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Prediction problem for continuous time stochastic processes with periodically correlated increments observed with noise

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Abstract We propose solution of the problem of the mean square optimal estimation of linear functionals which depend on the unobserved values of a continuous time stochastic process with periodically correlated increments based on observations of this process with periodically stationary noise. To solve the problem, we transform the processes to the sequences of stochastic functions which form an infinite dimensional vector stationary sequences. In the case of known spectral densities of these sequences, we obtain formulas for calculating values of the mean square errors and the spectral characteristics of the optimal estimates of the functionals. Formulas determining the least favorable spectral densities and the minimax (robust) spectral characteristics of the optimal linear estimates of functionals are derived in the case where the sets of admissible spectral densities are given.

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1. Introduction

In this paper, we study the prediction problem for a continuous time stochastic process $\xi(t)$, $t\in\mathbb{R}$, with periodically correlated increments $\xi^{(d)}(t,\tau T)=\Delta^d_{T\tau}\xi(t)$ of order d and period T, where $\Delta_s\xi(t)=\xi(t)-\xi(t-s)$, based on observations of the process $\xi(t)$ with a periodically correlated noise stochastic process $\eta(t)$, $t\in\mathbb{R}$. The resent studies, for example, by Basawa et al. [1], Dudek et al. [6], Reisen et al. [40], show a constant interest to the non-stationary models and robust methods of estimation.

1.1. A brief review of the previous results and the literature

Kolmogorov [18], Wiener [47] and Yaglom [49] developed effective methods of solution of interpolation, extrapolation (prediction) and filtering problems for stationary stochastic sequences and processes. For a particular problem, they developed methods of finding an estimate $\widetilde{x}(t)$ constructed from available observations that minimizes the mean square error $\Delta(\widetilde{x}(t),f)=\mathsf{E}|x(t)-\widetilde{x}(t)|^2$ in the case where the spectral density $f(\lambda)$ of the stationary process or sequence $x(t)$ is exactly known and fixed. Such estimates are called optimal linear estimates within this article.

The developed classical estimation methods are not directly applicable in practice since the exact spectral structure of the processes is not usually available. In this case the estimated spectral densities can be considered as the true ones. However, Vastola and Poor [46] showed with the help of the concrete examples, that such substitution can result in a significant increase of the estimate error. Therefore it is reasonable to consider the estimates, called

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minimax-robust, which minimize the maximum of the mean-square errors for all spectral densities from a given set of admissible spectral densities simultaneously. The minimax-robust method of extrapolation was proposed by Grenander [12] who considered the estimation of the functional $Ax = \int_0^1 a(t)x(t)dt$ as a game between two players, one of which minimizes the mean square error $\Delta(f, \widetilde{A}x)$ by $\widetilde{A}\zeta$ and another one maximizes the error by f. He showed that the game has a saddle point solution under proper conditions. For more details see the further study by Franke and Poor [7] and the survey paper by Kassam and Poor [17]. A wide range of results has been obtained by Moklyachuk [26, 27, 28, 29, 30]. These results have been extended on the vector-valued stationary processes and sequences by Moklyachuk and Masyutka [32].

The concept of stationarity admits some generalizations, a combination of two of which – stationary dth increments and periodical correlation – is in scope of this article. Random processes with stationary dth increments $x(t)$ were introduced by Yaglom and Pinsker [38]. The increment sequence $x^{(d)}(t,\tau) = \Delta_\tau^d x(t)$ generated by such process is stationary by the variable t, namely, the mathematical expectations $\mathsf{E}x^{(d)}(t,\tau)$ and $\mathsf{E}x^{(d)}(t+s,\tau_1)\overline{x^{(d)}(t,\tau_2)}$ do not depend on t. Yaglom and Pinsker [38] described the spectral representation of such process and the spectral density canonical factorization, and they also solved the extrapolation problem for these processes. The minimax-robust extrapolation, interpolation and filtering problems for stochastic processes with stationary increments were investigated by Luz and Moklyachuk [21].

Dubovetska and Moklyachuk [5] derived the classical and minimax-robust estimates for another generalization of stationary processes – periodically correlated (cyclostationary) processes, introduced by Gladyshev [11]. The correlation function $K(t,s) = \mathsf{E}x(t)\overline{x(s)}$ of such processes is a T-periodic function: $K(t,s) = K(t+T, s+T)$, which implies a time-dependent spectrum.

Periodically correlated processes are widely used in signal processing and communications, see the books by Gardner [8], Hurd and Miamee [14], Napolitano [34] and the reviews by Napolitano [36, 35], Gardner et al. [9], Serpedin et al. [42]. In the recent decade a major contribution to the topic was made by the Workshops on Cyclostationary Systems and Their Applications, Grodek, Poland, [2, 3].

Periodic time series are often considered as an extension of SARIMA model [1, 25, 37] and are used for forecasting stream flows with quarterly, monthly or weekly cycles, see Osborn [37]. Particularly, Lund [25] proposed a test assessing if a PARMA model is preferable to a SARMA one. He also showed that the model's performance improves when it includes a fractional integration. A long-range dependence is widely investigated and confirmed as a reasonable assumption for practical applications. As an example, see Porter-Hudak [39] for the investigation of the seasonal ARFIMA model with application to the monetary aggregates used by U.S. Federal Reserve, or Reisen, et al. [40] for a semiparametric robust fractional parameters estimation in the SARFIMA model illustrated by the forecasting of SO_2 pollutant concentrations. The long-range dependence defined in terms of the third order cumulants of the time series has been studied by Terdik [44, 45].

In this paper, a continuous time stochastic process $\xi(t)$ with periodically stationary dth increments is studied. Its structural function $D^{(d)}(t,s;\tau_1 T,\tau_2 T) := \mathsf{E}\xi^{(d)}(t,\tau_1 T)\overline{\xi^{(d)}(s,\tau_2 T)}$ is a T-periodic function by the variables t and s: $D^{(d)}(t,s;\tau_1 T,\tau_2 T) = D^{(d)}(t+T,s+T;\tau_1 T,\tau_2 T)$ [24]. We deal with the problem of the mean-square optimal estimation of the linear functionals $A\xi = \int_0^\infty a(t)\xi(t)dt$ and $A_{NT}\xi = \int_0^{(N+1)T} a(t)\xi(t)dt$ which depend on the unobserved values of the process $\xi(t)$ based on observations of this process with periodically stationary noise at points $t<0$.

Similar problems for discrete time processes have been studied by Kozak and Moklyachuk [19], Luz and Moklyachuk [22, 23]. The problem of estimation of continuous time stochastic process $\xi(t)$ with periodically stationary dth increments based on observations of the process without noise at points $t<0$ was studied by Luz and Moklyachuk [24].

1.2. Contributions

The main contribution of this paper is the developed classical and minimax solutions to the prediction problem for the continuous time stochastic process with periodically stationary increments observed with a periodically correlated noise. It is presented in sections 4 – 6.

1.3. Organization

The paper is organized as follows. In section 2, we describe a presentation of a continuous time periodically stationary process as a stationary H-valued sequence. This approach is extended on the periodically stationary increments in section 3. The traditional Hilbert space projection method of prediction is developed in section 4. Particularly, formulas for calculating the mean-square errors and the spectral characteristics of the optimal linear estimates of the functionals $A\xi$ and $A_{NT}\xi$ are derived under some conditions on spectral densities. An approach to solution of the prediction problem which is based on factorizations of spectral densities is developed in section 5. In section 6 we present our results on minimax-robust prediction for the studied processes: relations that determine the least favourable spectral densities and the minimax spectral characteristics are derived for some classes of spectral densities.

2. Continuous time periodically correlated processes and generated vector stationary sequences

In this section, we present a brief review of properties of periodically correlated processes and describe an approach to presenting it as stationary H-valued sequences. In the next section, this approach is applied to develop the spectral theory for periodically correlated increment processes.

Definition 2.1 (Gladyshev [11])
A mean-square continuous stochastic process $\eta : \mathbb{R} \to H = L_2(\Omega, \mathcal{F}, \mathbb{P})$, with $\mathsf{E}\eta(t) = 0$, is called periodically correlated (PC) with period T, if its correlation function $K(t,s) = \mathsf{E}\eta(t)\overline{\eta(s)}$ for all $t, s \in \mathbb{R}$ and some fixed $T > 0$ is such that

$$K(t,s) = \mathsf{E}\eta(t)\overline{\eta(s)} = \mathsf{E}\eta(t+T)\overline{\eta(s+T)} = K(t+T, s+T).$$

For a periodically correlated stochastic process $\eta(t)$, one can construct the following sequence of stochastic functions [4], [31]

$$\{\eta_j(u) = \eta(u + jT), u \in [0,T), j \in \mathbb{Z}\}. \tag{1}$$

The sequence (1) forms a $L_2([0,T); H)$-valued stationary sequence $\{\eta_j, j \in \mathbb{Z}\}$ with the correlation function

$$B_\eta(l,j) = \langle \eta_l, \eta_j \rangle_H = \int_0^T \mathsf{E}[\eta(u + lT)\overline{\eta(u + jT)}]du = \int_0^T K_\eta(u + (l-j)T, u)du = B_\eta(l-j),$$

where $K_\eta(t,s) = \mathsf{E}\eta(t)\overline{\eta(s)}$ is the correlation function of the PC process $\eta(t)$. Chose the following orthonormal basis in the space $L_2([0,T); \mathbb{R})$

$$\{\widetilde{e}_k = \frac{1}{\sqrt{T}} e^{2\pi i\{(-1)^k \left[\frac{k}{2}\right]\}u/T}, k = 1, 2, 3, \dots\}, \quad \langle \widetilde{e}_j, \widetilde{e}_k \rangle = \delta_{kj}. \tag{2}$$

Making use of this basis the stationary sequence $\{\eta_j, j \in \mathbb{Z}\}$ can be represented in the form

$$\eta_j = \sum_{k=1}^{\infty} \eta_{kj} \widetilde{e}_k, \tag{3}$$

where

$$\eta_{kj} = \langle \eta_j, \widetilde{e}_k \rangle = \frac{1}{\sqrt{T}} \int_0^T \eta_j(v) e^{-2\pi i\{(-1)^k \left[\frac{k}{2}\right]\}v/T} dv.$$

The sequence $\{\eta_j, j \in \mathbb{Z}\}$, or the corresponding to it vector sequence

$$\{\vec{\eta}_j = (\eta_{kj}, k = 1, 2, \dots)^\top, j \in \mathbb{Z}\},$$

is called a generated by the process $\{\eta(t), t \in \mathbb{R}\}$ vector stationary sequence. The components $\{\eta_{kj}\} : k = 1, 2, \dots; j \in \mathbb{Z}$ of the generated stationary sequence $\{\eta_j, j \in \mathbb{Z}\}$ satisfy the relations [15], [26]

$$\mathsf{E}\eta_{kj} = 0, \quad \|\eta_j\|_H^2 = \sum_{k=1}^{\infty} \mathsf{E}|\eta_{kj}|^2 \leq P_\eta = B_\eta(0), \quad \mathsf{E}\eta_{kl}\overline{\eta}_{nj} = \langle R_\eta(l-j)\widetilde{e}_k, \widetilde{e}_n \rangle.$$

The correlation function $R_\eta(j)$ of the generated stationary sequence $\{\eta_j, j \in \mathbb{Z}\}$ is a correlation operator function. The correlation operator $R_\eta(0) = R_\eta$ is a kernel operator and its kernel norm satisfies the following properties:

$$\|\eta_j\|_H^2 = \sum_{k=1}^{\infty} \langle R_\eta \widetilde{e}_k, \widetilde{e}_k \rangle \leq P_\eta.$$

The generated stationary sequence $\{\eta_j, j \in \mathbb{Z}\}$ has the spectral density function $g(\lambda) = \{g_{kn}(\lambda)\}_{k,n=1}^{\infty}$, that is positive valued operator function of variable $\lambda \in [-\pi, \pi)$, if its correlation function $R_\eta(j)$ can be represented in the form

$$\langle R_\eta(j)\widetilde{e}_k, \widetilde{e}_n \rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{ij\lambda} \langle g(\lambda)\widetilde{e}_k, \widetilde{e}_n \rangle d\lambda.$$

We finish our review by the statement, that for almost all $\lambda \in [-\pi, \pi)$ the spectral density $f(\lambda)$ is a kernel operator with an integrable kernel norm

$$\sum_{k=1}^{\infty} \frac{1}{2\pi} \int_{-\pi}^{\pi} \langle g(\lambda)\widetilde{e}_k, \widetilde{e}_k \rangle d\lambda = \sum_{k=1}^{\infty} \langle R_\eta \widetilde{e}_k, \widetilde{e}_k \rangle = \|\eta_j\|_H^2 \leq P_\eta.$$

3. Stochastic processes with periodically correlated dth increments

For a given stochastic process $\{\xi(t), t \in \mathbb{R}\}$, consider the stochastic dth increment process

$$\xi^{(d)}(t, \tau) = (1 - B_\tau)^d \xi(t) = \sum_{l=0}^{d} (-1)^l \binom{d}{l} \xi(t - l\tau), \tag{4}$$

with the step $\tau \in \mathbb{R}$, generated by the stochastic process $\xi(t)$. Here B_τ is the backward shift operator: $B_\tau \xi(t) = \xi(t - \tau)$, $\tau \in \mathbb{R}$.

We prefer to use the notation $\xi^{(d)}(t, \tau)$ instead of widely used $\Delta_\tau^d \xi(t)$ to avoid a duplicate with the mean square error notation.

Definition 3.1
A stochastic process $\{\xi(t), t \in \mathbb{R}\}$ is called a stochastic process with periodically stationary (periodically correlated) increments with the step $\tau \in \mathbb{Z}$ and the period $T > 0$ if the mathematical expectations exist and satisfy the relations

$$\begin{aligned} \mathsf{E}\xi^{(d)}(t + T, \tau T) &= \mathsf{E}\xi^{(d)}(t, \tau T) = c^{(d)}(t, \tau T), \\ \mathsf{E}\xi^{(d)}(t + T, \tau_1 T)\overline{\xi^{(d)}(s + T, \tau_2 T)} &= D^{(d)}(t + T, s + T; \tau_1 T, \tau_2 T) = D^{(d)}(t, s; \tau_1 T, \tau_2 T) \end{aligned}$$

for every $t, s \in \mathbb{R}$, $\tau_1, \tau_2 \in \mathbb{Z}$ and for some fixed $T > 0$.

The functions $c^{(d)}(t, \tau T)$ and $D^{(d)}(t, s; \tau_1 T, \tau_2 T)$ from the Definition 3.1 are called the *mean value* and the *structural function* of the stochastic process $\xi(t)$ with periodically stationary (periodically correlated) increments.

For the stochastic process $\{\xi(t), t \in \mathbb{R}\}$ with periodically correlated increments $\xi^{(d)}(t, \tau T)$ and the integer step τ, we follow the procedure described in the Section 2 and construct a sequence of stochastic functions

$$\{\xi_j^{(d)}(u) := \xi_{j,\tau}^{(d)}(u) = \xi_j^{(d)}(u + jT, \tau T),\ u \in [0, T), j \in \mathbb{Z}\}. \tag{5}$$

Sequence (5) forms a $L_2([0,T);H)$-valued stationary increment sequence $\{\xi^{(d)}_j, j\in\mathbb{Z}\}$ with the structural function

$$\begin{aligned} B_{\xi^{(d)}}(l,j) &= \langle \xi^{(d)}_l, \xi^{(d)}_j\rangle_H = \int_0^T \mathsf{E}[\xi^{(d)}_j(u+lT,\tau_1 T)\overline{\xi^{(d)}_j(u+jT,\tau_2 T)}]du \\ &= \int_0^T D^{(d)}(u+(l-j)T,u;\tau_1 T,\tau_2 T)du = B_{\xi^{(d)}}(l-j). \end{aligned}$$

Making use of the orthonormal basis (2) the stationary increment sequence $\{\xi^{(d)}_j, j\in\mathbb{Z}\}$ can be represented in the form

$$\xi^{(d)}_j = \sum_{k=1}^{\infty} \xi^{(d)}_{kj}\widetilde{e}_k, \tag{6}$$

where

$$\xi^{(d)}_{kj} = \langle \xi^{(d)}_j, \widetilde{e}_k\rangle = \frac{1}{\sqrt{T}}\int_0^T \xi^{(d)}_j(v)e^{-2\pi i\{(-1)^k\left[\frac{k}{2}\right]\}v/T}dv.$$

We call this sequence $\{\xi^{(d)}_j, j\in\mathbb{Z}\}$, or the corresponding to it vector sequence

$$\{\vec{\xi}^{(d)}(j,\tau)=\vec{\xi}^{(d)}_j=(\xi^{(d)}_{kj},k=1,2,\dots)^\top=(\xi^{(d)}_k(j,\tau),k=1,2,\dots)^\top, j\in\mathbb{Z}\}, \tag{7}$$

an infinite dimension vector stationary increment sequence generated by the increment process $\{\xi^{(d)}(t,\tau T), t\in\mathbb{R}\}$. Further, we will omit the word vector in the notion generated vector stationary increment sequence.

Components $\{\xi^{(d)}_{kj}\}: k=1,2,\dots; j\in\mathbb{Z}$ of the generated stationary increment sequence $\{\xi^{(d)}_j, j\in\mathbb{Z}\}$ are such that, [15], [26]

$$\mathsf{E}\xi^{(d)}_{kj}=0, \quad \|\xi^{(d)}_j\|^2_H=\sum_{k=1}^{\infty}\mathsf{E}|\xi^{(d)}_{kj}|^2 \le P_{\xi^{(d)}} = B_{\xi^{(d)}}(0),$$

and

$$\mathsf{E}\xi^{(d)}_{kl}\overline{\xi^{(d)}_{nj}} = \langle R_{\xi^{(d)}}(l-j;\tau_1,\tau_2)\widetilde{e}_k,\widetilde{e}_n\rangle.$$

The *structural function* $R_{\xi^{(d)}}(j):=R_{\xi^{(d)}}(j;\tau_1,\tau_2)$ of the generated stationary increment sequence $\{\xi^{(d)}_j, j\in\mathbb{Z}\}$ is a correlation operator function. The correlation operator $R_{\xi^{(d)}}(0)=R_{\xi^{(d)}}$ is a kernel operator and its kernel norm satisfies the following limitations:

$$\|\xi^{(d)}_j\|^2_H=\sum_{k=1}^{\infty}\langle R_{\xi^{(d)}}\widetilde{e}_k,\widetilde{e}_k\rangle \le P_{\xi^{(d)}}.$$

Suppose that the structural function $R_{\xi^{(d)}}(j)$ admits a representation

$$\langle R_{\xi^{(d)}}(j;\tau_1,\tau_2)\widetilde{e}_k,\widetilde{e}_n\rangle = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{ij\lambda}(1-e^{-i\tau_1\lambda})^d(1-e^{i\tau_2\lambda})^d\frac{1}{\lambda^{2d}}\langle f(\lambda)\widetilde{e}_k,\widetilde{e}_n\rangle d\lambda.$$

Then $f(\lambda)=\{f_{kn}(\lambda)\}_{k,n=1}^{\infty}$ is a *spectral density function* of the generated stationary increment sequence $\{\xi^{(d)}_j, j\in\mathbb{Z}\}$. It is a positive valued operator functions of variable $\lambda\in[-\pi,\pi)$, and for almost all $\lambda\in[-\pi,\pi)$ it is a kernel operator with an integrable kernel norm

$$\sum_{k=1}^{\infty}\frac{1}{2\pi}\int_{-\pi}^{\pi}(1-e^{-i\tau_1\lambda})^d(1-e^{i\tau_2\lambda})^d\frac{1}{\lambda^{2d}}\langle f(\lambda)\widetilde{e}_k,\widetilde{e}_k\rangle d\lambda = \sum_{k=1}^{\infty}\langle R_{\xi^{(d)}}\widetilde{e}_k,\widetilde{e}_k\rangle = \|\zeta_j\|^2_H \le P_{\xi^{(d)}}. \tag{8}$$

The stationary dth increment sequence $\vec{\xi}^{(d)}_j$ admits the spectral representation

$$\vec{\xi}^{(d)}_j = \int_{-\pi}^{\pi} e^{i\lambda j}(1-e^{-i\tau\lambda})^d \frac{1}{(i\lambda)^d} d\vec{Z}_{\xi^{(d)}}(\lambda),$$

where $\vec{Z}_{\xi^{(d)}}(\lambda) = \{Z_k(\lambda)\}_{k=1}^{\infty}$ is a vector-valued random process with uncorrelated increments on $[-\pi,\pi)$.

Consider the generated stationary stochastic sequence $\vec{\eta}_j$ defined in Section **??**, which is uncorrelated with the increment sequence $\vec{\xi}^{(d)}_j$. It admits the spectral representation

$$\vec{\eta}_j = \int_{-\pi}^{\pi} e^{i\lambda j} d\vec{Z}_{\eta}(\lambda), \tag{9}$$

where $\vec{Z}_{\eta}(\lambda)$ is a vector-valued random process with uncorrelated increments on $[-\pi,\pi)$. The spectral representation of the sequence $\vec{\zeta}^{(d)}_j$, generated by the process $\zeta(t) = \xi(t) + \eta(t)$, is determined by the spectral densities $f(\lambda)$ and $g(\lambda)$ by the relation

$$\vec{\zeta}^{(d)}_j = \int_{-\pi}^{\pi} e^{i\lambda j}(1-e^{-i\tau\lambda})^d \frac{1}{(i\lambda)^d} d\vec{Z}_{\xi^{(d)}+\eta^{(d)}}(\lambda). \tag{10}$$

The random processes $\vec{Z}_{\eta}(\lambda)$ and $\vec{Z}_{\eta^{(d)}}(\lambda)$ are connected by the relation $d\vec{Z}_{\eta^{(d)}}(\lambda) = (i\lambda)^d d\vec{Z}_{\eta}(\lambda)$, $\lambda \in [-\pi,\pi)$, see [21]. The spectral density $p(\lambda) = \{p_{kn}(\lambda)\}_{k,n=1}^{\infty}$ of the sequence $\vec{\zeta}^{(d)}_j$ is determined by the spectral densities $f(\lambda)$ and $g(\lambda)$ by the relation

$$p(\lambda) = f(\lambda) + \lambda^{2d} g(\lambda).$$

In the space $H = L_2(\Omega, \mathcal{F}, \mathsf{P})$, consider a closed linear subspace

$$H(\vec{\xi}^{(d)}) = \overline{span}\{\xi^{(d)}_{kj} : k = 1,2,\dots;\ j \in \mathbb{Z}\}$$

generated by the components of the generated stationary increment sequence $\vec{\xi}^{(d)} = \{\xi^{(d)}_{kj} = \xi^{(d)}_k(j,\tau),\ \tau > 0\}$. For $q \in \mathbb{Z}$, consider also a closed linear subspace

$$H^q(\vec{\xi}^{(d)}) = \overline{span}\{\xi^{(d)}_{kj} : k = 1,2,\dots;\ j \le q\}.$$

Define a subspace

$$S(\vec{\xi}^{(d)}) = \bigcap_{q\in\mathbb{Z}} H^q(\vec{\xi}^{(d)})$$

of the Hilbert space $H(\vec{\xi}^{(d)})$. The space $H(\vec{\xi}^{(d)})$ admits a decomposition $H(\vec{\xi}^{(d)}) = S(\vec{\xi}^{(d)}) \oplus R(\vec{\xi}^{(d)})$ where $R(\vec{\xi}^{(d)})$ is the orthogonal complement of the subspace $S(\vec{\xi}^{(d)})$ in the space $H(\vec{\xi}^{(d)})$.

Definition 3.2
A stationary (wide sense) increment sequence $\vec{\xi}^{(d)}_j = \{\xi^{(d)}_{kj}\}_{k=1}^{\infty}$ is called regular if $H(\vec{\xi}^{(d)}) = R(\vec{\xi}^{(d)})$, and it is called singular if $H(\vec{\xi}^{(d)}) = S(\vec{\xi}^{(d)})$.

Theorem 3.1
A stationary increment sequence $\xi^{(d)}_j$ is uniquely represented in the form

$$\xi^{(d)}_{kj} = \xi^{(d)}_{S,kj} + \xi^{(d)}_{R,kj} \tag{11}$$

where $\xi^{(d)}_{R,kj}, k = 1,\dots,\infty$, is a regular stationary increment sequence and $\xi^{(d)}_{S,kj}, k = 1,\dots,\infty$, is a singular stationary increment sequence. The increment sequences $\xi^{(d)}_{R,kj}$ and $\xi^{(d)}_{S,kj}$ are orthogonal for all $j \in \mathbb{Z}$. They are defined by the formulas

$$\xi^{(d)}_{S,kj} = \mathsf{E}[\xi^{(d)}_{kj} | S(\vec{\xi}^{(d)})],$$
$$\xi^{(d)}_{R,kj} = \xi^{(d)}_{kj} - \xi^{(d)}_{S,kj}.$$

Consider an innovation sequence $\vec{\varepsilon}(u)=\{\varepsilon_m(u)\}_{m=1}^{M}, u\in\mathbb{Z}$ for a regular stationary increment, namely, the sequence of uncorrelated random variables such that $\mathsf{E}\varepsilon_m(u)\overline{\varepsilon}_j(v)=\delta_{mj}\delta_{uv}$, $\mathsf{E}|\varepsilon_m(u)|^2=1, m,j=1,\dots,M; u\in\mathbb{Z}$, and $H^r(\vec{\xi}^{(d)})=H^r(\vec{\varepsilon})$ holds true for all $r\in\mathbb{Z}$, where $H^r(\vec{\varepsilon})$ is the Hilbert space generated by elements $\{\varepsilon_m(u): m=1,\dots,M; u\le r\}$, δ_{mj} and δ_{uv} are Kronecker symbols.

Theorem 3.2
A stationary increment sequence $\vec{\xi}_j^{(d)}$ is regular if and only if there exists an innovation sequence $\vec{\varepsilon}(u)=\{\varepsilon_m(u)\}_{m=1}^{M}, u\in\mathbb{Z}$ and a sequence of matrix-valued functions $\varphi^{(d)}(l,\tau)=\{\varphi_{km}^{(d)}(l,\tau)\}_{k=\overline{1,\infty}}^{m=\overline{1,M}}, l\ge 0$, such that

$$\sum_{l=0}^{\infty}\sum_{k=1}^{\infty}\sum_{m=1}^{M}|\varphi_{km}^{(d)}(l,\tau)|^2<\infty,\quad \xi_{kj}^{(d)}=\sum_{l=0}^{\infty}\sum_{m=0}^{M}\varphi_{km}^{(d)}(l,\tau)\vec{\varepsilon}_m(j-l). \tag{12}$$

Representation (12) is called the canonical moving average representation of the generated stationary increment sequence $\vec{\xi}_j^{(d)}$.

The spectral function $F(\lambda)$ of a stationary increment sequence $\vec{\xi}_j^{(d)}$ which admits canonical representation (12) has the spectral density $f(\lambda)=\{f_{ij}(\lambda)\}_{i,j=1}^{\infty}$ admitting the canonical factorization

$$f(\lambda)=\varphi(e^{-i\lambda})\varphi^*(e^{-i\lambda}), \tag{13}$$

where the function $\varphi(z)=\sum_{k=0}^{\infty}\varphi(k)z^k$ has analytic in the unit circle $\{z:|z|\le 1\}$ components $\varphi_{ij}(z)=\sum_{k=0}^{\infty}\varphi_{ij}(k)z^k; i=1,\dots,\infty; j=1,\dots,M$. Based on moving average representation (12) define

$$\varphi_\tau(z)=\sum_{k=0}^{\infty}\varphi^{(d)}(k,\tau)z^k=\sum_{k=0}^{\infty}\varphi_\tau(k)z^k.$$

Then the following factorization holds true:

$$\frac{|1-e^{i\lambda\tau}|^{2d}}{\lambda^{2d}}f(\lambda)=\varphi_\tau(e^{-i\lambda})\varphi_\tau^*(e^{-i\lambda}),\quad \varphi_\tau(e^{-i\lambda})=\sum_{k=0}^{\infty}\varphi_\tau(k)e^{-i\lambda k}. \tag{14}$$

4. Hilbert space projection method of prediction

Let a periodically correlated increment process $\xi^{(d)}(t,\tau T)$, $t\in\mathbb{R}$, generates by formula (6) an infinite dimension vector stationary increment sequence $\{\vec{\xi}_j^{(d)}, j\in\mathbb{Z}\}$ which has the spectral density matrix $f(\lambda)=\{f_{ij}(\lambda)\}_{i,j=1}^{\infty}$. As a noise process, consider a periodically stationary stochastic process $\eta(t)$, $t\in\mathbb{R}$, uncorrelated with the process $\xi(t)$. Let the process $\eta(t)$ generates by formula (3) an infinite dimension vector stationary sequence $\{\vec{\eta}_j, j\in\mathbb{Z}\}$ with the spectral density matrix $g(\lambda)=\{g_{ij}(\lambda)\}_{i,j=1}^{\infty}$.

By the classical **prediction problem** we understand the problem of the mean square optimal linear estimation of the functionals

$$A\xi=\int_0^{\infty}a(t)\xi(t)dt,\quad A_{NT}\xi=\int_0^{(N+1)T}a(t)\xi(t)dt$$

which depend on the unknown values of the stochastic process $\xi(t)$. Estimates are based on observations of the process $\zeta(t)=\xi(t)+\eta(t)$ at points $t<0$.

Assumptions:

- the mean values of the increment sequence $\vec{\xi}_j^{(d)}$ and stationary sequence $\vec{\eta}_j$ equal to 0; the increment step $\tau>0$;

- the spectral densities $f(\lambda)$ and $g(\lambda)$ satisfy the *minimality condition*

$$\int_{-\pi}^{\pi} \mathrm{Tr}\left[\frac{\lambda^{2d}}{|1-e^{i\lambda\tau}|^{2d}}(f(\lambda)+\lambda^{2d}g(\lambda))^{-1}\right]d\lambda<\infty. \quad (15)$$

The latter assumption is the necessary and sufficient condition under which the mean square errors of the optimal estimates of the functional $A\vec{\xi}$ to be defined below is not equal to 0.

The Hilbert space projection method of estimation may be applied under the condition that the element, which we want to estimate, belongs to the Hilbert space $H=L_2(\Omega,\mathcal{F},\mathsf{P})$ of random variables with a zero mean value and a finite variance. That is not the case for the functional $A\xi$. To overcome this difficulty we find a representation of the functional $A\xi$ as a sum of a functional with finite second moment from the space H and a functional depended on the observed values of the process $\zeta(t)=\xi(t)+\eta(t)$. This representation is described by the following two lammas.

Lemma 4.1 ([24])
The linear functional

$$A\zeta=\int_0^{\infty}a(t)\zeta(t)dt$$

allows the representation

$$A\zeta=B\zeta-V\zeta,$$

where

$$B\zeta=\int_0^{\infty}b^{\tau}(t)\zeta^{(d)}(t,\tau T)dt,\quad V\zeta=\int_{-\tau Td}^{0}v^{\tau}(t)\zeta(t)dt,$$

and

$$v^{\tau}(t) = \sum_{l=\lceil -\frac{t}{\tau T}\rceil}^{d}(-1)^l\binom{d}{l}b^{\tau}(t+l\tau T),\quad t\in[-\tau Td;0), \quad (16)$$

$$b^{\tau}(t) = \sum_{k=0}^{\infty}a(t+\tau Tk)d(k)=D^{\tau T}\mathbf{a}(t),\ t\geq 0, \quad (17)$$

Here $\lceil x\rceil$ denotes the least integer greater than or equal to x, $[x]$ denotes the integer part of x, coefficients $\{d(k):k\geq 0\}$ are determined by the relation

$$\sum_{k=0}^{\infty}d(k)x^k=\left(\sum_{j=0}^{\infty}x^j\right)^d,$$

$D^{\tau T}$ is the linear transformation acting on an arbitrary function $x(t)$, $t\geq 0$, as follows:

$$D^{\tau T}\mathbf{x}(t)=\sum_{k=0}^{\infty}x(t+\tau Tk)d(k).$$

From Lemma 4.1, we obtain the following representation of the functional $A\xi$:

$$A\xi=A\zeta-A\eta=B\zeta-A\eta-V\zeta=H\xi-V\zeta,$$

where

$$H\xi:=B\zeta-A\eta,$$

and

$$A\zeta=\int_0^{\infty}a(t)\zeta(t)dt,\quad A\eta=\int_0^{\infty}a(t)\eta(t)dt,$$

$$B\zeta=\int_0^{\infty}b^{\tau}(t)\zeta^{(d)}(t,\tau T)dt,\quad V\zeta=\int_{-\tau Td}^{0}v^{\tau}(t)\zeta(t)dt,$$

the functions $b_{\tau}(t)$, $t\in[0;\infty)$, and $v_{\tau}(t)$, $t\in[-\tau Td;0)$, are calculated by formulas (17) and (16) respectively.

The functional $H\xi$ allows a representation in terms of the sequences $\vec{\eta}_j=(\eta_{kj},k=1,2,\dots)^{\top}$ and $\vec{\zeta}_j^{(d)}=\vec{\xi}_j^{(d)}+\vec{\eta}_j^{(d)}=(\zeta_{kj}^{(d)},k=1,2,\dots)^{\top}$, $j\in\mathbb{Z}$, which is described in the following lemma.

Lemma 4.2
The functional $H\xi=B\zeta-A\eta$ can be represented in the form

$$H\xi=\sum_{j=0}^{\infty}(\vec{b}_j^{\tau})^{\top}\vec{\zeta}_j^{(d)}-\sum_{j=0}^{\infty}(\vec{a}_j)^{\top}\vec{\eta}_j=B\vec{\zeta}-V\vec{\eta}=:H\vec{\xi},$$

where the vector

$$\vec{b}_j^{\tau}=(b_{kj}^{\tau},k=1,2,\dots)^{\top}=(b_{1j}^{\tau},b_{3j}^{\tau},b_{2j}^{\tau},\dots,b_{2k+1,j}^{\tau},b_{2k,j}^{\tau},\dots)^{\top},$$

with the entries

$$b_{kj}^{\tau}=\langle b_j^{\tau},\widetilde{e}_k\rangle=\frac{1}{\sqrt{T}}\int_0^T b_j^{\tau}(v)e^{-2\pi i\{(-1)^k\left[\frac{k}{2}\right]\}v/T}dv,$$

and the vector

$$\vec{a}_j=(a_{kj},k=1,2,\dots)^{\top}=(a_{1j},a_{3j},a_{2j},\dots,a_{2k+1,j},a_{2k,j},\dots)^{\top},$$

with the entries

$$a_{kj}=\langle a_j,\widetilde{e}_k\rangle=\frac{1}{\sqrt{T}}\int_0^T a_j(v)e^{-2\pi i\{(-1)^k\left[\frac{k}{2}\right]\}v/T}dv,\quad k=1,2,\dots,\ j=0,1,\dots,\infty.$$

The coefficients $\{\vec{a}_j,j=0,1,\dots,\infty\}$ and $\{\vec{b}_j^{\tau},j=0,1,\dots,\infty\}$ are related as

$$\vec{b}_j^{\tau}=\sum_{m=j}^{\infty}\mathrm{diag}_{\infty}(d_{\tau}(m-j))\vec{a}_m=(D^{\tau}\mathbf{a})_j,\quad j=0,1,\dots,\infty. \tag{18}$$

where $\mathbf{a}=((\vec{a}_0)^{\top},(\vec{a}_1)^{\top},\dots)^{\top}$, the coefficients $\{d_{\tau}(k):k\geq 0\}$ are determined by the relationship

$$\sum_{k=0}^{\infty}d_{\tau}(k)x^k=\left(\sum_{j=0}^{\infty}x^{\tau j}\right)^d,$$

D^{τ} is a linear transformation determined by a matrix with the infinite dimension matrix entries $D^{\tau}(k,j),k,j=0,1,\dots$ such that $D^{\tau}(k,j)=\mathrm{diag}_{\infty}(d_{\tau}(j-k))$ if $0\leq k\leq j\leq\infty$ and $D^{\tau}(k,j)=\mathrm{diag}_{\infty}(0)$ for $0\leq j<k\leq\infty$; $\mathrm{diag}_{\infty}(x)$ denotes an infinite dimensional diagonal matrix with the entry x on its diagonal.

Proof
See Appendix. □

Assume, that coefficients $\{\vec{a}_j,j=0,1,\dots\}$ and $\{\vec{b}_j^{\tau},j=0,1,\dots\}$, that determine the functional $H\vec{\xi}$, satisfy the conditions

$$\sum_{j=0}^{\infty}\|\vec{a}_j\|<\infty,\quad\sum_{j=0}^{\infty}(j+1)\|\vec{a}_j\|^2<\infty,\quad\|\vec{a}_j\|^2=\sum_{k=1}^{\infty}|a_{kj}|^2, \tag{19}$$

$$\sum_{j=0}^{\infty}\|\vec{b}_j^{\tau}\|<\infty,\quad\sum_{j=0}^{\infty}(j+1)\|\vec{b}_j^{\tau}\|^2<\infty,\quad\|\vec{b}_j^{\tau}\|^2=\sum_{k=1}^{\infty}|b_{kj}^{\tau}|^2. \tag{20}$$

Under conditions (19) - (20) the functional $H\vec{\xi}$ has finite second moment. Since the functional $V\zeta$ depends on the observations $\{\xi(t)+\eta(t):t\in[\tau Td;0)\}$, the estimates $\widehat{A}\xi$ and $\widehat{H}\vec{\xi}$ of the functionals $A\xi$ and $H\vec{\xi}$, as well as the mean-square errors $\Delta(f,g;\widehat{A}\xi)=\mathsf{E}|A\xi-\widehat{A}\xi|^2$ and $\Delta(f,g;\widehat{H}\vec{\xi})=\Delta(f,g;\widehat{H}\vec{\xi})=\mathsf{E}|H\vec{\xi}-\widehat{H}\vec{\xi}|^2$ of the estimates $\widehat{A}\xi$ and $\widehat{H}\vec{\xi}$ satisfy the relations

$$\widehat{A}\xi=\widehat{H}\vec{\xi}-V\zeta \tag{21}$$

and

$$\Delta(f,g;\widehat{A}\xi)=\mathsf{E}|A\xi-\widehat{A}\xi|^2=\mathsf{E}|H\vec{\xi}-V\zeta-\widehat{H}\vec{\xi}+V\zeta|^2=\mathsf{E}|H\vec{\xi}-\widehat{H}\vec{\xi}|^2=\Delta(f,g;\widehat{H}\vec{\xi}).$$

Thus, the functional $H\vec{\xi}$ is a target element to be estimated. Let us describe its spectral representation. Making use of representations (10) and (9), we obtain

$$H\vec{\xi}=\int_{-\pi}^{\pi}(\vec{B}_\tau(e^{i\lambda}))^{\top}\frac{(1-e^{-i\lambda\tau})^d}{(i\lambda)^d}d\vec{Z}_{\xi^{(d)}+\eta^{(d)}}(\lambda)-\int_{-\pi}^{\pi}(\vec{A}(e^{i\lambda}))^{\top}d\vec{Z}_{\eta}(\lambda),$$

where

$$\vec{B}_\tau(e^{i\lambda})=\sum_{j=0}^{\infty}\vec{b}_j^{\tau}e^{i\lambda j}=\sum_{j=0}^{\infty}(D^{\tau}\mathbf{a})_je^{i\lambda j},\quad \vec{A}(e^{i\lambda})=\sum_{j=0}^{\infty}\vec{a}_je^{i\lambda j}.$$

The classical approach of estimation consists in finding a projection of the element $H\vec{\xi}$ on the closed linear subspace of $H=L_2(\Omega,\mathcal{F},\mathsf{P})$ generated by the observations. Let us define this subspace as

$$H^{0-}(\xi_\tau^{(d)}+\eta_\tau^{(d)})=\overline{span}\{\vec{\xi}_{kj}^{(d)}+\vec{\eta}_{kj}^{(d)}:k=1,\dots,\infty;j=-1,-2,-3,\dots\}.$$

Define also the closed linear subspaces of the Hilbert space $L_2(f(\lambda)+\lambda^{2d}g(\lambda))$ of vector-valued functions endowed by the inner product

$$\langle g_1;g_2\rangle=\int_{-\pi}^{\pi}(g_1(\lambda))^{\top}(f(\lambda)+\lambda^{2d}g(\lambda))\overline{g_2(\lambda)}d\lambda$$

as

$$L_2^{0-}(f(\lambda)+\lambda^{2d}g(\lambda))=\overline{span}\{e^{i\lambda j}(1-e^{-i\lambda\tau})^d\frac{1}{(i\lambda)^d}\vec{\delta}_k,\ k=1,2,3,\dots;\ j=-1,-2,-3,\dots\},$$

where $\vec{\delta}_k=\{\delta_{kl}\}_{l=1}^{\infty}$, δ_{kl} are Kronecker symbols.

Remark 4.1
Representation (10) yields a map between the elements $e^{i\lambda j}(1-e^{-i\lambda\tau})^d(i\lambda)^{-d}\vec{\delta}_k$ of the space $L_2^{0-}(f(\lambda)+\lambda^{2d}g(\lambda))$ and the elements $\vec{\xi}_{kj}^{(d)}+\vec{\eta}_{kj}^{(d)}$ of the space $H^{0-}(\xi_\tau^{(d)}+\eta_\tau^{(d)})$.

The mean square optimal estimate $\widehat{H}\vec{\xi}$ is found as a projection of the element $H\vec{\xi}$ on the subspace $H^{0-}(\xi_\tau^{(d)}+\eta_\tau^{(d)})$:

$$\widehat{H}\vec{\xi}=\mathsf{Proj}_{H^{0-}(\xi_\tau^{(d)}+\eta_\tau^{(d)})}H\vec{\xi}.$$

Relation (21) let us write the optimal estimate $\widehat{A}\xi$ in the form

$$\widehat{A}\xi=\mathsf{Proj}_{H^{0-}(\xi_\tau^{(d)}+\eta_\tau^{(d)})}H\vec{\xi}-V\zeta$$

or in the form

$$\widehat{A}\xi=\int_{-\pi}^{\pi}(\vec{h}_\tau(\lambda))^{\top}d\vec{Z}_{\xi^{(d)}+\eta^{(d)}}(\lambda)-\int_{-\tau Td}^{0}v^{\tau}(t)\zeta(t)dt, \tag{22}$$

where $\vec{h}_\tau(\lambda)=\{h_k^{\tau}(\lambda)\}_{k=1}^{\infty}$ is the spectral characteristic of the estimate $\widehat{H}\vec{\xi}$.

Denote

$$\vec{A}_\tau(e^{i\lambda})=(1-e^{i\lambda\tau})^d\vec{A}(e^{i\lambda})=\sum_{j=0}^{\infty}\vec{a}_j^\tau e^{i\lambda j},$$

where

$$\vec{a}_j^\tau=\sum_{l=0}^{\min\{[j/\tau],d\}}(-1)^l\binom{d}{l}\vec{a}(j-\tau l),\quad j\geq 0. \tag{23}$$

Define the vector

$$\mathbf{a}^\tau=((\vec{a}_0^\tau)^\top,(\vec{a}_1^\tau)^\top,(\vec{a}_2^\tau)^\top,\dots)^\top.$$

With the help of the Fourier coefficients

$$T_{l,j}^\tau=\frac{1}{2\pi}\int_{-\pi}^{\pi}e^{i\lambda(j-l)}\frac{\lambda^{2d}}{|1-e^{i\lambda\tau}|^{2d}}\left(g(\lambda)(f(\lambda)+\lambda^{2d}g(\lambda))^{-1}\right)^\top d\lambda,\quad l,j\geq 0,$$
$$P_{l,j}^\tau=\frac{1}{2\pi}\int_{-\pi}^{\pi}e^{i\lambda(j-l)}\frac{\lambda^{2d}}{|1-e^{i\lambda\tau}|^{2d}}\left((f(\lambda)+\lambda^{2d}g(\lambda))^{-1}\right)^\top d\lambda,\quad l,j\geq 0,$$
$$Q_{l,j}=\frac{1}{2\pi}\int_{-\pi}^{\pi}e^{i\lambda(j-l)}\left(f(\lambda)(f(\lambda)+\lambda^{2d}g(\lambda))^{-1}g(\lambda)\right)^\top d\lambda,\quad l,j\geq 0,$$

of the corresponding matrix functions, define the linear operators $\mathbf{P}_\tau$, $\mathbf{T}_\tau$ and $\mathbf{Q}$ in the space ℓ_2 by matrices with the infinite dimensional matrix entries $(\mathbf{P}_\tau)_{l,j}=P_{l,j}^\tau$, $(\mathbf{T}_\tau)_{l,j}=T_{l,j}^\tau$ and $(\mathbf{Q})_{l,j}=Q_{l,j}$, $l,j\geq 0$.

Notation: $\langle\vec{x},\vec{y}\rangle=\sum_{j=0}^{\infty}(\vec{x}_j)^\top\overline{\vec{y}_j}$ for vectors $\vec{x}=((\vec{x}_0)^\top,(\vec{x}_1)^\top,(\vec{x}_2)^\top,\dots)^\top$, $\vec{y}=((\vec{y}_0)^\top,(\vec{y}_1)^\top,(\vec{y}_2)^\top,\dots)^\top$.

Theorem 4.1
Consider two uncorrelated processes: a stochastic process $\xi(t)$, $t\in\mathbb{R}$ with a periodically stationary increments, which determines a generated stationary dth increment sequence $\vec{\xi}_j^{(d)}$ with the spectral density matrix $f(\lambda)=\{f_{kn}(\lambda)\}_{k,n=1}^{\infty}$, and a periodically stationary stochastic process $\eta(t)$, $t\in\mathbb{R}$, which determines a generated stationary sequence $\vec{\eta}_j$ with the spectral density matrix $g(\lambda)=\{g_{kn}(\lambda)\}_{k,n=1}^{\infty}$. Let the coefficients $\vec{a}_j$, $\vec{b}_j^\tau$, $j=0,1,\dots$, generated by the function $a(t)$, $t\geq 0$, satisfy conditions (19) – (20). Let minimality condition (15) be satisfied. The optimal linear estimate $\widehat{A}\xi$ of the functional $A\xi$ based on observations of the process $\xi(t)+\eta(t)$ at points $t<0$ is calculated by formula (22). The spectral characteristic $\vec{h}_\tau(\lambda)$ is calculated by formula

$$(\vec{h}_\tau(\lambda))^\top=(\vec{B}_\tau(e^{i\lambda}))^\top\frac{(1-e^{-i\lambda\tau})^d}{(i\lambda)^d}$$
$$-\left((\vec{A}_\tau(e^{i\lambda}))^\top g(\lambda)+(\vec{C}_\tau(e^{i\lambda}))^\top\right)\frac{(-i\lambda)^d}{(1-e^{i\lambda\tau})^d}(f(\lambda)+\lambda^{2d}g(\lambda))^{-1}, \tag{24}$$

where

$$\vec{C}_\tau(e^{i\lambda})=\sum_{j=0}^{\infty}(\mathbf{P}_\tau^{-1}D^\tau\mathbf{a}-\mathbf{P}_\tau^{-1}\mathbf{T}_\tau\mathbf{a}^\tau)_je^{i\lambda j}.$$

The value of the mean-square error is calculated by the formula

$$\Delta(f,g;\widehat{A}\xi)=\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{\lambda^{2d}}{|1-e^{i\lambda\tau}|^{2d}}(\mathbf{C}_\tau^f(e^{i\lambda}))^\top(f(\lambda)+\lambda^{2d}g(\lambda))^{-1}f(\lambda)(f(\lambda)+\lambda^{2d}g(\lambda))^{-1}\overline{\mathbf{C}_\tau^f(e^{i\lambda})}d\lambda$$
$$+\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{\lambda^{4n}}{|1-e^{i\lambda\tau}|^{4n}}(\mathbf{C}_\tau^g(e^{i\lambda}))^\top(f(\lambda)+\lambda^{2d}g(\lambda))^{-1}g(\lambda)(f(\lambda)+\lambda^{2d}g(\lambda))^{-1}\overline{\mathbf{C}_\tau^g(e^{i\lambda})}d\lambda$$
$$=\langle D^\tau\mathbf{a}-\mathbf{T}_\tau\mathbf{a}^\tau,\mathbf{P}_\tau^{-1}D^\tau\mathbf{a}-\mathbf{P}_\tau^{-1}\mathbf{T}_\tau\mathbf{a}^\tau\rangle+\langle\mathbf{Q}\mathbf{a},\mathbf{a}\rangle, \tag{25}$$

where

$$\mathbf{C}^f_\tau(e^{i\lambda}) := \overline{g(\lambda)}\vec{A}_\tau(e^{i\lambda}) + \vec{C}_\tau(e^{i\lambda}),$$
$$\mathbf{C}^g_\tau(e^{i\lambda}) := |1-e^{i\lambda\tau}|^{2d}\lambda^{-2d}\overline{f^0(\lambda)}\vec{A}(e^{i\lambda}) - (1-e^{-i\lambda\tau})^d\vec{C}_\tau(e^{i\lambda}).$$

Proof
See Appendix. □

Corollary 4.1
The spectral characteristics $\vec{h}^1_\tau(\lambda)$ and $\vec{h}^2_\tau(\lambda)$ of the optimal estimates $\widehat{B}\vec{\zeta}$ and $\widehat{A}\vec{\eta}$ of the functionals $B\vec{\zeta}$ and $A\vec{\eta}$ based on observations $\xi(t)+\eta(t)$ at points $t<0$ are calculated by the formulas

$$\begin{aligned}(\vec{h}^1_\tau(\lambda))^\top &= (\vec{B}_\tau(e^{i\lambda}))^\top\frac{(1-e^{-i\lambda\tau})^d}{(i\lambda)^d}\\ &\quad-\frac{(-i\lambda)^d}{(1-e^{i\lambda\tau})^d}\left(\sum_{j=0}^{\infty}(\mathbf{P}_\tau^{-1}D^\tau\mathbf{a})_je^{i\lambda j}\right)^\top(f(\lambda)+\lambda^{2d}g(\lambda)^{-1},\\ (\vec{h}^2_\tau(\lambda))^\top &= (\vec{A}_\tau(e^{i\lambda}))^\top g(\lambda)\frac{(-i\lambda)^d}{(1-e^{i\lambda\tau})^d}(f(\lambda)+\lambda^{2d}g(\lambda))^{-1}\\ &\quad-\frac{(-i\lambda)^d}{(1-e^{i\lambda\tau})^d}\left(\sum_{j=0}^{\infty}(\mathbf{P}_\tau^{-1}\mathbf{T}_\tau\mathbf{a}^\tau)_je^{i\lambda j}\right)^\top(f(\lambda)+\lambda^{2d}g(\lambda))^{-1},\end{aligned}$$

respectively.

The optimal estimate $\widehat{A}_{NT}\xi$ of the functional $A_{NT}\xi$ which depend on the unknown values of the process $\xi(t)$ at points $t\in[0,T(N+1)]$, based on observations of the process $\xi(t)+\eta(t)$ at points $t<0$ can be obtained by using Theorem 4.1.
We first formulate the following corollaries from Lemma 4.1 and Lemma 4.2.

Corollary 4.2
The linear functional

$$A_{NT}\xi=\int_0^{(N+1)T}a(t)\xi(t)dt$$

allows the representation

$$A_{NT}\xi=B_{NT}\xi-V_{NT}\xi,$$

where

$$B_{NT}\xi=\int_0^{(N+1)T}b^{\tau,N}(t)\xi^{(d)}(t,\tau T)dt,\quad V_{NT}\xi=\int_{-\tau Td}^0v^{\tau,N}(t)\xi(t)dt,$$

and

$$v^{\tau,N}(t)=\sum_{l=\lceil-\frac{t}{\tau T}\rceil}^{\min\{[\frac{(N+1)T-t}{\tau T}],d\}}(-1)^l\binom{d}{l}b^{\tau,N}(t+l\tau T),\quad t\in[-\tau Td;0),\tag{26}$$

$$b^{\tau,N}(t)=\sum_{k=0}^{[\frac{(N+1)T-t}{\tau T}]}a(t+\tau Tk)d(k)=D^{\tau T,N}\mathbf{a}(t).t\in[0;(N+1)T],\tag{27}$$

The linear transformation $D^{\tau T,N}$ acts on an arbitrary function $x(t)$, $t\in[0;(N+1)T]$, as follows

$$D^{\tau T,N}\mathbf{x}(t)=\sum_{k=0}^{[\frac{(N+1)T-t}{\tau T}]}x(t+\tau Tk)d(k).$$

Corollary 4.3
The functional

$$H_{NT}\xi = B_{NT}\zeta - A_{NT}\eta,$$

$$B_{NT}\zeta = \int_0^{T(N+1)} b^{\tau,N}(t)\zeta^{(d)}(t,\tau T)dt, \quad A_{NT}\eta = \int_0^{T(N+1)} a(t)\eta(t)dt,$$

can be represented in the form

$$H_{NT}\xi = \sum_{j=0}^{N} (\vec{b}_j^{\tau,N})^\top \vec{\zeta}_j^{(d)} - \sum_{j=0}^{N} (\vec{a}_j)^\top \vec{\eta}_j = B_N\vec{\zeta} - V_N\vec{\eta} =: H_N\vec{\xi},$$

where the vector

$$\vec{b}_j^{\tau,N} = (b_{kj}^{\tau,N}, k = 1,2,\dots)^\top = (b_{1j}^{\tau,N}, b_{3j}^{\tau,N}, b_{2j}^{\tau,N}, \dots, b_{2k+1,j}^{\tau,N}, b_{2k,j}^{\tau,N}, \dots)^\top,$$

with the entries

$$b_{kj}^{\tau,N} = \langle b_j^{\tau,N}, \widetilde{e}_k\rangle = \frac{1}{\sqrt{T}}\int_0^T b_j^{\tau,N}(v)e^{-2\pi i\{(-1)^k\left[\frac{k}{2}\right]\}v/T}dv.$$

The coefficients $\{\vec{a}_j, j = 0,1,\dots,N\}$ and $\{\vec{b}_j^{\tau,N}, j = 0,1,\dots,N\}$ are related as

$$\vec{b}_j^{\tau,N} = \sum_{m=j}^{N} \mathsf{diag}_\infty(d_\tau(m-j))\vec{a}_m = (D_N^\tau \mathbf{a}_N)_j, \quad j = 0,1,\dots,N. \tag{28}$$

where $\mathbf{a}_N = ((\vec{a}_0)^\top, (\vec{a}_1)^\top, \dots, (\vec{a}_N)^\top, 0, \dots)^\top$, D_N^τ is a linear transformation determined by a matrix with the infinite dimension matrix entries $D_N^\tau(k,j), k,j \geq 0$ such that $D_N^\tau(k,j) = \mathsf{diag}_\infty(d_\tau(j-k))$ if $0 \leq k \leq j \leq N$ and $D_N^\tau(k,j) = \mathsf{diag}_\infty(0)$ for $0 \leq j < k$ or $j,k > N$.

Put $a(t) = 0$, $t > T(N+1)$. Define vector coefficients $\{\vec{a}_j^{\tau,N} : 0 \leq j \leq N + \tau d\}$ by the formula

$$\vec{a}_j^{\tau,N} = \sum_{l=\max\left\{\left\lceil\frac{j-N}{\tau}\right\rceil,0\right\}}^{\min\left\{\left[\frac{j}{\tau}\right],d\right\}} (-1)^l \binom{d}{l} \vec{a}(j-\tau l), \quad 0 \leq j \leq N + \tau d,$$

and a vector

$$\mathbf{a}_N^\tau = ((\vec{a}_0^{\tau,N})^\top, (\vec{a}_1^{\tau,N})^\top, (\vec{a}_2^{\tau,N})^\top, \dots, (\vec{a}_{N+\tau d}^{\tau,N})^\top, 0\dots)^\top.$$

The following theorem holds true.

Theorem 4.2
Consider two uncorrelated processes: a stochastic process $\xi(t)$, $t \in \mathbb{R}$ with periodically stationary increments, which determines a generated stationary dth increment sequence $\vec{\xi}_j^{(d)}$ with the spectral density matrix $f(\lambda) = \{f_{kn}(\lambda)\}_{k,n=1}^\infty$, and a periodically stationary stochastic process $\eta(t)$, $t \in \mathbb{R}$, which determines a generated stationary sequence $\vec{\eta}_j$ with the spectral density matrix $g(\lambda) = \{g_{kn}(\lambda)\}_{k,n=1}^\infty$. Let the coefficients $\vec{a}_j$, $\vec{b}_j^\tau$, $j = 0,1,\dots,N$ generated by the function $a(t)$, $0 \leq t \leq T(N+1)$, satisfy conditions

$$\|\vec{a}_j\|^2 = \sum_{k=1}^\infty |a_{kj}|^2 < \infty, \quad j = 0,1,\dots,N,$$

and

$$\|\vec{b}_j^{\tau,N}\|^2 = \sum_{k=1}^\infty |b_{kj}^{\tau,N}|^2 < \infty, \quad j = 0,1,\dots,N.$$

Let minimality condition (15) be satisfied. The optimal linear estimate $\widehat{A}_{NT}\xi$ of the functional $A_{NT}\xi$ based on observations of the process $\xi(t)+\eta(t)$ at points $t<0$ is calculated by formula

$$\widehat{A}_{NT}\xi=\int_{-\pi}^{\pi}(\vec{h}_{\tau,N}(\lambda))^{\top}d\vec{Z}_{\xi^{(d)}+\eta^{(d)}}(\lambda)-\int_{-\tau Td}^{0}v_{\tau,N}(t)\zeta(t)dt,$$

where the spectral characteristic $\vec{h}_{\tau,N}(\lambda)$ of the optimal estimate $\widehat{A}_{NT}\xi$ is calculated by formula

$$\begin{aligned}(\vec{h}_{\tau,N}(\lambda))^{\top}=&\frac{(1-e^{-i\lambda\tau})^d}{(i\lambda)^d}\left(\sum_{j=0}^{N}(D^{\tau}_{N}\mathbf{a}_N)_je^{i\lambda j}\right)^{\top}\\&-\frac{(-i\lambda)^d}{(1-e^{i\lambda\tau})^d}\left(\sum_{j=0}^{N+\tau d}\vec{a}^{\tau,N}_je^{i\lambda j}\right)^{\top}g(\lambda)(f(\lambda)+\lambda^{2d}g(\lambda))^{-1}\\&-\frac{(-i\lambda)^d}{(1-e^{i\lambda\tau})^d}\left(\sum_{j=0}^{\infty}(\mathbf{P}^{-1}_{\tau}D^{\tau}_{N}\mathbf{a}_N-\mathbf{P}^{-1}_{\tau}\mathbf{T}_{\tau}\mathbf{a}_{\tau,N})_je^{i\lambda j}\right)^{\top}(f(\lambda)+\lambda^{2d}g(\lambda))^{-1},\end{aligned}$$

The value of the mean-square error is calculated by formula

$$\begin{aligned}\Delta(f,g;\widehat{A}_{NT}\xi)&=\Delta(f,g;\widehat{H}_N\vec{\xi})=\mathsf{E}|H_N\vec{\xi}-\widehat{H}_N\vec{\xi}|^2\\&=\langle D^{\tau}_{N}\mathbf{a}_N-\mathbf{T}_{\tau}\mathbf{a}_{\tau,N},\mathbf{P}^{-1}_{\tau}D^{\tau}_{N}\mathbf{a}_N-\mathbf{P}^{-1}_{\tau}\mathbf{T}_{\tau}\mathbf{a}_{\tau,N}\rangle+\langle\mathbf{Q}_N\mathbf{a}_N,\mathbf{a}_N\rangle,\end{aligned}$$

where $\mathbf{Q}_N$ is a linear operator in the space ℓ_2 defined by the matrix with the infinite dimensional matrix entries $(\mathbf{Q}_N)_{l,j}=Q_{l,j}$, $0\leq l,j\leq N$, and $(\mathbf{Q}_N)_{l,j}=0$ otherwise.

5. Prediction based on factorizations of spectral densities

Assume that the following canonical factorizations take place

$$\frac{|1-e^{i\lambda\tau}|^{2d}}{\lambda^{2d}}(f(\lambda)+\lambda^{2d}g(\lambda))=\Theta_{\tau}(e^{-i\lambda})\Theta^*_{\tau}(e^{-i\lambda}),\Theta_{\tau}(e^{-i\lambda})=\sum_{k=0}^{\infty}\Theta_{\tau}(k)e^{-i\lambda k},\tag{29}$$

$$g(\lambda)=\sum_{k=-\infty}^{\infty}g(k)e^{i\lambda k}=\Phi(e^{-i\lambda})\Phi^*(e^{-i\lambda}),\quad\Phi(e^{-i\lambda})=\sum_{k=0}^{\infty}\phi(k)e^{-i\lambda k}.\tag{30}$$

Define the matrix-valued function $\Psi_{\tau}(e^{-i\lambda})=\{\Psi_{ij}(e^{-i\lambda})\}_{i=\overline{1,M}}^{j=\overline{1,\infty}}$ by the equation

$$\Psi_{\tau}(e^{-i\lambda})\Theta_{\tau}(e^{-i\lambda})=E_M,$$

where E_M is an identity $M\times M$ matrix. Then the following factorization takes place

$$\frac{\lambda^{2d}}{|1-e^{i\lambda\tau}|^{2d}}(f(\lambda)+\lambda^{2d}g(\lambda))^{-1}=\Psi^*_{\tau}(e^{-i\lambda})\Psi_{\tau}(e^{-i\lambda}),\ \Psi_{\tau}(e^{-i\lambda})=\sum_{k=0}^{\infty}\psi_{\tau}(k)e^{-i\lambda k},\tag{31}$$

Remark 5.1
Any spectral density matrix $f(\lambda)$ is self-adjoint: $f(\lambda)=f^*(\lambda)$. Thus, $(f(\lambda))^{\top}=\overline{f(\lambda)}$. One can check that an inverse spectral density $f^{-1}(\lambda)$ is also self-adjoint $f^{-1}(\lambda)=(f^{-1}(\lambda))^*$ and $(f^{-1}(\lambda))^{\top}=\overline{f^{-1}(\lambda)}$.

The following Lemmas provide factorizations of the operators $\mathbf{P}_{\tau}$ and $\mathbf{T}_{\tau}$, which contain coefficients of factorizations (29) – (31).

Lemma 5.1
Let factorization (29) takes place and let $M\times\infty$ matrix function $\Psi_\tau(e^{-i\lambda})$ satisfy equation $\Psi_\tau(e^{-i\lambda})\Theta_\tau(e^{-i\lambda})=E_M$. Define the linear operators Ψ_τ and Θ_τ in the space ℓ_2 by the matrices with the matrix entries $(\Psi_\tau)_{k,j}=\psi_\tau(k-j)$, $(\Theta_\tau)_{k,j}=\Theta_\tau(k-j)$ for $0\leq j\leq k$, $(\Psi_\tau)_{k,j}=0$, $(\Theta_\tau)_{k,j}=0$ for $0\leq k<j$. Then:
a) the linear operator $\mathbf{P}_\tau$ admits the factorization

$$\mathbf{P}_\tau=(\Psi_\tau)^\top\overline{\Psi}_\tau;$$

b) the inverse operator $(\mathbf{P}_\tau)^{-1}$ admits the factorization

$$(\mathbf{P}_\tau)^{-1}=\overline{\Theta}_\tau(\Theta_\tau)^\top.$$

Lemma 5.2
Let factorizations (29) and (30) take place. Then the operator $\mathbf{T}_\tau$ admits the representation

$$\mathbf{T}_\tau=(\Psi_\tau)^\top\mathbf{Z}_\tau,$$

where $\mathbf{Z}_\tau$ is a linear operator in the space ℓ_2 defined by a matrix with the entries

$$\begin{aligned}(\mathbf{Z}_\tau)_{k,j}&=\sum_{l=j}^{\infty}\overline{\psi}_\tau(l-j)\overline{g}(l-k),\quad k,j\geq 0\\ g(k)&=\sum_{m=\max\{0,-k\}}^{\infty}\phi(m)\phi^*(k+m),\quad k\in\mathbb{Z}.\end{aligned}$$

Remark 5.2
Lemma 5.1 and Lemma 5.2 imply the factorization

$$(\mathbf{P}_\tau)^{-1}\mathbf{T}_\tau\mathbf{a}^\tau=\overline{\Theta}_\tau(\Theta_\tau)^\top(\Psi_\tau)^\top\mathbf{Z}_\tau\mathbf{a}^\tau=\overline{\Theta}_\tau\mathbf{Z}_\tau\mathbf{a}^\tau=\overline{\Theta}_\tau\mathbf{e}_\tau,$$

where $\mathbf{e}_\tau:=\mathbf{Z}_\tau\mathbf{a}^\tau$.

The proofs of Lemma 5.1 and Lemma 5.2, as well as the justification of the following representations of the spectral characteristics $\vec{h}^1_\tau(\lambda)$ and $\vec{h}^2_\tau(\lambda)$, correspond to the ones in [23] for the finite-dimensional vector stationary increment sequences.

The spectral characteristic $\vec{h}^2_\tau(\lambda)$ of the optimal estimate $\widehat{A}\vec{\eta}$ from Corollary 4.1 can be presented as

$$\vec{h}^2_\tau(\lambda)=\frac{(1-e^{-i\lambda\tau})^d}{(i\lambda)^d}\Psi^\top_\tau(e^{-i\lambda})\vec{C}_{\tau,g}(e^{-i\lambda}),$$

where

$$\vec{C}_{\tau,g}(e^{-i\lambda})=\sum_{m=1}^{\infty}(\overline{\psi}_\tau\mathbf{C}_{\tau,g})_m e^{-i\lambda m}.$$

Here $\overline{\psi}_\tau=(\overline{\psi}_\tau(0),\overline{\psi}_\tau(1),\overline{\psi}_\tau(2),\dots)$,

$$\begin{aligned}(\overline{\psi}_\tau\mathbf{C}_{\tau,g})_m&=\sum_{k=0}^{\infty}\overline{\psi}_\tau(k)\mathbf{c}_{\tau,g}(k+m),\\ \mathbf{c}_{\tau,g}(m)&=\sum_{k=0}^{\infty}\overline{g}(m+k)\vec{a}^\tau_k=\sum_{l=0}^{\infty}\overline{\phi}(l)\sum_{k=0}^{\infty}\phi^\top(l+m+k)\vec{a}^\tau_k=\sum_{l=0}^{\infty}\overline{\phi}(l)(\widetilde{\Phi}\mathbf{a}^\tau)_{l+m},\\ (\widetilde{\Phi}\mathbf{a}^\tau)_m&=\sum_{k=0}^{\infty}\phi^\top(m+k)\vec{a}^\tau_k.\end{aligned}$$

The spectral characteristic $\vec{h}^1_\tau(\lambda)$ of the optimal estimate $\widehat{B}\vec{\xi}$ from Corollary 4.1 can be presented in the form

$$\vec{h}^1_\tau(\lambda)=\frac{(1-e^{-i\lambda\tau})^d}{(i\lambda)^d}\left(\vec{B}_\tau(e^{i\lambda})-\Psi_\tau^\top(e^{-i\lambda})\vec{r}_\tau(e^{i\lambda})\right)=\frac{(1-e^{-i\lambda\tau})^d}{(i\lambda)^d}\Psi_\tau^\top(e^{-i\lambda})\vec{C}_{\tau,1}(e^{-i\lambda}),$$

where

$$\vec{r}_\tau(e^{i\lambda})=\sum_{m=0}^\infty(\theta_\tau^\top D^\tau\mathbf{A})_m e^{i\lambda m}=\sum_{m=0}^\infty\sum_{p=0}^\infty\theta_\tau^\top(p)\vec{b}^\tau_{p+m}e^{i\lambda m},$$

$$\vec{C}_{\tau,1}(e^{-i\lambda})=\sum_{m=1}^\infty(\theta_\tau^\top\widetilde{\mathbf{B}}_\tau)_m e^{-i\lambda m}=\sum_{m=1}^\infty\sum_{p=m}^\infty\theta_\tau^\top(p)\vec{b}^\tau_{p-m}e^{-i\lambda m}$$
$$=\sum_{m=1}^\infty\sum_{p=0}^\infty\theta_\tau^\top(m+p)\vec{b}^\tau_{p}e^{-i\lambda m},$$

a vector $\theta_\tau^\top=((\Theta_\tau(0))^\top,(\Theta_\tau(1))^\top,(\Theta_\tau(2))^\top,\ldots)$; $\mathbf{A}$ is a linear symmetric operator determined by the matrix with the vector entries $(\mathbf{A})_{k,j}=\vec{a}_{k+j}$, $k,j\geq 0$; $\widetilde{\mathbf{B}}_\tau$ is a linear operator, which is determined by a matrix with the vector entries $(\widetilde{\mathbf{B}}_\tau)_{k,j}=\vec{b}^\tau_{k-j}$ for $0\leq j\leq k$, $(\widetilde{\mathbf{B}}_\tau)_{k,j}=0$ for $0\leq k<j$.

Then the spectral characteristic $\vec{h}_\tau(\lambda)$ of the estimate $\widehat{A}\xi$ can be calculated by the formula

$$\begin{aligned}\vec{h}_\tau(\lambda)&=\frac{(1-e^{-i\lambda\tau})^d}{(i\lambda)^d}\left(\sum_{k=0}^\infty\psi_\tau^\top(k)e^{-i\lambda k}\right)\sum_{m=1}^\infty\left(\theta_\tau^\top\widetilde{\mathbf{B}}_\tau-\overline{\psi}_\tau\mathbf{C}_{\tau,g}\right)_m e^{-i\lambda m}\\&=\frac{(1-e^{-i\lambda\tau})^d}{(i\lambda)^d}\Psi_\tau^\top(e^{-i\lambda})\left(\vec{C}_{\tau,1}(e^{-i\lambda})-\vec{C}_{\tau,g}(e^{-i\lambda})\right)\\&=\vec{B}_\tau(e^{i\lambda})\frac{(1-e^{-i\lambda\tau})^d}{(i\lambda)^d}-\widetilde{\vec{h}}_\tau(\lambda),\end{aligned}\tag{32}$$

where

$$\begin{aligned}\widetilde{\vec{h}}_\tau(\lambda)&=\frac{(1-e^{-i\lambda\tau})^d}{(i\lambda)^d}\left(\sum_{k=0}^\infty\psi_\tau^\top(k)e^{-i\lambda k}\right)\left(\sum_{m=0}^\infty(\theta_\tau^\top D^\tau\mathbf{A})_m e^{i\lambda m}+\sum_{m=1}^\infty(\overline{\psi}_\tau\mathbf{C}_{\tau,g})_m e^{-i\lambda m}\right)\\&=\frac{(1-e^{-i\lambda\tau})^d}{(i\lambda)^d}\Psi_\tau^\top(e^{-i\lambda})\left(\vec{r}_\tau(e^{i\lambda})+\vec{C}_{\tau,g}(e^{-i\lambda})\right),\end{aligned}$$

The value of the mean square error of the estimate $\widehat{A}\xi$ is calculated by the formula

$$\begin{aligned}\Delta(f,g;\widehat{A}\xi)&=\Delta(f,g;\widehat{H}\vec{\xi})=\mathsf{E}|H\vec{\xi}-\widehat{H}\vec{\xi}|^2\\&=\frac{1}{2\pi}\int_{-\pi}^{\pi}(\vec{A}(e^{i\lambda}))^\top g(\lambda)\overline{\vec{A}(e^{i\lambda})}d\lambda\\&\quad+\frac{1}{2\pi}\int_{-\pi}^{\pi}(\widetilde{\vec{h}}_\tau(e^{i\lambda}))^\top(f(\lambda)+\lambda^{2d}g(\lambda))\overline{\widetilde{\vec{h}}_\tau(e^{i\lambda})}d\lambda\\&\quad-\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{(i\lambda)^d}{(1-e^{-i\lambda\tau})^d}(\widetilde{\vec{h}}_\tau(e^{i\lambda}))^\top g(\lambda)\overline{\vec{A}^\tau(e^{i\lambda})}d\lambda\\&\quad-\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{(-i\lambda)^d}{(1-e^{i\lambda\tau})^d}(\vec{A}^\tau(e^{i\lambda}))^\top g(\lambda)\overline{\widetilde{\vec{h}}_\tau(e^{i\lambda})}d\lambda\\&=\|\Phi^\top\mathbf{a}^\tau\|^2+\|\widetilde{\Phi}\mathbf{a}^\tau\|_1^2+\left\langle\theta_\tau^\top D^\tau\mathbf{A}-\overline{\psi}_\tau\mathbf{C}_{\tau,g},\theta_\tau^\top D^\tau\mathbf{A}\right\rangle\\&\quad-\left\langle\theta_\tau^\top D^\tau\mathbf{A},\mathbf{Z}_\tau\mathbf{a}^\tau\right\rangle-\left\langle\mathbf{Z}_\tau\mathbf{a}^\tau,\overline{\psi}_\tau\mathbf{C}_{\tau,g}\right\rangle_1,\end{aligned}\tag{33}$$

where $\|\vec{x}\|_1^2 = \langle\vec{x},\vec{x}\rangle_1$, $\langle\vec{x},\vec{y}\rangle_1 = \sum_{j=1}^{\infty}(\vec{x}_j)^\top\overline{\vec{y}}_j$ for the vectors

$$\vec{x} = ((\vec{x}_0)^\top, (\vec{x}_1)^\top, (\vec{x}_2)^\top, \dots)^\top, \quad \vec{y} = ((\vec{y}_0)^\top, (\vec{y}_1)^\top, (\vec{y}_2)^\top, \dots)^\top.$$

The obtained results are summarized in the form of the following theorem.

Theorem 5.1
Let the conditions of Theorem 4.1 be fulfilled and the spectral densities $f(\lambda)$ and $g(\lambda)$ of the stochastic processes $\xi(t)$ and $\eta(t)$ admit canonical factorizations (29) – (31). Then the spectral characteristic $\vec{h}_\tau(\lambda)$ and the value of the mean square error $\Delta(f,g;\widehat{A}\xi)$ of the optimal estimate $\widehat{A}\xi$ of the functional $A\xi$ based on observations of the processes $\xi(t)+\eta(t)$ at points $t<0$ can be calculated by formulas (32) and (33) respectively.

6. Minimax (robust) method of prediction

Consider the estimation problem for the functional $A\xi$ based on the observations $\xi(t)+\eta(t)$ at points $t<0$ when the spectral densities of sequences are not exactly known while a set $\mathcal{D}=\mathcal{D}_f\times\mathcal{D}_g$ of admissible spectral densities is defined. The minimax (robust) approach of estimation is applied. It is formalized by the following two definitions.

Definition 6.1
For a given class of spectral densities $\mathcal{D}=\mathcal{D}_f\times\mathcal{D}_g$ the spectral densities $f^0(\lambda)\in\mathcal{D}_f$, $g^0(\lambda)\in\mathcal{D}_g$ are called least favorable in the class $\mathcal{D}$ for the optimal linear prediction of the functional $A\xi$ if the following relation holds true:

$$\Delta(f^0,g^0) = \Delta(h(f^0,g^0);f^0,g^0) = \max_{(f,g)\in\mathcal{D}_f\times\mathcal{D}_g}\Delta(h(f,g);f,g).$$

Definition 6.2
For a given class of spectral densities $\mathcal{D}=\mathcal{D}_f\times\mathcal{D}_g$ the spectral characteristic $h^0(\lambda)$ of the optimal linear estimate of the functional $A\xi$ is called minimax-robust if there are satisfied the conditions

$$h^0(\lambda)\in H_{\mathcal{D}} = \bigcap_{(f,g)\in\mathcal{D}_f\times\mathcal{D}_g} L_2^{0-}(f(\lambda)+\lambda^{2d}g(\lambda)),$$

and

$$\min_{h\in H_{\mathcal{D}}}\max_{(f,g)\in\mathcal{D}_f\times\mathcal{D}_g}\Delta(h;f,g) = \max_{(f,g)\in\mathcal{D}_f\times\mathcal{D}_g}\Delta(h^0;f,g).$$

Taking into account the introduced definitions and the derived relations we can verify that the following lemmas hold true.

Lemma 6.1
Spectral densities $f^0\in\mathcal{D}_f$, $g^0\in\mathcal{D}_g$ which satisfy condition (15) are least favorable in the class $\mathcal{D}=\mathcal{D}_f\times\mathcal{D}_g$ for the optimal linear prediction of the functional $A\vec{\xi}$ if operators $\mathbf{P}_\tau^0$, $\mathbf{T}_\tau^0$, $\mathbf{Q}^0$ defined by the Fourier coefficients of the functions

$$\frac{\lambda^{2d}}{|1-e^{i\lambda\tau}|^{2d}}\left(g^0(\lambda)(f^0(\lambda)+\lambda^{2d}g^0(\lambda))^{-1}\right)^\top,$$

$$\frac{\lambda^{2d}}{|1-e^{i\lambda\tau}|^{2d}}\left((f^0(\lambda)+\lambda^{2d}g^0(\lambda))^{-1}\right)^\top,$$

$$\left(f^0(\lambda)(f^0(\lambda)+\lambda^{2d}g^0(\lambda))^{-1}g^0(\lambda)\right)^\top$$

determine a solution of the constrained optimization problem

$$\max_{(f,g)\in\mathcal{D}_f\times\mathcal{D}_g}\left(\langle D^\tau\mathbf{a}-\mathbf{T}_\tau\mathbf{a}^\tau,\mathbf{P}_\tau^{-1}D^\tau\mathbf{a}-\mathbf{P}_\tau^{-1}\mathbf{T}_\tau\mathbf{a}^\tau\rangle+\langle\mathbf{Q}\mathbf{a},\mathbf{a}\rangle\right)$$
$$=\langle D^\tau\mathbf{a}-\mathbf{T}_\tau^0\mathbf{a}^\tau,(\mathbf{P}_\tau^0)^{-1}D^\tau\mathbf{a}-(\mathbf{P}_\tau^0)^{-1}\mathbf{T}_\tau^0\mathbf{a}^\tau\rangle+\langle\mathbf{Q}^0\mathbf{a},\mathbf{a}\rangle. \quad (34)$$

The minimax spectral characteristic $h^0=h_\tau(f^0,g^0)$ is calculated by formula (24) if $h_\tau(f^0,g^0)\in H_{\mathcal{D}}$.

Lemma 6.2
The spectral densities $f^0 \in \mathcal{D}_f$, $g^0 \in \mathcal{D}_g$ which admit canonical factorizations (14), (29) and (30) are least favourable densities in the class $\mathcal{D}$ for the optimal linear prediction of the functional $A\xi$ based on observations of the process $\xi(t) + \eta(t)$ at points $t < 0$ if the matrix coefficients of canonical factorizations (29) and (30) determine a solution to the constrained optimization problem

$$\|\Phi^\top \mathbf{a}^\tau\|^2 + \|\widetilde{\Phi}\mathbf{a}^\tau\|_1^2 + \left\langle \theta_\tau^\top D^\tau \mathbf{A} - \overline{\psi}_\tau \mathbf{C}_{\tau,g}, \theta_\tau^\top D^\tau \mathbf{A} \right\rangle - \left\langle \theta_\tau^\top D^\tau \mathbf{A}, \mathbf{Z}_\tau \mathbf{a}^\tau \right\rangle - \left\langle \mathbf{Z}_\tau \mathbf{a}^\tau, \overline{\psi}_\tau \mathbf{C}_{\tau,g} \right\rangle_1 \to \sup, \quad (35)$$

for

$$f(\lambda) = \frac{\lambda^{2d}}{|1 - e^{i\lambda\tau}|^{2d}} \Theta_\tau(e^{-i\lambda}) \Theta_\tau^*(e^{-i\lambda}) - \lambda^{2d} \Phi(e^{-i\lambda}) \Phi^*(e^{-i\lambda}) \in \mathcal{D}_f,$$
$$g(\lambda) = \Phi(e^{-i\lambda}) \Phi^*(e^{-i\lambda}) \in \mathcal{D}_g.$$

The minimax spectral characteristic $\vec{h}^0 = \vec{h}_\tau(f^0, g^0)$ is calculated by formula (32) if $\vec{h}_\tau(f^0, g^0) \in H_\mathcal{D}$.

Lemma 6.3
The spectral density $g^0 \in \mathcal{D}_g$ which admits canonical factorizations (29), (30) with the known spectral density $f(\lambda)$ is the least favourable in the class $\mathcal{D}_g$ for the optimal linear prediction of the functional $A\xi$ based on observations of the process $\xi(t) + \eta(t)$ at points $t < 0$ if the matrix coefficients of the canonical factorizations

$$f(\lambda) + \lambda^{2d} g^0(\lambda) = \frac{\lambda^{2d}}{|1 - e^{i\lambda\tau}|^{2d}} \left(\sum_{k=0}^{\infty} \theta_\tau^0(k) e^{-i\lambda k} \right) \left(\sum_{k=0}^{\infty} \theta_\tau^0(k) e^{-i\lambda k} \right)^*, \quad (36)$$

$$g^0(\lambda) = \left(\sum_{k=0}^{\infty} \phi^0(k) e^{-i\lambda k} \right) \left(\sum_{k=0}^{\infty} \phi^0(k) e^{-i\lambda k} \right)^* \quad (37)$$

and the equation $\Psi_\tau^0(e^{-i\lambda}) \Theta_\tau^0(e^{-i\lambda}) = E_M$ determine a solution to the constrained optimization problem

$$\|\Phi^\top \mathbf{a}^\tau\|^2 + \|\widetilde{\Phi}\mathbf{a}^\tau\|_1^2 + \left\langle \theta_\tau^\top D^\tau \mathbf{A} - \overline{\psi}_\tau \mathbf{C}_{\tau,g}, \theta_\tau^\top D^\tau \mathbf{A} \right\rangle - \left\langle \theta_\tau^\top D^\tau \mathbf{A}, \mathbf{Z}_\tau \mathbf{a}^\tau \right\rangle - \left\langle \mathbf{Z}_\tau \mathbf{a}^\tau, \overline{\psi}_\tau \mathbf{C}_{\tau,g} \right\rangle_1 \to \sup, \quad (38)$$

for

$$g(\lambda) = \Phi(e^{-i\lambda}) \Phi^*(e^{-i\lambda}) \in \mathcal{D}_g.$$

The minimax spectral characteristic $\vec{h}^0 = \vec{h}_\tau(f, g^0)$ is calculated by formula (32) if $\vec{h}_\tau(f, g^0) \in H_\mathcal{D}$.

Lemma 6.4
The spectral density $f^0 \in \mathcal{D}_f$ which admits canonical factorizations (14), (29) with the known spectral density $g(\lambda)$ is the least favourable spectral density in the class $\mathcal{D}_f$ for the optimal linear prediction of the functional $A\xi$ based on observations of the process $\xi(t) + \eta(t)$ at points $t < 0$ if matrix coefficients of the canonical factorization

$$f^0(\lambda) + \lambda^{2d} g(\lambda) = \frac{\lambda^{2d}}{|1 - e^{i\lambda\tau}|^{2d}} \left(\sum_{k=0}^{\infty} \theta_\tau^0(k) e^{-i\lambda k} \right) \left(\sum_{k=0}^{\infty} \theta_\tau^0(k) e^{-i\lambda k} \right)^*, \quad (39)$$

and the equation $\Psi_\tau^0(e^{-i\lambda}) \Theta_\tau^0(e^{-i\lambda}) = E_M$ determine a solution to the constrained optimization problem

$$\left\langle \theta_\tau^\top D^\tau \mathbf{A} - \overline{\psi}_\tau \mathbf{C}_{\tau,g}, \theta_\tau^\top D^\tau \mathbf{A} \right\rangle - \left\langle \theta_\tau^\top D^\tau \mathbf{A}, \mathbf{Z}_\tau \mathbf{a}^\tau \right\rangle - \left\langle \mathbf{Z}_\tau \mathbf{a}^\tau, \overline{\psi}_\tau \mathbf{C}_{\tau,g} \right\rangle_1 \to \sup, \quad (40)$$

for

$$f(\lambda) = \frac{\lambda^{2d}}{|1 - e^{i\lambda\tau}|^{2d}} \Theta_\tau(e^{-i\lambda}) \Theta_\tau^*(e^{-i\lambda}) - \lambda^{2d} \Phi(e^{-i\lambda}) \Phi^*(e^{-i\lambda}) \in \mathcal{D}_f$$

for the fixed matrix coefficients $\{\phi(k) : k \geq 0\}$. The minimax spectral characteristic $\vec{h}^0 = \vec{h}_\tau(f^0, g)$ is calculated by formula (32) if $\vec{h}_\tau(f^0, g) \in H_\mathcal{D}$.

For more detailed analysis of properties of the least favorable spectral densities and minimax-robust spectral characteristics we observe that the minimax spectral characteristic h^0 and the least favourable spectral densities (f^0, g^0) form a saddle point of the function $\Delta(h; f, g)$ on the set $H_{\mathcal{D}} \times \mathcal{D}$.

The saddle point inequalities

$$\Delta(h; f^0, g^0) \geq \Delta(h^0; f^0, g^0) \geq \Delta(h^0; f, g) \quad \forall f \in \mathcal{D}_f, \forall g \in \mathcal{D}_g, \forall h \in H_{\mathcal{D}}$$

hold true if $h^0 = \vec{h}_\tau(f^0, g^0)$ and $\vec{h}_\tau(f^0, g^0) \in H_{\mathcal{D}}$, where (f^0, g^0) is a solution of the constrained optimisation problem

$$\widetilde{\Delta}(f, g) = -\Delta(\vec{h}_\tau(f^0, g^0); f, g) \to \inf, \quad (f, g) \in \mathcal{D}, \tag{41}$$

where the functional $\Delta(\vec{h}_\tau(f^0, g^0); f, g)$ is calculated by the formula

$$\begin{aligned}
&\Delta(\vec{h}_\tau(f^0, g^0); f, g) = \\
&= \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{\lambda^{2d}}{|1 - e^{i\lambda\tau}|^{2d}} (\mathbf{C}_\tau^{f0}(e^{i\lambda}))^\top (f^0(\lambda) + \lambda^{2d} g^0(\lambda))^{-1} f(\lambda) (f^0(\lambda) + \lambda^{2d} g^0(\lambda))^{-1} \overline{\mathbf{C}_\tau^{f0}(e^{i\lambda})} d\lambda \\
&\quad + \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{\lambda^{4n}}{|1 - e^{i\lambda\tau}|^{4n}} (\mathbf{C}_\tau^{g0}(e^{i\lambda}))^\top (f^0(\lambda) + \lambda^{2d} g^0(\lambda))^{-1} g(\lambda) (f^0(\lambda) + \lambda^{2d} g^0(\lambda))^{-1} \overline{\mathbf{C}_\tau^{g0}(e^{i\lambda})} d\lambda,
\end{aligned}$$

where

$$\mathbf{C}_\tau^{f0}(e^{i\lambda}) := \overline{g^0(\lambda)} \vec{A}_\tau(e^{i\lambda}) + \sum_{j=0}^{\infty} ((\mathbf{P}_\tau^0)^{-1} D^\tau \mathbf{a} - (\mathbf{P}_\tau^0)^{-1} \mathbf{T}_\tau^0 \mathbf{a}^\tau)_j e^{i\lambda j},$$

$$\mathbf{C}_\tau^{g0}(e^{i\lambda}) := |1 - e^{i\lambda\tau}|^{2d} \lambda^{-2d} \overline{f^0(\lambda)} \vec{A}(e^{i\lambda}) - (1 - e^{-i\lambda\tau})^d \sum_{j=0}^{\infty} ((\mathbf{P}_\tau^0)^{-1} D^\tau \mathbf{a} - (\mathbf{P}_\tau^0)^{-1} \mathbf{T}_\tau^0 \mathbf{a}^\tau)_j e^{i\lambda j}.$$

or it is calculated by the formula

$$\begin{aligned}
\Delta(\vec{h}_\tau(f^0, g^0); f, g) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{|1 - e^{i\lambda\tau}|^{2d}}{\lambda^{2d}} (\mathbf{r}_{\tau,f}^0(e^{-i\lambda}))^\top \Psi_\tau^0(e^{-i\lambda}) f(\lambda) (\Psi_\tau^0(e^{-i\lambda}))^* \overline{\mathbf{r}_{\tau,f}^0(e^{-i\lambda})} d\lambda \\
&\quad + \frac{1}{2\pi} \int_{-\pi}^{\pi} (\mathbf{r}_{\tau,g}^0(e^{-i\lambda}))^\top \Psi_\tau^0(e^{-i\lambda}) g(\lambda) (\Psi_\tau^0(e^{-i\lambda}))^* \overline{\mathbf{r}_{\tau,g}^0(e^{-i\lambda})} d\lambda,
\end{aligned}$$

where

$$\mathbf{r}_{\tau,f}^0(e^{-i\lambda}) = \sum_{m=0}^{\infty} ((\theta_\tau^0)^\top D^\tau \mathbf{A})_m e^{i\lambda m} + \sum_{m=1}^{\infty} (\overline{\psi}_\tau^0 \mathbf{C}_{\tau,g}^0)_m e^{-i\lambda m},$$

$$\mathbf{r}_{\tau,g}^0(e^{-i\lambda}) = (1 - e^{-i\lambda\tau})^d \left(\sum_{m=0}^{\infty} ((\theta_\tau^0)^\top D^\tau \mathbf{A})_m e^{i\lambda m} + \sum_{m=1}^{\infty} (\overline{\psi}_\tau^0 \mathbf{C}_{\tau,g}^0)_m e^{-i\lambda m} \right) - (\Theta_\tau^0(e^{-i\lambda}))^\top A(e^{i\lambda}).$$

The constrained optimization problem (41) is equivalent to the unconstrained optimisation problem

$$\Delta_{\mathcal{D}}(f, g) = \widetilde{\Delta}(f, g) + \delta(f, g | \mathcal{D}_f \times \mathcal{D}_g) \to \inf, \tag{42}$$

where $\delta(f, g | \mathcal{D}_f \times \mathcal{D}_g)$ is the indicator function of the set $\mathcal{D} = \mathcal{D}_f \times \mathcal{D}_g$. Solution (f^0, g^0) of this unconstrained optimization problem is characterized by the condition $0 \in \partial\Delta_{\mathcal{D}}(f^0, g^0)$, where $\partial\Delta_{\mathcal{D}}(f^0, g^0)$ is the subdifferential of the functional $\Delta_{\mathcal{D}}(f, g)$ at point $(f^0, g^0) \in \mathcal{D} = \mathcal{D}_f \times \mathcal{D}_g$, that is the set of all continuous linear functionals Λ on $L_1 \times L_1$ which satisfy the inequality $\Delta_{\mathcal{D}}(f, g) - \Delta_{\mathcal{D}}(f^0, g^0) \geq \Lambda((f, g) - (f^0, g^0)), (f, g) \in \mathcal{D}$ (see [30, 41] for more details). This condition makes it possible to find the least favourable spectral densities in some special classes of spectral densities $\mathcal{D} = \mathcal{D}_f \times \mathcal{D}_g$.

The form of the functional $\Delta(\vec{h}_\tau(f^0,g^0);f,g)$ is convenient for application the Lagrange method of indefinite multipliers for finding solution to the problem (42). Making use of the method of Lagrange multipliers and the form of subdifferential of the indicator function $\delta(f,g|\mathcal{D}_f\times\mathcal{D}_g)$ of the set $\mathcal{D}_f\times\mathcal{D}_g$ of spectral densities, we describe relations that determine least favourable spectral densities in some special classes of spectral densities (see [21, 30] for additional details).

6.1. Least favorable spectral densities in classes $\mathcal{D}_0\times\mathcal{D}_{1\delta}$

Consider the prediction problem for the functional $A\xi$ which depends on unobserved values of a process $\xi(t)$ with stationary increments based on observations of the process $\xi(t)+\eta(t)$ at points $t<0$ under the condition that the sets of admissible spectral densities $\mathcal{D}_0^k, D_{1\delta}^k, k=1,2,3,4$ are defined as follows:

$$\mathcal{D}_0^1=\left\{f(\lambda)\left|\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{|1-e^{i\lambda\tau}|^{2d}}{\lambda^{2d}}f(\lambda)d\lambda=P\right.\right\},$$

$$\mathcal{D}_0^2=\left\{f(\lambda)\left|\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{|1-e^{i\lambda\tau}|^{2d}}{\lambda^{2d}}\mathrm{Tr}\,[f(\lambda)]d\lambda=p\right.\right\},$$

$$\mathcal{D}_0^3=\left\{f(\lambda)\left|\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{|1-e^{i\lambda\tau}|^{2d}}{\lambda^{2d}}f_{kk}(\lambda)d\lambda=p_k, k=\overline{1,\infty}\right.\right\},$$

$$\mathcal{D}_0^4=\left\{f(\lambda)\left|\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{|1-e^{i\lambda\tau}|^{2d}}{\lambda^{2d}}\langle B_1,f(\lambda)\rangle\,d\lambda=p\right.\right\},$$

and

$$\mathcal{D}_{1\delta}^1=\left\{g(\lambda)\left|\frac{1}{2\pi}\int_{-\pi}^{\pi}\left|g_{ij}(\lambda)-g_{ij}^1(\lambda)\right|d\lambda\le\delta_i^j, i,j=\overline{1,\infty}\right.\right\}.$$

$$\mathcal{D}_{1\delta}^2=\left\{g(\lambda)\left|\frac{1}{2\pi}\int_{-\pi}^{\pi}\left|\mathrm{Tr}(g(\lambda)-g_1(\lambda))\right|d\lambda\le\delta\right.\right\};$$

$$\mathcal{D}_{1\delta}^3=\left\{g(\lambda)\left|\frac{1}{2\pi}\int_{-\pi}^{\pi}\left|g_{kk}(\lambda)-g_{kk}^1(\lambda)\right|d\lambda\le\delta_k, k=\overline{1,\infty}\right.\right\};$$

$$\mathcal{D}_{1\delta}^4=\left\{g(\lambda)\left|\frac{1}{2\pi}\int_{-\pi}^{\pi}\left|\langle B_2,g(\lambda)-g_1(\lambda)\rangle\right|d\lambda\le\delta\right.\right\};$$

Here $g_1(\lambda)=\{g_{ij}^1(\lambda)\}_{i,j=1}^{\infty}$ is a fixed spectral density, $p,p_k,k=\overline{1,\infty}$, are given numbers, P, B_2 are a given positive-definite Hermitian matrices, $\delta,\delta_k,k=\overline{1,\infty}$, $\delta_i^j,i,j=\overline{1,\infty}$, are given numbers.

From the condition $0\in\partial\Delta_{\mathcal{D}}(f^0,g^0)$ we find the following equations which determine the least favourable spectral densities for these given sets of admissible spectral densities.

For the first set of admissible spectral densities $\mathcal{D}_0^1\times\mathcal{D}_{1\delta}^1$, we have equations

$$\left(\mathbf{C}_\tau^{f0}(e^{i\lambda})\right)\left(\mathbf{C}_\tau^{f0}(e^{i\lambda})\right)^*=\left(\frac{|1-e^{i\lambda\tau}|^{2d}}{\lambda^{2d}}(f^0(\lambda)+\lambda^{2d}g^0(\lambda))\right)\vec{\alpha}_f\cdot\vec{\alpha}_f^*\times$$
$$\times\left(\frac{|1-e^{i\lambda\tau}|^{2d}}{\lambda^{2d}}(f^0(\lambda)+\lambda^{2d}g^0(\lambda))\right),\quad(43)$$

$$\left(\mathbf{C}_\tau^{g0}(e^{i\lambda})\right)\left(\mathbf{C}_\tau^{g0}(e^{i\lambda})\right)^*=\left(\frac{|1-e^{i\lambda\tau}|^{2d}}{\lambda^{2d}}(f^0(\lambda)+\lambda^{2d}g^0(\lambda))\right)\{\beta_{ij}\gamma_{ij}(\lambda)\}_{i,j=1}^{\infty}\times$$
$$\times\left(\frac{|1-e^{i\lambda\tau}|^{2d}}{\lambda^{2d}}(f^0(\lambda)+\lambda^{2d}g^0(\lambda))\right),\quad(44)$$

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\left|g_{ij}^{0}(\lambda)-g_{ij}^{1}(\lambda)\right|d\lambda=\delta_{i}^{j}, \tag{45}$$

where $\vec{\alpha}_f$, β_{ij} are Lagrange multipliers, functions $|\gamma_{ij}(\lambda)|\leq 1$ and

$$\gamma_{ij}(\lambda)=\frac{g_{ij}^{0}(\lambda)-g_{ij}^{1}(\lambda)}{\left|g_{ij}^{0}(\lambda)-g_{ij}^{1}(\lambda)\right|}:\ g_{ij}^{0}(\lambda)-g_{ij}^{1}(\lambda)\neq 0,\ i,j=\overline{1,\infty}.$$

For the second set of admissible spectral densities $\mathcal{D}_0^2\times\mathcal{D}_{1\delta}^2$, we have equations

$$\left(\mathbf{C}_{\tau}^{f0}(e^{i\lambda})\right)\left(\mathbf{C}_{\tau}^{f0}(e^{i\lambda})\right)^{*}=\alpha_f^2\left(\frac{|1-e^{i\lambda\tau}|^{2d}}{\lambda^{2d}}(f^0(\lambda)+\lambda^{2d}g^0(\lambda))\right)^2, \tag{46}$$

$$\left(\mathbf{C}_{\tau}^{g0}(e^{i\lambda})\right)\left(\mathbf{C}_{\tau}^{g0}(e^{i\lambda})\right)^{*}=\beta^2\gamma_2(\lambda)\left(\frac{|1-e^{i\lambda\tau}|^{2d}}{\lambda^{2d}}(f^0(\lambda)+\lambda^{2d}g^0(\lambda))\right)^2, \tag{47}$$

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\left|\mathrm{Tr}\,(g_0(\lambda)-g_1(\lambda))\right|d\lambda=\delta, \tag{48}$$

where α_f^2, β^2 are Lagrange multipliers, the function $|\gamma_2(\lambda)|\leq 1$ and

$$\gamma_2(\lambda)=\mathrm{sign}\,\left(\mathrm{Tr}\,(g_0(\lambda)-g_1(\lambda))\right):\ \mathrm{Tr}\,(g_0(\lambda)-g_1(\lambda))\neq 0.$$

For the third set of admissible spectral densities $\mathcal{D}_0^3\times\mathcal{D}_{1\delta}^3$, we have equations

$$\left(\mathbf{C}_{\tau}^{f0}(e^{i\lambda})\right)\left(\mathbf{C}_{\tau}^{f0}(e^{i\lambda})\right)^{*}=\left(\frac{|1-e^{i\lambda\tau}|^{2d}}{\lambda^{2d}}(f^0(\lambda)+\lambda^{2d}g^0(\lambda))\right)\left\{\alpha_{fk}^2\delta_{kl}\right\}_{k,l=1}^{\infty}\times \\ \times\left(\frac{|1-e^{i\lambda\tau}|^{2d}}{\lambda^{2d}}(f^0(\lambda)+\lambda^{2d}g^0(\lambda))\right), \tag{49}$$

$$\left(\mathbf{C}_{\tau}^{g0}(e^{i\lambda})\right)\left(\mathbf{C}_{\tau}^{g0}(e^{i\lambda})\right)^{*}=\left(\frac{|1-e^{i\lambda\tau}|^{2d}}{\lambda^{2d}}(f^0(\lambda)+\lambda^{2d}g^0(\lambda))\right)\left\{\beta_k^2\gamma_k^2(\lambda)\delta_{kl}\right\}_{k,l=1}^{\infty}\times \\ \times\left(\frac{|1-e^{i\lambda\tau}|^{2d}}{\lambda^{2d}}(f^0(\lambda)+\lambda^{2d}g^0(\lambda))\right), \tag{50}$$

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\left|g_{kk}^{0}(\lambda)-g_{kk}^{1}(\lambda)\right|d\lambda=\delta_k, \tag{51}$$

where α_{fk}^2, β_k^2 are Lagrange multipliers, δ_{kl} are Kronecker symbols, functions $\left|\gamma_k^2(\lambda)\right|\leq 1$ and

$$\gamma_k^2(\lambda)=\mathrm{sign}\,(g_{kk}^0(\lambda)-g_{kk}^1(\lambda)):\ g_{kk}^0(\lambda)-g_{kk}^1(\lambda)\neq 0,\ k=\overline{1,\infty}.$$

For the fourth set of admissible spectral densities $\mathcal{D}_0^4\times\mathcal{D}_{1\delta}^4$, we have equations

$$\left(\mathbf{C}_{\tau}^{f0}(e^{i\lambda})\right)\left(\mathbf{C}_{\tau}^{f0}(e^{i\lambda})\right)^{*}=\alpha_f^2\left(\frac{|1-e^{i\lambda\tau}|^{2d}}{\lambda^{2d}}(f^0(\lambda)+\lambda^{2d}g^0(\lambda))\right)B_1^{\top}\times \\ \times\left(\frac{|1-e^{i\lambda\tau}|^{2d}}{\lambda^{2d}}(f^0(\lambda)+\lambda^{2d}g^0(\lambda))\right), \tag{52}$$

$$\left(\mathbf{C}_{\tau}^{g0}(e^{i\lambda})\right)\left(\mathbf{C}_{\tau}^{g0}(e^{i\lambda})\right)^{*}=\beta^2\gamma_2'(\lambda)\left(\frac{|1-e^{i\lambda\tau}|^{2d}}{\lambda^{2d}}(f^0(\lambda)+\lambda^{2d}g^0(\lambda))\right)B_2^{\top}\times$$
$$\times\left(\frac{|1-e^{i\lambda\tau}|^{2d}}{\lambda^{2d}}(f^0(\lambda)+\lambda^{2d}g^0(\lambda))\right), \quad (53)$$

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\left|\left\langle B_2,g^0(\lambda)-g_1(\lambda)\right\rangle\right|d\lambda=\delta, \quad (54)$$

where α_f^2, β^2 are Lagrange multipliers, function $|\gamma_2'(\lambda)|\leq 1$ and

$$\gamma_2'(\lambda)=\text{sign }\left\langle B_2,g^0(\lambda)-g_1(\lambda)\right\rangle: \ \left\langle B_2,g^0(\lambda)-g_1(\lambda)\right\rangle\neq 0.$$

The derived results are summarized in the following theorem.

Theorem 6.1
The least favorable spectral densities $f^0(\lambda)$, $g^0(\lambda)$, in the classes $\mathcal{D}_0^k\times\mathcal{D}_{1\delta}^k$, $k=1,2,3,4$ for the optimal linear predictionof the functional $A\xi$ from observations of the process $\xi(t)+\eta(t)$ at points $t<0$ are determined by equations (43)–(45), (46)–(48), (49)–(51), (52)–(54), respectively, the minimality condition (15), the constrained optimization problem (34) and restrictions on densities from the corresponding classes $\mathcal{D}_0^k,\mathcal{D}_{1\delta}^k$, $k=1,2,3,4$. The minimax-robust spectral characteristic of the optimal estimate of the functional $A\xi$ is determined by formula (24).

In the case where the spectral densities $f(\lambda)$ and $g(\lambda)$ admit canonical factorizations (29) and (30), the equation for the least favourable spectral densities are described below.

For the first set of admissible spectral densities $\mathcal{D}_0^1\times\mathcal{D}_{1\delta}^1$:

$$\left(\mathbf{r}_{\tau,f}^0(e^{i\lambda})\right)\left(\mathbf{r}_{\tau,f}^0(e^{i\lambda})\right)^*=(\Theta_\tau(e^{-i\lambda}))^{\top}\vec{\alpha}_f\cdot\vec{\alpha}_f^*\overline{\Theta_\tau(e^{-i\lambda})}, \quad (55)$$
$$\left(\mathbf{r}_{\tau,g}^0(e^{i\lambda})\right)\left(\mathbf{r}_{\tau,g}^0(e^{i\lambda})\right)^*=(\Theta_\tau(e^{-i\lambda}))^{\top}\left\{\beta_{ij}\gamma_{ij}(\lambda)\right\}_{i,j=1}^{T}\overline{\Theta_\tau(e^{-i\lambda})}, \quad (56)$$

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\left|g_{ij}^0(\lambda)-g_{ij}^1(\lambda)\right|d\lambda=\delta_i^j, \quad (57)$$

where $\vec{\alpha}_f$, β_{ij} are Lagrange multipliers, functions $|\gamma_{ij}(\lambda)|\leq 1$ and

$$\gamma_{ij}(\lambda)=\frac{g_{ij}^0(\lambda)-g_{ij}^1(\lambda)}{\left|g_{ij}^0(\lambda)-g_{ij}^1(\lambda)\right|}: \ g_{ij}^0(\lambda)-g_{ij}^1(\lambda)\neq 0, \ i,j=\overline{1,\infty}.$$

For the second set of admissible spectral densities $\mathcal{D}_0^2\times\mathcal{D}_{1\delta}^2$:

$$\left(\mathbf{r}_{\tau,f}^0(e^{i\lambda})\right)\left(\mathbf{r}_{\tau,f}^0(e^{i\lambda})\right)^*=\alpha_f^2(\Theta_\tau(e^{-i\lambda}))^{\top}\overline{\Theta_\tau(e^{-i\lambda})}, \quad (58)$$
$$\left(\mathbf{r}_{\tau,g}^0(e^{i\lambda})\right)\left(\mathbf{r}_{\tau,g}^0(e^{i\lambda})\right)^*=\beta^2\gamma_2(\lambda)(\Theta_\tau(e^{-i\lambda}))^{\top}\overline{\Theta_\tau(e^{-i\lambda})}, \quad (59)$$

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\left|\mathrm{Tr}\,(g^0(\lambda)-g_1(\lambda))\right|d\lambda=\delta, \quad (60)$$

where α_f^2, β^2 are Lagrange multipliers, function $|\gamma_2(\lambda)|\leq 1$ and

$$\gamma_2(\lambda)=\text{sign }(\mathrm{Tr}\,(g^0(\lambda)-g_1(\lambda))): \ \mathrm{Tr}\,(g^0(\lambda)-g_1(\lambda))\neq 0.$$

For the third set of admissible spectral densities $\mathcal{D}_0^3\times\mathcal{D}_{1\delta}^3$:

$$\left(\mathbf{r}_{\tau,f}^0(e^{i\lambda})\right)\left(\mathbf{r}_{\tau,f}^0(e^{i\lambda})\right)^*=(\Theta_\tau(e^{-i\lambda}))^{\top}\left\{\alpha_{fk}^2\delta_{kl}\right\}_{k,l=1}^{T}\overline{\Theta_\tau(e^{-i\lambda})}, \quad (61)$$
$$\left(\mathbf{r}_{\tau,g}^0(e^{i\lambda})\right)\left(\mathbf{r}_{\tau,g}^0(e^{i\lambda})\right)^*=(\Theta_\tau(e^{-i\lambda}))^{\top}\left\{\beta_k^2\gamma_k^2(\lambda)\delta_{kl}\right\}_{k,l=1}^{T}\overline{\Theta_\tau(e^{-i\lambda})}, \quad (62)$$

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\left|g_{kk}^{0}(\lambda)-g_{kk}^{1}(\lambda)\right|d\lambda=\delta_{k}, \tag{63}$$

where α_{fk}^2, β_k^2 are Lagrange multipliers, δ_{kl} are Kronecker symbols, functions $\left|\gamma_k^2(\lambda)\right|\leq 1$ and

$$\gamma_k^2(\lambda)=\mathrm{sign}\;(g_{kk}^0(\lambda)-g_{kk}^1(\lambda)):\;g_{kk}^0(\lambda)-g_{kk}^1(\lambda)\neq 0,\;k=\overline{1,\infty}.$$

For the fourth set of admissible spectral densities $\mathcal{D}_0^4\times\mathcal{D}_{1\delta}^4$:

$$\left(\mathbf{r}_{\tau,f}^{0}(e^{i\lambda})\right)\left(\mathbf{r}_{\tau,f}^{0}(e^{i\lambda})\right)^{*}=\alpha_f^2(\Theta_\tau(e^{-i\lambda}))^{\top}B_1\overline{\Theta_\tau(e^{-i\lambda})}, \tag{64}$$

$$\left(\mathbf{r}_{\tau,g}^{0}(e^{i\lambda})\right)\left(\mathbf{r}_{\tau,g}^{0}(e^{i\lambda})\right)^{*}=\beta^2\gamma_2'(\lambda)(\Theta_\tau(e^{-i\lambda}))^{\top}B_2\overline{\Theta_\tau(e^{-i\lambda})}, \tag{65}$$

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\left|\left\langle B_2,g^0(\lambda)-g_1(\lambda)\right\rangle\right|d\lambda=\delta, \tag{66}$$

where α_f^2, β^2 are Lagrange multipliers, function $|\gamma_2'(\lambda)|\leq 1$ and

$$\gamma_2'(\lambda)=\mathrm{sign}\;\left\langle B_2,g^0(\lambda)-g_1(\lambda)\right\rangle:\;\left\langle B_2,g^0(\lambda)-g_1(\lambda)\right\rangle\neq 0.$$

The following theorems hold true.

Theorem 6.2
The least favorable spectral densities $f^0(\lambda)$, $g^0(\lambda)$ in the classes $\mathcal{D}_0^k\times\mathcal{D}_{1\delta}^k$, $k=1,2,3,4$ for the optimal linear prediction of the functional $A\xi$ from observations of the process $\xi(t)+\eta(t)$ at points $t<0$ are determined by canonical factorizations (29) and (30), equations (55)–(57), (58)–(60), (61)–(63), (64)–(66), respectively, constrained optimization problem (35) and restrictions on densities from the corresponding classes $\mathcal{D}_0^k,\mathcal{D}_{1\delta}^k,k=1,2,3,4$. The minimax-robust spectral characteristic of the optimal estimate of the functional $A\vec{\xi}$ is determined by formula (32).

Theorem 6.3
In the case where the spectral density $g(\lambda)$ is known, the least favorable spectral density $f^0(\lambda)$ in the classes $\mathcal{D}_0^k$, $k=1,2,3,4$ for the optimal linear predictionof the functional $A\xi$ from observations of the process $\xi(t)+\eta(t)$ at points $t<0$ is determined by canonical factorizations (30) and (29), equations (55), (58), (61), (64), respectively, constrained optimization problem (40) and restrictions on density from the corresponding classes $\mathcal{D}_0^k$, $k=1,2,3,4$. The minimax-robust spectral characteristic of the optimal estimate of the functional $A\xi$ is determined by formula (32).

6.2. Least favorable spectral densities in classes $\mathcal{D}_\varepsilon\times\mathcal{D}_V^U$

Consider the prediction problem for the functional $A\xi$ depending on unobserved values of the process $\vec{\xi}(m)$ with stationary increments based on observations of the process $\xi(t)+\eta(t)$ at points $t<0$ under the condition that the sets of admissible spectral densities $\mathcal{D}_\varepsilon^k,\mathcal{D}_V^{Uk},k=1,2,3,4$ are defined as follows:

$$\mathcal{D}_\varepsilon^1=\left\{f(\lambda)\middle|f(\lambda)=(1-\varepsilon)f_1(\lambda)+\varepsilon W(\lambda),\;\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{|1-e^{i\lambda\tau}|^{2d}}{\lambda^{2d}}f(\lambda)d\lambda=P\right\},$$

$$\mathcal{D}_\varepsilon^2=\left\{f(\lambda)\middle|\mathrm{Tr}\,[f(\lambda)]=(1-\varepsilon)\mathrm{Tr}\,[f_1(\lambda)]+\varepsilon\mathrm{Tr}\,[W(\lambda)],\;\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{|1-e^{i\lambda\tau}|^{2d}}{\lambda^{2d}}\mathrm{Tr}\,[f(\lambda)]d\lambda=p\right\};$$

$$\mathcal{D}_\varepsilon^3=\left\{f(\lambda)\middle|f_{kk}(\lambda)=(1-\varepsilon)f_{kk}^1(\lambda)+\varepsilon w_{kk}(\lambda),\;\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{|1-e^{i\lambda\tau}|^{2d}}{\lambda^{2d}}f_{kk}(\lambda)d\lambda=p_k,k=\overline{1,\infty}\right\};$$

$$\mathcal{D}_\varepsilon^4=\left\{f(\lambda)\middle|\left\langle B_1,f(\lambda)\right\rangle=(1-\varepsilon)\left\langle B_1,f_1(\lambda)\right\rangle+\varepsilon\left\langle B_1,W(\lambda)\right\rangle,\;\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{|1-e^{i\lambda\tau}|^{2d}}{\lambda^{2d}}\left\langle B_1,f(\lambda)\right\rangle d\lambda=p\right\};$$

and

$$\mathcal{D}_V^{U1}=\left\{g(\lambda)\middle|V(\lambda)\leq g(\lambda)\leq U(\lambda),\frac{1}{2\pi}\int_{-\pi}^{\pi}g(\lambda)d\lambda=Q\right\},$$
$$\mathcal{D}_V^{U2}=\left\{g(\lambda)\middle|\mathrm{Tr}\,[V(\lambda)]\leq \mathrm{Tr}\,[g(\lambda)]\leq \mathrm{Tr}\,[U(\lambda)],\frac{1}{2\pi}\int_{-\pi}^{\pi}\mathrm{Tr}\,[g(\lambda)]d\lambda=q\right\},$$
$$\mathcal{D}_V^{U3}=\left\{g(\lambda)\middle|v_{kk}(\lambda)\leq g_{kk}(\lambda)\leq u_{kk}(\lambda),\frac{1}{2\pi}\int_{-\pi}^{\pi}g_{kk}(\lambda)d\lambda=q_k,k=\overline{1,\infty}\right\},$$
$$\mathcal{D}_V^{U4}=\left\{g(\lambda)\middle|\left\langle B_2,V(\lambda)\right\rangle\leq\left\langle B_2,g(\lambda)\right\rangle\leq\left\langle B_2,U(\lambda)\right\rangle,\frac{1}{2\pi}\int_{-\pi}^{\pi}\left\langle B_2,g(\lambda)\right\rangle d\lambda=q\right\}.$$

Here $f_1(\lambda)$, $V(\lambda)$, $U(\lambda)$ are known and fixed spectral densities, $W(\lambda)$ is an unknown spectral density, $p,p_k,q,q_k,k=\overline{1,\infty}$, are given numbers, P,B_1,Q,B_2 are given positive-definite Hermitian matrices.

The condition $0\in\partial\Delta_{\mathcal{D}}(f^0,g^0)$ implies the following equations determining the least favourable spectral densities for these given sets of admissible spectral densities.

For the first set of admissible spectral densities $\mathcal{D}_\varepsilon^1\times\mathcal{D}_V^{U1}$, we have equations

$$\left(\mathbf{C}_\tau^{f0}(e^{i\lambda})\right)\left(\mathbf{C}_\tau^{f0}(e^{i\lambda})\right)^*=\left(\frac{|1-e^{i\lambda\tau}|^{2d}}{\lambda^{2d}}(f^0(\lambda)+\lambda^{2d}g^0(\lambda))\right)(\vec{\alpha}_f\cdot\vec{\alpha}^*+\Gamma(\lambda))\times$$
$$\times\left(\frac{|1-e^{i\lambda\tau}|^{2d}}{\lambda^{2d}}(f^0(\lambda)+\lambda^{2d}g^0(\lambda))\right),\quad(67)$$

$$\left(\mathbf{C}_\tau^{g0}(e^{i\lambda})\right)\left(\mathbf{C}_\tau^{g0}(e^{i\lambda})\right)^*=\left(\frac{|1-e^{i\lambda\tau}|^{2d}}{\lambda^{2d}}(f^0(\lambda)+\lambda^{2d}g^0(\lambda))\right)(\vec{\beta}\cdot\vec{\beta}^*+\Gamma_1(\lambda)+\Gamma_2(\lambda))\times$$
$$\times\left(\frac{|1-e^{i\lambda\tau}|^{2d}}{\lambda^{2d}}(f^0(\lambda)+\lambda^{2d}g^0(\lambda))\right),\quad(68)$$

where $\vec{\alpha}_f$ and $\vec{\beta}$ are vectors of Lagrange multipliers, function $\Gamma(\lambda)\leq 0$ and $\Gamma(\lambda)=0$ if $f_0(\lambda)>(1-\varepsilon)f_1(\lambda)$, the matrix $\Gamma_1(\lambda)\leq 0$ and $\Gamma_1(\lambda)=0$ if $g_0(\lambda)>V(\lambda)$, the matrix $\Gamma_2(\lambda)\geq 0$ and $\Gamma_2(\lambda)=0$ if $g_0(\lambda)<U(\lambda)$.

For the second set of admissible spectral densities $\mathcal{D}_\varepsilon^2\times\mathcal{D}_V^{U2}$, we have equations

$$\left(\mathbf{C}_\tau^{f0}(e^{i\lambda})\right)\left(\mathbf{C}_\tau^{f0}(e^{i\lambda})\right)^*=(\alpha_f^2+\gamma(\lambda))\left(\frac{|1-e^{i\lambda\tau}|^{2d}}{\lambda^{2d}}(f^0(\lambda)+\lambda^{2d}g^0(\lambda))\right)^2,\quad(69)$$

$$\left(\mathbf{C}_\tau^{g0}(e^{i\lambda})\right)\left(\mathbf{C}_\tau^{g0}(e^{i\lambda})\right)^*=(\beta^2+\gamma_1(\lambda)+\gamma_2(\lambda))\left(\frac{|1-e^{i\lambda\tau}|^{2d}}{\lambda^{2d}}(f^0(\lambda)+\lambda^{2d}g^0(\lambda))\right)^2,\quad(70)$$

where α_f^2, β^2 are Lagrange multipliers, function $\gamma(\lambda)\leq 0$ and $\gamma(\lambda)=0$ if $\mathrm{Tr}\,[f^0(\lambda)]>(1-\varepsilon)\mathrm{Tr}\,[f_1(\lambda)]$, function $\gamma_1(\lambda)\leq 0$ and $\gamma_1(\lambda)=0$ if $\mathrm{Tr}\,[g^0(\lambda)]>\mathrm{Tr}\,[V(\lambda)]$, the function $\gamma_2(\lambda)\geq 0$ and $\gamma_2(\lambda)=0$ if $\mathrm{Tr}\,[g^0(\lambda)]<\mathrm{Tr}\,[U(\lambda)]$.

For the third set of admissible spectral densities $\mathcal{D}_\varepsilon^3\times\mathcal{D}_V^{U3}$, we have equation

$$\left(\mathbf{C}_\tau^{f0}(e^{i\lambda})\right)\left(\mathbf{C}_\tau^{f0}(e^{i\lambda})\right)^*=\left(\frac{|1-e^{i\lambda\tau}|^{2d}}{\lambda^{2d}}(f^0(\lambda)+\lambda^{2d}g^0(\lambda))\right)\left\{(\alpha_{fk}^2+\gamma_k(\lambda))\delta_{kl}\right\}_{k,l=1}^{\infty}\times$$
$$\times\left(\frac{|1-e^{i\lambda\tau}|^{2d}}{\lambda^{2d}}(f^0(\lambda)+\lambda^{2d}g^0(\lambda))\right),\quad(71)$$

$$\left(\mathbf{C}_\tau^{g0}(e^{i\lambda})\right)\left(\mathbf{C}_\tau^{g0}(e^{i\lambda})\right)^*=\left(\frac{|1-e^{i\lambda\tau}|^{2d}}{\lambda^{2d}}(f^0(\lambda)+\lambda^{2d}g^0(\lambda))\right)\left\{(\beta_k^2+\gamma_{1k}(\lambda)+\gamma_{2k}(\lambda))\delta_{kl}\right\}_{k,l=1}^{\infty}\times$$
$$\times\left(\frac{|1-e^{i\lambda\tau}|^{2d}}{\lambda^{2d}}(f^0(\lambda)+\lambda^{2d}g^0(\lambda))\right),\quad(72)$$

where α_{fk}^2, β_k^2 are Lagrange multipliers, δ_{kl} are Kronecker symbols, functions $\gamma_k(\lambda)\leq 0$ and $\gamma_k(\lambda)=0$ if $f_{kk}^0(\lambda)>(1-\varepsilon)f_{kk}^1(\lambda)$, functions $\gamma_{1k}(\lambda)\leq 0$ and $\gamma_{1k}(\lambda)=0$ if $g_{kk}^0(\lambda)>v_{kk}(\lambda)$, functions $\gamma_{2k}(\lambda)\geq 0$ and $\gamma_{2k}(\lambda)=0$ if $g_{kk}^0(\lambda)<u_{kk}(\lambda)$.

For the forth set of admissible spectral densities $\mathcal{D}_\varepsilon^4\times\mathcal{D}_V^{U4}$, we have equation

$$\left(\mathbf{C}_\tau^{f0}(e^{i\lambda})\right)\left(\mathbf{C}_\tau^{f0}(e^{i\lambda})\right)^*=(\alpha_f^2+\gamma'(\lambda))\left(\frac{|1-e^{i\lambda\tau}|^{2d}}{\lambda^{2d}}(f^0(\lambda)+\lambda^{2d}g^0(\lambda))\right)B_1^\top\times \times\left(\frac{|1-e^{i\lambda\tau}|^{2d}}{\lambda^{2d}}(f^0(\lambda)+\lambda^{2d}g^0(\lambda))\right), \quad (73)$$

$$\left(\mathbf{C}_\tau^{g0}(e^{i\lambda})\right)\left(\mathbf{C}_\tau^{g0}(e^{i\lambda})\right)^*=(\beta^2+\gamma_1'(\lambda)+\gamma_2'(\lambda))\left(\frac{|1-e^{i\lambda\tau}|^{2d}}{\lambda^{2d}}(f^0(\lambda)+\lambda^{2d}g^0(\lambda))\right)B_2^\top\times \times\left(\frac{|1-e^{i\lambda\tau}|^{2d}}{\lambda^{2d}}(f^0(\lambda)+\lambda^{2d}g^0(\lambda))\right), \quad (74)$$

where α_f^2, β^2, are Lagrange multipliers, function $\gamma'(\lambda)\leq 0$ and $\gamma'(\lambda)=0$ if $\langle B_1,f^0(\lambda)\rangle>(1-\varepsilon)\langle B_1,f_1(\lambda)\rangle$, functions $\gamma_1'(\lambda)\leq 0$ and $\gamma_1'(\lambda)=0$ if $\langle B_2,g^0(\lambda)\rangle>\langle B_2,V(\lambda)\rangle$, functions $\gamma_2'(\lambda)\geq 0$ and $\gamma_2'(\lambda)=0$ if $\langle B_2,g^0(\lambda)\rangle<\langle B_2,U(\lambda)\rangle$.

The following theorem holds true.

Theorem 6.4
The least favorable spectral densities $f^0(\lambda)$, $g^0(\lambda)$ in classes $\mathcal{D}_\varepsilon^k\times\mathcal{D}_V^{Uk}, k=1,2,3,4$ for the optimal linear extrapolation of the functional $A\xi$ from observations of the process $\xi(t)+\eta(t)$ at points $t<0$ are determined by equations (67)–(68), (69)–(70), (71)–(72), (73)–(74), respectively, the minimality condition (15), the constrained optimization problem (34) and restrictions on densities from the corresponding classes $\mathcal{D}_\varepsilon^k,\mathcal{D}_V^{Uk}, k=1,2,3,4$. The minimax-robust spectral characteristic of the optimal estimate of the functional $A\xi$ is determined by formula (24).

Let the spectral densities $f(\lambda)$ and $g(\lambda)$ admit the canonical factorizations (29) and (30). The equations for the least favourable spectral densities are described below.

For the first set of admissible spectral densities $\mathcal{D}_\varepsilon^1\times\mathcal{D}_V^{U1}$, we have equation

$$\left(\mathbf{r}_{\tau,f}^0(e^{i\lambda})\right)\left(\mathbf{r}_{\tau,f}^0(e^{i\lambda})\right)^*=(\Theta_\tau(e^{-i\lambda}))^\top(\vec{\alpha}_f\cdot\vec{\alpha}_f^*+\Gamma(\lambda))\overline{\Theta_\tau(e^{-i\lambda})}, \quad (75)$$

$$\left(\mathbf{r}_{\tau,g}^0(e^{i\lambda})\right)\left(\mathbf{r}_{\tau,g}^0(e^{i\lambda})\right)^*=(\Theta_\tau(e^{-i\lambda}))^\top(\vec{\beta}\cdot\vec{\beta}^*+\Gamma_1(\lambda)+\Gamma_2(\lambda))\overline{\Theta_\tau(e^{-i\lambda})} \quad (76)$$

where $\vec{\alpha}_f$ and $\vec{\beta}$ are vectors of Lagrange multipliers, matrix $\Gamma(\lambda)\leq 0$ and $\Gamma(\lambda)=0$ if $f^0(\lambda)>(1-\varepsilon)f_1(\lambda)$, matrix $\Gamma_1(\lambda)\leq 0$ and $\Gamma_1(\lambda)=0$ if $g^0(\lambda)>V(\lambda)$, matrix $\Gamma_2(\lambda)\geq 0$ and $\Gamma_2(\lambda)=0$ if $g^0(\lambda)<U(\lambda)$.

For the second set of admissible spectral densities $\mathcal{D}_\varepsilon^2\times\mathcal{D}_V^{U2}$, we have equations

$$\left(\mathbf{r}_{\tau,f}^0(e^{i\lambda})\right)\left(\mathbf{r}_{\tau,f}^0(e^{i\lambda})\right)^*=(\alpha_f^2+\gamma(\lambda))(\Theta_\tau(e^{-i\lambda}))^\top\overline{\Theta_\tau(e^{-i\lambda})}, \quad (77)$$

$$\left(\mathbf{r}_{\tau,g}^0(e^{i\lambda})\right)\left(\mathbf{r}_{\tau,g}^0(e^{i\lambda})\right)^*=(\beta^2+\gamma_1(\lambda)+\gamma_2(\lambda))(\Theta_\tau(e^{-i\lambda}))^\top\overline{\Theta_\tau(e^{-i\lambda})}, \quad (78)$$

where α_f^2, β^2 are Lagrange multipliers, function $\gamma(\lambda)\leq 0$ and $\gamma(\lambda)=0$ if $\mathrm{Tr}\,[f^0(\lambda)]>(1-\varepsilon)\mathrm{Tr}\,[f_1(\lambda)]$, function $\gamma_1(\lambda)\leq 0$ and $\gamma_1(\lambda)=0$ if $\mathrm{Tr}\,[g^0(\lambda)]>\mathrm{Tr}\,[V(\lambda)]$, the function $\gamma_2(\lambda)\geq 0$ and $\gamma_2(\lambda)=0$ if $\mathrm{Tr}\,[g^0(\lambda)]<\mathrm{Tr}\,[U(\lambda)]$.

For the third set of admissible spectral densities $\mathcal{D}_\varepsilon^3\times\mathcal{D}_V^{U3}$, we have equation

$$\left(\mathbf{r}_{\tau,f}^0(e^{i\lambda})\right)\left(\mathbf{r}_{\tau,f}^0(e^{i\lambda})\right)^*=(\Theta_\tau(e^{-i\lambda}))^\top\left\{(\alpha_{fk}^2+\gamma_k(\lambda))\delta_{kl}\right\}_{k,l=1}^\infty\overline{\Theta_\tau(e^{-i\lambda})}, \quad (79)$$

$$\left(\mathbf{r}_{\tau,g}^0(e^{i\lambda})\right)\left(\mathbf{r}_{\tau,g}^0(e^{i\lambda})\right)^*=(\Theta_\tau(e^{-i\lambda}))^\top\left\{(\beta_k^2+\gamma_{1k}(\lambda)+\gamma_{2k}(\lambda))\delta_{kl}\right\}_{k,l=1}^\infty\overline{\Theta_\tau(e^{-i\lambda})}, \quad (80)$$

where α_{fk}^2, β_k^2 are Lagrange multipliers, δ_{kl} are Kronecker symbols, functions $\gamma_k(\lambda) \leq 0$ and $\gamma_k(\lambda) = 0$ if $f_{kk}^0(\lambda) > (1-\varepsilon) f_{kk}^1(\lambda)$, functions $\gamma_{1k}(\lambda) \leq 0$ and $\gamma_{1k}(\lambda) = 0$ if $g_{kk}^0(\lambda) > v_{kk}(\lambda)$, functions $\gamma_{2k}(\lambda) \geq 0$ and $\gamma_{2k}(\lambda) = 0$ if $g_{kk}^0(\lambda) < u_{kk}(\lambda)$.

For the fourth set of admissible spectral densities $\mathcal{D}_{\varepsilon}^4 \times \mathcal{D}_V^{U4}$, we have equation

$$\left(\mathbf{r}_{\tau,f}^0(e^{i\lambda})\right)\left(\mathbf{r}_{\tau,f}^0(e^{i\lambda})\right)^* = (\alpha_f^2 + \gamma'(\lambda))(\Theta_\tau(e^{-i\lambda}))^\top B_1 \overline{\Theta_\tau(e^{-i\lambda})}, \tag{81}$$

$$\left(\mathbf{r}_{\tau,g}^0(e^{i\lambda})\right)\left(\mathbf{r}_{\tau,g}^0(e^{i\lambda})\right)^* = (\beta^2 + \gamma_1'(\lambda) + \gamma_2'(\lambda))(\Theta_\tau(e^{-i\lambda}))^\top B_2 \overline{\Theta_\tau(e^{-i\lambda})}, \tag{82}$$

where α_f^2, β^2, are Lagrange multipliers, function $\gamma'(\lambda) \leq 0$ and $\gamma'(\lambda) = 0$ if $\langle B_1, f^0(\lambda)\rangle > (1-\varepsilon)\langle B_1, f_1(\lambda)\rangle$, functions $\gamma_1'(\lambda) \leq 0$ and $\gamma_1'(\lambda) = 0$ if $\langle B_2, g^0(\lambda)\rangle > \langle B_2, V(\lambda)\rangle$, functions $\gamma_2'(\lambda) \geq 0$ and $\gamma_2'(\lambda) = 0$ if $\langle B_2, g^0(\lambda)\rangle < \langle B_2, U(\lambda)\rangle$.

The following theorems hold true.

Theorem 6.5
The least favorable spectral densities $f^0(\lambda)$, $g^0(\lambda)$ in the classes $\mathcal{D}_{\varepsilon}^k \times \mathcal{D}_V^{Uk}, k = 1,2,3,4$ for the optimal linear prediction of the functional $A\xi$ from observations of the process $\xi(t) + \eta(t)$ at points $t < 0$ by canonical factorizations (29) and (30), equations (75)–(76), (77)–(78), (79)–(80), (81)–(82), respectively, constrained optimization problem (35) and restrictions on densities from the corresponding classes $\mathcal{D}_{\varepsilon}^k, \mathcal{D}_V^{Uk}, k = 1,2,3,4$. The minimax-robust spectral characteristic of the optimal estimate of the functional $A\xi$ is determined by formula (32).

Theorem 6.6
If the spectral density $g(\lambda)$ is known, the least favorable spectral density $f^0(\lambda)$ in the classes $\mathcal{D}_{\varepsilon}^k$, $k = 1,2,3,4$ for the optimal linear prediction of the functional $A\xi$ from observations of the process $\xi(t) + \eta(t)$ at points $t < 0$ is determined by canonical factorizations (30) and (29), equations (75), (77), (79), (81), respectively, constrained optimization problem (40) and restrictions on density from the corresponding classes $\mathcal{D}_{f\varepsilon}^k$, $k = 1,2,3,4$. The minimax-robust spectral characteristic of the optimal estimate of the functional $A\xi$ is determined by formula (32).

Conclusions

In this article, we dealt with continuous time stochastic processes with periodically correlated dth increments. These stochastic processes form a class of non-stationary stochastic processes that combine periodic structure of covariation functions of processes as well as integrating one.

We derived solutions of the problem of estimation of the linear functionals constructed from the unobserved values of a continuous time stochastic process with periodically correlated dth increments. Estimates are based on observations of this process with periodically stationary noise at points $t < 0$. We obtained the estimates by representing the process under investigation as a vector-valued sequence with stationary increments. Based on the solutions for these type of sequences, we solved the corresponding problem for the considered class of continuous time stochastic processes. The problem is investigated in the case of spectral certainty, where spectral densities of sequences are exactly known. In this case we propose an approach based on the Hilbert space projection method. We derive formulas for calculating the spectral characteristics and the mean-square errors of the optimal estimates of the functionals. In the case of spectral uncertainty where the spectral densities are not exactly known while, instead, some sets of admissible spectral densities are specified, the minimax-robust method is applied. We propose a representation of the mean square error in the form of a linear functional in L_1 with respect to spectral densities, which allows us to solve the corresponding constrained optimization problem and describe the minimax-robust estimates of the functionals. Formulas that determine the least favorable spectral densities and minimax-robust spectral characteristics of the optimal linear estimates of the functionals are derived for a wide list of specific classes of admissible spectral densities.

The further steps of the study consist in practical application of the developed estimates and techniques in the fields of environmental researches, financial and econometrics forecasting, signal processing etc., as well as an investigation of their limitations.

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Appendix

Proof of Lemma 4.2.

Define

$$b_j^\tau(u) = b^\tau(u + jT), \quad \zeta_j^{(d)}(u) = \zeta_j^{(d)}(u + jT, \tau T),\ u \in [0, T),$$

and

$$a_j(u) = a(u + jT), \quad \eta_j(u) = \eta_j(u + jT),\ u \in [0, T).$$

Making use of decomposition (6) for the increment sequence $\{\zeta_j^{(d)}, j \in \mathbb{Z}\}$ and the solution of equation

$$(-1)^k \left[\frac{k}{2}\right] + (-1)^m \left[\frac{m}{2}\right] = 0 \tag{83}$$

of two variables (k, m), which is given by pairs $(1, 1)$, $(2l + 1, 2l)$ and $(2l, 2l + 1)$ for $l = 2, 3, \ldots$, rewrite the functional $B\zeta$ as [31]

$$\begin{aligned}
B\zeta &= \int_0^\infty b^\tau(t)\zeta^{(d)}(t, \tau T)dt = \sum_{j=0}^\infty \int_0^T b_j^\tau(u)\zeta_j^{(d)}(u)du \\
&= \sum_{j=0}^\infty \frac{1}{T}\int_0^T \left(\sum_{k=1}^\infty b_{kj}^\tau e^{2\pi i\{(-1)^k[\frac{k}{2}]\}u/T}\right)\left(\sum_{m=1}^\infty \zeta_{mj}^{(d)} e^{2\pi i\{(-1)^m[\frac{m}{2}]\}u/T}\right) du \\
&= \sum_{j=0}^\infty \sum_{k=1}^\infty \sum_{m=1}^\infty b_{kj}^\tau \zeta_{mj}^{(d)} \frac{1}{T}\int_0^T e^{2\pi i\left\{(-1)^k[\frac{k}{2}]+(-1)^m[\frac{m}{2}]\right\}u/T} du \\
&= \sum_{j=0}^\infty \sum_{k=1}^\infty b_{kj}^\tau \zeta_{kj}^{(d)} = \sum_{j=0}^\infty (\vec{b}_j^\tau)^\top \vec{\zeta}_j^{(d)} \\
&= B\vec{\zeta}.
\end{aligned}$$

The representation of the functional $V\eta$ is obtained in the same way:

$$V\eta = \int_0^\infty a(t)\eta(t)dt = \sum_{j=0}^\infty \int_0^T a_j(u)\eta_j(u)du = \sum_{j=0}^\infty \sum_{k=1}^\infty a_{kj}\eta_{kj} = \sum_{j=0}^\infty (\vec{a}_j)^\top \vec{\eta}_j = V\vec{\eta}.$$

From Lemma 4.1 we obtain

$$b_j^\tau(u) = \sum_{l=0}^\infty a(u + jT + \tau Tl)d(l) = D^{\tau T}a(u),\ u \in [0; T),\ j = 0, 1, \ldots,$$

and

$$b_{kj}^\tau = \sum_{l=0}^\infty a_{kj+\tau l}d(l),\ j = 0, 1, \ldots,$$

which finalizes the proof of Lemma 4.2.

Proof of Theorem 4.1.
A projection $\widehat{H}\vec{\xi}$ of the element $H\vec{\xi}$ on the subspace $H^{0-}(\xi^{(d)}_{\tau}+\eta^{(d)}_{\tau})$ is characterized by two conditions:
1) $\widehat{H}\vec{\xi}\in H^{0-}(\xi^{(d)}_{\tau}+\eta^{(d)}_{\tau})$;
2) $(H\vec{\xi}-\widehat{H}\vec{\xi})\perp H^{0-}(\xi^{(d)}_{\tau}+\eta^{(d)}_{\tau})$.
The second condition implies the following relation which holds true for all $j\leq -1$ and for all $k\geq 1$

$$\int_{-\pi}^{\pi}\Bigg((\vec{B}_{\tau}(e^{i\lambda}))^{\top}\frac{(1-e^{-i\lambda\tau})^{d}}{(i\lambda)^{d}}(f(\lambda)+\lambda^{2d}g(\lambda))-(\vec{h}_{\tau}(\lambda))^{\top}(f(\lambda)+\lambda^{2d}g(\lambda))-$$
$$-(\vec{A}(e^{i\lambda}))^{\top}g(\lambda)(-i\lambda)^{d}\Bigg)\vec{\delta}_{k}\frac{(1-e^{i\lambda\tau})^{d}}{(-i\lambda)^{d}}e^{-i\lambda j}d\lambda=0.$$

From these relations, we conclude that the spectral characteristic $\vec{h}_{\tau}(\lambda)$ of the estimate $\widehat{H}\vec{\xi}$ allow a representation in the form (24) where

$$\vec{C}_{\tau}(e^{i\lambda})=\sum_{j=0}^{\infty}\vec{c}^{\,\tau}_{j}e^{i\lambda j},$$

and $\vec{c}^{\,\tau}_{j}=\{c^{\tau}_{kj}\}_{k=1}^{\infty}$, $j=0,1,2,\dots$, are unknown coefficients to be found.

Condition 1) implies $(i\lambda)^{d}(1-e^{-i\lambda\tau})^{-d}\vec{h}_{\tau}(\lambda)\in L_{2}^{0-}$, and thus,

$$\int_{-\pi}^{\pi}\Bigg((\vec{B}_{\tau}(e^{i\lambda}))^{\top}-(\vec{A}_{\tau}(e^{i\lambda}))^{\top}g(\lambda)\frac{\lambda^{2d}}{|1-e^{i\lambda\tau}|^{2d}}(f(\lambda)+\lambda^{2d}g(\lambda))^{-1}$$
$$-(\vec{C}_{\tau}(e^{i\lambda}))^{\top}\frac{\lambda^{2d}}{|1-e^{i\lambda\tau}|^{2d}}(f(\lambda)+\lambda^{2d}g(\lambda))^{-1}\Bigg)e^{-i\lambda j}d\lambda=\vec{0},\quad j\geq 0,$$

which can be presented as a system of linear equations

$$\vec{b}^{\,\tau}_{l}-\sum_{j=0}^{\infty}T^{\tau}_{l,j}\vec{a}^{\,\tau}_{j}=\sum_{j=0}^{\infty}P^{\tau}_{l,j}\vec{c}^{\,\tau}_{j},\quad l\geq 0,\tag{84}$$

determining the unknown coefficients $\vec{c}^{\,\tau}_{j}$, $j\geq 0$.

Rewrite the system of equations (84) in the matrix form

$$D^{\tau}\mathbf{a}-\mathbf{T}_{\tau}\mathbf{a}^{\tau}=\mathbf{P}_{\tau}\mathbf{c}^{\tau},$$

where

$$\mathbf{c}^{\tau}=((\vec{c}^{\,\tau}_{0})^{\top},(\vec{c}^{\,\tau}_{1})^{\top},(\vec{c}^{\,\tau}_{2})^{\top},\dots)^{\top}.$$

Consequently, the unknown coefficients $\vec{c}^{\,\tau}_{j}$, $j\geq 0$, determining the spectral characteristic $\vec{h}_{\tau}(\lambda)$ are as follows

$$\vec{c}^{\,\tau}_{j}=(\mathbf{P}_{\tau}^{-1}D^{\tau}\mathbf{a}-\mathbf{P}_{\tau}^{-1}\mathbf{T}_{\tau}\mathbf{a}^{\tau})_{j},\quad j\geq 0,$$

where $(\mathbf{P}_{\tau}^{-1}D^{\tau}\mathbf{a}-\mathbf{P}_{\tau}^{-1}\mathbf{T}_{\tau}\mathbf{a}^{\tau})_{j}$, $j\geq 0$, is the jth infinite dimension vector entry of the vector $\mathbf{P}_{\tau}^{-1}D^{\tau}\mathbf{a}-\mathbf{P}_{\tau}^{-1}\mathbf{T}_{\tau}\mathbf{a}^{\tau}$. The existence of the inverse matrix $(\mathbf{P}^{\tau})^{-1}$ is justified in [21] under condition (15). Thus, the function $\vec{C}_{\tau}(e^{i\lambda})$ is calculated as

$$\vec{C}_{\tau}(e^{i\lambda})=\sum_{j=0}^{\infty}(\mathbf{P}_{\tau}^{-1}D^{\tau}\mathbf{a}-\mathbf{P}_{\tau}^{-1}\mathbf{T}_{\tau}\mathbf{a}^{\tau})_{j}e^{i\lambda j}$$

and the spectral characteristic $\vec{h}_\tau(\lambda)$ is calculated by the formula

$$
\begin{aligned}
(\vec{h}_\tau(\lambda))^\top &= (\vec{B}_\tau(e^{i\lambda}))^\top \frac{(1-e^{-i\lambda\tau})^d}{(i\lambda)^d} \\
&\quad - (\vec{A}_\tau(e^{i\lambda}))^\top g(\lambda)\frac{(-i\lambda)^d}{(1-e^{i\lambda\tau})^d}(f(\lambda)+\lambda^{2d}g(\lambda))^{-1} \\
&\quad - \frac{(-i\lambda)^d}{(1-e^{i\lambda\tau})^d}\left(\sum_{k=0}^{\infty}(\mathbf{P}_\tau^{-1}D^\tau\mathbf{a}-\mathbf{P}_\tau^{-1}\mathbf{T}_\tau\mathbf{a}^\tau)_k e^{i\lambda j}\right)^\top (f(\lambda)+\lambda^{2d}g(\lambda))^{-1}.
\end{aligned}
$$

The value of the mean square error of the estimate $\widehat{A}\vec{\xi}$ is calculated by the formula

$$
\begin{aligned}
\Delta(f,g;\widehat{A}\vec{\xi}) &= \Delta(f,g;\widehat{H}\vec{\xi}) = \mathsf{E}|H\vec{\xi}-\widehat{H}\vec{\xi}|^2 \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{\lambda^{2d}}{|1-e^{i\lambda\tau}|^{2d}}\left((\vec{A}_\tau(e^{i\lambda}))^\top g(\lambda)+(\vec{C}_\tau(e^{i\lambda}))^\top\right) \\
&\quad \times (f(\lambda)+\lambda^{2d}g(\lambda))^{-1} f(\lambda)\,(f(\lambda)+\lambda^{2d}g(\lambda))^{-1} \\
&\quad \times \left(g(\lambda)\overline{\vec{A}_\tau(e^{i\lambda})}+\overline{\vec{C}_\tau(e^{i\lambda})}\right)d\lambda \\
&\quad + \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{\lambda^{4d}}{|1-e^{i\lambda\tau}|^{4d}}\left(\frac{|1-e^{i\lambda\tau}|^{2d}}{\lambda^{2d}}(\vec{A}(e^{i\lambda}))^\top f(\lambda)-(1-e^{-i\lambda\tau})^d(\vec{C}_\tau(e^{i\lambda}))^\top\right) \\
&\quad \times (f(\lambda)+\lambda^{2d}g(\lambda))^{-1} g(\lambda)\,(f(\lambda)+\lambda^{2d}g(\lambda))^{-1} \\
&\quad \times \left(\frac{|1-e^{i\lambda\tau}|^{2d}}{\lambda^{2d}}f(\lambda)\overline{\vec{A}(e^{i\lambda})}-(1-e^{i\lambda\tau})^d\overline{\vec{C}_\tau(e^{i\lambda})}\right)d\lambda \\
&= \langle D^\tau\mathbf{a}-\mathbf{T}_\tau\mathbf{a}^\tau, \mathbf{P}_\tau^{-1}D^\tau\mathbf{a}-\mathbf{P}_\tau^{-1}\mathbf{T}_\tau\mathbf{a}^\tau\rangle+\langle\mathbf{Q}\mathbf{a},\mathbf{a}\rangle,
\end{aligned}
$$

which finalizes the proof of Theorem 4.1.

REFERENCES

1. I.V. Basawa, R. Lund, and Q. Shao, *First-order seasonal autoregressive processes with periodically varying parameters*, Statistics & Probability Letters, vol. 67, no. 4, pp. 299–306, 2004.
2. F. Chaari, J. Leskow, A. Napolitano, and A. Sanchez-Ramirez, *Cyclostationarity: Theory and Methods*, Springer Cham, 2014.
3. F. Chaari, J. Leskow, R. Zimroz, A. Wylomanska, and A. Dudek, *Cyclostationarity: Theory and Methods – IV. Contributions to the 10th Workshop on Cyclostationary Systems and Their Applications, February 2017, Grodek, Poland*, Springer Cham, 2020.
4. I. I. Dubovets'ka, and M. P. Moklyachuk, *Minimax estimation problem for periodically correlated stochastic processes*, Journal of Mathematics and System Science, vol. 3, no.1, pp. 26–30, 2013.
5. I. I. Dubovets'ka, and M. P. Moklyachuk, *Extrapolation of periodically correlated processes from observations with noise*, Theory of Probability and Mathematical Statistics, vol. 88, pp. 67–83, 2014.
6. A. Dudek, H. Hurd, and W. Wojtowicz, *PARMA methods based on Fourier representation of periodic coefficients*, Wiley Interdisciplinary Reviews: Computational Statistics, vol. 8, no. 3, pp. 130–149, 2016.
7. J. Franke, and H. V. Poor, *Minimax-robust filtering and finite-length robust predictors*, In: Robust and Nonlinear Time Series Analysis, Lecture Notes in Statistics, Springer-Verlag, No.26, pp. 87–126, 1984.
8. W.A. Gardner, *Cyclostationarity in communications and signal processing*, New York, NY: IEEE, 1994.
9. W.A. Gardner, A. Napolitano, and L.Paura, *Cyclostationarity: half a century of research*, Signal Process., vol. 86, no. 4, pp. 639–697, 2006.
10. I. I. Gikhman, and A. V. Skorokhod, *The theory of stochastic processes. I.*, Berlin: Springer, 574 p., 2004.
11. E. G. Gladyshev, *Periodically and almost-periodically correlated random processes with continuous time parameter*, Theory Probab. Appl. vol. 8, pp. 173–177, 1963.
12. U. Grenander, *A prediction problem in game theory*, Arkiv för Matematik, vol. 3, pp. 371–379, 1957.
13. Y. Hosoya, *Robust linear extrapolations of second-order stationary processes*, Annals of Probability, vol. 6, no. 4, pp. 574–584, 1978.
14. H.L. Hurd, and A. Miamee, *Periodically correlated random sequences. Spectral theory and practice*, Wiley Series in Probability and Statistics, Hoboken, NJ: John Wiley & Sons, 2007.
15. G. Kallianpur, and V. Mandrekar,: *Spectral theory of stationary H-valued processes*, J. Multivariate Analysis, vol. 1, pp. 1–16, 1971.

16. K. Karhunen, *Uber lineare Methoden in der Wahrscheinlichkeitsrechnung*, Annales Academiae Scientiarum Fennicae. Ser. A I, no. 37, 1947.
17. S. A. Kassam, and H. V. Poor, *Robust techniques for signal processing: A survey*, Proceedings of the IEEE No.73, pp. 433–481, 1985.
18. A. N. Kolmogorov, *Selected works by A. N. Kolmogorov. Vol. II: Probability theory and mathematical statistics. Ed. by A. N. Shiryayev. Mathematics and Its Applications. Soviet Series. 26. Dordrecht etc.* Kluwer Academic Publishers, 1992.
19. P. S. Kozak, M. M. Luz and M. P. Moklyachuk, *Minimax prediction of sequences with periodically stationary increments*, Carpathian Mathematical Publications, vol.13, iss.2 pp. 352–376, 2021.
20. M. Luz, and M. Moklyachuk, *Minimax-robust prediction problem for stochastic sequences with stationary increments and cointegrated sequences*, Statistics, Optimization and Information Compututing, vol. 3, no. 2, pp. 160–188, 2015.
21. M. Luz, and M. Moklyachuk, *Estimation of Stochastic Processes with Stationary Increments and Cointegrated Sequences*, London: ISTE; Hoboken, NJ: John Wiley & Sons, 282 p., 2019.
22. M. Luz, and M. Moklyachuk, *Minimax-robust forecasting of sequences with periodically stationary long memory multiple seasonal increments*, Statistics, Optimization and Information Computing, vol. 8, no. 3, pp. 684–721, 2020.
23. M. Luz, and M. Moklyachuk, *Minimax prediction of sequences with periodically stationary increments observed with noise and cointegrated sequences*, In: M. Moklyachuk (ed.) Stochastic Processes: Fundamentals and Emerging Applications. Nova Science Publishers, New York, pp. 189–247, 2023.
24. M. Luz, and M. Moklyachuk, *Estimation problem for continuous time stochastic processes with periodically correlated increments*, Statistics, Optimization and Information Computing, vol.11, no. 4, pp. 811–828, 2023.
25. R. Lund, *Choosing seasonal autocovariance structures: PARMA or SARMA*, In: Bell WR, Holan SH, McElroy TS (eds) Economic time series: modelling and seasonality. Chapman and Hall, London, pp. 63–80, 2011.
26. M. P. Moklyachuk, *Estimation of linear functionals of stationary stochastic processes and a two-person zero-sum game*, Stanford University Technical Report, no. 169, pp. 1–82, 1981.
27. M. P. Moklyachuk, *On a problem of game theory and the extrapolation of stochastic processes with values in a Hilbert space*, Theory Probability and Mathematical Statistics, vol. 24, pp. 115–124, 1981.
28. M. P. Moklyachuk, *On an antagonistic game and prediction of stationary random sequences in a Hilbert space*, Theory Probability and Mathematical Statistics, vol. 25, pp. 107–113, 1982.
29. M. P. Moklyachuk, *Robust estimations of functionals of stochastic processes*, Kyïv: Vydavnychyj Tsentr "Kyïvs'kyĭ Universytet", 320 p., 2008. (in Ukrainian)
30. M. P. Moklyachuk, *Minimax-robust estimation problems for stationary stochastic sequences*, Statistics, Optimization and Information Computing, vol. 3, no. 4, pp. 348–419, 2015.
31. M. P. Moklyachuk, and I. I. Golichenko, *Periodically correlated processes estimates*, Saarbrücken: LAP Lambert Academic Publishing, 308 p., 2016.
32. M.P. Moklyachuk, and A.Yu. Masyutka, *Minimax-robust estimation technique: For stationary stochastic processes*, LAP Lambert Academic Publishing, 296 p. 2012.
33. M. Moklyachuk, M. Sidei, and O. Masyutka, *Estimation of stochastic processes with missing observations*, Mathematics Research Developments. New York, NY: Nova Science Publishers, 336 p., 2019.
34. A. Napolitano, *Generalizations of cyclostationary signal processing. Spectral analysis and applications.*, Hoboken, NJ: John Wiley & Sons, 2012.
35. A. Napolitano, *Cyclostationarity: New trends and applications*, Signal Processing, vol. 120, pp. 385–408, 2016.
36. A. Napolitano, *Cyclostationarity: Limits and generalizations*, Signal Process., vol. 120, pp. 323–347, 2016.
37. D. Osborn, *The implications of periodically varying coefficients for seasonal time-series processes*, Journal of Econometrics, vol. 48, no. 3, pp. 373–384, 1991.
38. M. S. Pinsker, and A. M. Yaglom, *On linear extrapolaion of random processes with nth stationary incremens*, Doklady Akademii Nauk SSSR, n. Ser. vol. 94, pp. 385–388, 1954.
39. S. Porter-Hudak, *An application of the seasonal fractionally differenced model to the monetary aggegrates*, Journal of the American Statistical Association, vol.85, no. 410, pp. 338–344, 1990.
40. V. A. Reisen, E. Z. Monte, G. C. Franco, A. M. Sgrancio, F. A. F. Molinares, P. Bondond, F. A. Ziegelmann, and B. Abraham, *Robust estimation of fractional seasonal processes: Modeling and forecasting daily average SO2 concentrations*, Mathematics and Computers in Simulation, vol. 146, pp. 27–43, 2018.
41. R. T. Rockafellar, *Convex Analysis*, Princeton Landmarks in Mathematics. Princeton, NJ: Princeton University Press, 451 p., 1997.
42. E. Serpedin, F. Panduru, I. Sarı, and G.B. Giannakis, *Bibliography on cyclostationarity*, Signal Process., vol. 85, no. 12, pp. 2233–2303, 2005.
43. C. C. Solci, V. A. Reisen, A. J. Q. Sarnaglia, and P. Bondon, *Empirical study of robust estimation methods for PAR models with application to the air quality area*, Communication in Statistics - Theory and Methods, vol. 48, no. 1, pp. 152–168, 2020.
44. G. Terdik, *Long-range dependence in higher order for non-Gaussian time series*, Acta Sci. Math., vol. 74, no. 1-2, pp. 425–447, 2008.
45. G. Terdik, *Long-range dependence and asymptotic self-similarity in third order*, Publ. Math. Debr., vol. 76, no. 3, pp. 379–393, 2010.
46. S. K. Vastola, and H. V. Poor, *An analysis of the effects of spectral uncertainty on Wiener filtering*, Automatica vol.28, 289–293, 1983.
47. N. Wiener, *Extrapolation, interpolation, and smoothing of stationary time series. With engineering applications*. Cambridge, Mass.: The M. I. T. Press, Massachusetts Institute of Technology, 163 p. 1966.
48. A. M. Yaglom, *Correlation theory of stationary and related random processes with stationary nth increments*, Mat. Sbornik, vol. 37, no. 1, pp. 141–196, 1955.
49. A. M. Yaglom, *Correlation theory of stationary and related random functions. Vol. 1: Basic results; Vol. 2: Suplementary notes and references*, Springer Series in Statistics, Springer-Verlag, New York etc., 1987.