STATISTICS, OPTIMIZATION AND INFORMATION COMPUTING
Stat., Optim. Inf. Comput., Vol. **12**, Month 2024, pp 0–10.
Published online in International Academic Press (www.IAPress.org)

Reliability estimation of a multicomponent stress-strength model based on copula function under progressive first failure censoring

Ola Abuelamayem *

Department of Statistics, Faculty of Economics and Political Science, Cairo University, Egypt

Abstract In reliability analysis of a multicomponent stress-strength model, most studies assumed independence between stress and strength variable. However, this assumption may not be realistic. To account for dependency, copula approach can be used. Although it is important, only few studies considered this case and usually under complete study. Observing the failures for all units may be difficult due to cost and time limitation. Recently, progressive first failure censoring scheme has attracted attention in the literature due to its ability to save time and money. To the best of our knowledge, dependent multicomponent stress-strength model under progressive first failure censoring was not considered yet. In this article, we derived the likelihood function for progressive first failure censored sample under copula and multicomponent stress strength model. A simulation study is performed and a real dataset is analyzed to test the applicability of the model. Maximum likelihood estimates, asymptotic confidence interval and bootstrap confidence intervals are obtained. The results illustrated that the proposed censoring scheme under copula provides a good estimate for the reliability.

Keywords Stress-strength model, Copula, Progressive first failure censoring, Maximum likelihood estimation, bootstrap.

AMS 2010 subject classifications 62F10; 62F12; 62F40; 62H12

DOI: 10.19139/soic-2310-5070-1894

1. Introduction

A common feature of lifetime data is that, the exact failure time for all units may not be observable, this can be due to cost, time limitation or the nature of the study. Those units for which the exact failure time is unknown are called censored data. There are different types of censoring like type-I, type-II, random, progressive type I, progressive type II, hybrid, first failure and progressive first failure.

Cohen [5] introduced progressive censoring, where the units are allowed to be randomly removed during the experiment to reduce cost and time. Since then several authors applied progressive censoring, see for example, Balakrishnan and Aggarwala [3], Kundu and Joarder [12] and Aly *et al* [1]. Balasooriya [2] introduced first failure censoring scheme. It is useful in a situation where the lifetime of a product is quite high and test facilities are scarce but test material is relatively cheap. This censoring approach saves cost and time. In first failure censoring, one combines the test units into several groups and each group contains a set of test units. The experiment then performed on each unit until the first failure occurs in each group. Wu and Kus [24] mixed both progressive and first failure censoring to introduce progressive first failure censoring scheme which can be explained as follows

Suppose n groups are in the experiment each with k units. At first failure $x_{(1)}$, R_1 groups are randomly removed from the experiment along with the group which contains the first failure unit. The remaining groups continue

*Correspondence to: Ola Abuelamayem (Email: ola.abuelamayem@feps.edu.eg). Department of Statistics, Faculty of Economics and Political Science, Cairo University, Egypt.

ISSN 2310-5070 (online) ISSN 2311-004X (print)

the experiment and R_2 groups are randomly removed at the second failure along with the group which contains the second failure. And so on, till the m^{th} failure where the experiment terminates and all remaining groups are removed from the experiment along with the group which contains the m^{th} failure. The likelihood function has the following form

$$L = ck^m \prod_{i=1}^{m} f(x_{(i)})(1 - F(x_{(i)}))^{k(R_i+1)-1}.$$

where

$$c = n(n - R_1 - 1)(n - R_1 - R_2 - 2)...(n - R_1 - R_2 - ...R_{m-1} - m + 1).$$

Many censoring schemes are special cases of progressive first failure:

- First failure when R=[0,0,...0].
- Progressive type II when k=1.
- Type-II when k=1 and R=[0,0,...,n-m]
- Complete case when k=1 and R=[0,0,...0].

Since then, several authors applied progressive first failure censoring scheme. For example, Soliman *et al* [21], Dube *et al* [7], Maurya *et al* [15] and Mahmoud *et al* [17] estimated the parameters for Gombertz, Lindely, exponentiated Rayleigh and generalized linear exponential distributions, respectively.

In reliability studies, a system fails if the stress (Y) of a component is higher than the strength (X). This is called stress - strength models. For example, Krishna *et al* [13] and Jia *et al* [11] estimated the reliability using generalized inverted exponential under progressive first failure censoring and Weibull distribution under first failure progressive unified hybrid censoring scheme, respectively. When the system has several components, it is called multicomponent stress-strength models. It consists of j independent and identical strength components and a common stress component. This has several applications in real life, for example, a bridge having j number of vertical cable representing its strength and load traffic represents its strength. Several authors studied multicomponent systems under different censoring schemes. For example, Kayal *et al* [14] studied the case of chen distribution in the complete case. Jha *et al* [10] analyzed the system using Gompertz distribution under progressive type-II censoring. Ma *et al* [16] estimated the model using Gompertz distribution and generalized hybrid censoring scheme. Saini *et al* [22] estimated multicomponent stress-strength model using Burr XII under progressive first failure. Recently, Sharma and Kumar [23] analyzed a multicomponent stress-strength model using weighted exponential-Lindely distribution in complete case using Bayesian. All illustrated studies, assumed independence between stress and strength variables. However, this assumption is less realistic.

There are several cases when stress and strength are dependent. For example, container discharge/loading operations on a vessel are usually performed by two or more cranes along the quay. Since the cranes must share the same shuttle vehicles (SV), the completion times, X and Y, of the cranes are dependent. It is relevant to investigate probability R that one crane completes its operations before the other, so as to allocate the SV efficiently (cited in Domma and Giordano [6]). In economics, measuring a household financial afford-ability where X and Y are disposable household income and consumption, respectively. For other examples in engineering, education, quality control and finance see (Domma and Giordano [6]).

To take dependence between stress and strength into consideration, bivariate distribution of stress - strength model is introduced. For example, Nadarajah and Kotz [18] used bivariate exponential distribution to take dependence into consideration. Gupta *et al* [8] considered bivariate lognormal distribution. However, a shortcoming of this approach is that it assumes a certain dependence form between X and Y. Sometimes it is hard to find a dataset that follows exactly this dependence structure. When dependence structure is unknown or it is hard to imply a bivariate distribution, copula approach can be used. Copula is a function that model the joint cumulative distribution function through a link function and marginal distributions.

In complete case, Domma and Giordano [6] applied different copulas to handle dependence between X and Y. Gao *et al* [9] considered a multicomponent step-stress model using linear combination of Gumbel, Clayton and Frank copula and call it mixed copula approach. Zhu [26] considered reliability estimation using unit Gompertz and Kumaraswamy marginals and Clayton copula. Patil *et al* [20] considered copula approach with exponential marginals and different estimation techniques. Under censoring, Bai *et al* [4] considered multicomponent stress-strength model under progressive hybrid censoring. The experiment is defined as follows

Suppose there are n identical systems put into life test with pre-fixed removals $(R_1, R_2, ..., R_m)$. The test is terminated at the time point τ_0 which is pre-fixed. When the first observed time t_1 occurs, R_1 systems are randomly removed from the experiment. At the second failure t_2, R_2 systems are randomly removed from the remaining systems and so on, if the m^{th} observed failure t_m occurs before time τ_0, all the remaining systems are randomly removed and the test terminates at time t_m $(m^* = m)$. Otherwise, if the m^{th} failure occurs after time τ_0 the test terminates at time τ_0 with $r < m$ observed failures $(m^* = r)$ and all the remaining systems are removed.

The likelihood function has the following form

$$L_1 = \prod_{i=1}^{m^*}\{[\frac{\partial C(u,v)}{\partial u} f_{X^*}(t_i)]^{\delta_i}[\frac{\partial C(u,v)}{\partial v} f_Y(t_i)]^{1-\delta_i}[S(t_i)]^{R_i}\}S(t_{m*})^{n-m^*-\sum_{i=1}^{m^*}R_i}$$

where

$u = S_{X^*}(t_i)$

$v = S_Y(t_i)$

Progressive type-II censoring can be obtained as special case if the test terminates at the m^{th} failure with the following likelihood function

$$L_2 = \prod_{i=1}^{m}\{[\frac{\partial C(u,v)}{\partial u} f_{X^*}(t_i)]^{\delta_i}[\frac{\partial C(u,v)}{\partial v} f_Y(t_i)]^{1-\delta_i}[S(t_i)]^{R_i}\} \qquad (1.1)$$

where

$u = S_{X^*}(t_i)$

$v = S_Y(t_i)$

$R_m = n - m - \sum_{s=1}^{m-1} R_s$

To the best of our knowledge, multicomponent stress-strength model under progressive first failure censoring scheme was not considered yet.

1.1. Contributions

The main contribution of this paper is deriving the likelihood function of a multicomponent stress-strength model under progressive first failure censoring scheme. The model takes into consideration the dependency between stress and strength by applying the copula function.

1.2. Organization

The paper is organized as follows. In section 2, copula theory is explained. In section 3, the model is explained and the likelihood function is derived under progressive first failure censoring and copula. Maximum likelihood estimators are illustrated in section 4. In section 5, numerical analysis is performed using both simulation and a real dataset. The paper is concluded in section 6.

2. Copula

Copula is a function that connects marginal distributions to define a bivariate distribution. There are two main steps to define the required bivariate distribution. First, the marginal distributions should be properly defined. Second, select a suitable copula to define the dependence structure. Nelsen [19] illustrated that different copulas with the same marginals resulted in different dependence structure. To illustrate the main concepts of copula, we first need to explain Sklar's theorem.

2.1. *Sklar's theorem*

Let X and Y be two continuous random variables with distribution functions $F_1(x)$ and $F_2(y)$, and let F be a joint distribution function with marginals F_1 and F_2. Then there exists a copula function (C) such that

$$F(x, y) = C(F_1(x), F_2(y)).$$

Let F_1^{-1} and F_2^{-1} be the quasi- inverses of F_1 and F_2, respectively. Then, the previous equation can be inverted to express copula in terms of a joint distribution function and the inverses of the marginals as follows

for any $u \in [0, 1]^2$

$$C(u_1, u_2) = F(F_1^{-1}(u_1), F_1^{-1}(u_2)).$$

from the above definition, copula could be considered as a multivariate distribution whose marginals are uniform (0, 1). Thus, it has a density function, and survival function that can be explained as follows

2.2. *Survival copula*

Let $S(x, y)$ be a two dimensional survival function with survival marginals $S_1(x) = 1 - F_1(x)$ and $S_2(y) = 1 - F_2(y)$. Then, there exists a survival copula ($\hat{C}$) such that

$$S(x, y) = \hat{C}(S_1(x), S_2(y)).$$

Accordingly, for any u$\in [0, 1]^2$ and quasi inverses S_1^{-1} and S_2^{-1} for S_1 and S_2, respectively. The previous relation can be inverted as follows

$$\hat{C}(u_1, u_2) = S(S_1^{-1}(x), S_2^{-1}(y)).$$

Note that there is a difference between survival copula ($\hat{C}$) and joint survival function ($\bar{C}$). The relation between them can be explained as follows

$$\bar{C}(u_1, u_2) = P(U_1 > u_1, U_2 > u_2) = \hat{C}(1 - u_1, 1 - u_2).$$

For more details see, Nelsen (2006).

2.3. *Copula density*

A copula density (c) can be obtained as follows

$$c(u_1, u_2) = \frac{\partial^2 C(u_1, u_2)}{\partial u \partial v}.$$

Let c, f, f_1 and f_2 be the density function of C, F, F_1 and F_2, respectively. Then

$$f(x, y) = c(F_1(x), F_2(y))f_1(x)f_2(y).$$

2.4. Archimedean Copula

There are different types of copulas, the most commonly used one is Archimedean copula. This is due to a number of reasons, the ease with which they can be constructed, the great variety of families of copulas and many useful properties possessed by the members of this class.

The Archimedean copula is constructed by a continuous, strictly decreasing convex function ϕ from [0,1] to [0, ∞], such that $\phi(1) = 0$ and

$$C(u_1, u_2) = \phi^{[-1]}(\phi(u_1) + \phi(u_2)).$$

where $\phi^{[-1]}$ is the quasi-inverse of ϕ. ϕ is called the copula generator and is indexed by a dependence parameter α (ϕ_α) (For more details see, Nelsen [19]).

In our study, we are interested in Gumbel copula. The privilege of Gumbel copula over others is that it is an extreme value copula. Here, we considered extreme value copula due to the application of the real dataset. In the data we are analyzing excessive water drought which is an application to extreme value theory. However, other copulas can be used in case of other applications. The generator of Gumbel copula in $\phi(t) = (-log(t))^\alpha$ and it has the following form

$$C_\alpha(u, v) = e^{(-[(-log(u))^\alpha + (-log(v))^\alpha]^{1/\alpha})}, \alpha \geq 1. \qquad (2.1)$$

The Kendall's tau coefficient (τ_α) can be obtained using the following relation

$$\tau_\alpha = \frac{\alpha - 1}{\alpha}. \qquad (2.2)$$

The copula will be used in our model to reflect the dependency between stress and strength. This will be illustrated more in the next section

3. Model description and likelihood derivation

Suppose a system consists of l series components with a common external stress. The strength of the j^{th} component is denoted by $X_j, j = 1, 2, ..., l$, with a cumulative distribution function $F_{X_j}(x_j)$ and a probability density function $f_{X_j}(x_j)$. The stress is denoted by Y with cumulative distribution and probability density functions $F_{(Y)}(y)$ and $f_Y(y)$, respectively. The lifetime of a series system is defined by the minimum of X_j. Let $X^{(*)} = min(X_j), j = 1, 2, ..., l$ with cumulative function defined as follows

$$F_{X^*}(x) = 1 - \prod_{j=1}^{l} [1 - F_{X_j}(x_j)].$$

The system fails if stress (Y) exceeds the minimum strength (X^*). Define an indicator δ such that

$$\delta = \begin{cases} 1 & if \quad X^* \leq Y \\ 0 & if \quad X^* > Y \end{cases}.$$

Now, we will define the experiment under progressive first failure censoring.

Suppose n groups are in the experiment each with k units. At first observed failure t_1, R_1 groups are randomly removed from the experiment along with the group which contains the first failure unit. The remaining groups continue the experiment and R_2 groups are randomly removed at the second failure along with the group which contains the second failure. And so on, till the m^{th} failure where the experiment terminates and all remaining groups are removed from the experiment along with the group which contains the m^{th} failure. To derive the likelihood function under progressive first failure, we generalized the likelihood function of progressive type-II censoring in eq (1.1). The likelihood function has the following form

$$L \propto \prod_{i=1}^{m}\{[\frac{\partial C(u,v)}{\partial u}f_{X^*}(t_i)]^{\delta_i}[\frac{\partial C(u,v)}{\partial v}f_Y(t_i)]^{1-\delta_i}[S(t_i)]^{k(R_i+1)-1}\} \quad (3.1)$$

where

$u = S_{X^*}(t_i)$

$v = S_Y(t_i)$

Special cases are

- First failure when R=[0,0,...0].
- Progressive type II when k=1.(which matches the likelihood function in eq 1.1)
- Type-II when k=1 and R=[0,0,...,n-m]
- Complete case when k=1 and R=[0,0,...0].

Model assumptions

In our study, we assumed

1) X_j has an exponential distribution with the following distribution and density functions, respectively.

$$\begin{aligned} F_{X_j}(x_j) &= 1 - e^{-\lambda_j^* x_j}, x_j > 0, \lambda_j^* > 0, \\ f_{X_j}(x_j) &= \lambda_j^* e^{-\lambda_j^* x_j}, x_j > 0, \lambda_j^* > 0. \end{aligned}$$

Accordingly the minimum strength X^* has an exponential distribution with the following distribution and density functions, respectively.

$$\begin{aligned} F_{X^*}(x^*) &= 1 - e^{-\lambda_1 x^*}, x_j > 0, \lambda_1 > 0, \\ f_{X^*}(x^*) &= \lambda_1 e^{-\lambda_1 x^*}, x_j > 0, \lambda_1 > 0. \end{aligned} \quad (3.2)$$

where

$\lambda_1 = \sum_{j=1}^{l} \lambda_j^*$

2) The common stress Y follows a Weibull distribution with the following distribution and density functions, respectively.

$$\begin{aligned} F_Y(y) &= 1 - e^{-\lambda_2 y^\beta}, y > 0, \lambda_2 > 0, \beta > 0, \\ f_Y(y) &= \lambda_2 e^{-\lambda_2 y^\beta}, y > 0, \lambda_2 > 0, \beta > 0. \end{aligned} \quad (3.3)$$

3) The dependence between X^* and Y is represented by Gumbel copula (eq 2.1). Using equation 3.2 and 3.3, the survival function for observed failures (t) can be written as

$$S(t) = e^{(-[(\lambda_1 t)^\alpha + (\lambda_2 t^\beta)^\alpha]^{1/\alpha})} \quad (3.4)$$

4) Systems are independent, that is the failure of system i has no effect on the failure of system j, $i \neq j$.

Accordingly the reliability of the system be obtained as follows

$$R^{**} = P(Y < X^*) = \int_0^\infty \int_0^{x^*} c(F_{X^*}(x^*), F_Y(y)) f_{X^*}(x^*) f_Y(y). \quad (3.5)$$

Using these assumptions, the likelihood function in eq 3.1 can be written as

$$\begin{aligned} L \quad \propto \quad & [e^{(-[(\lambda_1 t)^{\alpha}+(\lambda_2 t^{\beta})^{\alpha}]^{1/\alpha})}([\lambda_1 t]^{\alpha}+[\lambda_2 t^{\beta}]^{\alpha})^{(\frac{1}{\alpha}-1)}[\lambda_1 t]^{\alpha-1}\lambda_1]^{\delta_i} \\ & [e^{(-[(\lambda_1 t)^{\alpha}+(\lambda_2 t^{\beta})^{\alpha}]^{1/\alpha})}([\lambda_1 t]^{\alpha}+[\lambda_2 t^{\beta}]^{\alpha})^{(\frac{1}{\alpha}-1)}[\lambda_2 t^{\beta}]^{\alpha-1}\lambda_2\beta t^{\beta-1}]^{1-\delta_i} \\ & [e^{(-[(\lambda_1 t)^{\alpha}+(\lambda_2 t^{\beta})^{\alpha}]^{1/\alpha})}]^{k(R_i+1)-1} \qquad (3.6) \end{aligned}$$

To obtain the estimators using maximum likelihood approach, the first derivatives are obtained and equated to zero. However, no closed formulas were reached. Accordingly, we used numerical analysis that is illustrated in details in the next section.

4. Numerical Analysis

In this section, a simulation study is performed to evaluate the performance of the presented likelihood function. Also, a real dataset is analyzed to test its applicability.

4.1. simulation

Monte Carlo simulation is performed using R package with 1000 replications. The objective of the simulation study is to test the efficiency and consistency of the estimates under progressive first failure censoring. This is done by taking different sample sizes and different number of failures. Also, studying the effect and importance of considering the dependency in the model by taking different values for τ. Moreover, a comparison with other types of censoring is presented. The following different combinations will be considered

- M1: τ = 0.75, N = 500 , n = 100 , k=5, m=25 , R=(3,3,3,...,3).
- M2: τ = 0.75, N = 100 , n = 20 , k=5, m=5 , R=(3,3,3,3,3).
- M3: τ = 0.75, N = 250, n = 50 , k=5, m=25 , R=(1,1,1,...,1).
- M4: τ = 0.75, N = 50 , n = 10 , k=5, m=5 , R=(5,0,0,0,0).
- M1: τ = 0.5, N = 500 , n = 100 , k=5, m=25 , R=(3,3,3,...,3).
- M2: τ = 0.5, N =100 , n = 20 , k=5, m=5 , R=(3,3,3,3,3).
- M3: τ = 0.5, N = 250, n = 50, k=5, m=25 , R=(1,1,1,...,1).
- M4: τ = 0.5, N = 50 , n = 10 , k=5, m=5 , R=(5,0,0,0,0).

The following steps are used to perform the simulation study

Step1: Generate N independent samples from Gumbel copula using Weibull and exponential marginals.

Step 2: divide the observations to n groups with k units within each group.

Step 3: At first observed failure t_1, R_1 groups are randomly removed along with the group which contains the first failure. R_2 groups are randomly removed at the second failure t_2 along with the group which contains the second failure and so on.

Step 4: Maximize the likelihood function in equation 3.6.

Step 5: Calculate the reliability using equation 3.5 after substituting by the maximum likelihood estimates obtained from the previous step.

Step 6: Repeat the five previous steps 1000 times.

To obtain the confidence interval for the reliability function, bootstrap method is used with the following steps:

Step 1: Given N, n, R ,k and m, compute the maximum likelihood estimates $\hat{\lambda}_1$, $\hat{\lambda}_2$, $\hat{\beta}$ and $\hat{\alpha}$.

Step 2: Generate a bootstrap sample using the maximum likelihood estimates $\hat{\lambda}_1$, $\hat{\lambda}_2$, $\hat{\beta}$ and $\hat{\alpha}$. Then obtain the maximum likelihood estimates for this new sample $\hat{\lambda}_1^*$, $\hat{\lambda}_2^*$, $\hat{\beta}^*$ and $\hat{\alpha}^*$.

Step 3: Repeat the second step N_1 times and obtain N_1 estimates $\hat{\lambda}_1^{*(z)}$, $\hat{\lambda}_2^{*(z)}$, $\hat{\beta}^{*(z)}$ and $\hat{\alpha}^{*(z)}$, $z = 1, 2, ..., N_1$.

Step 4: Calculate the reliability using equation 3.5 with $\hat{\lambda}_1^{*(z)}$, $\hat{\lambda}_2^{*(z)}$, $\hat{\beta}^{*(z)}$ and $\hat{\alpha}^{*(z)}$ say $\hat{R}^{**(z)}$.

Step 5: Arrange $\hat{R}^{**(z)}$ in an ascending order.

Step 6: The two sided 100 (1-α) $\%$ confidence interval is given by
($\hat{R}_L^{**}$, $\hat{R}_U^{**}$) = ($\hat{R}^{**N_1(\alpha/2)}$, $\hat{R}^{**N_1(1-\alpha/2)}$).

Table 5.1: The maximum likelihood estimates.

			λ_1	λ_2	β	α
M1	τ=0.75	ABias	0.066	0.061	0.237	0.052
		MSE	0.120	0.471	0.566	0.412
		C.I	(0.898,2.234)	(0.600,3.278)	(1.120,3.626)	(2.799, 5.305)
M1	τ=0.5	ABias	0.059	0.185	0.819	0.443
		MSE	0.142	0.487	1.452	0.357
		C.I	(0.828,2.290)	(0.496,3.134)	(1.087,4.551)	(1.657, 3.229)
M2	τ=0.75	ABias	0.312	0.037	0.151	0.029
		MSE	0.743	1.374	1.802	2.169
		C.I	(0.237, 3.387)	(-0.334, 4.260)	(0.463, 4.765)	(1.144, 6.916)
M2	τ=0.5	ABias	0.371	0.140	0.692	0.366
		MSE	1.549	1.583	2.951	1.159
		C.I	(-0.568, 4.310)	(-0.605, 4.327)	(-0.674, 6.060)	(0.256, 4.476)
			λ_1	λ_2	β	α
M3	τ=0.75	ABias	0.109	0.294	0.584	0.139
		MSE	0.116	0.669	1.387	0.559
		C.I	(0.977, 2.241)	(0.297, 3.291)	(0.579, 4.589)	(2.699, 5.579)
M3	τ=0.5	ABias	0.064	0.474	1.291	0.743
		MSE	0.149	1.729	3.712	0.760
		C.I	(0.818, 2.310)	(-0.878, 3.930)	(0.488, 6.094)	(1.849, 3.637)
M4	τ=0.75	ABias	0.853	0.138	0.412	0.102
		MSE	1.779	2.335	3.008	2.808
		C.I	(-0.231, 4.997)	(-1.133, 4.857)	(-0.986, 5.812)	(0.819, 7.387)
M4	τ=0.5	ABias	0.418	0.312	1.054	0.613
		MSE	2.575	2.237	6.259	2.279
		C.I	(-0.618, 5.454)	(-1.179, 4.555)	(-1.410, 7.50)	(-0.091, 5.317)

The maximum likelihood estimates of the model parameters are obtained. The absolute bias (ABias= $|$estimate $-$ real value$|$), mean square error (MSE $=$ variance $+$ Bias^2) and 95% confidence interval(estimate -(+) 1.96$\sqrt{(var)}$) are obtained and presented in Table 5.1. Reliability estimates and 95$\%$ bootstrap confidence intervals are illustrated in table 5.2. The results are analyzed in three different ways

First, consider the following division to study the effect of increasing τ.

- Set 1: M1, M2, M3 and M4 at which $\tau = 0.75$.
- Set 2: M1, M2, M3 and M4 at which $\tau = 0.5$.

It can be seen that for the majority of cases the MSE, ABias and reliability are less in set 1 than that in set 2. For example, in table 5.1 (parameter λ_2 and M2), the ABias decreases from 0.140 at $\tau = 0.5$ to 0.037 at $\tau = 0.75$. Also, the MSE decreases from 1.583 to 1.374. Moreover, from table 5.2, the reliability decreased from 0.389 to 0.330. Accordingly, as the association increases, the estimates becomes closer to the real value. This indicates that the estimation in case of considering dependence is more precise.

Table 5.2:Reliability estimates

M1	τ=0.75	Estimate	0.367
		bootstrap C.I	(0.127, 0.503)
M1	τ=0.5	Estimate	0.433
		bootstrap C.I	(0.141, 0.487)
M2	τ=0.75	Estimate	0.330
		bootstrap C.I	(0.036, 0.503)
M2	τ=0.5	Estimate	0.389
		bootstrap C.I	(0.0008, 0.496)
M3	τ=0.75	Estimate	0.310
		bootstrap C.I	(0.066, 0.450)
M3	τ=0.5	Estimate	0.362
		bootstrap C.I	(0.0004, 0.440)
M4	τ=0.75	Estimate	0.188
		bootstrap C.I	(0.002, 0.308)
M4	τ=0.5	Estimate	0.243
		bootstrap C.I	(0.106, 0.468)
Progressive type II	τ=0.75	Estimate	0.373
		bootstrap C.I	(0.007,0.523)
Progressive type II	τ=0.5	Estimate	0.446
		bootstrap C.I	(0.008,0.523)
Type II	τ=0.75	Estimate	0.407
		bootstrap C.I	(0.013,0.531)
Type II	τ=0.5	Estimate	0.483
		bootstrap C.I	(0.126,0.521)

Second, the following division is used to study the effect of increasing the number of failures (m).

- Set 1: M1.
- Set 2: M2.

It can be seen that for different values of τ as m increases, the MSE decreases and the reliability increases. For example, in table 5.1, the MSE for λ_1 decreased from 0.743 at $m = 5$ to 0.120 at $m = 25$. Also, from table 5.2, it can be seen that the reliability increases from 0.330 to 0.367 at $m = 5$ and $m = 25$, respectively. This illustrates that increasing the number of analyzed units, gives better estimates. Although the ratio relative to the total is the same, but the information gained from analyzing more units as absolute numbers resulted in better estimates.

Third, consider the following division to study the effect of increasing the number of groups and accordingly the total sample size.

- Set 1: M1 and M2.
- Set 2: M3 and M4.

By comparing M1 with M3 and M2 with M4, it is seen that, for different values of τ, as the number of groups increases and accordingly the total sample size increases, the ABias and MSE decrease. For example, in table 5.1

at $\tau = 0.5$ and λ_2, the ABias decreased from 0.312 to 0.140 for M4 and M2, respectively. Also, the MSE decreased from 2.237 to 1.583 for M4 and M2, respectively. This illustrates the consistency of the parameters.

To study how progressive first failure performs relative to other types of censoring. A comparison is performed with progressive type II and type II censoring schemes with same assumptions as M1. It can be seen from table 5.2 that the reliability estimates are very close in the three schemes.

Finally, by summarizing all tables, it is clear that the CIs includes the true values of the proposed parameters.

4.2. A real data

This data set represents the monthly water capacity for Shasta reservoir in California, USA (see Wang *etal* [25]). The data illustrates the monthly water capacity from 1981 till 1985. The aim is to infer about excessive drought. The water level will not cause excessive drought if the water capacity in July is less than the minimum water capacity from January till June in the same year. $X_{1J}, j = 1, 2, ..., 6$ are water capacities from January till June in 1981, and Y_{1J} is water capacity in November 1981. $X_{2J}, j = 1, 2, ..., 6$ are water capacities from January till June in 1982, and Y_{1J} is water capacity in November 1982, and so on till 1985. The data is divided by the maximum capacity 4526800.

$$X = \begin{pmatrix} 0.763 & 0.854 & 0.954 & 0.949 & 0.882 & 0.797 \\ 0.786 & 0.803 & 0.897 & 0.988 & 0.996 & 0.967 \\ 0.826 & 0.791 & 0.823 & 0.947 & 1 & 0.988 \\ 0.752 & 0.837 & 0.913 & 0.959 & 0.949 & 0.899 \\ 0.689 & 0.716 & 0.761 & 0.784 & 0.713 & 0.631 \end{pmatrix} \qquad Y = \begin{pmatrix} 0.670 \\ 0.899 \\ 0.921 \\ 0.792 \\ 0.506 \end{pmatrix}$$

First, we perform Anderson-Darling goodness of fit test to check if the data in X and Y follow exponential and Weibull distributions, respectively. The p-value for X and Y respectively are 0.084 and 0.928, Hence, we can assume the data follow exponential and Weibull distribution at 5% level of significance.

Second, we test the correlation between X and Y using Pearson correlation coefficient, and the p-value of the test is 0.023. Therefor, we can conclude there is significant correlation between X and Y at 5% level of significance and the copula model can be applied.

Now, we obtained the reliability estimates using both complete data and progressive first failure with R=[2,0]. The results are presented in table 5.3, and it can be seen that progressive first failure provides a good estimate for the reliability function as it is close to that from the complete case.

Table 5.3:Reliability estimates for real data

	Reliability estimate
Complete case	0.370
Progressive first failure	0.297

5. Conclusion

In lifetime data, it is difficult to observe the failure time for all units of the study due to time and cost limitation and sometimes the nature of the study. Accordingly, performing the study under censoring is more realistic. Recently, progressive first failure censoring has attracted attention in the literature due to it is ability to save cost and time. To the best of our knowledge, a dependent multicomponent stress-strength model under progressive first failure was not performed yet. In our study, we derived the likelihood function under copula progressive first failure censoring scheme and multicomponent stress-strength model. A simulation study is performed and a real dataset is analyzed. The results indicated that progressive first failure censoring provided a good reliability estimate but with a fewer number of analyzed units. An extension of this work can be done by taking into consideration any prior information about the parameters through Bayesian estimation.

REFERENCES

1. H. Aly, H. Muhammed, and O. Abuelamayem, *Estimation of the Bivariate Kumaraswamy Lifetime Distribution under Progressive Type-I Censoring*, Journal of Data Sience, vol. 18, no. 4, pp. 739–749, 2020.
2. U. Balasooriya, *Failurecensored reliability sampling plans for the exponential distribution*, Journal of Statistical Computation and Simulation, vol. 52, no. 4, pp. 337–349, 1995.
3. N. Balakrishnan, and R. Aggarwala, *Progressive Censoring: Theory, Methods and Applications*, 1^{st} Edition, Springer Science, New York, 2000.
4. X. Bai, Y. Shi, Y. Liu, and B. Liu, *Reliability estimation of multicomponent stressstrength model based on copula function under progressively hybrid censoring*, Journal of Computational and Applied Mathematics, vol. 344, pp. 100–114, 2018.
5. A. Cohen, *Progressively censored samples in life testing*, Technometrics, vol. 5, no. 3, pp. 327–339, 1963.
6. F. Domma, and S. Giordano, *A copula-based approach to account for dependence in stress-strength models*, Statistical Papers, vol. 54, pp. 807–826, 2013.
7. M. Dube, R. Garg, and H. Krishna, *On progressively first failure censored Lindley distribution*, Computational Statistics, vol. 31, pp. 139–163, 2016.
8. R. Gupta, M. Ghitany, and Al-Mutairi *Estimation of reliability from a bivariate log-normal data*, Journal of Statistical Computation and Simulation, vol. 83, no. 6, pp. 1068–1081, 2013.
9. J. Gao, Z. An, B. Liu, *A dependent stressstrength interference model based on mixed copula function*, Journal of Mechanical Science and Technology, vol. 30, no. 10, pp. 4443–4446, 2016.
10. M. Jha, S. Dey, R. Alotaibi, and Y. Tripathi, *Reliability estimation of a multicomponent stress-strength model for unit Gompertz distribution under progressive Type II censoring*, Quality and Reliability Engineering International, vol. 36, no. 3, pp. 965–987, 2020.
11. J. Jia, Z. Yan, and X. Peng, *Inferences on Stress-Strength Reliability from Inverse Weibull Distribution based on First-Failure Progressively Unified Hybrid Censoring Schemes*, International Journal of Applied Mathematics, vol. 51, no. 4, pp. 1–9, 2021.
12. D. Kundu, and A. Joarder, *Analysis of type-II progressively hybrid censored data*, Computational Statistics and Data Analysis, vol. 50, no. 10, pp. 2509–2528, 2006.
13. H. Krishna, M. Dube, and R. Garg, *Estimation of $P(Y < X)$ for progressively first-failure-censored generalized inverted exponential distribution*, Journal Of Statistical Computation and Simulation, vol. 87, no. 11, pp. 2274–2289, 2017.
14. T. Kayal, Y. Tripathi, and L. Wang, *Inference for the Chen Distribution Under Progressive First Failure Censoring*, Journal of Statistical Theory and Practice, vol. 13, no. 52, 2019.
15. R. Maurya, Y. Tripathi, and M. Rastogi, *Estimation and Prediction for a Progressively First-Failure Censored Inverted Exponentiated Rayleigh Distribution*, Journal of Statistical Theory and Practice, vol. 13, no. 39, 2019.
16. H. Ma, Z. Yan, and J. Jia, *Multicomponent stressstrength model based on generalized progressive hybrid censorins Scheme: a statistical analysis*, Entropy, vol. 24, no. 5, pp. 619, 2022.
17. M. Mahmoud, M. Ghazal, and H. Radwan, *Bayesian estimation and optimal censoring of inverted generalized linear exponential distribution using progressive first failure censoring*, Annals of Data Science, vol. 10, pp. 527–554, 2023.
18. S. Nadarajah, and S. Kotz, *Reliability for Some Bivariate Exponential Distribution*, Mathematical Problems in Engineering, https://doi.org/10.1155/MPE /2006/41652., 2006.
19. R. Nelsen, *An Introduction to Copulas*, 2nd Edition, New York: springer, 2006.
20. D. Patil, U. Naik-Nimbalkar, M. Kale, *Effect of Dependency on the Estimation of $P[Y < X]$ in Exponential Stress-strength Models*, Austrian Journal of Statistics, vol. 51, no. 4, pp. 10–34, 2022.
21. A. Soliman, A. Abd-Ellah, N. Abou-Elheggag, and G. Abd-Elmougod, *Estimation of the parameters of life for Gompertz distribution using progressive first-failure censored data*, Computational Statistics and Data Analysis, vol. 56, no. 8, pp. 2471–2485, 2012.
22. S. Saini, S. Tomer, and R. Garg, *On the reliability estimation of multicomponent stressstrength model for Burr XII distribution using progressively first-failure censored samples*, Journal of Statistical Computation and Simulation, vol. 92, no. 4, pp. 667–704, 2022.
23. S. sharma, and V. Kumar, *Reliability estimation in multicomponent stress-strength model using weighted exponential-Lindley distribution*, Journal of Statistical Computation and Simulation, pp. 1–27, 2024.
24. S. Wu, and C. Kus, *On estimation based on progressive first-failure-censored sampling*, Computational Statistics and Data Analysis, vol. 53, no. 10, pp. 3659–3670, 2009.
25. L. Wang, H. Lin, K. Ahmadi, and Y. Lio, *Estimation of Stress-Strength Reliability for Multicomponent System with Rayleigh Data*, Energies, vol. 14, no. 23, pp. 7917, 2021.
26. T. Zhu, *Reliability estimation of s-out-of-k system in a multicomponent stressstrength dependent model based on copula function*, Journal of Computational and Applied Mathematics, vol. 404, no. 113920, 2016.