\gamma}\right]\right), \tag{4}$$

respectively, for λ, $\gamma>0$, where ξ is a parameter vector from the baseline distribution. Generalizations of the Gompertz distributions include Marshall-Olkin extended generalized Gompertz distribution by Benkhelifa [12], beta generalized Gompertz distribution by Benkhelifa [11], generalized gamma-generalized Gompertz distribution by Bosi et al. [14], Type II exponentiated half-logistic Gompertz-G family of distributions by Moakofi and Oluyede [43], type II exponentiated half logistic-Gompertz-G power series class of distributions by Chamunorwa et al. [16], Topp-Leone-Gompertz exponentiated half logistic-G family of distributions by Dingalo et al. [22], and bivariate generalized Weibull Gompertz distribution by Bassiouny et al. [10].

Gupta et al. [31] presented results on exponentiated distributions and Lehman alternatives. Some of the generalized exponentiated distributions include among others the work by Ali et al. [3], exponentiated Weibull distribution by Nadarajah et al. [45], exponentiated generalized inverse Gaussian distribution by Lemonte and Cordeiro [41], exponentiated generalized alpha power family of distributions by Elsherpieny and Almetwally [25], exponentiated half-logistic odd Lindley-G family of distributions by Sengweni et al. [54], exponentiated power alpha index generalized family of distributions by Hussain et al. [34], exponentiated odd Lindley-X family by Abd El-Bar and Maria do Carmo [1], and exponentiated generalized-G Poisson family of distributions by Aryal and Yousof [7].

This paper aims to develop a new generalized family of distributions, explore some mathematical properties and its applications to real data. We are motivated to develop this new family of distributions because of its monotone and non-monotone hazard rate function which are widely applied in modeling data from different areas including insurance risk premiums, and credit risk portfolios. Furthermore, we seek to improve density shapes of some specified baseline distributions as well as achieve versatile skewness and kurtosis for these baseline distributions. The new family of distribution can assume almost symmetric, reverse-J, left and right-skewed shapes showing that it can be fitted to a wide range of real data including heavy-tailed data.

This paper is organised in the following manner. Section 2 presents the new family of distributions called the exponentiated-Gompertz-Marshall Olkin-G (EGom-MO-G) distribution and its sub-families. Sections 3 and 4 presents special models and mathematical properties, respectively. Estimation methods, simulation results and applications are presented in Sections 5, 7 and 8, respectively.

2. The New Generalized Family of Distributions

The section presents the new generalized family of distributions and its sub-families of distributions. In this note, we set $\lambda = 1$ in the Gom-G family of distributions to avoid overparameterization and redundancy. The cdf, pdf and hazard rate function (hrf) of the EGom-MO-G family of distributions are obtained using equations (1), (2), (3) and (4).The resulting cdf, pdf and hazard rate function (hrf) are given by:

$$F(x;\alpha,\gamma,\delta,\xi) = \left[1 - exp\left(\frac{1}{\gamma}\left[1 - \left(\frac{\delta\bar{G}(x;\xi)}{1-\bar{\delta}\bar{G}(x;\xi)}\right)^{-\gamma}\right]\right)\right]^{\alpha}, \tag{5}$$

$$\begin{aligned} f(x;\alpha,\gamma,\delta,\xi) &= \frac{\alpha\delta g(x;\xi)}{[1-\bar{\delta}\bar{G}(x;\xi)]^2}\left[\frac{\delta\bar{G}(x;\xi)}{1-\bar{\delta}\bar{G}(x;\xi)}\right]^{-\gamma-1} \\ &\times \; exp\left(\frac{1}{\gamma}\left[1 - \left(\frac{\delta\bar{G}(x;\xi)}{1-\bar{\delta}\bar{G}(x;\xi)}\right)^{-\gamma}\right]\right) \\ &\times \; \left[1 - exp\left(\frac{1}{\gamma}\left[1 - \left(\frac{\delta\bar{G}(x;\xi)}{1-\bar{\delta}\bar{G}(x;\xi)}\right)^{-\gamma}\right]\right)\right]^{\alpha-1}, \end{aligned} \tag{6}$$

and

$$\begin{aligned} h(x;\alpha,\gamma,\delta,\xi) &= \frac{\alpha\delta g(x;\xi)}{\left[1-\bar{\delta}\bar{G}(x;\xi)\right]^2}\left[\frac{\delta\bar{G}(x;\xi)}{1-\bar{\delta}\bar{G}(x;\xi)}\right]^{-\gamma-1} \\ &\times \; exp\left(\frac{1}{\gamma}\left[1 - \left(\frac{\delta\bar{G}(x;\xi)}{1-\bar{\delta}\bar{G}(x;\xi)}\right)^{-\gamma}\right]\right) \\ &\times \; \left[1 - exp\left(\frac{1}{\gamma}\left[1 - \left(\frac{\delta\bar{G}(x;\xi)}{1-\bar{\delta}\bar{G}(x;\xi)}\right)^{-\gamma}\right]\right)\right]^{\alpha-1} \\ &\times \; \left(1 - \left[1 - exp\left(\frac{1}{\gamma}\left[1 - \left(\frac{\delta\bar{G}(x;\xi)}{1-\bar{\delta}\bar{G}(x;\xi)}\right)^{-\gamma}\right]\right)\right]^{\alpha}\right)^{-1}, \end{aligned} \tag{7}$$

for $\alpha, \gamma, \delta > 0, \bar{\delta} = 1 - \delta$ and baseline parameter vector ξ, where $\bar{G}(x;\xi) = 1 - G(x;\xi)$. Note that γ, α and δ are shape parameters.

2.1. Sub-Families

In this section, sub-families of the EGom-MO-G family of distributions are presented.

- When $\alpha = 1$, we attain the Gompertz-Marshall-Olkin-G (Gom-MO-G) family of distributions with cdf given by

  $F(x;\gamma,\delta,\xi) = 1 - exp\left(\frac{1}{\gamma}\left[1 - \left(\frac{\delta\bar{G}(x;\xi)}{1-\bar{\delta}\bar{G}(x;\xi)}\right)^{-\gamma}\right]\right),$

  for $\gamma, \delta > 0$ and parameter vector ξ. This family of distributions is new.

- When $\gamma = 1$, we have a new family of distributions with the following cdf

  $F(x;\alpha,\delta,\xi) = \left[1 - exp\left(1 - \left[\frac{\delta\bar{G}(x;\xi)}{1-\bar{\delta}\bar{G}(x;\xi)}\right]^{-1}\right)\right]^{\alpha},$

  for $\alpha, \delta > 0$ and parameter vector ξ.

- When $\delta = 1$, we have the exponentiated Gompertz-G (EGom-G) family of distributions with the cdf

  $F(x;\alpha,\gamma,\xi) = \left[1 - exp\left(\frac{1}{\gamma}\left[1 - \left(\bar{G}(x;\xi)\right)^{-\gamma}\right]\right)\right]^{\alpha},$
  for $\gamma, \alpha > 0$ and parameter vector ξ.

- When $\alpha = \gamma = 1$, another family of distributions is obtained with the following cdf

  $F(x;\delta,\xi) = 1 - exp\left[1 - \left(\frac{\delta\bar{G}(x;\xi)}{1-\bar{\delta}\bar{G}(x;\xi)}\right)^{-1}\right],$

  for $\delta > 0$ and parameter vector ξ.

- When $\alpha = \delta = 1$, the Gom-G family of distributions (Alizadeh et al. [4]) is obtained with the cdf

  $F(x;\gamma,\xi) = 1 - exp\left(\frac{1}{\gamma}\left[1 - \left(\bar{G}(x;\xi)\right)^{-\gamma}\right]\right),$

  for $\gamma > 0$ and parameter vector ξ.

- When $\gamma = \delta = 1$, we have another family of distributions with the following cdf

  $F(x;\alpha;\xi) = \left[1 - exp\left(1 - [1 - G(x;\xi)]^{-1}\right)\right]^{\alpha},$

  for $\alpha > 0$ and parameter vector ξ.

- A reduced family of distributions is obtained when $\alpha = \delta = \gamma = 1$ and it is given by the cdf

  $$F(x;\xi) = 1 - exp\left(1 - [1 - G(x;\xi)]^{-1}\right) = 1 - exp\left[-\left(\frac{G(x;\xi)}{1 - G(x;\xi)}\right)\right],$$

  for parameter vector ξ. This is a unit odd exponential distribution.

3. Some Special Cases

Some special models of the EGom-MO-G family of distributions are given in this section.

3.1. Exponentiated Gompertz-Marshall-Olkin-Log-Logistic (EGom-MO-LLog) Distribution

Suppose the baseline distribution is a log-logistic distribution with cdf and pdf given by $G(x;\lambda) = 1 - (1 + x^{\lambda})^{-1}$ and $g(x;\lambda) = \lambda x^{\lambda-1}(1 + x^{-\lambda})^{-2}$, respectively for $\lambda > 0$ and $x > 0$. The cdf, pdf and hrf of the EGom-MO-LLog distribution are given as

$$F(x;\alpha,\gamma,\delta,\lambda) = \left[1 - exp\left(\frac{1}{\gamma}\left[1 - \left(\frac{\delta(1+x^{\lambda})^{-1}}{1-\bar{\delta}(1+x^{\lambda})^{-1}}\right)^{-\gamma}\right]\right)\right]^{\alpha},$$

$$\begin{aligned} f(x;\alpha,\gamma,\delta,\lambda) &= \frac{\alpha\delta[\lambda x^{\lambda-1}(1+x^{\lambda})^{-2}]}{[1-(1+x^{\lambda})^{-1}]^2} \times \left(\frac{\delta(1+x^{\lambda})^{-1}}{1-\bar{\delta}(1+x^{\lambda})^{-1}}\right)^{-\gamma-1} \\ &\times \; exp\left(\frac{1}{\gamma}\left[1 - \left(\frac{\delta(1+x^{\lambda})^{-1}}{1-\bar{\delta}(1+x^{\lambda})^{-1}}\right)^{\gamma}\right]\right) \\ &\times \; \left[1 - exp\left(\frac{1}{\gamma}\left[1 - \left(\frac{\delta(1+x^{\lambda})^{-1}}{1-\bar{\delta}(1+x^{\lambda})^{-1}}\right)^{\gamma}\right]\right)\right]^{\alpha-1}, \end{aligned}$$

and

$$\begin{aligned} h(x;\alpha,\gamma,\delta,\lambda) &= \frac{\alpha\delta[\lambda x^{\lambda-1}(1+x^{\lambda})^{-2}]}{[1-(1+x^{\lambda})^{-1}]^2} \times \left(\frac{\delta(1+x^{\lambda})^{-1}}{1-\bar{\delta}(1+x^{\lambda})^{-1}}\right)^{-\gamma-1} \\ &\times \; exp\left(\frac{1}{\gamma}\left[1 - \left(\frac{\delta(1+x^{\lambda})^{-1}}{1-\bar{\delta}(1+x^{\lambda})^{-1}}\right)^{\gamma}\right]\right) \\ &\times \; \left[1 - exp\left(\frac{1}{\gamma}\left[1 - \left(\frac{\delta(1+x^{\lambda})^{-1}}{1-\bar{\delta}(1+x^{\lambda})^{-1}}\right)^{\gamma}\right]\right)\right]^{\alpha-1} \\ &\times \; \left(1 - \left[1 - exp\left(\frac{1}{\gamma}\left[1 - \left(\frac{\delta(1+x^{\lambda})^{-1}}{1-\bar{\delta}(1+x^{\lambda})^{-1}}\right)^{\gamma}\right]\right)\right]^{\alpha}\right)^{-1}, \end{aligned}$$

respectively, for $\alpha, \gamma, \delta, \lambda > 0$.

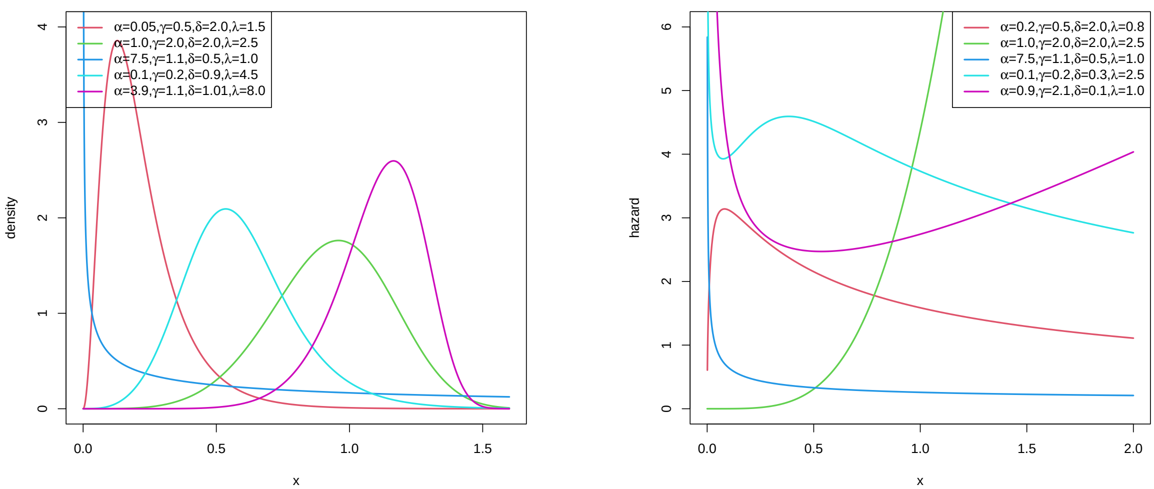

Figure 1. Density Function and Hazard Rate Function plots for the EGom-MO-LLog Distribution

Figure 1 shows that the pdf of the EGom-MO-LLog distribution can assume almost symmetric, reverse-J, left-skewed and right-skewed shapes. The hrf on the other hand can assume bathtub followed by upside down bathtub, bathtub, upside down bathtub, decreasing and increasing shapes.

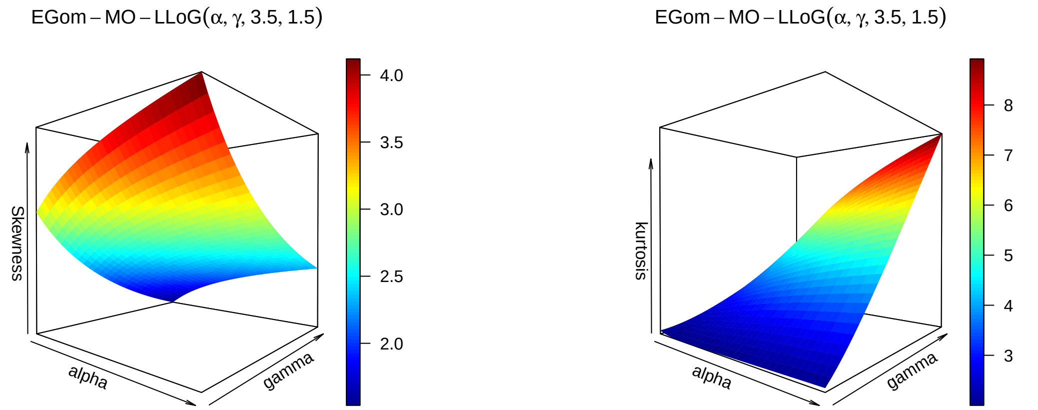

Figure 2. Skewness and Kurtosis plots for EGom-MO-LLog Distribution

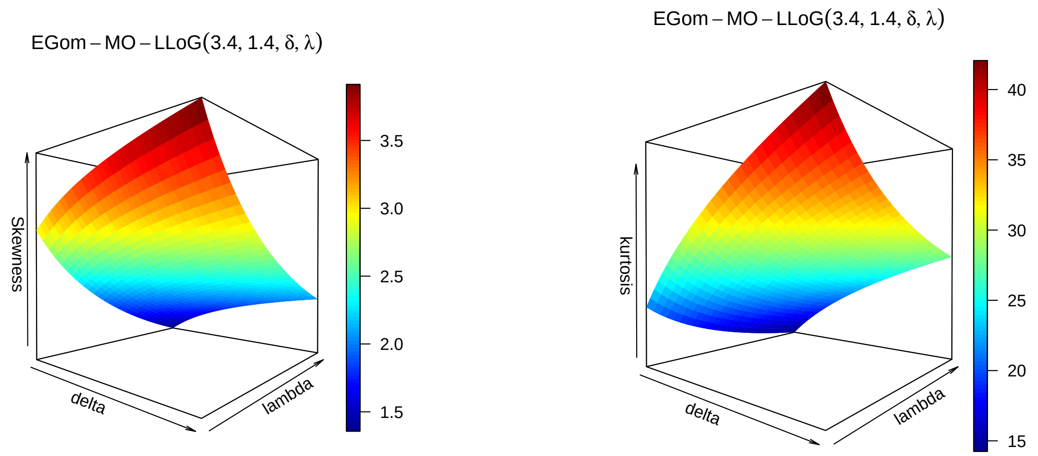

Figure 3. Skewness and Kurtosis plots for EGom-MO-LLog Distribution

We notice from Figures 2 and 3 that when we fix the parameters δ and λ the kurtosis and skewness of the EGom-MO-LLog distribution increases as α and γ increases. Furthermore, when we fix α and γ, the kurtosis and

skewness of the EGom-MO-LLog distribution increases as δ and λ increases.
Table 1 and 2 below presents the quantile values and moments for the EGom-MO-LLog distribution, respectively obtained using equation (8).

Table 1. Quantiles for the EGom-MO-LLog Distribution

	$(\alpha, \gamma, \delta, \lambda)$				
u	(1,1.5,1.3,3.5)	(1.7,1.1,1.8,2.5)	(1,0.5,2.5,4.5)	(2,0.3,2.9,5.4)	(9.9,1,1.5,6.5)
0.1	0.5614	0.7770	0.7484	1.0445	1.1407
0.2	0.6918	0.9436	0.8897	1.1491	1.1738
0.3	0.7847	1.0708	0.9935	1.2261	1.1996
0.4	0.8630	1.1858	1.0830	1.2961	1.2207
0.5	0.9334	1.2879	1.1710	1.3650	1.2397
0.6	1.0050	1.4008	1.2576	1.4343	1.2599
0.7	1.0752	1.5168	1.3552	1.5161	1.2820
0.8	1.1528	1.6555	1.4687	1.6140	1.3065
0.9	1.2541	1.8541	1.6331	1.7617	1.3416

Table 2. Moments for the EGom-MO-LLog Distribution

	$(\alpha, \gamma, \delta, \lambda)$				
	(2,1.5,1.3,3.5)	(2.7,1.9,1.8,2.5)	(3.4,2.5,2.5,4.5)	(4,1.3,2.9,5.4)	(3.3,1,1.5,4.5)
E(X)	1.0716	1.2728	1.2005	1.3187	1.2248
$E(X^2)$	1.1912	1.6899	1.4539	1.7547	1.5290
$E(X^3)$	1.3666	2.3273	1.7753	2.3553	1.9430
$E(X^4)$	1.6118	3.3098	2.1844	3.1880	2.5114
$E(X^5)$	1.9483	4.8447	2.7075	4.3497	3.2986
$E(X^6)$	2.4079	7.2786	3.3789	5.9809	4.3993
SD	0.2071	0.2645	0.1127	0.1259	0.1696
CV	0.1932	0.2078	0.0939	0.0955	0.1384
CS	-0.2045	-0.0921	-0.4338	-0.1693	-0.0195
CK	2.9496	2.9140	3.3620	3.0726	2.9954

3.2. Exponentiated Gompertz-Marshall-Olkin-Weibull (EGom-MO-W) Distribution

Suppose the baseline distribution is a Weibull distribution with the cdf and pdf given by $G(x;\lambda) = 1 - e^{-x^{\lambda}}$ and $g(x;\lambda) = \lambda x^{\lambda-1} e^{-x^{\lambda}}$, respectively, for $\lambda > 0$, then the cdf, pdf and hrf of the EGom-MO-W distribution are obtained as

$$F(x;\alpha,\gamma,\delta,\lambda) = \left[1 - exp\left(\frac{1}{\gamma}\left[1 - \left(\frac{\delta e^{-x^{\lambda}}}{1-\bar{\delta}e^{-x^{\lambda}}}\right)^{-\gamma}\right]\right)\right]^{\alpha},$$

$$\begin{aligned} f(x;\alpha,\gamma,\delta,\lambda) &= \frac{\alpha\delta[\lambda x^{\lambda-1}e^{-x^{\lambda}}]}{[1-\bar{\delta}e^{-x^{\lambda}}]^2}\left(\frac{\delta e^{-x^{\lambda}}}{1-\bar{\delta}e^{-x^{\lambda}}}\right)^{-\gamma-1} exp\left(\frac{1}{\gamma}\left[1 - \left(\frac{\delta e^{-x^{\lambda}}}{1-\bar{\delta}e^{-x^{\lambda}}}\right)^{-\gamma}\right]\right) \\ &\times \left[1 - exp\left(\frac{1}{\gamma}\left[1 - \left(\frac{\delta e^{-x^{\lambda}}}{1-\bar{\delta}e^{-x^{\lambda}}}\right)^{-\gamma}\right]\right)\right]^{\alpha-1}, \end{aligned}$$

and

$$
\begin{aligned}
h(x;\alpha,\gamma,\delta,\lambda) &= \frac{\alpha\delta[\lambda x^{\lambda-1}e^{-x^{\lambda}}]}{[1-\bar{\delta}e^{-x^{\lambda}}]^2}\left(\frac{\delta e^{-x^{\lambda}}}{1-\bar{\delta}e^{-x^{\lambda}}}\right)^{-\gamma-1} exp\left(\frac{1}{\gamma}\left[1-\left(\frac{\delta e^{-x^{\lambda}}}{1-\bar{\delta}e^{-x^{\lambda}}}\right)^{-\gamma}\right]\right)\\
&\times \left[1-exp\left(\frac{1}{\gamma}\left[1-\left(\frac{\delta e^{-x^{\lambda}}}{1-\bar{\delta}e^{-x^{\lambda}}}\right)^{-\gamma}\right]\right)\right]^{\alpha-1}\\
&\times \left(1-\left[1-exp\left(\frac{1}{\gamma}\left[1-\left(\frac{\delta e^{-x^{\lambda}}}{1-\bar{\delta}e^{-x^{\lambda}}}\right)^{-\gamma}\right]\right)\right]^{\alpha}\right)^{-1},
\end{aligned}
$$

respectively, for $\alpha,\gamma,\delta,\lambda > 0$.

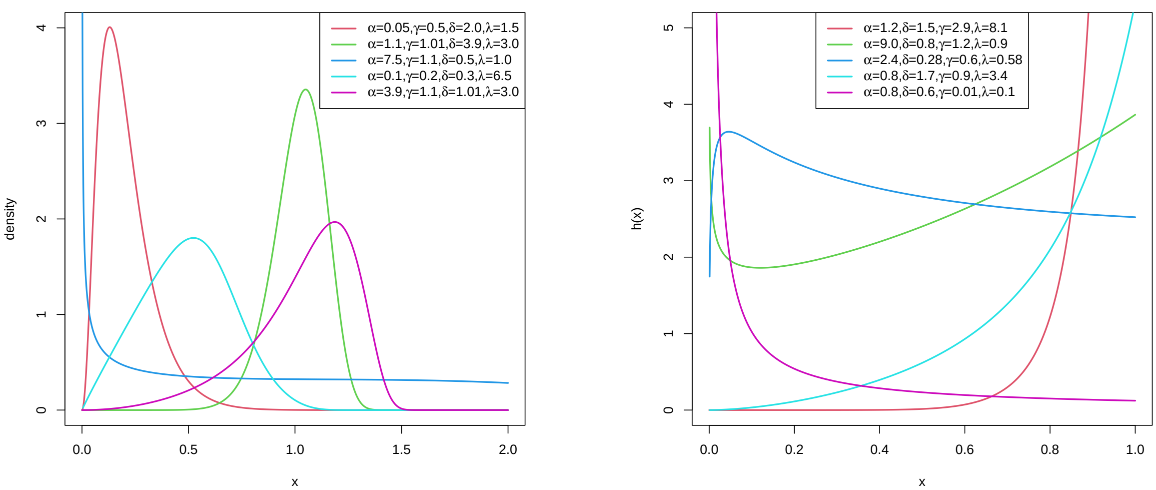

Figure 4. Plot of the Density Function and Hazard Rate Function for the EGom-MO-Weibull Distribution

We observe from Figure 4 that the pdf of the EGom-MO-W distribution takes shapes such as reverse-J, almost symmetric, negatively-skewed and positively-skewed, while the hrf can assume bathtub, upside down bathtub, increasing and decreasing shapes.

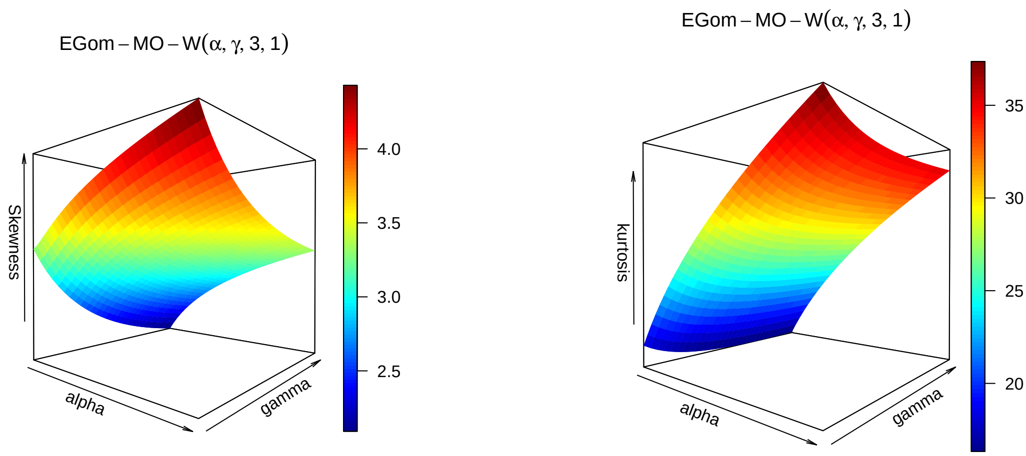

Figure 5. Skewness and Kurtosis plots for EGom-MO-Weibull Distribution

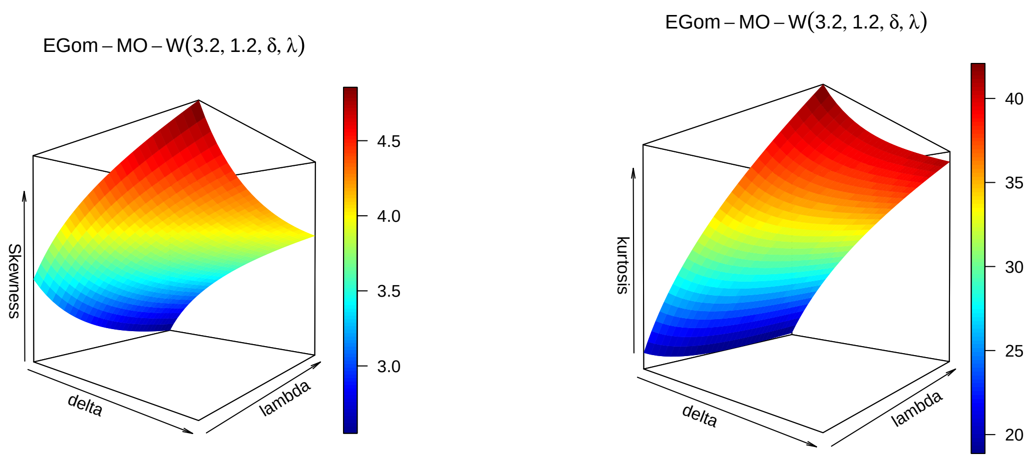

Figure 6. Skewness and Kurtosis plots for EGom-MO-Weibull Distribution

We observe from Figures 5 and 6 that when we fix the parameters δ and λ the skewness and kurtosis of the EGom-MO-W distribution increases as α and γ increases. Furthermore, when we fix α and γ, the skewness and kurtosis of the EGom-MO-W distribution increases as δ and λ increases.

Table 5 and 6 below presents the quantile values and moments for the EGom-MO-W distribution, respectively.

Table 3. Quantiles for the EGom-MO-W Distribution

	$(\alpha, \gamma, \delta, \lambda)$				
u	(1,1.5,1.3,3.5)	(1.7,1.1,1.8,2.5)	(1,1.2,2.5,0.5)	(2,1.8,2.9,0.4)	(9.9,1.6,1.5,0.5)
0.1	0.5522	0.7127	0.0550	0.3815	1.0539
0.2	0.6685	0.8274	0.1896	0.7526	1.2703
0.3	0.7459	0.9068	0.3869	1.1200	1.4417
0.4	0.8037	0.9709	0.6386	1.4981	1.6001
0.5	0.8588	1.0234	0.9407	1.9012	1.7541
0.6	0.9031	1.0750	1.3075	2.3487	1.9166
0.7	0.9476	1.1250	1.7575	2.8651	2.0994
0.8	0.9925	1.1802	2.3459	3.5171	2.3275
0.9	1.0447	1.2470	3.2362	4.4937	2.6658

Table 4. Moments for the EGom-MO-W Distribution

	$(\gamma, \alpha, \delta, \lambda)$				
	(2,1.5,0.5,1.9)	(1.3,1.2,0.6,2.3)	(2.1,2.9,2.7,1.8)	(1.1,1.5,2.5,1.6)	(2.1,1.5,1.7,1.9)
E(X)	0.6170	0.6621	1.0089	0.9500	0.9859
$E(X^2)$	0.4112	0.4827	1.0527	1.0264	1.0198
$E(X^3)$	0.2905	0.3763	1.1273	1.1960	1.0946
$E(X^4)$	0.2151	0.3087	1.2331	1.4687	1.2103
$E(X^5)$	0.1655	0.2637	1.3729	1.8762	1.3719
$E(X^6)$	0.1316	0.2328	1.5523	2.4738	1.5885
SD	0.1746	0.2106	0.1866	0.3520	0.2188
CV	0.2830	0.3180	0.1850	0.3706	0.2220
CS	-0.1535	-0.2038	-0.7716	-0.3308	-0.5011
CK	2.6963	2.5703	3.7137	2.4808	3.0907

3.3. Exponentiated Gompertz-Marshall-Olkin-Standard half logistic (EGom-MO-SHL) Distribution

Suppose the baseline distribution is a standard half logistic distribution with the cdf and pdf given by $G(x) = \frac{1-e^{-x}}{1+e^{-x}}$ and $g(x) = \frac{2e^{-x}}{(1+e^{-x})^2}$, for $x > 0$. Then the EGom-MO-SHL cdf, pdf and hrf are given as follows:

$$F(x;\alpha,\gamma,\delta) = \left[1 - exp\left(\frac{1}{\gamma}\left[1 - \left(\frac{\delta\left[1-\left(\frac{1-e^{-x}}{1+e^{-x}}\right)\right]}{1-\bar{\delta}\left[1-\left(\frac{1-e^{-x}}{1+e^{-x}}\right)\right]}\right)^{-\gamma}\right]\right)\right]^{\alpha},$$

$$
\begin{aligned}
f(x;\alpha,\gamma,\delta) &= \frac{\alpha\delta\frac{2e^{-x}}{(1+e^{-x})^2}}{\left[1-\bar{\delta}\left[1-\left(\frac{1-e^{-x}}{1+e^{-x}}\right)\right]\right]^2}\left(\frac{\delta\left[1-\left(\frac{1-e^{-x}}{1+e^{-x}}\right)\right]}{1-\bar{\delta}\left[1-\left(\frac{1-e^{-x}}{1+e^{-x}}\right)\right]}\right)^{-(\gamma+1)} \\
&\times\ exp\left(\frac{1}{\gamma}\left[1-\left(\frac{\delta\left[1-\left(\frac{1-e^{-x}}{1+e^{-x}}\right)\right]}{1-\bar{\delta}\left[1-\left(\frac{1-e^{-x}}{1+e^{-x}}\right)\right]}\right)^{-\gamma}\right]\right) \\
&\times \left[1-exp\left(\frac{1}{\gamma}\left[1-\left(\frac{\delta\left[1-\left(\frac{1-e^{-x}}{1+e^{-x}}\right)\right]}{1-\bar{\delta}\left[1-\left(\frac{1-e^{-x}}{1+e^{-x}}\right)\right]}\right)^{-\gamma}\right]\right)\right]^{\alpha-1},
\end{aligned}
$$

and

$$
\begin{aligned}
h(x;\alpha,\gamma,\delta) &= \frac{\alpha\delta\frac{2e^{-x}}{(1+e^{-x})^2}}{\left[1-\bar{\delta}\left[1-\left(\frac{1-e^{-x}}{1+e^{-x}}\right)\right]\right]^2}\left(\frac{\delta\left[1-\left(\frac{1-e^{-x}}{1+e^{-x}}\right)\right]}{1-\bar{\delta}\left[1-\left(\frac{1-e^{-x}}{1+e^{-x}}\right)\right]}\right)^{-(\gamma+1)} \\
&\times\ exp\left(\frac{1}{\gamma}\left[1-\left(\frac{\delta\left[1-\left(\frac{1-e^{-x}}{1+e^{-x}}\right)\right]}{1-\bar{\delta}\left[1-\left(\frac{1-e^{-x}}{1+e^{-x}}\right)\right]}\right)^{-\gamma}\right]\right) \\
&\times \left[1-exp\left(\frac{1}{\gamma}\left[1-\left(\frac{\delta\left[1-\left(\frac{1-e^{-x}}{1+e^{-x}}\right)\right]}{1-\bar{\delta}\left[1-\left(\frac{1-e^{-x}}{1+e^{-x}}\right)\right]}\right)^{-\gamma}\right]\right)\right]^{\alpha-1} \\
&\times \left(1-\left[1-exp\left(\frac{1}{\gamma}\left[1-\left(\frac{\delta\left[1-\left(\frac{1-e^{-x}}{1+e^{-x}}\right)\right]}{1-\bar{\delta}\left[1-\left(\frac{1-e^{-x}}{1+e^{-x}}\right)\right]}\right)^{-\gamma}\right]\right)\right]^{\alpha}\right)^{-1},
\end{aligned}
$$

respectively, for $\alpha,\gamma,\delta > 0$.

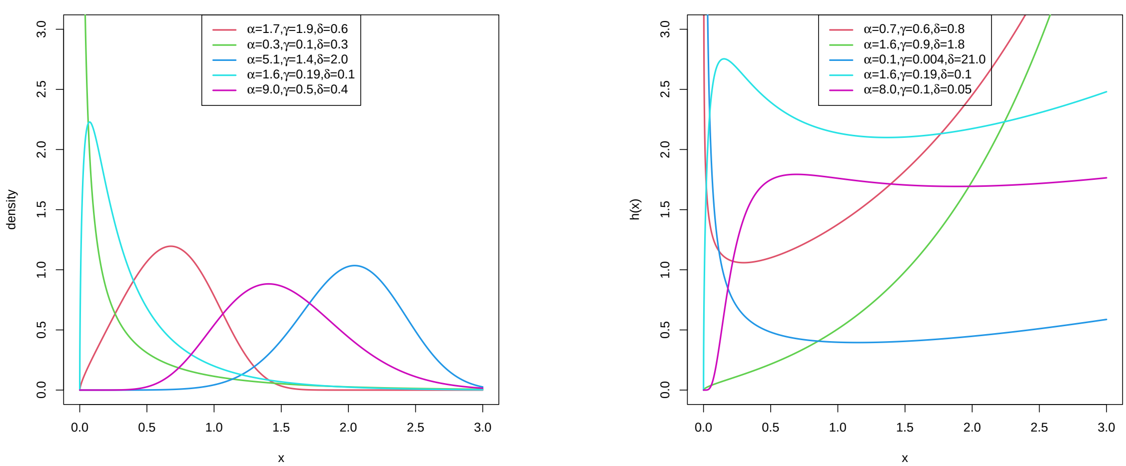

Figure 7. Density Function and Hazard Rate Function plots for EGom-MO-SHL Distribution

We observe from Figure 7, that the pdf of the EGom-MO-SHL distribution can take reverse-J, almost symmetric, positively-skewed and negatively-skewed shapes, while the hrf can take increasing, bathtub, decreasing, and upside down bathtub followed by bathtub shapes.

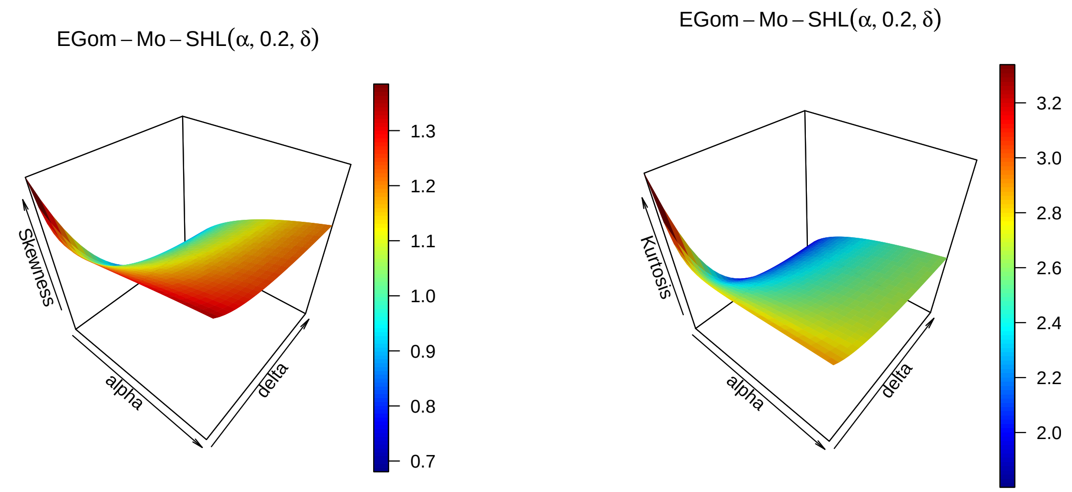

Figure 8. Plots of Skewness and Kurtosis for EGom-MO-SHL Distribution

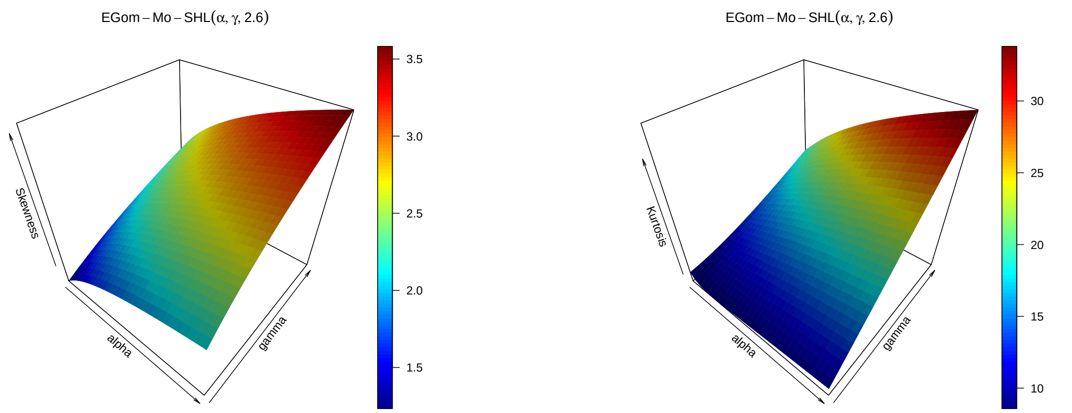

Figure 9. Plots of Skewness and Kurtosis for EGom-MO-SHL Distribution

Figures 8 and 9 shows that when we fix γ, the skewness and kurtosis of the EGom-MO-SHL increases and decreases then increases again as α and δ increases. Furthermore, when we fix δ the skewness and kurtosis of the EGom-MO-SHL distribution increases as α and γ increases.

Table 5. Quantiles for the EGom-MO-SHL Distribution

	(α, γ, δ)				
u	(1,1.5,1.3,3.5)	(1.7,1.1,1.8,2.5)	(1,1.2,2.5,0.5)	(2,1.8,2.9,0.4)	(9.9,1.6,1.5,0.5)
0.1	0.2387	0.7213	0.4193	1.0834	1.5226
0.2	0.4380	1.0050	0.7387	1.3572	1.6403
0.3	0.6220	1.2187	1.0050	1.5484	1.7285
0.4	0.7873	1.3959	1.2380	1.7013	1.8075
0.5	0.9484	1.5659	1.4545	1.8372	1.8739
0.6	1.1082	1.7289	1.6612	1.9692	1.9452
0.7	1.2732	1.9001	1.8754	2.0998	2.0187
0.8	1.4572	2.0885	2.1091	2.2481	2.1040
0.9	1.6959	2.3387	2.4071	2.4306	2.2254

Table 6. Moments for the EGom-MO-W Distribution

	$(\alpha, \gamma, \delta,)$				
	(2,1.5,1.3)	(0.7,1.9,1.8)	(0.4,2.5,2.5)	(1,1.3,2.9)	(1.3,1,1.5)
E(X)	0.1998	0.2655	0.2306	0.1496	0.2042
$E(X^2)$	0.1544	0.1769	0.1450	0.1056	0.1466
$E(X^3)$	0.1259	0.1331	0.1065	0.0819	0.1144
$E(X^4)$	0.1062	0.1068	0.0843	0.0670	0.0938
$E(X^5)$	0.0918	0.0892	0.0698	0.0567	0.0795
$E(X^6)$	0.0809	0.0765	0.0596	0.0491	0.0689
SD	0.3384	0.3262	0.3030	0.2884	0.3238
CV	1.6939	1.2286	1.3139	1.9281	1.5854
CS	1.2711	0.8528	1.1016	1.7164	1.2255
CK	2.8840	2.2369	2.8276	4.4264	2.8932

4. Mathematical Properties

Mathematical properties of the EGom-MO-G family of distributions are explored in this section, including linear representation of the density function, quantile function, probability weighted moments, distribution of order statistics, Rényi entropy and Stochastic ordering.

4.1. Quantile Function

The quantile function of the EGom-MO-G family of distributions is obtained by inverting $F_{EGom-MO-G}(x; \alpha, \gamma, \delta, \xi) = u, 0 \leq u \leq 1$, that is,

$$\left\{1 - exp\left[\frac{1}{\gamma}\left(1 - \left[\frac{\delta\bar{G}(x;\xi)}{1-\bar{\delta}\bar{G}(x;\xi)}\right]^{-\gamma}\right)\right]\right\}^{\alpha} = u.$$

This simplifies to

$$\bar{G}(x;\xi) = \frac{\left[1 - \gamma ln\left(1 - u^{\frac{1}{\alpha}}\right)\right]^{\frac{-1}{\gamma}}}{\delta + \bar{\delta}\left[1 - \gamma ln\left(1 - u^{\frac{1}{\alpha}}\right)\right]^{-\frac{1}{\gamma}}}.$$

Consequently, the quantile function of EGom-MO-G family of distributions is given as:

$$Q_G(u;\alpha,\gamma,\delta,\xi) = G^{-1}\left(1-\left[\frac{\left(1-\gamma ln(1-u^{\frac{1}{\alpha}})\right)^{\frac{-1}{\gamma}}}{\delta+\bar{\delta}\left(1-\gamma ln(1-u^{\frac{1}{\alpha}})\right)^{-\frac{1}{\gamma}}}\right]\right). \tag{8}$$

The quantile function is used to generate random numbers for simulation studies. Quantile values are obtained using equation (8) via a specified baseline cdf G using R software.

4.2. Linear Representation of the Density Function

In this sub-section, we demonstrate that the EGom-MO-G density function can be expressed as an infinite linear combination of the Exponentiated-G densities. We consider the following expansions:
$e^x = \sum_{j=0}^{\infty}\frac{x^j}{j!}$, for $x>0$, and $(1-x)^k = \sum_{i=0}^{\infty}\binom{k}{i}(-1)^i x^i$, for $|x|<1$, and $k>0$. Using the series expansions

$$\begin{aligned}\left[1-exp\left(\frac{1}{\gamma}\left[1-\left(\frac{\delta\bar{G}(x;\xi)}{1-\bar{\delta}\bar{G}(x;\xi)}\right)^{-\gamma}\right]\right)\right]^{\alpha-1} &= \sum_{i=0}^{\infty}\binom{\alpha-1}{i}(-1)^i \\ &\times \left[exp\left(\frac{1}{\gamma}\left[1-\left(\frac{\delta\bar{G}(x;\xi)}{1-\bar{\delta}\bar{G}(x;\xi)}\right)^{-\gamma}\right]\right)\right]^i,\end{aligned}$$

$$exp\left(\frac{(i+1)}{\gamma}\left[1-\left(\frac{\delta\bar{G}(x;\xi)}{1-\bar{\delta}\bar{G}(x;\xi}\right)^{-\gamma}\right]\right) = \sum_{j=0}^{\infty}\frac{(i+1)^j}{j!\gamma^j}\left[1-\left(\frac{\delta\bar{G}(x;\xi)}{1-\bar{\delta}\bar{G}(x;\xi)}\right)^{-\gamma}\right]^j,$$

$$\left[1-\left(\frac{\delta\bar{G}(x;\xi)}{1-\bar{\delta}\bar{G}(x;\xi)}\right)^{-\gamma}\right]^j = \sum_{k=0}^{\infty}\binom{j}{k}(-1)^k\left[\frac{\delta\bar{G}(x;\xi)}{1-\bar{\delta}\bar{G}(x;\xi)}\right]^{-\gamma k},$$

and

$$[1-G(x;\xi)]^{l-\gamma k-\gamma-1} = \sum_{m=0}^{\infty}\binom{l-\gamma k-\gamma-1}{m}(-1)^m G^m(x;\xi),$$

we then have

$$\begin{aligned}f(x;\gamma;\lambda,\delta,\xi) &= \alpha\sum_{i,j,k,l,m=0}^{\infty}\frac{(-1)^{i+k+l+m}}{m+1}\binom{\alpha-1}{i}\binom{j}{k}\frac{(i+1)^j}{j!\gamma^j} \\ &\times \binom{\gamma k+\gamma-1}{l}\binom{l-\gamma k-\gamma-1}{m}\delta^{-\gamma k-\gamma}\bar{\delta}^l(m+1)g(x;\xi)G^m(x;\xi) \\ &= \sum_{m=0}^{\infty}C_{m+1}g_{m+1}(x;\xi),\end{aligned}$$

where

$$\begin{aligned}C_{m+1} &= \alpha\sum_{i,j,k,l=0}^{\infty}\frac{(-1)^{i+k+l+m}}{m+1}\binom{\alpha-1}{i}\binom{j}{k}\frac{(i+1)^j}{j!\gamma^j} \\ &\times \binom{\gamma k+\gamma-1}{l}\binom{l-\gamma k-\gamma-1}{m}\delta^{-\gamma k-\gamma}\bar{\delta}^l,\end{aligned}$$

and $g_{m+1}(x;\xi) = (m+1)g(x;\xi)G^m(x;\xi)$, is the exponentiated-G (Exp-G) density with power parametr $(m+1)$. Hence, the new density is capable of being expressed as an infinite linear combination of Exp-G densities. Other mathematical and statistical properties of the new family of distributions can be obtained directly from those of Exp-G family of distributions.

4.3. Probability Weighted Moments

The probability weighted moments (PWMs) of the EGom-MO-G family of distribution is presented in this subsection. The PWMs of a random variable X with cdf $F(x)$ and pdf $f(x)$ is given by

$$M_{p,r} = E[X^p(F(X))^r] = \int_{-\infty}^{\infty} x^p (F(x))^r f(x)dx.$$

Using equations number (5) and (6), we can write

$$\begin{aligned} f(x)(F(x))^r &= \frac{\alpha\delta g(x;\xi)}{[1-\bar{\delta}\bar{G}(x;\xi)]^2}\left[\frac{\delta\bar{G}(x;\xi)}{1-\bar{\delta}\bar{G}(x;\xi)}\right]^{-\gamma-1} \\ &\times \; exp\left[\frac{1}{\gamma}\left(1-\left[\frac{\delta\bar{G}(x;\xi)}{1-\bar{\delta}\bar{G}(x;\xi)}\right]^{-\gamma}\right)\right] \\ &\times \; \left(1-exp\left[\frac{1}{\gamma}\left(1-\left[\frac{\delta\bar{G}(x;\xi)}{1-\bar{\delta}\bar{G}(x;\xi)}\right]^{-\gamma}\right)\right]\right)^{\alpha(r+1)-1}, \end{aligned}$$

which reduces to

$$\begin{aligned} f(x)(F(x))^r &= \alpha\sum_{i,j,k,l,m=0}^{\infty}(-1)^{i+k+l+m}\frac{(i+1)^j}{\gamma^j j!}\binom{\alpha(1+r)-1}{i} \\ &\times \binom{j}{k}\binom{\gamma(1+j)-1}{l}\binom{l-\gamma(1+j)-1}{m}\delta^{-\gamma(1+j)}\bar{\delta}^l G^m(x;\xi)g(x;\xi) \\ &= \alpha\sum_{i,j,k,l,m=0}^{\infty}\binom{\alpha(1+r)-1}{i}\frac{(-1)^{i+k+l+m}}{m+1}\binom{j}{k}\frac{(i+1)^j}{\gamma^j j!} \\ &\times \binom{\gamma(1+j)-1}{l}\binom{l-\gamma(1+j)-1}{m}\delta^{-\gamma(1+j)}\bar{\delta}^l(m+1)G^m(x;\xi)g(x;\xi) \\ &= \sum_{m=0}^{\infty} w_{m+1}g_{m+1}(x;\xi), \end{aligned}$$

where $g_{m+1}(x;\xi) = (m+1)G^m(x;\xi)g(x;\xi)$, and

$$\begin{aligned} w_{m+1} &= \alpha\sum_{i,j,k,l=0}^{\infty}\binom{\alpha(1+r)-1}{i}\frac{(-1)^{i+k+l+m}}{m+1}\binom{j}{k}\frac{(i+1)^j}{\gamma^j j!} \\ &\times \binom{\gamma(1+j)-1}{l}\binom{l-\gamma(1+j)-1}{m}\delta^{-\gamma(1+j)}\bar{\delta}^l. \end{aligned}$$

Thus, the PWMs of the EGom-MO-G family of distributions is given by

$$M_{p,r} = \int_{-\infty}^{\infty} x^p \sum_{m=0}^{\infty} w_{m+1}g_{m+1}(x;\xi)dx = \sum_{m=0}^{\infty} w_{m+1}\int_{-\infty}^{\infty} x^p g_{m+1}(x;\xi)dx.$$

Therefore, the PWMs of the EGom-MO-G family of distributions can be expressed in terms of the moments of the Exp-G distributions. PWMs can be used to estimate parameters of a distribution for a specified baseline cdf.

4.4. Distribution of Order Statistics

Let $X_1, X_2, ..., X_n$ be a random sample from the EGom-MO-G family of distributions. Then the pdf of the i^{th} order statistic is given as

$$f_{i:n}(x) = \frac{n! f(x)}{(i-1)!(n-i)!} \sum_{j=0}^{n-i} (-1)^j \binom{n-i}{j} [F(x)]^{j+i-1}.$$

Using the result from the PWMs, we get

$$\begin{aligned} f(x)(F(x))^{i+j-1} &= \alpha \sum_{k,l,m,n,p=0}^{\infty} \binom{\alpha(i+j)}{k} \frac{(-1)^{k+m+n+p}}{p+1} \binom{l}{m} \frac{(k+1)^l}{\gamma^l l!} \\ &\times \binom{\gamma(1+l)-1}{n} \binom{n-\gamma(1+l)-1}{p} \delta^{-\gamma(1+l)} \bar{\delta}^n (p+1) G^p(x;\xi) g(x;\xi) \\ &= \sum_{p=0}^{\infty} b_{p+1} g_{p+1}(x;\xi), \end{aligned}$$

where $g_{p+1}(x;\xi) = (p+1)G^p(x;\xi)g(x;\xi)$ and

$$\begin{aligned} b_{p+1} &= \alpha \sum_{k,l,m,n=0}^{\infty} \binom{\alpha(i+j)}{k} \frac{(-1)^{k+m+n+p}}{p+1} \binom{l}{m} \frac{(k+1)^l}{\gamma^l l!} \\ &\times \binom{\gamma(1+l)-1}{n} \binom{n-\gamma(1+l)-1}{p} \delta^{-\gamma(1+l)} \bar{\delta}^n. \end{aligned}$$

Therefore, the pdf of the i^{th} order statistic from the EGom-MO-G family of distributions is presented as

$$f_{i:n}(x) = \frac{n!}{(i-1)!(n-i)!} \sum_{p=0}^{n-i} \sum_{j=0}^{\infty} (-1)^j \binom{n-i}{j} b_{p+1} g_{p+1}(x;\xi).$$

Order statistics is applied in quality control particularly optimization in acceptance sampling. The k^{th} smallest value of the distribution can be used to obtain the quality standard that products should meet.

4.5. Rényi Entropy

Rényi entropy of the EGom-MO-G family of distributions is presented in this subsection. Rényi entropy (Rényi [51]) is given as

$I_R(v) = \frac{1}{1-v} log \left(\int_0^\infty [f(x;\alpha,\gamma,\delta,\xi)]^v dx \right)$, for $v \neq 1$, and $v > 0$.

Using equation (6), $f^v_{EGom-MO-G}(x;\alpha,\gamma,\delta,\xi) = f^v(x)$ can be re-written as

$$\begin{aligned} f^v(x) &= \frac{\alpha^v \delta^v g^v(x;\xi)}{[1-\bar{\delta}\bar{G}(x;\xi)]^{2v}} \left(\frac{\delta \bar{G}(x;\xi)}{1-\bar{\delta}\bar{G}(x;\xi)} \right)^{-v(\gamma+1)} exp \left(\frac{v}{\gamma} \left[1 - \left(\frac{\delta \bar{G}(x;\xi)}{1-\bar{\delta}\bar{G}(x;\xi)} \right)^{-\gamma} \right] \right) \\ &\times \left[1 - exp \left(\frac{1}{\gamma} \left[1 - \left(\frac{\delta \bar{G}(x;\xi)}{1-\bar{\delta}\bar{G}(x;\xi)} \right)^{-\gamma} \right] \right) \right]^{v(\alpha-1)}. \end{aligned}$$

Thus, after some algebraic simplification Rényi entropy of the EGom-MO-G family of distributions is given as

$$I_R(v) = \frac{1}{1-v} log \left[\sum_{m=0}^{\infty} d_m \left(\int_0^\infty g^v(x;\xi) G^m(x;\xi) dx \right) \right],$$

where

$$\begin{aligned} d_m &= \alpha^v \sum_{i,j,k,l=0}^{\infty} \binom{v(\alpha-1)}{i} (-1)^{i+k+l+m} \frac{(i+1)^j}{\gamma^j j!} \binom{j}{k} \\ &\times \binom{\gamma(v+k)-v}{l} \binom{l-\gamma(v+k)-v}{m} \delta^{-\gamma(v+k)} \bar{\delta}^l. \end{aligned}$$

We note that $\int_0^\infty g^v(x;\xi) G^m(x;\xi) dx$ can be obtained numerically. Note that Rényi entropy of the EGom-MO-G family of distributions can be obtained from the Exp-G distribution as

$$I_R(v) = \frac{1}{1-v} log \left[\sum_{m=0}^{\infty} d_m^* e^{(1-v) I_{REG}} \right],$$

where

$$\begin{aligned} d_m^* &= \alpha^v \sum_{i,j,k,l=0}^{\infty} \binom{v(\alpha-1)}{i} (-1)^{i+k+l+m} \frac{(i+1)^j}{\gamma^j j!} \binom{j}{k} \\ &\times \binom{\gamma(v+k)-v}{l} \binom{l-\gamma(v+k)-v}{m} \delta^{-\gamma(v+k)} \bar{\delta}^l \left(\frac{m}{v}+1\right)^v, \end{aligned}$$

and $I_{REG} = \frac{1}{1-v} log \left[\int_0^\infty \left[(\frac{m}{v}+1) g(x;\xi) (G(x;\xi))^{\frac{m}{v}} \right]^v dx \right]$ is the Rényi entropy of the Exp-G distribution with power parameter $\frac{m}{v}$.

In risk management, Rényi entropy can be used to compare several asset portfolios to identify one that would minimize the risk. Complete derivation of the mathematical properties have been given in the appendix.

4.6. Stochastic Ordering

Lets define random variables X and Y with cumulative distribution functions (cdfs) $F_X(t)$ and $F_Y(t)$, respectively. Then we say X is smaller in the stochastic ordering than Y denoted $X \leq_{st} Y$ if $\bar{F}_X(t) < \bar{F}_Y(t)$ for all t, where $\bar{F}_X(t) = 1 - F_X(t)$ is the survival function. We denote $X \leq_{hr} Y$ to show that X is smaller in the hazard rate order than Y. We note that $X \leq_{hr} Y \implies X \leq_{st} Y$. The random variable X is said to be smaller in the likelihood ratio order than Y denoted by $X \leq_{lr} Y$ if $\frac{f_X(t)}{f_Y(t)}$ is decreasing in t. We note that $X \leq_{lr} Y \implies X \leq_{hr} Y$. Likelihood ratio ordering is the strongest ordering, hence $X \leq_{lr} Y \implies X \leq_{hr} Y \implies X \leq_{st} Y$ (Shaked and Shanthikumar [55]).

Theorem: Suppose $X_1 \sim EGom - MO - G(\alpha_1, \gamma, \delta, \xi)$ and $X_2 \sim EGom - MO - G(\alpha_2, \gamma, \delta, \xi)$, then $\frac{f_1(x;\alpha_1,\gamma,\delta,\xi)}{f_2(x;\alpha_2,\gamma,\delta,\xi)}$ is decreasing in x whenever $\alpha_1 < \alpha_2$ and $x_1 <_{st} x_2$.

Proof: Note that the ratio

$$\frac{f_1(x;\alpha_1,\gamma,\delta,\xi)}{f_2(x;\alpha_2,\gamma,\delta,\xi)} = \frac{\alpha_1}{\alpha_2} \left[1 - \left(exp \left[\frac{1}{\gamma} \left(1 - \left[\frac{\delta \bar{G}(x;\xi)}{1 - \bar{\delta} \bar{G}(x;\xi)} \right]^{-\gamma} \right) \right] \right) \right]^{\alpha_1 - \alpha_2}.$$

Let $W = \frac{\delta \bar{G}(x;\xi)}{1 - \bar{\delta} \bar{G}(x;\xi)}$, then

$$\begin{aligned} \frac{\partial}{\partial x} \left(\frac{f_1(x;\alpha_1,\gamma,\delta,\xi)}{f_2(x;\alpha_2,\gamma,\delta,\xi)} \right) &= \frac{\alpha_1(\alpha_1 - \alpha_2)}{\alpha_2} \left[1 - \left(exp \left[\frac{1}{\gamma} \left(1 - W^{-\gamma} \right) \right] \right) \right]^{\alpha_1 - \alpha_2 - 1} \\ &\times \ exp \left[\frac{1}{\gamma} \left(1 - W^{-\gamma} \right) \right] W^{-(\gamma+1)} \frac{\partial W}{\partial x}. \end{aligned}$$

Therefore, since $\frac{\partial}{\partial x}\left(\frac{f_1(x;\alpha_1,\gamma,\delta,\xi)}{f_2(x;\alpha_2,\gamma,\delta,\xi)}\right) < 0$ whenever $\alpha_1 < \alpha_2$, likelihood ratio order exists among X_1 and X_2. As a result, we conclude that X_1 and X_2 are stochastically ordered, that is $X_1 <_{st} X_2$.

In finance, investors can compare returns on investment portfolios based on their stochastic ordering.

5. Methods of Estimation

We present and discuss several estimation methods in this section including maximum likelihood estimation (MLE), ordinary least squares (OLS), weighted least squares (WLS), Anderson-Darling (AD), Right Tail Anderson-Darling (RAD), and Cramér-von Mises (CVM).

5.1. Maximum Likelihood Estimation

Maximum likelihood estimation (MLE) method for estimating the parameters of the EGom-MO-G family of distributions is presented in this subsection. Let $X \sim EGom-MO-G(\alpha,\gamma,\delta,\xi)$ and $\Delta = (\alpha,\gamma,\delta,\xi)^T$ be the vector of model parameters. The log-likelihood function $\ell = \ell n(L)$ based on a random sample of size n from EGom-MO-G family of distributions is given by

$$\begin{aligned}\ell = \ell n(L) &= nlog(\alpha\delta) + \sum log\left[g(x_i;\xi)\right] - 2\sum log\left[1-\bar{\delta}\bar{G}(x_i;\xi)\right]\\ &- (\gamma+1)\sum log(W) + \frac{1}{\gamma}\sum(1-W^{-\gamma})\\ &+ (\alpha-1)\sum log\left[1-exp\left(\frac{1}{\gamma}[1-W^{-\gamma}]\right)\right],\end{aligned}$$

where $W = \frac{\delta\bar{G}(x_i;\xi)}{1-\bar{\delta}\bar{G}(x_i;\xi)}$.

Elements of the score vector can be readily obtained by taking partial derivatives of the log-likelihood function with respect to each component of the parameter vector $\Delta = (\alpha,\gamma,\delta,\xi)^T$. Results are given in the Appendix.

5.2. Anderson-Darling and Right Tail Anderson-Darling Estimation

5.2.1. Anderson-Darling Method Let $x_{(1)},......,x_{(n)}$ be an ordered random sample from the EGom-MO-G family of distributions. Then the Anderson-Darling estimates (ADEs) of the parameters of the EGom-MO-G family of distributions are obtained through minimizing the following function

$$AD(\alpha,\gamma,\delta,\xi) = -n - \frac{1}{n}\sum_{k=1}^{n}(2k-1)\left[log(F(x_{(k)})) + log(\bar{F}(x_{(n-k+1)}))\right],$$

with respect to α,γ,δ,ξ, where $\bar{F}(.) = 1 - F(.)$. Substituting for the EGom-MO-G $F(.)$ and $\bar{F}(.)$, we obtain

$$\begin{aligned}AD(\alpha,\gamma,\delta,\xi) &= -n - \frac{1}{n}\sum_{k=1}^{n}(2k-1)log\left\{1-exp\left(\frac{1}{\gamma}\left[1-\left(\frac{\delta\bar{G}(x_{(k)};\xi)}{1-\bar{\delta}\bar{G}(x_{(k)};\xi)}\right)^{-\gamma}\right]\right)\right\}^{\alpha}\\ &+ log\left\{1-\left[1-exp\left(\frac{1}{\gamma}\left[1-\left(\frac{\delta\bar{G}(x_{(n-k+1)};\xi)}{1-\bar{\delta}\bar{G}(x_{(n-k+1)};\xi)}\right)^{-\gamma}\right]\right)\right]^{\alpha}\right\}.\end{aligned}$$

5.2.2. Right-Tail Anderson-Darling Method The Right-Tail Anderson-Darling estimates (RADEs) of the parameters of the EGom-MO-G family of distributions are obtained through minimizing the following function

$$RAD(\alpha,\gamma,\delta,\xi)=\frac{n}{2}-2\sum_{k=1}^{n}F(x_{(k)};\alpha,\gamma,\delta,\xi)-\frac{1}{n}\sum_{k=1}^{n}(2k-1)log\left[\bar{F}(x_{(n-k+1)};\alpha,\gamma,\delta,\xi)\right],$$

with respect to α,γ,δ and ξ.

5.3. Cramér-von Mises Method

The Cramér-von Mises (CVM) estimates of the parameters of the EGom-MO-G family of distributions are obtained by minimizing the function

$$CVM(\alpha,\gamma,\delta,\xi)=\frac{1}{12n}+\sum_{k=1}^{n}\left[F(x_{(k)};\alpha,\gamma,\delta,\xi)-\frac{2k-1}{2n}\right]^2,$$

with respect to α,γ,δ and ξ.

5.4. Ordinary Least Squares and Weighted Least Squares Methods

The Ordinary Least Squares estimates (OLSE) and Weighted Least Squares estimates (WLSE) of the parameters of the EGom-MO-G family of distributions are obtained by minimizing the following function

$$V(\alpha,\gamma,\delta,\xi)=\sum_{k=1}^{n}\omega_k\left[F(x_{(k)};\alpha,\gamma,\delta,\xi)-\frac{k}{n+1}\right]^2,$$

with respect to α,γ,δ,ξ, where $\omega_k=1$ for OLSE and $\omega_k=\frac{(n+2)(n+1)^2}{k(n-k+1)}$ for the WLSE.

6. Censoring

When data only occurs within a certain interval, then we say that censoring has occurred. We consider type 1 censoring.

6.1. Type I Censoring

Under this type of censoring, the experiment terminates before all subjects attain their events or have failed. The experiment may fail as a result of costs, etc. If we define x to be the lifetime associated with an individual subject, then we can define a fixed censoring time as c_i, such that under type I censoring, $x_i \leq c_i$. Let $T_{i1}, T_{i2}, ...T_{in}$ be a sample of independent random variables. The log-likelihoood function $L(\varDelta)$ from this random censored sample of size n from the EGom-MO-G family of distributions is then given as

$$L(\varDelta)=\prod_{i=1}^{n}\left[f_{EGom-MO-G}(x_i)\right]^{\epsilon_i}\left[S_{EGom-MO-G}(x_i)\right]^{1-\epsilon_i},$$

where $S_{EGom-MO-G}(x_i) = 1 - F_{EGom-MO-G}(x_i)$ and ϵ_i is an indicator variable. Let $\Delta = (\alpha, \gamma, \delta, \xi)^T$ be the vector of parameters of the model. The log-likelihood function $\ell(\Delta)$ from a random sample of size n from EGom-MO-G family of distributions is given by

$$\begin{aligned}
\ell(\Delta) &= \sum \epsilon_i log(\alpha\delta) + \sum \epsilon_i log\left[g(x_i;\xi)\right] - 2\sum \epsilon_i log\left[1-\bar{\delta}\bar{G}(x_i;\xi)\right] \\
&- (\gamma+1)\sum \epsilon_i log(W) + \frac{1}{\gamma}\sum \epsilon_i (1-W^{-\gamma}) \\
&+ (\alpha-1)\sum \epsilon_i log\left[1-exp\left(\frac{1}{\gamma}[1-W^{-\gamma}]\right)\right] \\
&+ \sum (1-\epsilon_i) log\left\{1-\left[1-exp\left(\frac{1}{\gamma}[1-W^{-\gamma}]\right)\right]^{\alpha}\right\},
\end{aligned}$$

where $W = \frac{\delta\bar{G}(x_i;\xi)}{1-\bar{\delta}\bar{G}(x_i;\xi)}$.

Elements of the score vector can be obtained by taking derivative of $\ell(\Delta)$ with respect to each member of the parameter vector $\Delta = (\alpha, \gamma, \delta, \xi)^T$. Results of the partial derivatives are presented in the Appendix.

7. Simulation Study

We assess the performance of the EGom-MO-W distribution through simulation studies for various sample sizes, ($n = 25, 50, 100, 200, 400, 800$) using R software. Samples were simulated for the true parameter values and the results are presented in Table 7 and 8 for parameter sets $(0.6, 0.6, 0.6, 2.2)$, and $(0.3, 1.4, 1.2, 2.0)$, respectively. Table 7 and 8 presents average bias (AB) and root mean square error (RMSEs) for different estimation techniques including maximum likelihood estimation (MLE), Cramér-von Mises (CVM), Least Square Estimation (LSE), Weighted Least Squares Estimation (WLSE), Anderson-Darling (AD) and Right-Tail Anderson-Darling (AD. Figures 10 to 13 presents graphs for the RMSE for different estimation methods. Figures 14 and 15 presents the heatmaps of RMSE values based on each technique for parameter values in Tables 7 and 8, respectively. For the heat map, the rectangles with the light colour represents a category with the highest values of RMSE, while the rectangles with dark colours represents categories with less values of RMSE. MLE1 represents the RMSE of α based on the MLE technique, MLE2 represents RMSE of γ based on the MLE method, MLE3 represents RMSE of δ based on MLE and MLE4 represents RMSE of λ based on the MLE technique. The same denotion holds for CVM, LSE,WLSE, AD and RAD such that CVM1 will represent RMSE of α based on the CVM method. The ABias and RMSE for the estimated parameter $\hat{\theta}$ are defined as ABias($\hat{\theta}$)=$\frac{\sum_{i=1}^{N}\hat{\theta}_i-\theta}{N}$ and RMSE($\hat{\theta}$)=$\sqrt{\frac{\sum(\hat{\theta}_i-\theta)^2}{N}}$, respectively.

Simulation results in Tables 7 and 8 show that the EGom-MO-W model is fairly stable. We notice that the RMSE and AB decline as the sample size increases.This is supported by Figures 10 to 13 which show graphs of RMSE going downward as the sample size increases for different parameters. Figures 14 and 15 shows heat maps for the various techniques employed based on Tables 7 and 8, respectively. From Table 7, high values of RMSE are observed for δ via the RAD, AD, LS, CVM, MLE and WLS technique. For Table 8, high values of RMSE are observed for γ via the RAD technique. Furthermore, It is clear from Table 9 that MLE method performed best when compared with other estimation methods registering an overall sum of ranks of 12, followed by CVM with a total sum of ranks of 29.5. The RADE and ADE are the least performing estimation methods with a total sum of ranks of 56.5 and 57, respectively. We therefore, conclude that the MLE technique is the best estimation method for the EGOM-MO-W distribution and subsequently adopt it for applications in Section 8.

Table 7. **Simulation Results**

			(0.6,0.6,0.6,2.2)					
Sample size(n)	**Estimate**	**Parameter**	**MLE**	**CVM**	**LSE**	**WLSE**	**ADE**	**RADE**
25	AB	α	0.1375$^{\{1\}}$	-0.5900$^{\{5\}}$	-0.4100$^{\{3\}}$	-0.5667$^{\{4\}}$	-1.9809$^{\{6\}}$	0.1486$^{\{2\}}$
		γ	0.2584$^{\{1\}}$	1.8000$^{\{5\}}$	1.9001$^{\{6\}}$	0.6886$^{\{2\}}$	1.0775$^{\{4\}}$	-0.9948$^{\{3\}}$
		δ	1.1304$^{\{1\}}$	-1.5885$^{\{3\}}$	-1.5890$^{\{4\}}$	-2.6129$^{\{5\}}$	3.1863$^{\{6\}}$	-1.5006$^{\{2\}}$
		λ	-0.1648$^{\{1\}}$	-0.2088$^{\{2\}}$	-0.2695$^{\{3\}}$	-1.6666$^{\{6\}}$	0.4373$^{\{4\}}$	1.0806$^{\{5\}}$
	RMSE	α	0.3433$^{\{1\}}$	0.5900$^{\{3.5\}}$	0.5900$^{\{3.5\}}$	0.5688$^{\{2\}}$	2.0283$^{\{6\}}$	0.9353$^{\{5\}}$
		γ	0.5942$^{\{1\}}$	1.8722$^{\{5\}}$	1.9783$^{\{6\}}$	0.8458$^{\{2\}}$	1.0826$^{\{3\}}$	1.6766$^{\{4\}}$
		δ	3.6999$^{\{2\}}$	2.5885$^{\{1\}}$	3.8898$^{\{3\}}$	4.4478$^{\{5\}}$	4.0216$^{\{4\}}$	4.8999$^{\{6\}}$
		λ	1.9438$^{\{2\}}$	1.9874$^{\{4\}}$	1.5708$^{\{1\}}$	2.3912$^{\{6\}}$	1.9614$^{\{3\}}$	2.1982$^{\{5\}}$
		Sum of the ranks	**10**$^{\{1\}}$	**28.5**$^{\{2\}}$	**29.5**$^{\{3\}}$	**32**$^{\{4.5\}}$	**36**$^{\{6\}}$	**32**$^{\{4.5\}}$
50	AB	α	0.0498$^{\{1\}}$	-0.5680$^{\{5\}}$	-0.3768$^{\{3\}}$	-0.5595$^{\{4\}}$	-0.8993$^{\{6\}}$	0.1245$^{\{2\}}$
		γ	0.1105$^{\{1\}}$	1.2316$^{\{5\}}$	1.6953$^{\{6\}}$	0.4446$^{\{2\}}$	0.6959$^{\{3\}}$	-0.7014$^{\{4\}}$
		δ	1.1304$^{\{2\}}$	-0.8544$^{\{1\}}$	-1.5525$^{\{5\}}$	-1.5456$^{\{4\}}$	3.1061$^{\{6\}}$	-1.4106$^{\{3\}}$
		λ	0.0413$^{\{1\}}$	0.1655$^{\{3\}}$	-0.1252$^{\{2\}}$	-0.6802$^{\{5\}}$	0.4089$^{\{4\}}$	1.0684$^{\{6\}}$
	RMSE	α	0.2739$^{\{1\}}$	0.5692$^{\{4\}}$	0.5687$^{\{3\}}$	0.5628$^{\{2\}}$	1.0237$^{\{6\}}$	0.7453$^{\{5\}}$
		γ	0.3575$^{\{1\}}$	1.2707$^{\{4\}}$	1.7439$^{\{6\}}$	0.4580$^{\{2\}}$	0.7330$^{\{3\}}$	1.2947$^{\{5\}}$
		δ	3.3464$^{\{2\}}$	2.4475$^{\{1\}}$	3.4495$^{\{3\}}$	3.6691$^{\{5\}}$	3.5311$^{\{4\}}$	3.9000$^{\{6\}}$
		λ	1.1637$^{\{1\}}$	1.8745$^{\{5\}}$	1.2601$^{\{2\}}$	1.5203$^{\{4\}}$	1.3941$^{\{3\}}$	2.0988$^{\{6\}}$
		Sum of the ranks	**10**$^{\{1\}}$	**28**$^{\{2.5\}}$	**30**$^{\{4\}}$	**28**$^{\{2.5\}}$	**35**$^{\{5\}}$	**37**$^{\{6\}}$
100	AB	α	0.0160$^{\{1\}}$	-0.2739$^{\{4\}}$	-0.2573$^{\{3\}}$	-0.5413$^{\{6\}}$	-0.4688$^{\{5\}}$	0.1096$^{\{2\}}$
		γ	0.0640$^{\{1\}}$	1.1391$^{\{5\}}$	1.3593$^{\{6\}}$	0.3470$^{\{2\}}$	0.5420$^{\{4\}}$	-0.4700$^{\{3\}}$
		δ	0.6569$^{\{1\}}$	-0.7665$^{\{3\}}$	-1.2276$^{\{5\}}$	-0.9852$^{\{4\}}$	2.1881$^{\{6\}}$	-0.7616$^{\{2\}}$
		λ	-0.0358$^{\{1\}}$	-0.1568$^{\{3\}}$	0.0566$^{\{2\}}$	-0.4927$^{\{5\}}$	0.2459$^{\{4\}}$	0.7443$^{\{6\}}$
	RMSE	α	0.2496$^{\{1\}}$	0.3306$^{\{3\}}$	0.3105$^{\{2\}}$	0.5482$^{\{5\}}$	0.5158$^{\{4\}}$	0.7091$^{\{6\}}$
		γ	0.3263$^{\{1\}}$	1.1468$^{\{4\}}$	1.3663$^{\{6\}}$	0.3577$^{\{2\}}$	0.5698$^{\{3\}}$	1.1975$^{\{5\}}$
		δ	2.2736$^{\{2\}}$	1.3361$^{\{1\}}$	2.3346$^{\{3\}}$	2.4612$^{\{4\}}$	3.1721$^{\{6\}}$	2.5943$^{\{5\}}$
		λ	1.0480$^{\{2\}}$	1.8002$^{\{5\}}$	1.1957$^{\{3\}}$	0.5177$^{\{1\}}$	1.2462$^{\{4\}}$	1.9059$^{\{6\}}$
		Sum of the ranks	**10**$^{\{1\}}$	**28**$^{\{2\}}$	**30**$^{\{4\}}$	**29**$^{\{3\}}$	**36**$^{\{6\}}$	**35**$^{\{5\}}$
200	AB	α	-0.0156$^{\{2\}}$	-0.1996$^{\{4\}}$	-0.2079$^{\{5\}}$	-0.3486$^{\{6\}}$	0.0125$^{\{1\}}$	-0.0933$^{\{3\}}$
		γ	0.0488$^{\{1\}}$	1.1299$^{\{5\}}$	1.3049$^{\{6\}}$	0.2934$^{\{3\}}$	0.0769$^{\{2\}}$	0.4378$^{\{4\}}$
		δ	0.4824$^{\{1\}}$	-0.5169$^{\{3\}}$	-0.5151$^{\{2\}}$	-0.7905$^{\{5\}}$	2.1795$^{\{6\}}$	-0.5780$^{\{4\}}$
		λ	0.0266$^{\{1\}}$	-0.0627$^{\{2\}}$	-0.9842$^{\{6\}}$	-0.1842$^{\{4\}}$	0.1239$^{\{3\}}$	0.6258$^{\{5\}}$
	RMSE	α	0.2219$^{\{1\}}$	0.2452$^{\{3\}}$	0.2335$^{\{2\}}$	0.3610$^{\{5\}}$	0.2498$^{\{4\}}$	0.5199$^{\{6\}}$
		γ	0.2806$^{\{1\}}$	1.1334$^{\{5\}}$	1.3276$^{\{6\}}$	0.2967$^{\{2\}}$	0.3783$^{\{3\}}$	0.7117$^{\{4\}}$
		δ	1.7399$^{\{3\}}$	0.5915$^{\{1\}}$	1.9347$^{\{5\}}$	1.8583$^{\{4\}}$	2.2197$^{\{6\}}$	1.6240$^{\{2\}}$
		λ	0.9855$^{\{2\}}$	1.1453$^{\{5\}}$	1.0825$^{\{4\}}$	0.2230$^{\{1\}}$	0.9941$^{\{3\}}$	1.3144$^{\{6\}}$
		Sum of the ranks	**12**$^{\{1\}}$	**28**$^{\{2.5\}}$	**36**$^{\{6\}}$	**30**$^{\{4\}}$	**28**$^{\{2.5\}}$	**34**$^{\{5\}}$
400	AB	α	-0.0141$^{\{1\}}$	-0.1150$^{\{4\}}$	-0.1292$^{\{5\}}$	-0.2652$^{\{6\}}$	0.0296$^{\{2\}}$	0.0361$^{\{3\}}$
		γ	0.0432$^{\{1\}}$	0.1220$^{\{3\}}$	1.2556$^{\{4\}}$	0.2386$^{\{5\}}$	0.0618$^{\{2\}}$	-0.3077$^{\{6\}}$
		δ	0.2793$^{\{1\}}$	-0.3240$^{\{4\}}$	-0.3114$^{\{3\}}$	-0.3353$^{\{5\}}$	1.5636$^{\{6\}}$	-0.3041$^{\{2\}}$
		λ	0.0119$^{\{1\}}$	0.0529$^{\{3\}}$	0.2349$^{\{5\}}$	-0.0849$^{\{4\}}$	0.0440$^{\{2\}}$	0.4908$^{\{6\}}$
	RMSE	α	0.2062$^{\{1\}}$	0.2350$^{\{3\}}$	0.2130$^{\{2\}}$	0.2670$^{\{5\}}$	0.2353$^{\{4\}}$	0.3687$^{\{6\}}$
		γ	0.2601$^{\{2\}}$	0.2599$^{\{1\}}$	1.3118$^{\{6\}}$	0.2951$^{\{4\}}$	0.2649$^{\{3\}}$	0.6939$^{\{5\}}$
		δ	1.0766$^{\{2\}}$	0.5604$^{\{1\}}$	1.5549$^{\{5\}}$	1.2485$^{\{3\}}$	1.6120$^{\{6\}}$	1.3232$^{\{4\}}$
		λ	0.8751$^{\{2\}}$	1.0737$^{\{5\}}$	1.0463$^{\{4\}}$	0.0996$^{\{1\}}$	0.9443$^{\{3\}}$	1.3090$^{\{6\}}$
		Sum of the ranks	**11**$^{\{1\}}$	**24**$^{\{2\}}$	**34**$^{\{5\}}$	**33**$^{\{4\}}$	**28**$^{\{3\}}$	**38**$^{\{6\}}$
800	AB	α	-0.0098$^{\{1\}}$	-0.1056$^{\{4\}}$	-0.1243$^{\{5\}}$	-0.2400$^{\{6\}}$	0.0293$^{\{3\}}$	0.0200$^{\{2\}}$
		γ	0.0341$^{\{1\}}$	0.0519$^{\{2\}}$	1.0497$^{\{6\}}$	0.2308$^{\{4\}}$	0.0614$^{\{3\}}$	-0.2680$^{\{5\}}$
		δ	0.1491$^{\{3\}}$	-0.0577$^{\{1\}}$	-0.2848$^{\{5\}}$	-0.1956$^{\{4\}}$	1.2049$^{\{6\}}$	0.1244$^{\{2\}}$
		λ	0.0323$^{\{2\}}$	-0.0347$^{\{4\}}$	0.2100$^{\{6\}}$	0.0042$^{\{1\}}$	0.0326$^{\{3\}}$	0.1898$^{\{5\}}$
	RMSE	α	0.1807$^{\{2\}}$	0.1922$^{\{3\}}$	0.1611$^{\{1\}}$	0.2414$^{\{5\}}$	0.2094$^{\{4\}}$	0.3123$^{\{6\}}$
		γ	0.2364$^{\{3\}}$	0.0828$^{\{1\}}$	1.2104$^{\{6\}}$	0.2352$^{\{2\}}$	0.2516$^{\{4\}}$	0.3109$^{\{5\}}$
		δ	0.5900$^{\{2\}}$	0.5210$^{\{1\}}$	0.7206$^{\{4\}}$	0.6060$^{\{3\}}$	1.5686$^{\{6\}}$	1.0788$^{\{5\}}$
		λ	0.7849$^{\{2\}}$	1.0069$^{\{5\}}$	0.8660$^{\{4\}}$	0.0479$^{\{1\}}$	0.8437$^{\{3\}}$	1.1898$^{\{6\}}$
		Sum of the ranks	**16**$^{\{1\}}$	**21**$^{\{2\}}$	**37**$^{\{6\}}$	**26**$^{\{3\}}$	**32**$^{\{4\}}$	**36**$^{\{5\}}$

Table 8. **Simulation Results**

			(0.3,1.4,1.2,2.0)					
Sample size(n)	**Estimate**	**Parameter**	**MLE**	**CVM**	**LSE**	**WLSE**	**ADE**	**RADE**
25	AB	α	-0.0397$^{\{1\}}$	0.3474$^{\{5\}}$	0.3444$^{\{4\}}$	0.1637$^{\{3\}}$	-0.7201$^{\{6\}}$	-0.0424$^{\{2\}}$
		γ	0.4027$^{\{1\}}$	1.7803$^{\{6\}}$	0.5963$^{\{4\}}$	0.5468$^{\{2\}}$	1.7747$^{\{5\}}$	5.4971$^{\{3\}}$
		δ	0.9113$^{\{1\}}$	1.2109$^{\{2\}}$	1.2661$^{\{5\}}$	1.2467$^{\{3\}}$	1.6407$^{\{6\}}$	1.2614$^{\{4\}}$
		λ	0.1764$^{\{1\}}$	1.2052$^{\{5\}}$	0.9433$^{\{2\}}$	0.9704$^{\{4\}}$	-0.9697$^{\{3\}}$	1.1300$^{\{6\}}$
	RMSE	α	0.0570$^{\{1\}}$	0.4685$^{\{2\}}$	0.5158$^{\{3\}}$	0.5435$^{\{4\}}$	0.8333$^{\{6\}}$	0.6997$^{\{5\}}$
		γ	0.9631$^{\{1\}}$	2.6733$^{\{5\}}$	1.9223$^{\{4\}}$	1.3587$^{\{2\}}$	1.7848$^{\{3\}}$	9.0860$^{\{6\}}$
		δ	2.2884$^{\{5\}}$	1.4621$^{\{2\}}$	1.6783$^{\{4\}}$	1.6473$^{\{3\}}$	2.6612$^{\{6\}}$	1.4121$^{\{1\}}$
		λ	0.6664$^{\{1\}}$	1.6368$^{\{4\}}$	1.6192$^{\{3\}}$	1.9083$^{\{6\}}$	1.9010$^{\{5\}}$	1.4836$^{\{2\}}$
		Sum of the ranks	**12**$^{\{1\}}$	**31**$^{\{5\}}$	**30**$^{\{4\}}$	**27**$^{\{2\}}$	**40**$^{\{6\}}$	29$^{\{3\}}$
50	AB	α	-0.0305$^{\{1\}}$	0.0668$^{\{3\}}$	0.2530$^{\{5\}}$	0.1282$^{\{4\}}$	-0.5591$^{\{6\}}$	0.0365$^{\{2\}}$
		γ	0.1775$^{\{1\}}$	0.7495$^{\{4\}}$	0.3448$^{\{3\}}$	0.3246$^{\{2\}}$	1.4771$^{\{5\}}$	2.8875$^{\{6\}}$
		δ	0.3449$^{\{1\}}$	0.7555$^{\{2\}}$	1.1629$^{\{4\}}$	1.1019$^{\{3\}}$	1.4431$^{\{6\}}$	1.2317$^{\{5\}}$
		λ	0.1352$^{\{1\}}$	1.1673$^{\{6\}}$	0.5059$^{\{2\}}$	0.7117$^{\{4\}}$	-0.9362$^{\{5\}}$	0.5454$^{\{3\}}$
	RMSE	α	0.0456$^{\{1\}}$	0.2699$^{\{2\}}$	0.5140$^{\{4\}}$	0.5088$^{\{3\}}$	0.5858$^{\{6\}}$	0.5223$^{\{5\}}$
		γ	0.7377$^{\{1\}}$	1.5566$^{\{4\}}$	1.4298$^{\{3\}}$	1.2582$^{\{2\}}$	1.7802$^{\{5\}}$	3.4494$^{\{6\}}$
		δ	1.0328$^{\{1\}}$	1.3576$^{\{2\}}$	1.6372$^{\{6\}}$	1.5626$^{\{5\}}$	1.4519$^{\{4\}}$	1.3947$^{\{3\}}$
		λ	0.5285$^{\{1\}}$	1.6086$^{\{6\}}$	1.1750$^{\{5\}}$	1.0017$^{\{3\}}$	0.9837$^{\{2\}}$	1.0136$^{\{4\}}$
		Sum of the ranks	**8**$^{\{1\}}$	**29**$^{\{3\}}$	**32**$^{\{4\}}$	**26**$^{\{2\}}$	**39**$^{\{6\}}$	**34**$^{\{5\}}$
100	AB	α	-0.0262$^{\{1\}}$	0.0611$^{\{2\}}$	0.1632$^{\{5\}}$	0.1257$^{\{3\}}$	-0.5569$^{\{6\}}$	0.1428$^{\{4\}}$
		γ	0.1517$^{\{1\}}$	0.5114$^{\{4\}}$	0.2968$^{\{3\}}$	0.2629$^{\{2\}}$	1.1647$^{\{5\}}$	1.4458$^{\{6\}}$
		δ	0.1541$^{\{1\}}$	0.3227$^{\{2\}}$	1.0570$^{\{5\}}$	1.0534$^{\{4\}}$	0.9061$^{\{3\}}$	1.1413$^{\{6\}}$
		λ	0.1310$^{\{1\}}$	1.0359$^{\{6\}}$	0.4518$^{\{3\}}$	0.6733$^{\{4\}}$	-0.9051$^{\{5\}}$	0.2416$^{\{2\}}$
	RMSE	α	0.0381$^{\{1\}}$	0.1825$^{\{2\}}$	0.3527$^{\{3\}}$	0.5031$^{\{5\}}$	0.5571$^{\{6\}}$	0.4500$^{\{4\}}$
		γ	0.6327$^{\{1\}}$	1.2975$^{\{4\}}$	0.8105$^{\{2\}}$	0.9446$^{\{3\}}$	1.6868$^{\{5\}}$	1.8317$^{\{6\}}$
		δ	0.4726$^{\{1\}}$	0.6744$^{\{2\}}$	1.5756$^{\{6\}}$	1.5213$^{\{5\}}$	0.9537$^{\{3\}}$	1.2526$^{\{4\}}$
		λ	0.4796$^{\{1\}}$	1.2729$^{\{6\}}$	1.0682$^{\{5\}}$	0.7363$^{\{3\}}$	0.9057$^{\{4\}}$	0.6345$^{\{2\}}$
		Sum of the ranks	**8**$^{\{1\}}$	**28**$^{\{2\}}$	**32**$^{\{4\}}$	**29**$^{\{3\}}$	**37**$^{\{6\}}$	**34**$^{\{5\}}$
200	AB	α	-0.0246$^{\{1\}}$	0.0475$^{\{3\}}$	0.1356$^{\{5\}}$	0.0345$^{\{2\}}$	-0.4868$^{\{6\}}$	0.0976$^{\{4\}}$
		γ	0.1371$^{\{1\}}$	0.3583$^{\{4\}}$	0.1567$^{\{2\}}$	0.2387$^{\{3\}}$	0.8846$^{\{6\}}$	0.7863$^{\{5\}}$
		δ	0.0528$^{\{1\}}$	0.2887$^{\{2\}}$	1.0224$^{\{6\}}$	1.0185$^{\{5\}}$	-0.8886$^{\{4\}}$	0.3145$^{\{3\}}$
		λ	0.1065$^{\{1\}}$	1.0266$^{\{6\}}$	0.3065$^{\{3\}}$	0.6293$^{\{5\}}$	-0.6169$^{\{4\}}$	0.1692$^{\{2\}}$
	RMSE	α	0.0374$^{\{1\}}$	0.0803$^{\{2\}}$	0.2464$^{\{4\}}$	0.4682$^{\{5\}}$	0.4879$^{\{6\}}$	0.2115$^{\{3\}}$
		γ	0.5718$^{\{1\}}$	0.6524$^{\{2\}}$	0.7984$^{\{3\}}$	0.8428$^{\{4\}}$	0.9889$^{\{6\}}$	0.9684$^{\{5\}}$
		δ	0.3030$^{\{1\}}$	0.6713$^{\{2\}}$	1.4559$^{\{6\}}$	1.2890$^{\{5\}}$	0.9029$^{\{4\}}$	0.7861$^{\{3\}}$
		λ	0.4430$^{\{1\}}$	1.0389$^{\{6\}}$	0.7824$^{\{5\}}$	0.6556$^{\{3\}}$	0.7169$^{\{4\}}$	0.5140$^{\{2\}}$
		Sum of the ranks	**8**$^{\{1\}}$	**27**$^{\{2.5\}}$	**34**$^{\{5\}}$	**32**$^{\{4\}}$	**40**$^{\{6\}}$	**27**$^{\{2.5\}}$
400	AB	α	-0.0178$^{\{1\}}$	0.0306$^{\{3\}}$	0.0877$^{\{4\}}$	0.0292$^{\{2\}}$	-0.3198$^{\{6\}}$	0.0897$^{\{5\}}$
		γ	0.1130$^{\{1\}}$	0.2790$^{\{4\}}$	0.1510$^{\{3\}}$	0.1274$^{\{2\}}$	0.4033$^{\{5\}}$	0.7610$^{\{6\}}$
		δ	-0.0376$^{\{1\}}$	0.1394$^{\{2\}}$	1.0199$^{\{6\}}$	0.7749$^{\{5\}}$	-0.1666$^{\{3\}}$	0.2185$^{\{4\}}$
		λ	0.1032$^{\{1\}}$	0.1193$^{\{2\}}$	0.3046$^{\{5\}}$	0.3938$^{\{6\}}$	-0.2686$^{\{4\}}$	0.1457$^{\{3\}}$
	RMSE	α	0.0335$^{\{1\}}$	0.0618$^{\{2\}}$	0.1655$^{\{3\}}$	0.2677$^{\{5\}}$	0.3547$^{\{6\}}$	0.1758$^{\{4\}}$
		γ	0.4845$^{\{1\}}$	0.5814$^{\{3\}}$	0.6858$^{\{4\}}$	0.6898$^{\{5\}}$	0.5034$^{\{2\}}$	0.7745$^{\{6\}}$
		δ	0.2271$^{\{1\}}$	0.5801$^{\{3\}}$	1.1482$^{\{6\}}$	0.8364$^{\{5\}}$	0.6693$^{\{4\}}$	0.3047$^{\{2\}}$
		λ	0.3747$^{\{2\}}$	0.8191$^{\{6\}}$	0.3967$^{\{3\}}$	0.4831$^{\{5\}}$	0.3269$^{\{1\}}$	0.4134$^{\{4\}}$
		Sum of the ranks	**9**$^{\{1\}}$	**25**$^{\{2\}}$	**34**$^{\{4.5\}}$	**35**$^{\{6\}}$	**31**$^{\{3\}}$	**34**$^{\{4.5\}}$
800	AB	α	-0.0149$^{\{1\}}$	0.0191$^{\{2\}}$	0.0603$^{\{4\}}$	0.0260$^{\{3\}}$	-0.2393$^{\{6\}}$	0.0618$^{\{5\}}$
		γ	0.1093$^{\{2\}}$	0.1930$^{\{4\}}$	0.0862$^{\{1\}}$	0.1139$^{\{3\}}$	0.2874$^{\{5\}}$	0.5654$^{\{6\}}$
		δ	-0.0309$^{\{1\}}$	0.0961$^{\{3\}}$	0.9187$^{\{6\}}$	0.1061$^{\{4\}}$	-0.1227$^{\{5\}}$	0.0910$^{\{2\}}$
		λ	0.0290$^{\{1\}}$	0.0422$^{\{2\}}$	0.1596$^{\{5\}}$	0.3312$^{\{6\}}$	-0.1027$^{\{4\}}$	0.0855$^{\{3\}}$
	RMSE	α	0.0236$^{\{1\}}$	0.03914$^{\{2\}}$	0.1453$^{\{6\}}$	0.0538$^{\{3\}}$	0.1060$^{\{5\}}$	0.0907$^{\{4\}}$
		γ	0.4251$^{\{4\}}$	0.3934$^{\{3\}}$	0.2741$^{\{1\}}$	0.5593$^{\{5\}}$	0.2902$^{\{2\}}$	0.6274$^{\{6\}}$
		δ	0.1792$^{\{1\}}$	0.1954$^{\{2\}}$	0.9909$^{\{6\}}$	0.2929$^{\{5\}}$	0.1985$^{\{3\}}$	0.2759$^{\{4\}}$
		λ	0.3637$^{\{5\}}$	0.1282$^{\{1\}}$	0.2661$^{\{3\}}$	0.3697$^{\{6\}}$	0.2104$^{\{2\}}$	0.2785$^{\{4\}}$
		Sum of the ranks	**16**$^{\{1\}}$	**19**$^{\{2\}}$	**32**$^{\{3.5\}}$	**35**$^{\{6\}}$	**32**$^{\{3.5\}}$	**34**$^{\{5\}}$

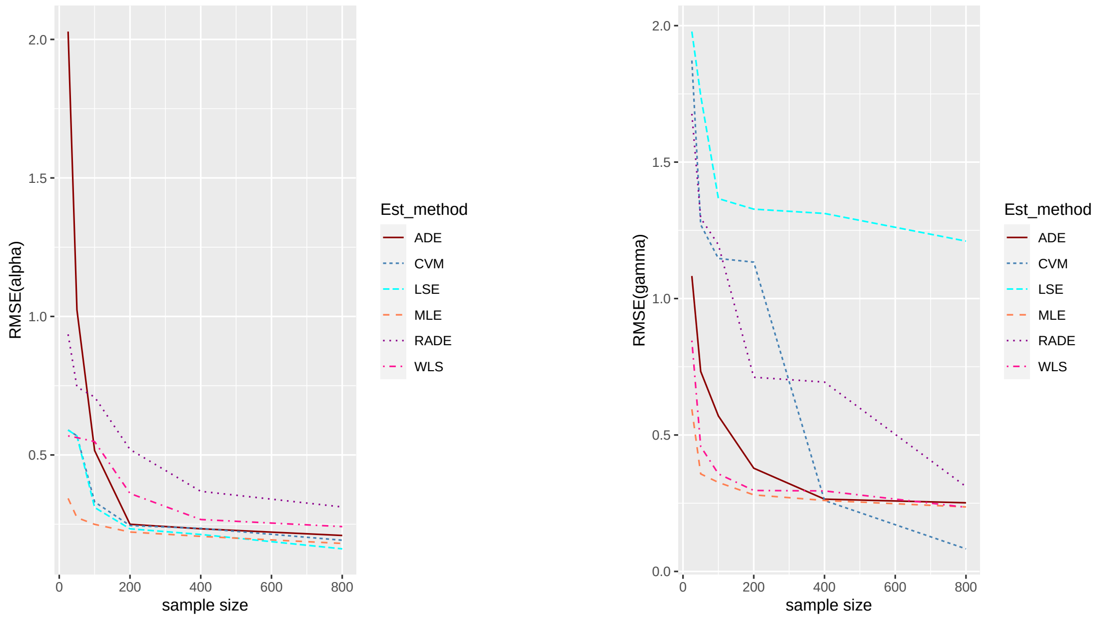

Figure 10. RMSE plots for parameters in Table 7

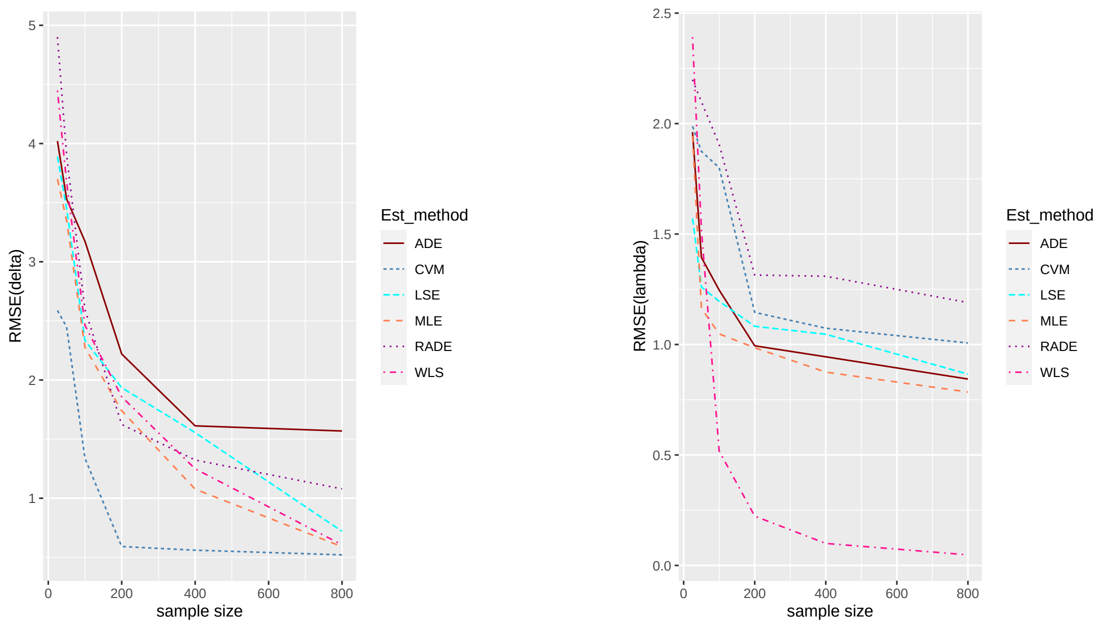

Figure 11. RMSE plots for parameters in Table 7

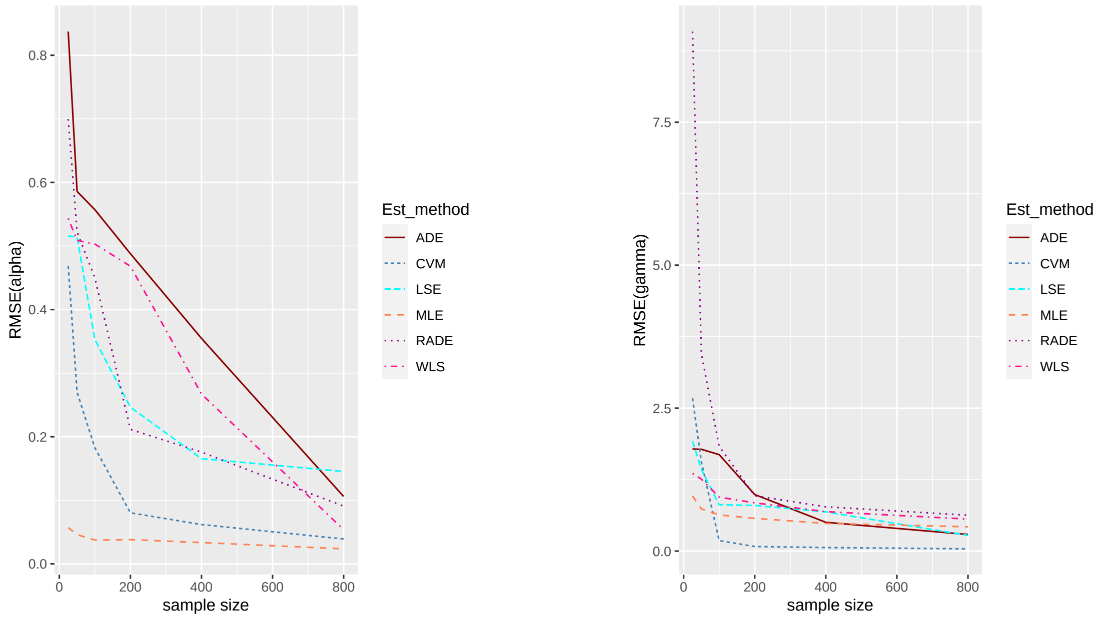

Figure 12. RMSE plots for parameters in Table 8

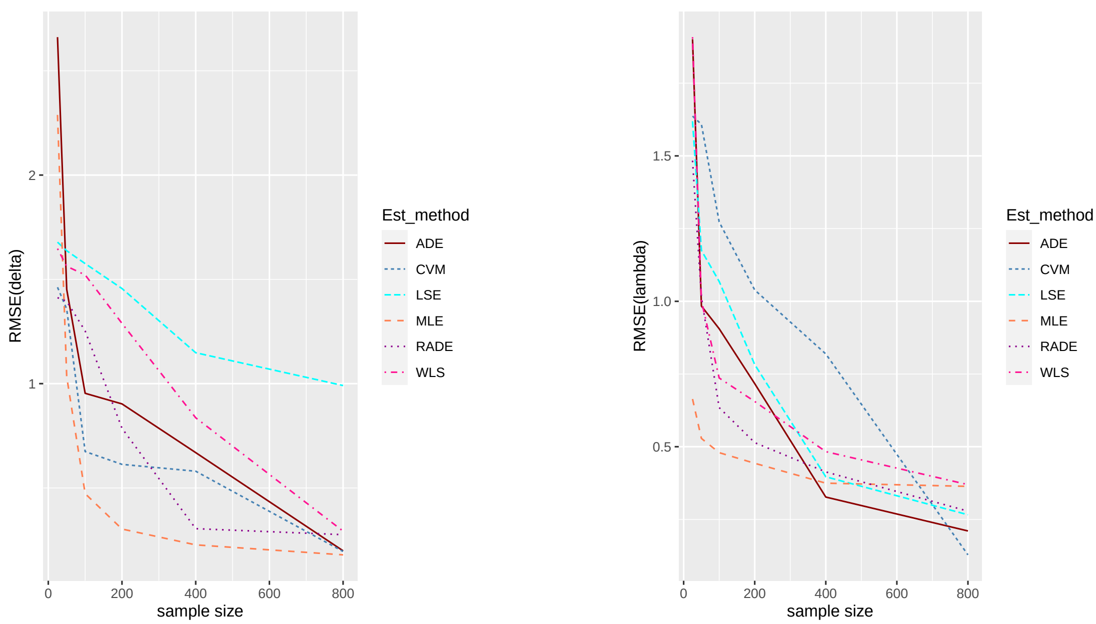

Figure 13. RMSE plots for parameters in Table 8

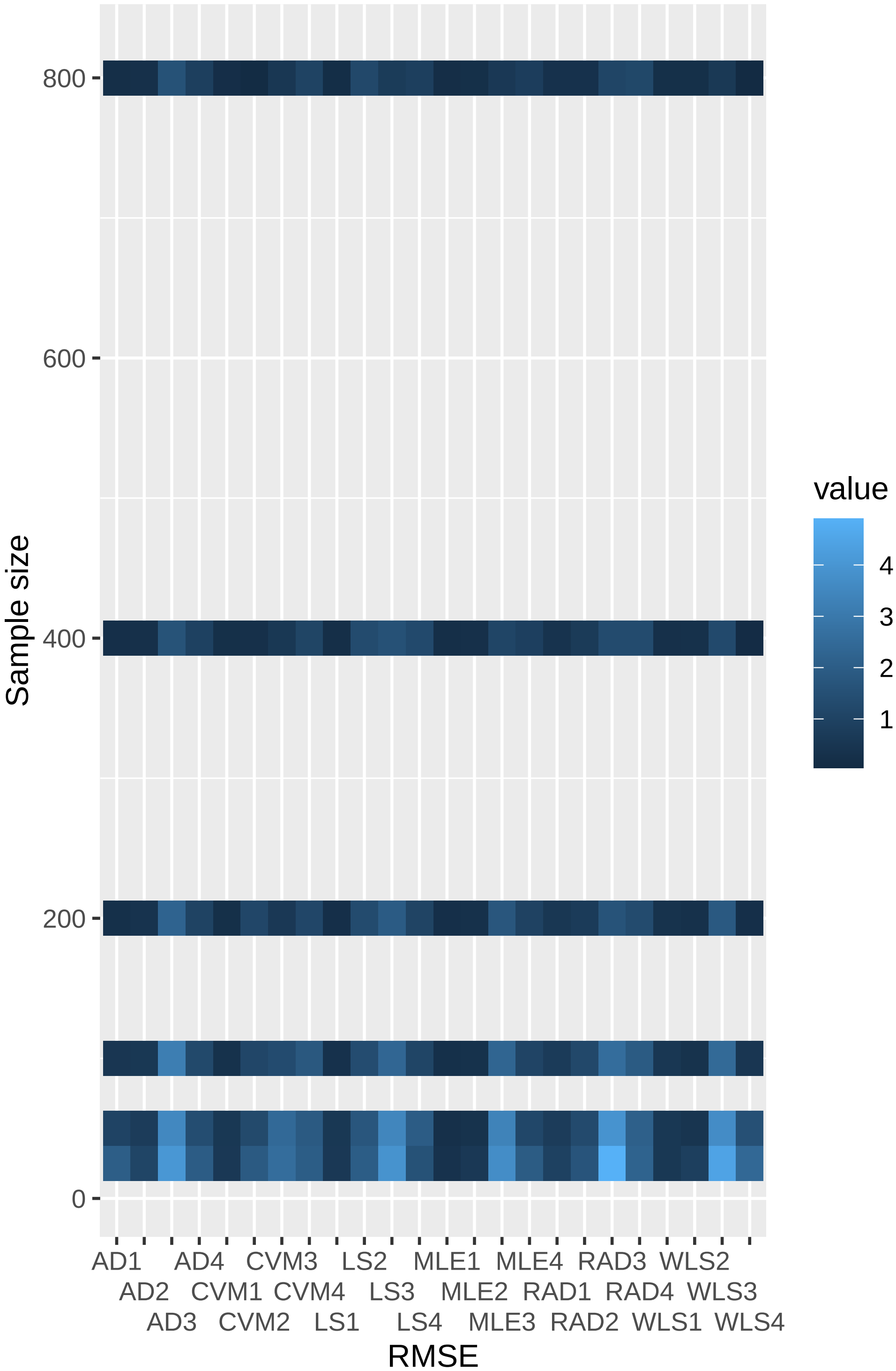

Figure 14. Heat map for parameters in Table 7

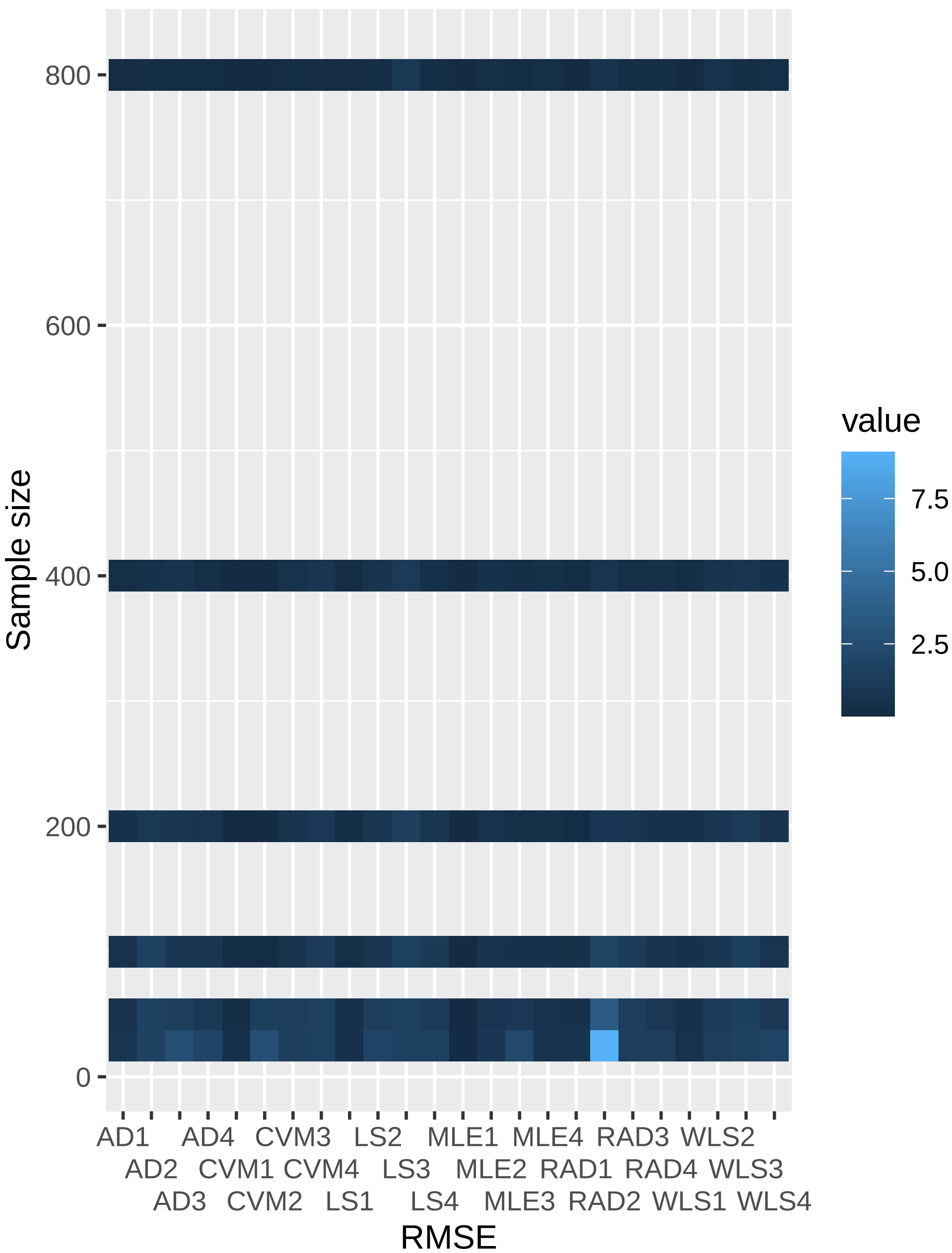

Figure 15. Heat map for parameters in Table 8

Table 9. **Partial and overall ranks of the estimation techniques for EGom-MO-W distribution**

		Partial ranks					
Parameter	**Sample size(n)**	**MLE**	**CVM**	**LSE**	**WLSE**	**ADE**	**RADE**
(0.6,0.6,0.6,2.2)	25	1	2	3	4.5	6	4.5
	50	1	2.5	4	2.5	5	6
	100	1	2	4	3	6	5
	200	1	2.5	6	4	2.5	5
	400	1	2	5	4	3	6
	800	1	2	6	3	4	5
(0.3,1.4,1.2,2.0)	25	1	5	4	2	6	3
	50	1	3	4	2	6	5
	100	1	2	4	3	6	5
	200	1	2.5	5	4	6	2.5
	400	1	2	4.5	6	3	4.5
	800	1	2	3.5	6	3.5	5
	Sum of the ranks	**12**	**29.5**	**53**	**44**	**57**	**56.5**
	Overall ranks	**1**	**2**	**4**	**3**	**6**	**5**

8. Applications

In this section, the EGom-MO-W distribution is applied to some real data sets. We will first consider application to uncensored data, then follow with application to censored data. Goodness-of-fit statistics including Akaike Information criterion (AIC), Bayesian Information Criterion (BIC), Consistent Akaike Information Criterion (CAIC), Kolmogorov-Smirnov (K-S) statistic, Cramér-von Mises (W*) and Anderson-Darling (A*) statistics have been used to compare distributions. The best performing model would be the one which assumes the smallest values of AIC, BIC, CAIC, W*, A*, and the highest p-value of the K-S statistic.

In addition, graphical measures will be explored including, probability plots, fitted pdfs, estimated cumulative distribution function (ECDF), Kaplan-Meier (K-M) survival plots, hazard rate function plots, profile plots and Total Time on Test (TTT) plots.

8.1. Uncensored Data

The EGom-MO-W distribution is fitted to two uncensored data sets. The results of the fitted EGom-MO-W distribution are then compared to some selected non-nested distributions. The distributions are: Marshall-Olkin Kappa (MOK) by Javed [36], Kumaraswamy Weibull (KwW) by Cordeiro et al. [21], Topp-Leone-Marshall-Olkin Weibull (TLMOW) by Ahmed et al. [2], Alpha power Topp-Leone Weibull (APTLW) by Benkhelifa [13], type II generalized inverse exponentiated Burr III (TIIGIBIII) by Jamal et al. [35], type II exponentiated half logistic-Weibull (TIIEHLW) by Hassan [33], Beta Odd Lindley-Exponential distribution (BOLE) by Chipepa et al. [18], type II generalized inverse exponentiated Lomax (TIIGIEL) by Jamal et al. [35], Marshall-Olkin-Gompertz (MO-G) by Eghwerido et al. [23], and type II Topp-Leone-Dagum (TIITLD) by Sakthivel & Dhivakar [53]. The pdfs of these distributions are given in the Appendix.

8.1.1. Recidivism Data We consider recidivism data studied by Stollmack & Harris [57]. The data is given in the appendix. Results of the fitted non-nested and nested models are presented in Tables 10 and 11, respectively.

Table 10. **Parameter Estimates and Goodness-of-Fit Statistics for various Models for Recidivism Data**

	Estimates of Parameters and Standard Error in Parenthesis				Goodness-of-Fit-Statistics								
Distribution	α	γ	δ	λ	-2log(L)	AIC	CAIC	BIC	W^*	A^*	K−S	P−value	SS
EGom-MO-W	6.4768×10^{-1} (8.8548×10^{-2})	1.9906×10^{4} (2.2904×10^{-6})	1.0224×10^{4} (5.6880×10^{-6})	1.0336×10^{-1} (1.2202×10^{-3})	760.4481	768.4481	776.8916	840.6905	0.0208	0.1549	0.0543	0.9938	0.0214
TLMOW	b 6.3079×10^{5} (6.2048×10^{-8})	δ 8.2895×10^{-4} (3.9485×10^{-4})	γ 2.2474×10^{-1} (3.1632×10^{-2})	λ 1.9644×10^{-1} (9.7714×10^{-2})	830.5825	838.5825	839.2968	847.0260	0.7440	4.3299	0.2429	0.0015	0.0483
MOK	a 2.1077×10^{2} (6.1284×10^{-9})	b 2.9871×10^{2} $(3.5522\times^{-8})$	λ 1.4961×10^{-2} $(7.2166\times^{-4})$	θ 7.5584×10 $(8.7230\times^{-8})$	769.7502	777.7501	778.4644	786.1936	0.0872	0.5863	0.0733	0.8980	20.2906
TIITLD	τ 1.6578×10^{5} (3.4070×10^{-9})	δ 2.7320×10^{4} (3.9645×10^{-8})	θ 7.2224×10^{-1} (4.6604×10^{-2})	β $9.5496\times^{-1}$ (4.1650×10^{-2})	767.5380	775.5380	776.2523	783.9815	0.1061	0.6724	0.0966	0.6199	0.0309
APTLW	θ 3.1484×10^{-1} (7.9183×10^{-11})	α 1.6346×10 (4.5648×10^{-13})	β 1.8322 (2.7913×10^{-10})	λ 1.8488×10^{-5} (2.5868×10^{-6})	761.1153	769.1152	769.8295	777.5587	0.0311	0.2117	0.0590	0.9837	20.2906
KwW	a 3.4817×10^{-1} (3.6715×10^{-2})	b 7.9314 (7.8835×10^{-5})	α $8.4208\times^{-4}$ (1.4013×10^{-4})	β 3.6986 (3.3020×10^{-3})	765.2955	3062.7820	3063.4960	3071.2250	0.4527	2.7036	1.0	2.2×10^{-16}	1.1453
MO-G	β 10.899 (1.3385×10^{-6})	μ 6.3952×10^{-12} (8.7860×10^{-4})	α 1.1514×10^{-2} (1.8482×10^{-3})		766.2612	773.397	773.818	779.7296	0.0405	0.2750	0.1201	0.3425	0.2077

Table 11. **Parameter Estimates and Goodness-of-Fit Statistics for nested Models for Recidivism Data**

	Estimates of Parameters and Standard Error in Parenthesis				Goodness-of-Fit Statistics							
Distribution	α	γ	δ	λ	-2log(L)	AIC	CAIC	BIC	W^*	A^*	K−S	P−value
EGom-MO-W	5.3309×10^{1} (6.6625×10^{-4})	1 -	9.6499×10^{-1} (7.9707×10^{-2})	9.9996×10^{-2} (7.7741×10^{-3})	798.1493	804.1493	804.5704	810.4819	0.4641	2.7635	0.1635	0.0768
EGom-MO-W	1 -	6.0016×10^{7} (1.0566×10^{-12})	7.0963×10^{6} (5.2942×10^{-7})	2.5255×10^{-2} (3.7016×10^{-3})	792.7892	798.7883	799.2094	805.1210	0.1571	0.9779	0.2619	0.0005
EGom-MO-W	53.9628 (9.3596)	1 -	1 -	0.1049 (0.0043)	797.0294	801.0294	801.2363	805.2512	0.4651	2.7683	0.1515	0.1217

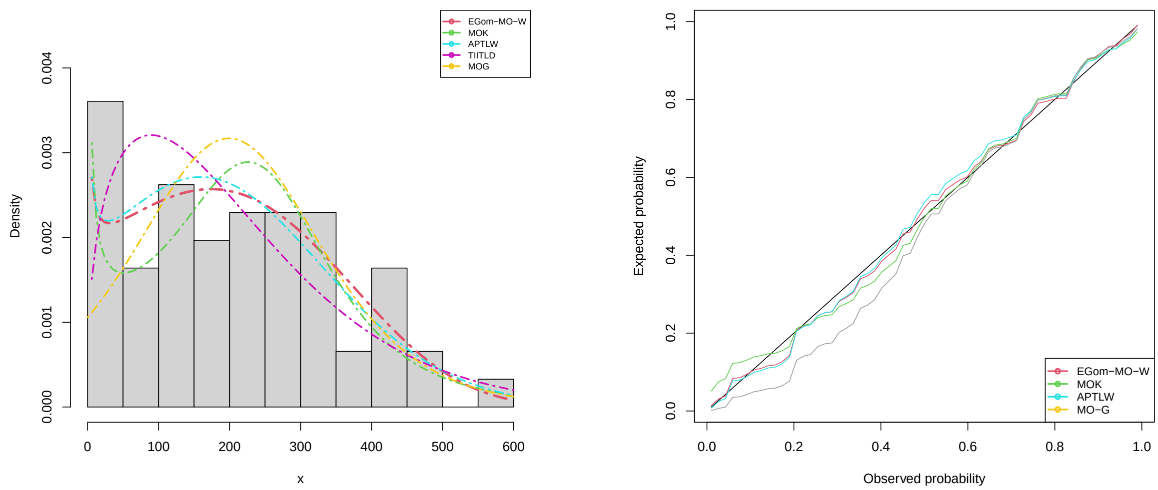

Figure 16. Fitted Densities and Probability Plots for Recidivism Data

From values of the goodness-of-fit statistics in Table 10, we observe that the EGom-MO-W family of distributions offers a better fit compared to the other models. Parameters of the EGom-MO-W distribution and their corresponding standard errors are significant, showing that addition of new parameters is important for the flexibility of the new distribution. Figure 16 shows that EGom-MO-W distribution fits the recidivism data well, as shown by the density curve taking the shape of the histogram and the probability curve not deviating much from an almost straight line.

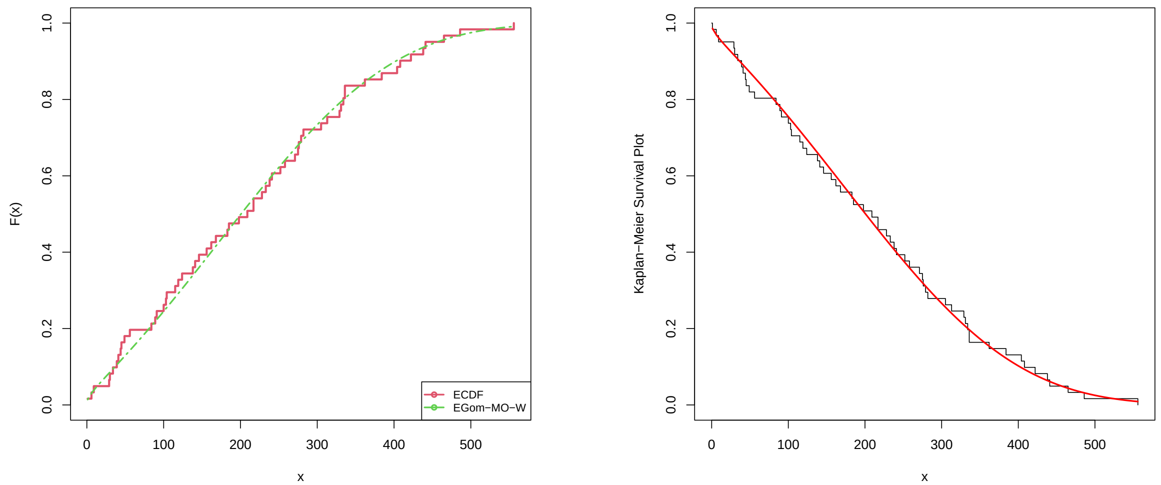

Figure 17. Estimated Cdf and Kaplan-Meier Survival Plots of the EGom-MO-W Distribution for Recidivism Data

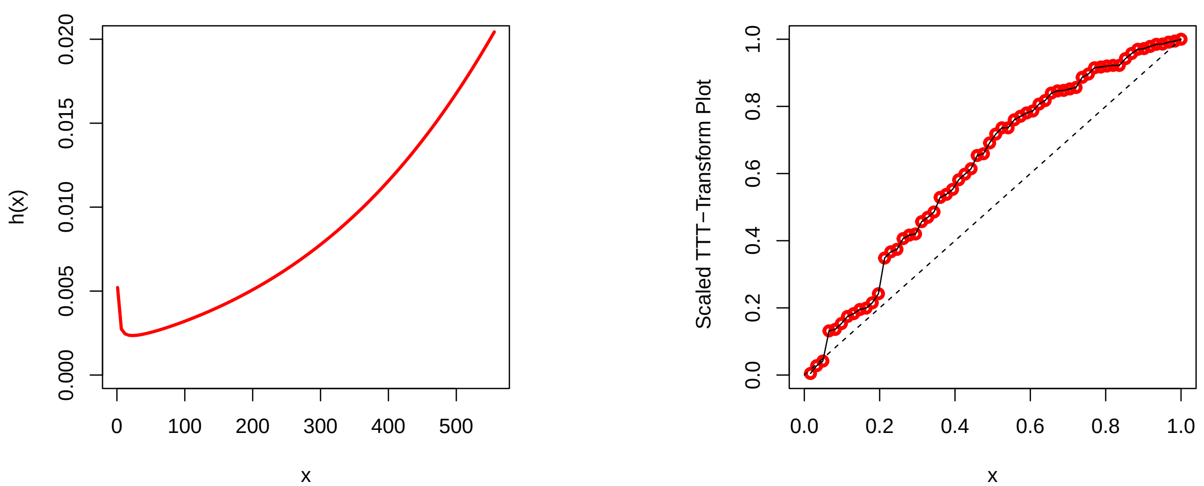

Figure 18. Estimated Hazard Rate Function and TTT Plot of the EGom-MO-W Distribution for Recidivism Data

Figure 17 shows that the ECDF and Kaplan -Meier survival plots follows the fitted EGom-MO-W distribution. The hazard rate function of the EGom-MO-W distribution is increasing, the TTT plot confirms this as observed from Figure 18.

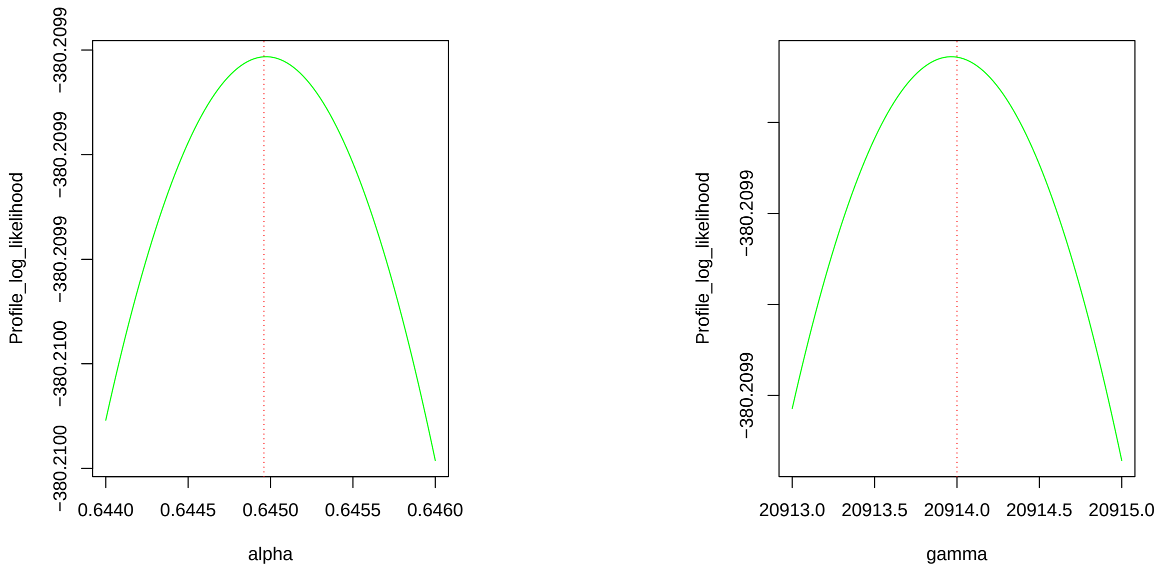

Figure 19. Profile log-likelihood Plots of the EGom-MO-W Distribution for Recidivism Data

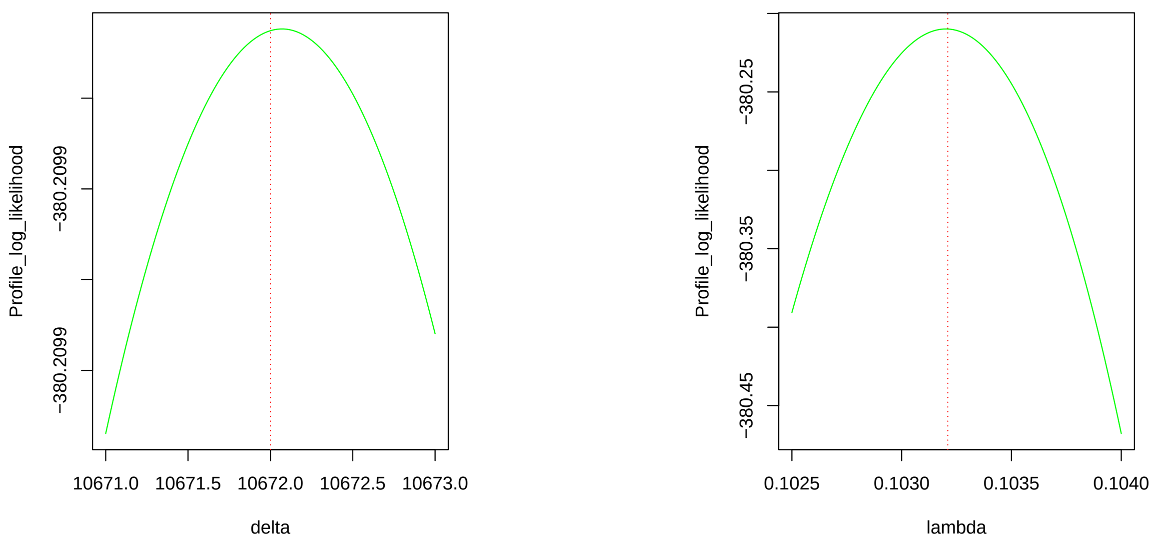

Figure 20. Profile log-likelihood Plots of the EGom-MO-W Distribution for Recidivism Data

We observe from profile log-likelihood plots in Figures 19 and 20 that estimates of the EGom-MO-W distribution are indeed at the maximum.

The estimated variance-covariance matrix for the recidivism data is presented as

$$\begin{bmatrix} 7.8429 \times 10^{-3} & -2.0291 \times 10^{-7} & 5.0389 \times 10^{-7} & 3.7958 \times 10^{-5} \\ -2.0291 \times 10^{-7} & 5.2499 \times 10^{-12} & -1.3037 \times 10^{-11} & -9.7774 \times 10^{-10} \\ 5.0389 \times 10^{-7} & -1.3037 \times 10^{-11} & 3.2373 \times 10^{-11} & 2.4295 \times 10^{-9} \\ 3.7958 \times 10^{-5} & -9.7774 \times 10^{-10} & 2.4295 \times 10^{-9} & 1.4914 \times 10^{-6} \end{bmatrix},$$

and the 95% asymptotic confidence intervals for the parameters α, γ, δ and λ are given as $6.4768 \times 10^{-1} \pm 1.7358 \times 10^{-1}, 1.9906 \times 10^{4} \pm 4.4909 \times 10^{-6}, 1.0224 \times 10^{4} \pm 1.1152 \times 10^{-5},$ and $1.0336 \times 10^{-1} \pm 2.3936 \times 10^{-3}$, respectively.

8.1.2. Endurance of Deep Groove Ball Bearings Data We consider the second data representing endurance of deep groove ball bearing by Gosh & Nadarajah [28]. The data is given in the Appendix.

Results of the fitted non-nested and nested models are presented in Table 12 and Table 13, respectively.

Table 12. **Parameter Estimates and Goodness-of-Fit Statistics for various Models for Endurance of Deep Groove Ball Bearings Data**

	Estimates of Parameters and Standard Error in Parenthesis				Goodness-of-Fit Statistics								
Distribution	α	γ	δ	λ	-2log(L)	AIC	CAIC	BIC	W^*	A^*	K−S	P−value	SS
EGom-MO-W	14.2210 (0.3638)	18.4098 (0.5522)	14.5052 (1.0903)	0.1000 (0.0121)	225.9365	233.9365	236.1588	238.4785	0.03250	0.1890	0.1072	0.9543	0.0294
TIIGIEBIII	λ 3.6448×10^{3} (2.5220×10^{-5})	θ 4.3648×10^{1} (7.4556×10^{-3})	c 1.4843×10^{-1} (1.9902×10^{-2})	k 3.9670 (2.4270×10^{-1})	229.6205	237.6205	239.8427	242.1624	7.3547	45.7687	0.9975	2.2×10^{-16}	6.9824
BOLE	a 3.9975 (3.9552×10^{-10})	b 1.8145×10 (8.9657×10^{-11})	λ 2.2544×10 (8.0887×10^{-11})	θ 1.3020×10^{-4} (1.3517×10^{-5})	226.0660	234.0660	236.2882	238.6080	0.0389	0.2171	0.1231	0.8765	0.0359
TIIEHLW	a 1.9050×10^{2} (1.3156×10^{-5})	λ 2.5324×10^{2} (5.8942×10^{-4})	δ 2.5087 (1.6033×10^{-1})	γ 9.2824×10^{-2} (1.4187×10^{-2})	226.243	234.2429	236.4651	238.7849	0.0432	0.2390	0.1273	0.8502	7.6221
TIITLD	τ 51.4063 (0.3414)	δ 1.4604 (2.1203)	θ 0.5501 (0.0880)	β 5.7502 (18.6346)	225.9654	233.9654	236.1876	238.5074	0.0335	0.1938	0.1092	0.9466	0.0301
TIIGIEL	λ 3.3081×10^{-1} (2.1529×10^{-1})	α 4.7833×10^{5} (1.4633×10^{-10})	a 6.5309×10^{-3} (1.8481×10^{-3})	b 1.9133×10^{5} $(7.6159 \times^{-10})$	231.9623	239.9623	242.1845	244.5043	0.1078	0.6436	0.1550	0.6386	0.0992
MO-G	β 18.267 (13.265)	μ 3.0100×10^{-10} (6.6126×10^{-3})	α 4.3590×10^{-2} (2.3557×10^{-2})		228.9341	234.7131	235.9763	238.1196	0.0782	0.4315	0.1406	0.7538	0.0555

Table 13. **Parameter Estimates and Goodness-of-Fit Statistics for Nested Models for Endurance of Deep Groove Ball Bearings Data**

	Estimates of Parameters and Standard Error in Parenthesis				Goodness-of-Fit Statistics							
Distribution	α	γ	δ	λ	-2log(L)	AIC	CAIC	BIC	W^*	A^*	K−S	P−value
EGom-MO-W	2.6889×10^{7} (4.1548×10^{-9})	1 -	1.4758×10^{-1} (1.2922×10^{-8})	5.8633×10^{-2} (1.9187×10^{-3})	230.0694	236.0683	237.3315	239.4748	0.0533	0.4032	0.1213	0.8873
EGom-MO-W	1 -	5.5354×10^{-9} (3.4030×10^{-2})	1 -	1.9134×10^{-1} (2.6136×10^{-2})	332.7376	339.0086	337.3376	339.0086	0.0298	0.1864	0.8238	5.5290×10^{-14}
EGom-MO-W	2.0×10^{-2} (4.1703×10^{-3})	1.0416×10^{-13} (1.3775×10^{-4})	1 -	1 -	3502.1080	3506.1120	3506.7120	3508.3830	1.4049	6.7253	1.0	2.20×10^{-16}

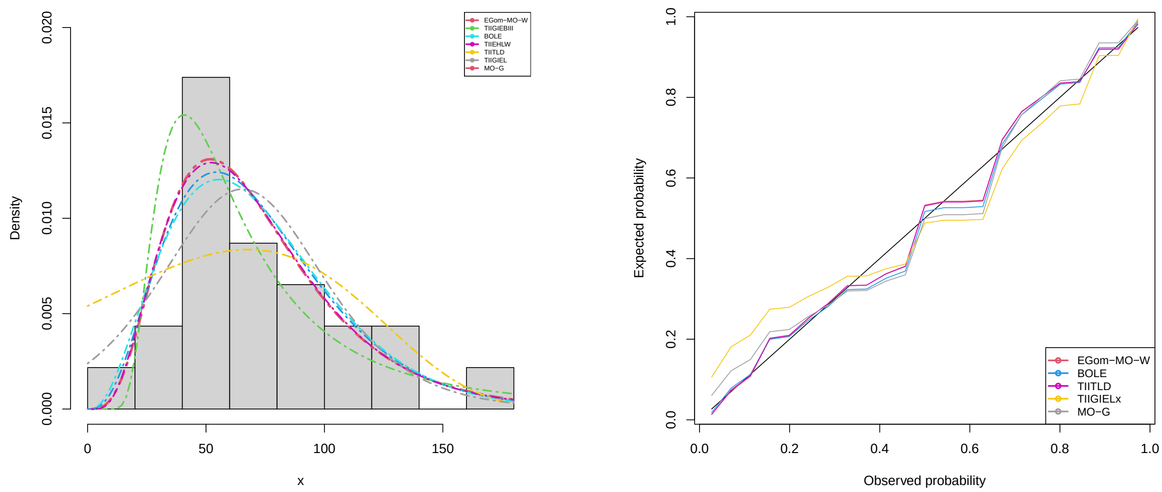

Figure 21. Fitted Densities and Probability Plots for Endurance of Deep Groove Ball Bearings Data

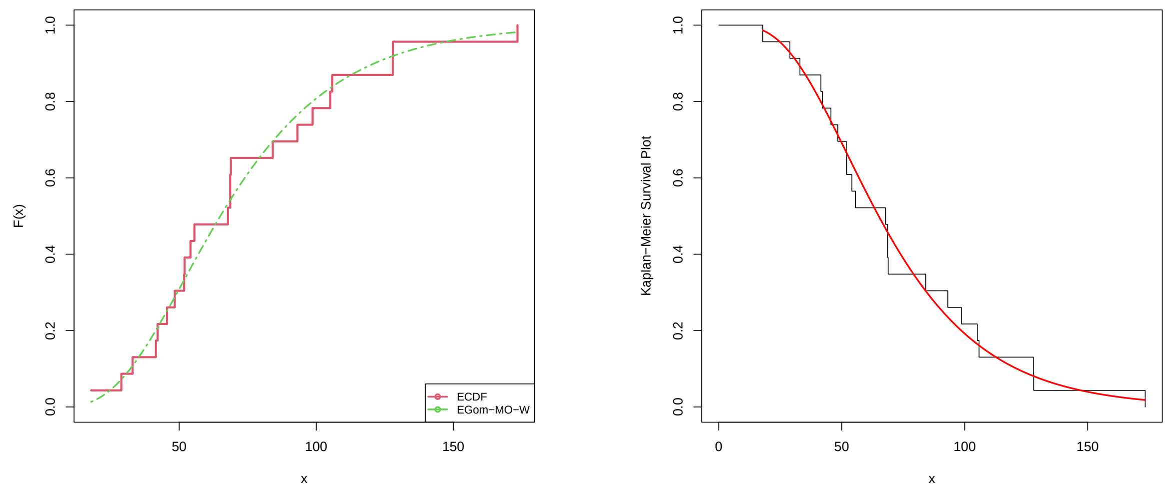

Figure 22. Estimated Cdf and Kaplan-Meier Survival Plots of the EGom-MO-W Distribution for Endurance of Deep Groove Ball Bearings Data

We observe from Table 12 that the EGom-MO-W distribution has the least values of AIC, CAIC, BIC, W*, A*, and the highest K-S p-value compared to the other models. In addition, Table 12 shows that parameters of the EGom-MO-W distribution and their corresponding standard errors are significant. This means adding extra parameters to the new distribution is considered important for its flexibility. Furthermore, the plots in Figure 21 show that the EGom-Mo-W distribution fits the deep groove ball bearings data well. The EGom-MO-W distribution takes the shape of the histogram, and the probability curve lies close to the probability plot line. In addition,the ECDF and Kaplan-Meier survival plots follows the fitted EGom-MO-W distribution as observed from Figure 22.

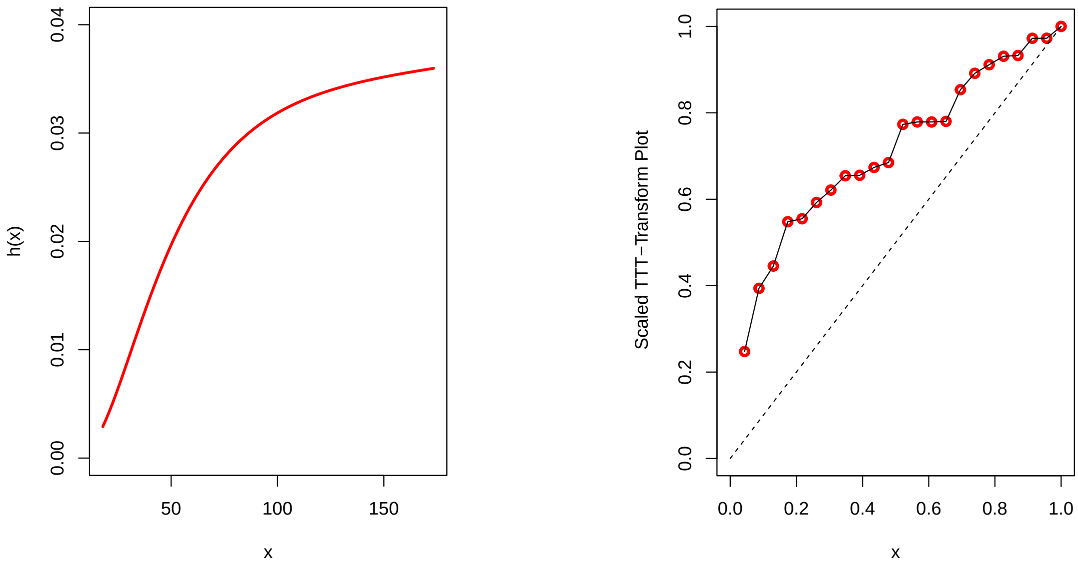

Figure 23. Estimated Hazard Rate Function and TTT Plot of the EGom-MO-W Distribution for Endurance of Deep Groove Ball Bearings Data

From Figure 23, the TTT plot confirms that the hazard rate function of the EGom-MO-W distribution is increasing.

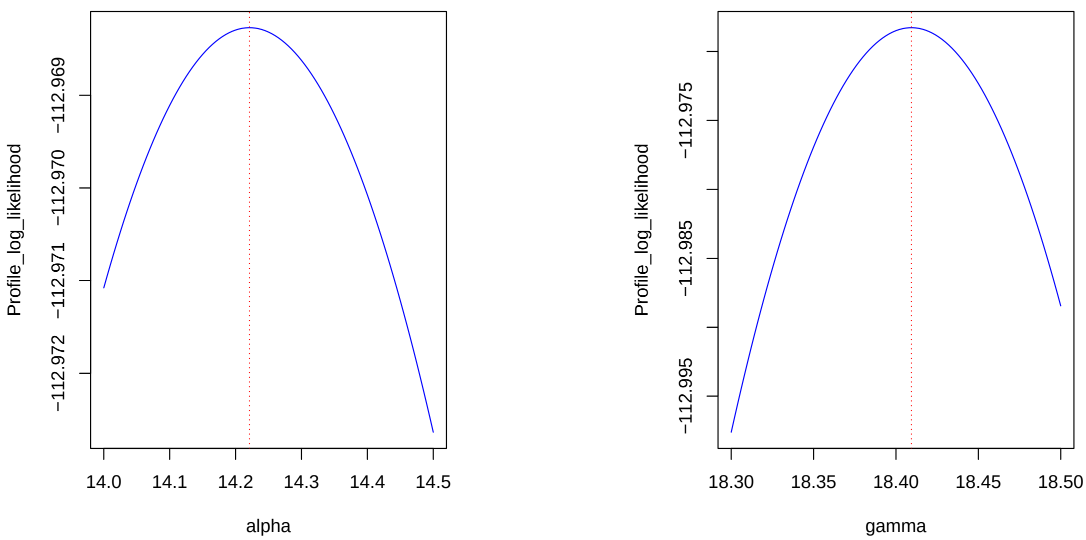

Figure 24. Profile log-likelihood Plots of the EGom-MO-W Distribution for Endurance of Deep Groove Ball Bearings Data

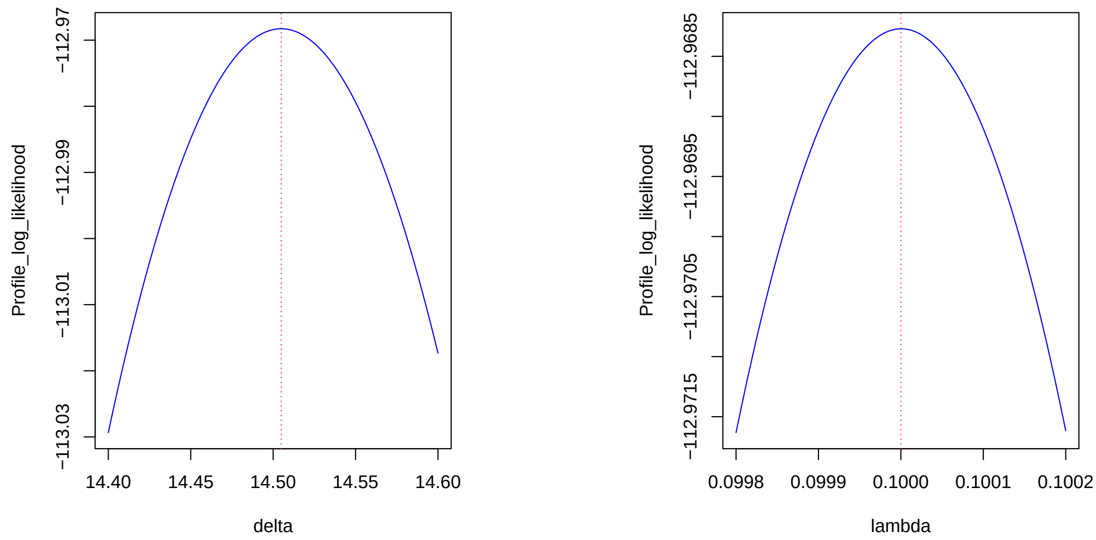

Figure 25. Profile log-likelihood Plots of the EGom-MO-W Distribution for Endurance of Deep Groove Ball Bearings Data

Figures 24 and 25 show that indeed parameter values of the EGom-MO-W distribution are at their maximum.

The estimated variance-covariance matrix is given as

$$\begin{bmatrix} 0.1324 & -0.2009 & 0.3967 & 0.0043 \\ -0.2009 & 0.3049 & -0.6021 & -0.0065 \\ 0.3967 & -0.6021 & 1.1887 & 0.0129 \\ 0.0043 & -0.0065 & 0.0129 & 0.0001 \end{bmatrix},$$

and the 95% asymptotic confidence intervals are then given as $14.2210 \pm 1.7358 \times 10^{-1}, 18.4098 \pm 4.4909 \times 10^{-6}, 14.5052 \pm 1.1152 \times 10^{-5}$ and $0.1 \pm 2.3936 \times 10^{-3}$, respectively.

8.1.3. Likelihood-Ratio Tests We present likelihood ratio test results for comparing the EGom-MO-W distribution with its nested distributions.

Table 14. **Likelihood Ratio Test Results**

Recidivism data			Endurance of deep groove ball bearings data		
Model	df	χ^2(P-value)	Model	df	χ^2(P-value)
EGom-MO-W$(\alpha, 1, \delta, \lambda)$	1	$37.7012(< 0.001)$	EGom-MO-W$(\alpha, 1, \delta, \lambda)$	1	4.1299(0.04)
EGom-MO-W$(1, \gamma, \delta, \lambda)$	1	$32.3411(< 0.001)$	EGom-MO-W$(1, \gamma, 1, \lambda)$	2	$106.8011(< 0.001)$
EGom-MO-W$(1, 1, \delta, \lambda)$	2	$36.5813(< 0.001)$	EGom-MO-W$(\alpha, \gamma, 1, 1)$	2	$3276(< 0.001)$

It is quite apparent from Table 14 that the EGom-MO-W distribution is signigficantly better than its nested models at 5% level of significance.

8.2. Censored Data

We further fitted the EGom-MO-W distribution to censored data and compared its performance against some existing non-nested model considered in Subsection 8.1.

8.2.1. Cancer data We consider data representing the survival times of 45 cancer patients who were put on a combined chemotherapy and radiotherapy treatment. We demonstrate type I right censoring using the data. The data was also studied by Shakhatreh et al. [56] and Klein & Moeschberger [39]. The data is presented in the Appendix.

Table 15. **Parameter Estimates and Goodness-of-Fit Statistics for various Models for Cancer Patients Data**

	Estimates of Parameters and Standard Error in Parenthesis				Goodness-of-Fit Statistics			
Distribution	α	γ	δ	λ	-2log(L)	AIC	CAIC	BIC
EGom-MO-W	5.8555×10^{2}	1.6313×10	6.0288	1.7169×10^{-2}	576.337	678.9049	679.9049	686.1316
	(6.4599×10^{-5})	(3.1142×10^{-2})	(9.7752×10^{-2})	(1.9581×10^{-3})				
	λ	θ	c	k				
TIIGIEBIII	6.2249	8.5324×10^{2}	8.8473×10^{-2}	1.2249	539.345	684.3715	685.3715	691.5982
	(1.1629×10^{-1})	(6.8614×10^{-4})	(8.4643×10^{-3})	(4.2725×10^{-1})				
	a	λ	δ	γ				
TIIEHLW	1.5222	8.5123×10^{2}	4.0105	7.8947×10^{-2}	575.6949	680.3408	681.3408	687.5674
	(7.6746×10^{-1})	(2.1129×10^{-4})	(2.7383×10^{-1})	(1.9033×10^{-2})				
	τ	δ	θ	β				
TIITLD	4.0118×10^{-1}	2.1462×10^{3}	1.6849	6.7618×10^{-1}	574.2418	681.9246	682.9246	689.1512
	(1.4032×10^{-1})	(1.4659×10^{-5})	(1.8872×10^{-1})	(2.6129×10^{-1})				
	λ	α	a	b				
TIIGIEL	1.7740×10^{2}	2.2952	2.4270×10^{-3}	3.2151×10^{2}	580.1892	684.4489	685.4489	691.6756
	(5.2632×10^{-9})	(4.0854×10^{-7})	(3.8710×10^{-4})	(1.6505×10^{-9})				
	a	b	λ	θ				
MOK	1.1922×10^{-1}	1.4695×10^{-2}	3.0952×10	1.7060×10^{3}	574.1477	683.7532	684.7532	690.9799
	(3.6622×10^{-2})	(1.1544×10^{-3})	(8.6639×10^{-7})	(1.9079×10^{-6})				

Table 15 reveals that the EGom-MO-W distribution performs better than the non-nested models as supported by smallest values of AIC, BIC and CAIC. Furthermore, since parameters of the EGom-MO-W distribution are significant, we can conclude that addition of extra parameters to the model was necessary.

9. Conclusions

In this paper, we obtained a new generalized distribution called exponentiated-Gompertz-Marshall-Olkin-G (EGom-MO-G) family of distributions. Some mathematical properties such as the linear representation of the density function, distribution of order statistics, Rényi entropy, probability weighted moments, and quantile function were derived. Furthermore, three special cases from this new family of distributions were investigated. Results from the Monte Carlo simulations showed that maximum likelihood estimation (MLE) technique outperformed other estimation methods. Subsequently, MLE technique was adopted for applications of three real data sets of which one was censored and two were complete data sets. For all the concerned data sets, the EGom-MO-W distribution demonstrated unparalleled performance against related equi-parameter non-nested models. This was supported by the visuals and goodness-of-fit statistics. Thus, results from this paper show that the new developed family of distributions can be useful in fitting data across disciplines.

Appendix

The appendix can be accessed via the following google drive link.

https://drive.google.com/file/d/1RjUwVNdlZrIY3F6Q5h6jevOIQe_51Ep8/view?usp=share_link

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