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Prediction of New Lifetimes of a Step-Stress Test Using Cumulative Exposure Model with Censored Gompertz Data

Mohammad A. Amleh *, Israa F. Al-Freihat

Department of Mathematics, Faculty of Science, Zarqa University, Zarqa, 13110, JORDAN

Abstract The paper involves predicting new lifetimes in step-stress accelerated life tests with Type-II censoring using the Gompertz distribution. It introduces point predictors and explores constructing prediction intervals for future lifetimes. The evaluation includes an extensive simulation study and an analysis of actual dataset. Predictors are compared based on biases and mean square prediction errors, and assessment of prediction intervals considers average lengths and coverage probabilities. Maximum likelihood predictor excels as a point predictor, while shortest-length based method outperforms in constructing prediction intervals.

Keywords Step-stress accelerated life test, Cumulative exposure model, Type-II censoring, Gompertz distribution, Maximum likelihood predictor, Conditional median predictor, Best unbiased predictor, Prediction intervals

AMS 2010 subject classifications 62G32, 62M20, 62N01

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1. Introduction

A step-stress accelerated life test (SSALT) is a reliability testing methodology used to assess the performance and longevity of a product or system by subjecting it to increasing levels of stress in discrete steps. Instead of applying a constant high stress level, SSALT involves gradually increasing the stress levels over a series of steps. The key steps in a SSALT include selecting stress levels, defining the durations for each step, collecting data on failures or performance degradation at each step, and then using statistical analysis to extrapolate the results to predict the product performance under normal operating conditions. SSALT is valuable for identifying weak points in a product design and estimating its expected lifespan under real-world usage. See Nelson [1]. Indeed, there has been limited discussion regarding the prediction issue associated with the step-stress model in the existing literature. Basak [2], as well as Basak and Balakrishnan ([3], [4]), examined the issue of predicting the failure times of items subjected to censoring within a step-stress model based on the Exponential distribution and involving progressive Type-I censoring, progressive Type-II censoring, and Type-II censoring, respectively. More recently, Amleh and Raqab ([5], [6]), and Amleh [7], discussed the prediction problem in the context of a step-stress plan using different distributions and models. These distributions include the Lomax distribution under the cumulative exposure (CE) model, the Weibull distribution under the Khamis-Higgins (KH) model, and the Rayleigh distribution under the CE model, respectively. Furthermore, Amleh and Raqab [8] considered the Bayesian prediction of new order statistics under the KH model for Type-II censored Weibull data.

*Correspondence to: Mohammed Amleh (Email: malamleh@zu.edu.jo). Department of Mathematics, Faculty of Science, Zarqa University, Zarqa, 13110, JORDAN

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1.1. Cumulative Exposure Model

Nelson [1] presented the most widely used model, commonly referred to as the CE model. This model assumes that the remaining lifespan of the experimental units relies solely on the cumulative exposure they have encountered, without any consideration of the specific manner in which the exposure was accumulated. Furthermore, the units that continue to operate will experience failure based on the cumulative distribution corresponding to the stress level currently under testing, and this process will initiate from the previously accumulated stress level. See Kundu and Ganguly [9]. Assuming the presence of three stress levels (S_1, S_2, S_3) undergoing fixed-time changes (τ_1, τ_2), we also presume that the lifetime distribution functions under stress levels S_1, S_2, and S_3 are denoted as F_1, F_2, and F_3, respectively. Additionally, these distribution functions are considered to be part of the same family of distributions. The experimental setup involves initiating n identical units, each initially exposed to stress level S_1, with lifetimes following the cumulative distribution function (CDF) F_1(t). The failure times of units are recorded, and the surviving units continue until time τ_1, when the stress is elevated to S_2. Subsequently, the units follow the CDFF_2(t), commencing from the previously accumulated fraction of failures. This process repeats as the stress levels transition from S_2 to S_3, with the corresponding lifetime distributions changing to F_3(t), and so forth. The CDF of lifetime in the CE model, derived from the segments of the CDFs corresponding to three stress levels, is represented as follows:

$$G(t) = \begin{cases} G_1(t) = F_1(t) & for 0 < t < \tau_1 \\ G_2(t) = F_2(t - \tau_1 + h_1) & for \tau_1 \leq t < \tau_2, \\ G_3(t) = F_3(t - \tau_2 + h_2) & for \tau_2 \leq t < \infty \end{cases} \tag{1}$$

where h_1 represents the equivalent starting time for F_2, can be obtained by solving the equation

$$F_1(\tau_1) = F_2(h_1),$$

and the equivalent starting time for F_3, denoted as h_2, can be acquired through the solution of the equation.

$$F_2(\tau_2) = F_3(h_2).$$

Continuing with the same approach, we eventually arrive at

$$G(t) = F_i(t - \tau_{i-1} + h_{i-1}), \tau_{i-1} \leq t < \tau_i, i = 1, 2, ..., m,$$

with $\tau_0 = h_0 = 0, \tau_m = \infty$, and $h_i, i = 1, 2, \ldots, m$, is a solution of the equation

$$F_i(\tau_i) = F_{i+1}(h_i).$$

1.2. Type-II Censored Samples

The Type-II censored sample is created by ending a life-testing experiment upon observing a designated number of failures, denoted as r, and censoring the remaining units $n - r$. Specifying the number of failures introduces randomness to the time-to-failure of a test unit, resulting in an unknown termination time before the experiment. While this is a drawback of Type-II censoring, it offers the advantage of ensuring the necessary number of failures is obtained during the life test. For additional information regarding Type-II censoring and the related inferential issues, one can consult references such as Lawless [10], Cohen and Whitten [11], and Cohen [12].

1.3. The Gompertz Distribution

The Gompertz distribution is a continuous probability distribution that is often used to model survival data or lifetime data. It is named after Gompertz [13], who introduced the distribution to model human mortality rates. Since then, it has found applications in various fields, including actuarial science, demography, biology, and reliability engineering.

Fig. 1 illustrates the CE model for a failure mode, considering the alteration of three stress levels at predetermined times.

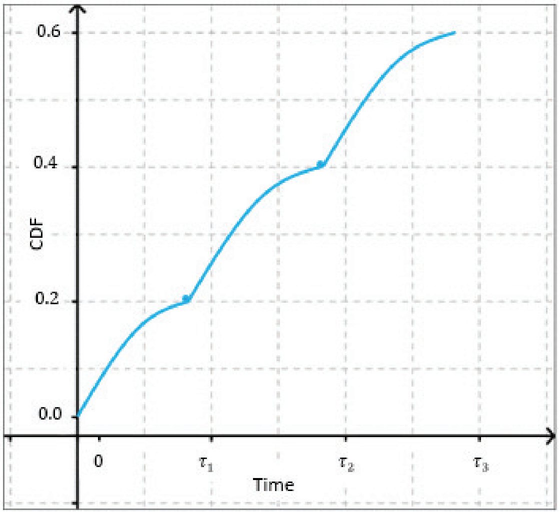

Figure 1. The plot of CE model.

Regarding the Gompertz distribution, Hakamipour and Rezaei [14] presented an optimal method for devising a SSALT using the Gompertz distribution, considering two stress variables and Type-II censoring. Almarashi and Abd-Elmougod [15] discussed the lifetime problem under the SSALT of two separate risks with Gompertz distribution. Alghamdi [16] conducted an analysis of the product's failure time by employing a constant-stress accelerated life test (CSALT) that considered two independent competitive risks, all within the framework of the Gompertz distribution. The probability density function (PDF), CDF, reliability function (RF) and hazard rate function (HRF) of the two-parameter Gompertz distribution are given, respectively, by

$$f(t) = \lambda\theta e^{\theta t - \lambda}(e^{\theta t} - 1), t \geq 0, \theta, \lambda \geq 0, \tag{2}$$

$$F(t) = 1 - e^{-\lambda}(e^{\theta t} - 1), t \geq 0, \theta, \lambda \geq 0, \tag{3}$$

$$R(t) = e^{-\lambda}(e^{\theta t} - 1), t \geq 0, \theta, \lambda \geq 0, \tag{4}$$

$$h(t) = \lambda\theta e^{\theta t}, t \geq 0, \theta, \lambda \geq 0, \tag{5}$$

Here, T denotes a random variable (typically representing time, age, or survival time), λ is the shape parameter and θ is the scale parameter. The Gompertz distribution exhibits a positively skewed pdf, featuring a longer tail on the right side, along with an increasing HRF over time. Originally, it is used for modeling human mortality rates, it demonstrates an exponential increase in mortality rates with age, making it suitable for analyzing failure or death rates of systems or populations over time. See Pollard and Valkovics [17]. Utilizing the R programming language, we have generated multiple figures to visually elucidate the Gompertz distribution. These include Fig. 2, illustrating the PDF, Fig. 3, displaying the CDF, Fig. 4, showcasing the RF and Fig. 5, presenting the HRF. Each of these figures portrays the effects of varying λ and θ values on the distribution's characteristics.

This paper addresses the problem of predicting the failure lifetimes of censored units based on the Gompertz distribution within a simple step-stress plan under a CE model. In Section 2, we provide a description of the

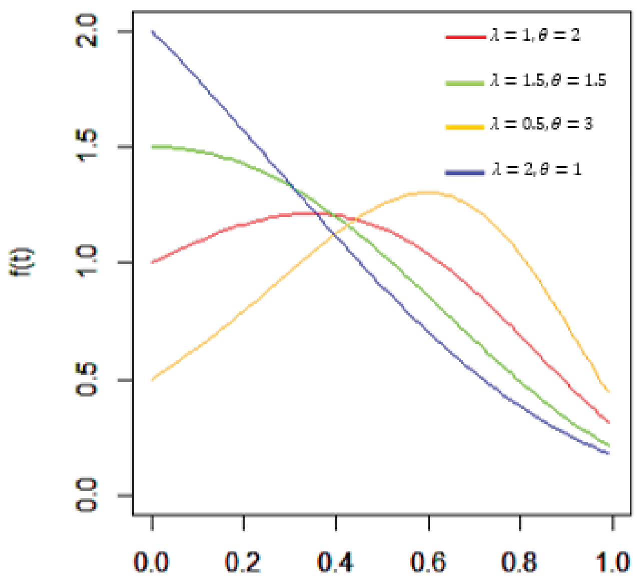

Figure 2. PDF Plot of the Gompertz distribution considering different values of λ and θ.

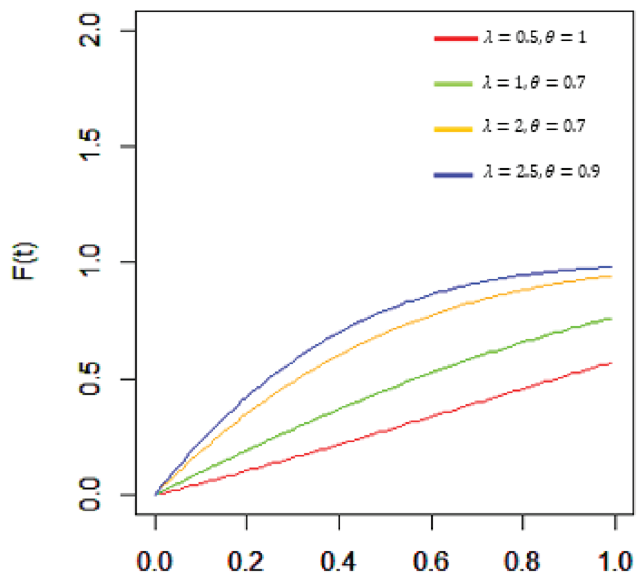

Figure 3. CDF Plot of the Gompertz distribution considering different values of λ and θ.

model under consideration. Numerical techniques are employed in Section 3 to acquire the maximum likelihood estimates (MLEs).

Section 4 introduces various point predictors, such as the maximum likelihood predictor (MLP), conditional median predictor (CMP), and best unbiased predictor (BUP). Moving on to Section 5, several approaches for

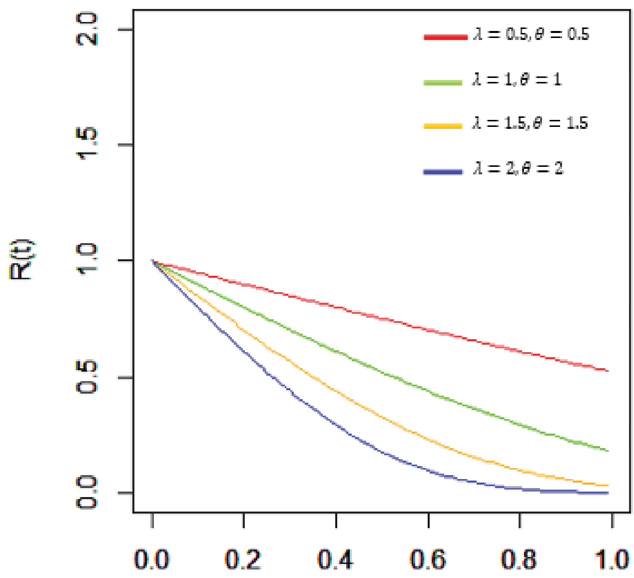

Figure 4. RF Plot of the Gompertz distribution considering different values of λ and θ.

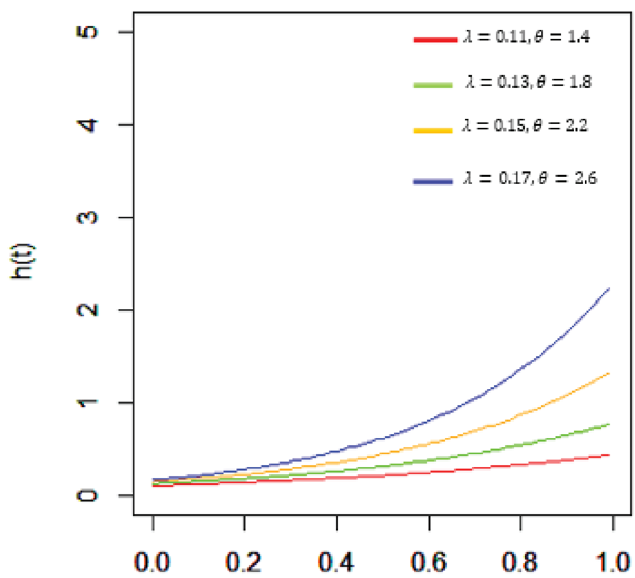

Figure 5. HRF Plot of the Gompertz distribution considering different values of λ and θ.

constructing prediction intervals (PIs) of censored lifetimes are proposed. In Section 6, a simulation study and real data analysis are conducted to evaluate the effectiveness of the prediction procedures. Ultimately, the conclusion of the paper is presented in Section 7.

2. Model Description

We posit that the failure time data originates from a CE model. Additionally, we are considering a simple step stress model with two stress levels, S_1 and S_2, and employing Type-II censoring. The lifetime distributions at stress levels S_1 and S_2 are presumed to adhere to a Gompertz distribution. This distribution shares a common shape parameter λ while having distinct scale parameters, θ_1 and θ_2, for the respective stress levels. Within the context of the simple step-stress model combined with Type-II censoring, we start with a set of n identical units that are simultaneously subjected to a life-test. Initially, each of these units is exposed to stress level S_1. Subsequently, the experiment continues until a predetermined time denoted as τ, at which point the stress level is switched to S_2. The experiment persists until a specified number of failures, denoted as r, are observed. Let n_1 represent the count of units failing before τ, and let n_2 represent the count of units failing after τ, so that the total number of failures is given by $r = n_1 + n_2$. If the count of failures, r, is reached before τ, the test is terminated; otherwise, the experiment persists after time τ until the required r failures are observed.
The ordered failure times that are observed will be represented as:

$$\{t_{1:n} < ... < t_{n_1:n} < \tau \leq t_{n_1+1:n} < ... < t_{r:n}\}. \tag{6}$$

Our model is based on the following fundamental assumptions:

1. For any level of stress, the life of test units follows a Gompertz distribution.
2. The scale parameters θ_i, at test step i, for $i = 1, 2$, are assumed to be corresponding to stress levels S_1 and S_2.
3. A CE model holds, i.e., the remaining life of a test product depends only on the CE it has seen.
4. The shape parameter λ is constant for both stress levels.

By applying Eq. (1), the CDF of the simple step-stress model with two stress levels, S_1 and S_2, is expressed as follows:

$$F(t) = \begin{cases} F_1(t), & 0 \leq t < \tau \\ F_2(t - \tau + h), & \tau \leq t < \infty \end{cases} \tag{7}$$

The alteration in stress level from S_1 to S_2 space leads to a change in the lifetime distribution at stress level S_2, shifting it from $F_2(t)$ to $F_2(t - \tau + h)$, where

$$F_1(\tau) = F_2(h),$$

after finding the solution for h in the aforementioned equation, we obtain $h = \dfrac{\theta_1}{\theta_2}\tau$.
Consequently, the Gompertz CE model for a simple step-stress test can be expressed as follows:

$$G(t) = \begin{cases} G_1(t) = 1 - e^{-\lambda(e^{\theta_1 t}-1)}, & 0 \leq t < \tau \\ G_2(t) = 1 - e^{\lambda(e^{\theta_1\tau+\theta_2(t-\tau)}-1)}, & \tau \leq t < \infty \end{cases} \tag{8}$$

with the corresponding PDF

$$g(t) = \begin{cases} g_1(t) = \lambda\theta_1 e^{\theta_1 t - \lambda\left(e^{\theta_1 t - 1}\right)}, & 0 \leq t < \tau \\ g_2(t) = \lambda\theta_2 e^{\theta_1\tau+\theta_2(t-\tau)-\lambda\left(e^{\theta_1\tau+\theta_2(t-\tau)-1}\right)}, & \tau \leq t < \infty \end{cases} \tag{9}$$

3. Maximum likelihood Estimation

Given the provided Type-II censored data in Eq. (6), it is possible to derive the likelihood function and subsequently determine the MLEs for the unknown parameters λ, θ_1 and θ_2. The likelihood function of this censored sample can be formulated as follows:

$$L(\lambda,\theta_1,\theta_2|t)=\frac{n!}{r!}\{\Pi_{i=1}^{r}g(t_{i:n})\{1-G(t_{r:n})\}^{n-r}\},0<t_{1:n}<...<t_{r:n\prime} \tag{10}$$

see Arnold et al. [18]. In this context, r represents the sum of n_1 and n_2, while $t=(t_{1:n},\ldots,t_{n_1:n},\tau,t_{n_1+1:n},\ldots,t_{r:n})$ stands for the collection of recorded Type-II censored data. The non-existence of the MLE for θ_1 is clear when $n_1=0$, and for θ_2 when $n_1=r$. MLEs for θ_1 and θ_2 are viable only within the condition that $1\le n_1\le r-1$. When the condition $1\le n_1\le r-1$ is satisfied, the likelihood function in Eq. (10) transforms to

$$\begin{aligned}L(\lambda,\theta_1,\theta_2|t)=&\frac{n!}{n_1!n_2!}\{\Pi_{i=1}^{n_1}g_1(t_{i:n})\}\{\Pi_{i=n_1+1}^{n_1}g_1(t_{i:n})\}\times\{1-G_2(t_{r:n})\}^{n-r},\\&0<t_{1:n}<...<t_{n_1:n}<\tau\le t_{n_1+1:n}<...<t_{r:n}<\infty,\end{aligned} \tag{11}$$

accordingly, the likelihood function can be written as:

$$\begin{aligned}&L(\lambda,\theta_1,\theta_2|t)\alpha\\&\Pi_{i=1}^{n_1}\{\lambda\theta_1e^{\theta_1t_{i:n}-\lambda\left(e^{\theta_1t_{i:n}-1}\right)}\}\\&\times\Pi_{i=n_1+1}^{r}\{\lambda\theta_2e^{\theta_2\tau+\theta_2(t_{i:n}-\tau)-\lambda\left(e^{\theta_1\tau\theta_2(t_{i:n}-\tau)-1}\right)}\}\\&\times\left[e^{-\lambda\left(e^{\theta_1\tau\theta_2(t_{r:n}-\tau)-1}\right)}\right]^{n-r},\end{aligned}$$

it can be streamlined in the following manner

$$\begin{aligned}&L(\lambda,\theta_1,\theta_2|t)\alpha\\&\theta^{n_1}\theta^{n_2}\lambda^re^{-(n-r)\lambda\left[e^{\theta_2(t_{r:n}-\tau)+\theta_1\tau-1}\right]}\\&\times e^{n_1\lambda+\theta_1\Sigma_{i=1}^{n_1}t_{i=n}-\lambda\Sigma_{i=1}^{n_1}e^{\theta_1t_{i:n}}}\\&\times e^{n_2\lambda+\theta_2\Sigma_{i=n_1+1}^{r}(t_{i:n}-\tau)+n_2\theta_1\tau-\lambda\Sigma_{i=n_1+1}^{r}e^{\theta_2(t_{i:n}-\tau)+\theta_1\tau}}.\end{aligned} \tag{12}$$

The expression for the log-likelihood function can be represented as follows:

$$\begin{aligned}&l(\lambda,\theta_1,\theta_2|t)\alpha\\&n_1log\theta_1+n_2log\theta_2+rlog\lambda-(n-r)\lambda\left[e^{\theta_2(t_{r:n}-\tau)+\theta_1\tau}-1\right]\\&+r\lambda+\theta_1\Sigma_{i=1}^{n_1}t_{i:n}-\lambda\Sigma_{i=1}^{n_1}e^{\theta_1t_{i:n}}\\&+\theta_2\Sigma_{i=n_1+1}^{r}(t_{i:n}-\tau)+n_2\theta_1\tau-\lambda\Sigma_{i=n_1+1}^{r}e^{\theta_2(t_{i:n}-\tau)+\theta_1\tau}.\end{aligned} \tag{13}$$

By taking the derivatives of the log-likelihood function with respect to λ,θ_1 and θ_2 as given in Eq. (13), we derive the subsequent set of likelihood equations. The likelihood equations are obtained as:

$$\frac{\partial l(\lambda,\theta_1,\theta_2|t)}{\partial\lambda}=\frac{r}{\lambda}+(r-n)\left[e^{\theta_2(t_{r:n}-\tau)+\theta_1\tau}-1\right]+r-\Sigma_{i=1}^{n_1}e^{\theta_1t_{i:n}}-\Sigma_{i=n_1+1}^{r}e^{\theta_2(t_{i:n}-\tau)+\theta_1\tau}1=0. \tag{14}$$

$$\begin{aligned}&\frac{\partial l(\lambda,\theta_1,\theta_2|t)}{\partial\theta_1}=(r-n)\tau\lambda e^{\theta_2(t_{r:n}-\tau)+\theta_1\tau}+\frac{n_1}{\theta_1}+\Sigma_{i=1}^{n_1}t_{i:n}\\&-\lambda\Sigma_{i=1}^{n_1}t_{i:n}e^{\theta_1t_{i:n}}+n_2\tau-\lambda\Sigma_{i=n_1+1}^{r}\tau e^{\theta_2(t_{i:n}-\tau)+\theta_1\tau}=0.\end{aligned} \tag{15}$$

$$\begin{aligned}&\frac{\partial l(\lambda,\theta_1,\theta_2|t)}{\partial\theta_2}=(r-n)(t_{r:n}-\tau)\lambda e^{\theta_2(t_{r:n}-\tau)+\theta_1\tau}+\frac{n_2}{\theta_2}\\&+\Sigma_{i=n_1+1}^{r}(t_{i:n}-\tau)-\lambda\Sigma_{i=n_1+1}^{r}(t_{i:n}-\tau)e^{\theta_2(t_{i:n}-\tau)+\theta_1\tau}=0.\end{aligned} \tag{16}$$

Solving these equations is necessary to determine the MLEs for the parameters λ, θ_1 and θ_2. The process of estimation, involving Eq.s (14), (15) and (16), cannot be solved analytically. As a result, these equations can be solved simultaneously using a numerical approach like the Newton-Raphson method or other similar techniques. The algorithm for generating the data and calculating the MLEs for the parameters λ, θ_1 and θ_2 is carried out using the subsequent procedure:

First step: Create a set of randomly chosen values with a total of n elements, following a uniform distribution $U(0,1)$. Then, derive the order statistics from these values:

$$U_{1:n} < U_{2:n} < ... < U_{n:n}.$$

Second step: Determine the random variable n_1 for which

$$U_{n_1} < P(T \leq \tau) = G_1(\tau) \leq U_{n_1+1:n},$$

where T symbolizes the time of failure, leading to the following :

$$U_{n_1} < 1 - e^{-\lambda(e^{\theta_1 \tau}-1)} \leq U_{n_1} + 1 : n.$$

Third step: Create the required censored sample using the order statistics $U_{i:n}$ in the subsequent manner:

$$t_{i:n} = \begin{cases} \frac{1}{\theta_1} log\left(\frac{-1}{\lambda} log(1-U_{i:n})+1\right), & i = 1, 2, ..., n_1 \\ \frac{1}{\theta_2} log\left(\frac{-1}{\lambda} log(1-U_{i:n}+1\right) - \frac{\theta_1}{\theta_2}\tau + \tau, & i = n_1+1, ..., r \end{cases} \tag{17}$$

Fourth step: Calculate the MLEs for λ, θ_1 and θ_2 using Eq.s (14), (15) and (16), relying on the censored data $t_{1:n}, t_{2:n}, \ldots, t_{n_1:n}, t_{n_1+1:n}, \ldots, t_{r:n}$, as described in Eq. (17). For more details, see Alkhalfan [19].

4. Prediction for Simple Step-Stress Model

In this discussion, we address the issue of predicting future failure times using observed ones within the framework of Gompertz CE model. Let $T_{1:n} < T_{2:n} < ... < T_{r:n}$ represent the observed ordered lifetime units, referred to as the informative sample. Additionally, $T_{s:n}$, where $s = r+1, \ldots, n$, denotes the yet-to-be observed future lifetimes drawn from the same sample. The prediction problem revolves around determining how we can predict the future lifetimes $T_{s:n}$, based on the observed ordered statistics $T_{i:n}, 0 < i \leq r$.

Due to the Markovian property of censored order statistics, it is established that the conditional distribution of $Y = T_{s:n}$ given $T = t$, where:

$$t = (t_{1:n}, ..., t_{n_1:n}, t_{n_1+1:n}, ..., t_{r:n}),$$

is equivalent to the distribution of $Y = T_{s:n}$ given $T_{r:n} = t_{r:n}$. Consequently, the density of Y given $T = t$ corresponds to the density of the $(s-r)-th$ order statistic among $(n-r)$ units from the population. This population has a left-truncated density $\dfrac{g(y)}{1-G(t_{r:n})}$, where $y > t_{r:n}$, and $G(y)$ and $g(y)$ are defined in Section 2 as given in Eq.s (8) and (9), respectively. Hence, we can represent the density of $Y = T_{s:n}$ given $T = t$ as follows:

$$\begin{aligned} g_{T_{s:n}|\tau(y|\lambda,\theta_1,\theta_2,data)} &= \frac{(n-r)!}{(s-r-1)!(n-s)!}\lambda\theta_2 e^{\theta_1\tau+\theta_2(y-\tau)} \\ &\times \{1 - e^{-\lambda\left[e^{\theta_1\tau+\theta_2(y-\tau)} - e^{\theta_1\tau+\theta_2(t_r-\tau)}\right]}\}^{s-r-1} \\ &\times e^{-\lambda(n-s+1)\left[e^{\theta_1\tau+\theta_2(y-\tau)} - e^{\theta_1\tau+\theta_2(t_r-\tau)}\right]}, y > t_{r:n}. \end{aligned} \tag{18}$$

Three point predictors are presented in the following subsections.

4.1. Maximum Likelihood Predictor

Kaminsky and Rhodin [20] proposed the MLP approach, which involves not only estimating the unknown parameters in the given model but also predicting future order statistics. The predictive likelihood function (PLF) of $Y = T_{s:n}$ is given by

$$L(y, \lambda, \theta_1, \theta_2, |t) = g_{T_{s:n}|T(y|t,\theta_1,\theta_2)} \cdot g_{T(t,\theta_1,\theta_2)} = g_{T_{s:n}|T_{r:n}(y|t_{r:n},\theta_1,\theta_2)} \cdot g_{T(t,\theta_1,\theta_2)}, \tag{19}$$

where $g_{T_{s:n}|T_{r:n}}(y|t_{r:n}, \theta_1, \theta_2$ is the conditional density of $T_{s:n}$ given the observed value of $T = t$, as in Eq. (18), and $g_T(t, \theta_1, \theta_2)$ is the density of T. Actually, the PLF of $Y = T_{s:n}$ can be expressed as:

$$\begin{aligned} &L(y, \lambda, \theta_1, \theta_2|t)\alpha\Pi_{i=1}^{n_1} g_1(t_{i:n}) \times [G_2(y) - G_2(t_{r:n})]^{s-r-1} \\ &g_2(y)\left[1 - G_2(y)\right]^{n-s}, 0 \le n_1 \le r, r+1 \le s \le n. \end{aligned} \tag{20}$$

taking the case when $1 \le n_1 < r \le n$, we obtain

$$\begin{aligned} L(y, \lambda, \theta_1, \theta_2|t)\alpha\Pi_{i=1}^{n_1}\{\lambda\theta_1 e^{\theta_1 t_{i:n} - \lambda(e^{\theta_1 t_{i:n}-1})}\}\Pi_{i=n_1+1}^{r}\{\lambda\theta_2 e^{\theta_1\tau\theta_2(t_{i:n}-\tau) - \lambda\left(e^{\theta_1\tau\theta_2(t_{i:n}-\tau)}-1\right)}\} \\ \times \left[e^{-\lambda(e^{\theta_1\tau+\theta_2(t_{r:n}-\tau)}-1)} - e^{-\lambda(e^{\theta_1\tau+\theta_2(y-\tau)}-1)}\right]^{s-r-1} \\ \times \lambda\theta_2 e^{\theta_1\tau+\theta_2(y-\tau)-\lambda(e^{\theta_1\tau+\theta_2(y-\tau)}-1)}\left[e^{-\lambda(e^{\theta_1\tau+\theta_2(y-\tau)}-1)}\right]^{n-s}, \end{aligned}$$

it can be simplified as follows.

$$\begin{aligned} &L(y, \lambda, \theta_1, \theta_2|t)\alpha{\theta_1}^{n_1}{\theta_2}^{n_2+1}\lambda^{r+1}e^{\theta_1\tau(n_2+1)+\theta_2(y-\tau)+n\lambda} \times e^{-\lambda(n-s+1)\left(e^{\theta_1\tau+\theta_2(y-\tau)}\right)} \\ &\times e^{\theta_1\Sigma_{i=1}^{n_1}t_{i:n} - \lambda\Sigma_{i=1}^{n_1}e^{\theta_1 t_{i:n}} + \theta_2\Sigma_{i=n_1+1}^{r}(t_{i:n}-\tau) - \lambda\Sigma_{i=n_1+1}^{r}e^{\theta_2(t_{i:n}-\tau)+\theta_1\tau}} \\ &\times \left[e^{-\lambda e^{\theta_1\tau+\theta_2(t_{r:n}-\tau)}} - e^{-\lambda e^{\theta_1\tau+\theta_2(y-\tau)}}\right]^{s-r-1}. \end{aligned} \tag{21}$$

The log PLF can be expressed as

$$\begin{aligned} &l(y, \lambda, \theta_1, \theta_2|t)\alpha n_1 log\theta_1 + (n_2+1)log\theta_2 + (r+1)log\lambda \\ &+ \theta_1\tau(n_2+1) + \theta_2(y-\tau) + n\lambda - \lambda(n-s+1)e^{\theta_2(y-\tau)+\theta_1\tau} \\ &+ \theta_1\Sigma_{i=1}^{n_1}t_{i:n} - \lambda\Sigma_{i=1}^{n_1}e^{\theta_1 t_{i:n}} + \theta_2\Sigma_{i=n_1+1}^{r}(t_{i:n}-\tau) - \lambda\Sigma_{i=n_1+1}^{r}e^{\theta_2(t_{i:n}-\tau)+\theta_1\tau} \\ &+ (s-r-1)log\left[e^{-\lambda e^{\theta_1\tau+\theta_2(t_{r:n}-\tau)}} - e^{-\lambda e^{\theta_1\tau+\theta_2(y-\tau)}}\right]. \end{aligned} \tag{22}$$

Taking the derivative of the log PLF in Eq. (22) with respect to y, λ, θ_1 and θ_2 yields the subsequent predictive likelihood equations (PLEs). These equations should be solved to determine the point predictors of $Y = T_{s:n}$ as well as the estimates of λ, θ_1 and θ_2. The likelihood equations are given as:

$$\begin{aligned} &\frac{\partial l(y, \lambda, \theta_1, \theta_2|t)}{\partial\lambda} = \frac{r+1}{\lambda} - (n-s+1)e^{\theta_2(y-\tau)+\theta_1\tau} + n \\ &+ \frac{(s-r-1)e^{\theta_1\tau}\left[e^{-\lambda e^{\theta_2(y-\tau)}+\theta_2(y-\tau)} - e^{-\lambda e^{\theta_2(t_{r:n}-\tau)}+\theta_2(t_{r:n}-\tau)}\right]}{e^{-\lambda e^{\theta_2(t_{r:n}-\tau)}} - e^{-\lambda e^{\theta_2(y-\tau)}}} \\ &- \Sigma_{i=1}^{n_1}e^{\theta_1 t_{i:n}} - \Sigma_{i=n_1+1}^{r}e^{\theta_1(t_{i:n}-\tau)+\theta_1\tau} = 0. \end{aligned} \tag{23}$$

$$\begin{aligned} &\frac{\partial l(y, \lambda, \theta_1, \theta_2|t)}{\partial\theta_1} = \frac{n_1}{\theta_1} - (n-s+1)\tau\lambda e^{\theta_2(y-\tau)+\theta_1\tau} \\ &+ \frac{(s-r-1)\lambda\tau e^{\theta_1\tau}\left[e^{-\lambda e^{\theta_2(y-\tau)}+\theta_2(y-\tau)} - e^{-\lambda e^{\theta_2(t_{r:n}-\tau)}+\theta_2(t_{r:n}-\tau)}\right]}{e^{-\lambda e^{\theta_2(t_{r:n}-\tau)}} - e^{-\lambda e^{\theta_2(y-\tau)}}} \\ &+ \Sigma_{i=1}^{n_1}t_{i:n} - \lambda\Sigma_{i=1}^{n_1}t_{i:n}e^{\theta_1 t_{i:n} - \lambda}\Sigma_{i=n_1+1}^{r}\tau e^{\theta_2(t_{i:n}-\tau)+\theta_1\tau} + (n_2+1)\tau = 0. \end{aligned} \tag{24}$$

$$
\begin{aligned}
\frac{\partial l(y,\lambda,\theta_1,\theta_2|t)}{\partial\theta_2} &= \frac{n_2+1}{\theta_2} - (n-s+1)(y-\tau)\lambda e^{\theta_2(y-\tau)+\theta_1\tau} \\
&+ \frac{(s-r-1)\lambda e^{\theta_1\tau}\left[(y-\tau)e^{-\lambda e^{\theta_2(y-\tau)}+\theta_2(y-\tau)} - (t_{r:n}-\tau)e^{-\lambda e^{\theta_2(t_{r:n}-\tau)}+\theta_2(t_{r:n}-\tau)}\right]}{e^{-\lambda e^{\theta_2(t_{r:n}-\tau)}} - e^{-\lambda e^{\theta_2(y-\tau)}}} \\
&+ \Sigma_{i=n_1+1}^{r}(t_{i:n}-\tau) - \lambda\Sigma_{i=n_1+1}^{r}(t_{i:n}-\tau)e^{\theta_2(t_{i:n}-\tau)+\theta_1\tau} + (y-\tau) = 0.
\end{aligned} \tag{25}
$$

$$
\begin{aligned}
\frac{\partial l(y,\lambda,\theta_1,\theta_2|t)}{\partial y} &= -(n-s+1)\theta_2\lambda e^{\theta_2(y-\tau)+\theta_1\tau} + \theta_2 \\
&+ \frac{(s-r-1)\lambda\theta_2\left[e^{-\lambda e^{\theta_2(y-\tau)}+\theta_{2(y-\tau)+\theta_1\tau}}\right]}{e^{-\lambda e^{\theta_2(t_{r:n}-\tau)}} - e^{-\lambda e^{\theta_2(y-\tau)}}} = 0.
\end{aligned} \tag{26}
$$

As it is not possible to solve Eq.s (23)-(26) directly, we use numerical methods to solve them concurrently. This process aims to determine the MLP of Y and the associated estimators of λ, θ_1 and θ_2 , known as the predictive maximum likelihood estimators (PMLEs). The point predictor obtained from this procedure will be denoted as $\widehat{Y}_M$, representing the resulting MLP of Y.

4.2. *Conditional Median Predictor*

Raqab and Nagaraja [21] introduced a point prediction method referred to as the CMP. A predictor $\widehat{Y}$ is termed the CMP of Y if it constitutes the median of the conditional Y distribution when T is equal to t, that is

$$P(Y \le \widehat{Y}|T=t) = P_\theta(Y \ge \widehat{Y}|T=t),$$

by utilizing the conditional distribution of Y given $T=t$, we can access

$$P(Y \le \widehat{Y}|T=t) = P\left(1 - e^{-\lambda\left[e^{\theta_1\tau+\theta_2(Y-\tau)} - e^{\theta_1\tau+\theta_2(t_r-\tau)}\right]} \ge 1 - e^{-\lambda\left[e^{\theta_1\tau+\theta_2(\widehat{Y}-\tau)} - e^{\theta_1\tau+\theta_2(t_r-\tau)}\right]}|T=t\right).$$

It can be demonstrated that, given $T=t$, the distribution of

$$W = 1 - e^{-\lambda\left[e^{\theta_1\tau+\theta_{2(Y-\tau)}} - e^{\theta_1\tau+\theta_2(t_r-\tau)}\right]},$$

represents a Beta distribution with the parameters $s-r$ and $n-s+1$, which is symbolized as Beta $(s-r, n-s+1)$. Therefore, let B denote a random distribution of Beta $(s-r, n-s+1)$, and M_B be the median of B, the CMP of Y can be derived as follows:

$$\widehat{Y}_{cmp} = \tau - \frac{\theta_1}{\theta_2}\tau + \frac{1}{\theta_2}log\left[e^{\theta_1\tau+\theta_2(t_r-\tau)} - \frac{1}{\lambda}log(1-M_B)\right]. \tag{27}$$

The CMP of Y can be approximated by substituting the MLEs of λ, θ_1, and θ_2 into Eq. (27).

4.3. *Best Unbiased Predictor*

A point predictor $\widehat{Y}$ for the random variable $Y = T_{s:n}$ is referred to as the BUP of Y if its prediction error's mean, $E(\widehat{Y} - Y)$, is zero, and the variance of its prediction error, $Var(\widehat{Y} - Y)$, is equal to or smaller than that of any other unbiased predictor for Y. Utilizing the conditional PDF of Y given $T=t$, as presented in Eq. (18), the BUP of Y is determined by the following expression:

$$\widehat{Y}_{BUP} = E(Y|T) = \int_{t_{r:n}}^{\infty} ygT_{s:n}|T(y|\lambda,\theta_1,\theta_2,data)dy.$$

Using the binomial expansion:

$$
\begin{aligned}
&\{1-e^{-\lambda[e^{\theta_1\tau+\theta_2(y-\tau)}-e^{\theta_1\tau+\theta_2(t_r-\tau)}]}\}^{s-r-1}\\
&=\Sigma_{k=0}^{s-r-1}\binom{s-r-1}{k}(-1)^{s-r-k-1}\\
&\times e^{-\lambda(s-r-k-1)\left[e^{\theta_1\tau+\theta_2(y-\tau)}-e^{\theta_1\tau+\theta_2(t_r-\tau)}\right]},
\end{aligned}
$$

We attain

$$
\begin{aligned}
&\widehat{Y}_{BUP}=(s-r)\binom{n-r}{s-r}\lambda\theta_2\\
&\times\Sigma_{k=0}^{s-r-1}\binom{s-r-1}{k}(-1)^{s-r-k-1}\times e^{\lambda(n-r-k)[e^{\theta_1\tau+\theta_2(t_r-\tau)}]}\\
&\times\int_{t_{r:n}}^{\infty}ye^{\theta_1\tau+\theta_2(y-\tau)}e^{-\lambda(n-r-k)[e^{\theta_1\tau+\theta_2(y-\tau)}]}dy.
\end{aligned}
\tag{28}
$$

Approximating the BUP of Y involves replacing the MLEs of the unknown parameters λ,θ_1, and θ_2 into Eq. (28).

5. Prediction Intervals

In the context of the prediction problem, one of the aspects is to predict the future lifetimes of unobserved events. This is done by creating PIs for $Y=T_{s:n}$, where s represents the time point from $r+1$ to n, based on the available Type-II censored sample $T=(T_{1:n},T_{2:n},\dots,T_{r:n})$. In this section, we explore three methods to obtain such PIs.

5.1. Pivotal Method

Let us consider the random variable

$$W=1-e^{-\lambda[e^{\theta_1\tau+\theta_2(Y-\tau)}-e^{\theta_1\tau+\theta_2(t_r-\tau)}]},Y>t_{r:n}. \tag{29}$$

Because the conditional distribution of W given $T=t$ follows a Beta distribution with parameters $s-r$ and $n-s+1,W$ can be regarded as a pivotal quantity to derive the PI of Y. When we take $(1-\alpha)$, where $0<\alpha<1$, as a prediction coefficient and apply Eq. (29), we acquire the following result:

$$P\left(B_{\frac{\alpha}{2}}<W<B_{1-\frac{\alpha}{2}}\right)=1-\alpha,$$

where B_α represents the 100 α-th percentile of the Beta distribution with parameters $(s-r,n-s+1)$. Thus, $a(1-\alpha)100\%$ pivotal PI for Y is denoted as $(L_1(T),U_1(T))$, where

$$L_1(T)=\tau-\frac{\theta_1}{\theta_2}\tau+\frac{1}{\theta_2}log\left[e^{\theta_1\tau+\theta_2(t_r-\tau)}-\frac{1}{\lambda}log(1-B_{\frac{\alpha}{2}})\right],$$

$$U_1(T)=\tau-\frac{\theta_1}{\theta_2}\tau+\frac{1}{\theta_2}log\left[e^{\theta_1\tau+\theta_2(t_r-\tau)}-\frac{1}{\lambda}log(1-B_{1-\frac{\alpha}{2}})\right].$$

The approximate evaluation of the prediction limits $L_1(T)$ and $U_1(T)$ can be achieved by substituting λ,θ_1 and θ_2 with their corresponding MLEs.

5.2. Highest Conditional Density Method

The conditional distribution of $W = 1 - e^{-\lambda[e^{\theta_1\tau+\theta_2(Y-\tau)} - e^{\theta_1\tau+\theta_2(t_r-\tau)}]}$ given $T = t$ is Beta $(s - r, n - s + 1)$. Consequently, the conditional pdf of W is:

$$g(w) = \frac{(n-r)!}{(s-r-1)!(n-s)!} w^{s-r-1}(1-w)^{n-s}, 0 < w < 1. \tag{30}$$

The density described in Eq. (30) forms a unimodal function in w. An interval (x_1, x_2) is referred to as the highest conditional density (HCD) with a content of $(1 - \alpha)$ if $(x_1, x_2) = \{x : x \in [0, 1], f(x) \geq k\} \subseteq [0, 1]$, where

$$\int_{x_1}^{x_2} f(u)du = 1 - \alpha,$$

for some $k > 0$. When $r + 1 < s < n$, the function $g(w)$ is unimodal, reaching its peak at $\delta = \dfrac{s-r-1}{n-r-1} \in (0, 1)$. Thus, acquiring the HCD PI involves identifying two percentiles, x_1 and x_2 , such that $P(W < x_1) = P(W > x_2) = \frac{\alpha}{2}$, with $x_1 \leq \delta \leq x_2$, satisfying

$$\int_{x_1}^{x_2} g(w)dw = 1 - \alpha, \tag{31}$$

$$g(x_1) = g(x_2). \tag{32}$$

See Casella and Berger [22]. Eq.s (31) and (32) can be expressed in a simplified form as

$$B_{x_2}(s - r, n - s + 1) - B_{x_1}(s - r, n - s + 1) = 1 - \alpha, \tag{33}$$

$$\left(\frac{1-x_2}{1-x_1}\right)^{n-s} = \left(\frac{x_1}{x_2}\right)^{s-r-1}. \tag{34}$$

Here, $B_\nu(a, b)$ represents the incomplete beta function, and $\Gamma(\cdot)$ stands for the gamma function.

$$B_\nu(a, b) = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \int_0^\nu u^{a-1}(1-u)^{b-1}du.$$

Hence, $a(1 - \alpha)100\%$ HCD PI for Y is defined as $(L_2(T), U_2(T))$, with

$$L_2(T) = \tau - \frac{\theta_1}{\theta_2}\tau + \frac{1}{\theta_2} log\left[e^{\theta_1\tau+\theta_2(t_r-\tau)} - \frac{1}{\lambda} log(1 - x_1)\right],$$

$$U_2(T) = \tau - \frac{\theta_1}{\theta_2}\tau + \frac{1}{\theta_2} log\left[e^{\theta_1\tau+\theta_2(t_r-\tau)} - \frac{1}{\lambda} log(1 - x_2)\right].$$

In the specific situation where $s = r + 1$ and $s < n$, the function $g(w)$ can be expressed as $(n - r)(1 - w)^{n-r-1}$, where $0 < w < 1$. This function is characterized by being decreasing with respect to w, and it takes the values $g(0) = n - r$ and $g(1) = 0$. Consequently, the PI for $Y = T_{s:n}$ can be represented as $(0, x_2)$, where $x_2 = 1 - \alpha^{1/(n-r)}$. This finding leads to the following conclusion:

$$L_2(T) = t_{r:n},$$

$$U_2(T) = \tau - \frac{\theta_1}{\theta_2}\tau + \frac{1}{\theta_2} log\left[e^{\theta_1\tau+\theta_2(t_r-\tau)} - \frac{1}{\lambda(n-r)} log(\alpha)\right].$$

When both $s = r + 1$ and $s = n$, the function $g(w)$ follows a uniform distribution $U(0, 1)$. We define x_1 and x_2 as $x_1 = \alpha/2$ and $x_2 = 1 - \alpha/2$, respectively. With these conditions in place, the following results are obtained

$$L_2(T) = \tau - \frac{\theta_1}{\theta_2}\tau + \frac{1}{\theta_2}log\left[e^{\theta_1\tau+\theta_2(t_r-\tau)} - \frac{1}{\lambda}log\left(1 - \frac{\alpha}{2}\right)\right],$$

$$U_2(T) = \tau - \frac{\theta_1}{\theta_2}\tau + \frac{1}{\theta_2}log\left[e^{\theta_1\tau+\theta_2(t_r-\tau)} - \frac{1}{\lambda}log\left(\frac{\alpha}{2}\right)\right].$$

At last, when $s = n$ and $s > r + 1$, the density $g(w) = (n - r)w^{(n-r-1)}$, with $0 < w < 1$, is an increasing function, and it satisfies $g(0) = 0$ and $g(1) = n - r$. In this case, we choose the PI for Y to be in the form $(x_1, 1)$, such that

$$\int_{x_1}^{1} g(w)dw = 1 - \alpha,$$

indicating that $x_1 = \alpha^{1/(n-r)}$. Consequently, $a(1 - \alpha)100\%$ HCD PI of Y can be expressed as

$$L_2(T) = \tau - \frac{\theta_1}{\theta_2}\tau + \frac{1}{\theta_2}log\left[e^{\theta_1\tau+\theta_2(t_r-\tau)} - \frac{1}{\lambda}log\left(1 - \alpha^{\overline{\frac{1}{n-r}}}\right)\right],$$

$$U_2(T) = \infty.$$

5.3. Shortest-Length Based Method

The conditional distribution of $W = 1 - e^{-\lambda\left[e^{\theta_1\tau+\theta_2(Y-\tau)} - e^{\theta_1\tau+\theta_2(t_r-\tau)}\right]}$ given $T = t$ is Beta $(s - r, n - s + 1)$, we choose the values of constants c and d that fulfill the equation:

$$P\left(c < 1 - e^{-\lambda\left[e^{\theta_1\tau+\theta_2(Y-\tau)} - e^{\theta_1\tau+\theta_2(t_r-\tau)}\right]} < d\right) = 1 - \alpha.$$

In this context, the values of constants c and d are selected to minimize the Length of PI $U_3(T) - L_3(T)$. The optimization problem for determining the shortest possible length of the $(1 - \alpha)100\%$ PI can be formulated as follows:
Minimize the Length = $U_3(T) - L_3(T)$, subject to

$$B_d(s - r, n - s + 1) - B_c(s - r, n - s + 1) = 1 - \alpha.$$

Creating the shortest length (SL) PI at a $(1 - \alpha)100\%$, involves the process of minimizing the Lagrangian function:

$$\begin{aligned} R(c, d, z) =& \frac{1}{\theta_2}log\left[e^{\theta_1\tau+\theta_2(t_r-\tau)} - \frac{1}{\lambda}log(1 - d)\right] - \frac{1}{\theta_2}log\left[e^{\theta_1\tau+\theta_2(t_r-\tau)} - \frac{1}{\lambda}log(1 - c)\right] \\ &- z\left[\{B_d(s - r, n - s + 1) - B_d(s - r, n - s + 1)\} - (1 - \alpha)\right], \end{aligned}$$

here, z represents the Lagrange multiplier. Upon taking the derivatives of R with respect to c, d, and z, respectively, we obtain:

$$\frac{\partial R}{\partial c} = -\frac{1}{\theta_2(c-1)(log(1-c) - \lambda e^{\theta_1\tau+\theta_2(t_r-\tau)})} + zp(c, s - r, n - s + 1) = 0.$$

$$\frac{\partial R}{\partial d} = \frac{1}{\theta_2(d-1)(log(1-d) - \lambda e^{\theta_1\tau+\theta_2(t_r-\tau)})} + zp(d, s - r, n - s + 1) = 0.$$

$$\frac{\partial R}{\partial d} = \left[\{B_d(s - r, n - s + 1) - B_c(s - r, n - s + 1)\} - (1 - \alpha)\right] = 0.$$

In this context, $p(x, a, b)$ symbolizes the density of the Beta distribution with parameters a and b . The expressions provided above can also be expressed equivalently as:

$$\frac{log(1-d) - \lambda e^{\theta_1\tau+\theta_2(t_r-\tau)}}{log(1-c) - \lambda e^{\theta_1\tau+\theta_2(t_r-\tau)}} = \frac{(c-1)p(c, s-r, n-s+1)}{(d-1)p(c, s-r, n-s+1)}, \tag{35}$$

$$B_d(s-r, n-s+1) - B_c(s-r, n-s+1) = 1-\alpha. \tag{36}$$

The values of c and d are determined through numerical solutions of Eq.s (35) and (36). Consequently, employing this setup, $a(1-\alpha)100\%$ PI of Y can be represented as $(L_3(T), U_3(T))$, and it is determined by:

$$L_3(T) = \tau - \frac{\theta_1}{\theta_2}\tau + \frac{1}{\theta_2} log\left[e^{\theta_1\tau+\theta_2(t_r-\tau)} - \frac{1}{\lambda}log(1-c)\right],$$

$$U_3(T) = \tau - \frac{\theta_1}{\theta_2}\tau + \frac{1}{\theta_2} log\left[e^{\theta_1\tau+\theta_2(t_r-\tau)} - \frac{1}{\lambda}log(1-d)\right].$$

6. Simulation Study and Data Analysis

In this section, an extensive simulation study is employed to assess the effectiveness of the prediction methods achieved in the preceding sections. Additionally, a real dataset is being used to display the accuracy and applicability of the various prediction methods introduced in this paper.

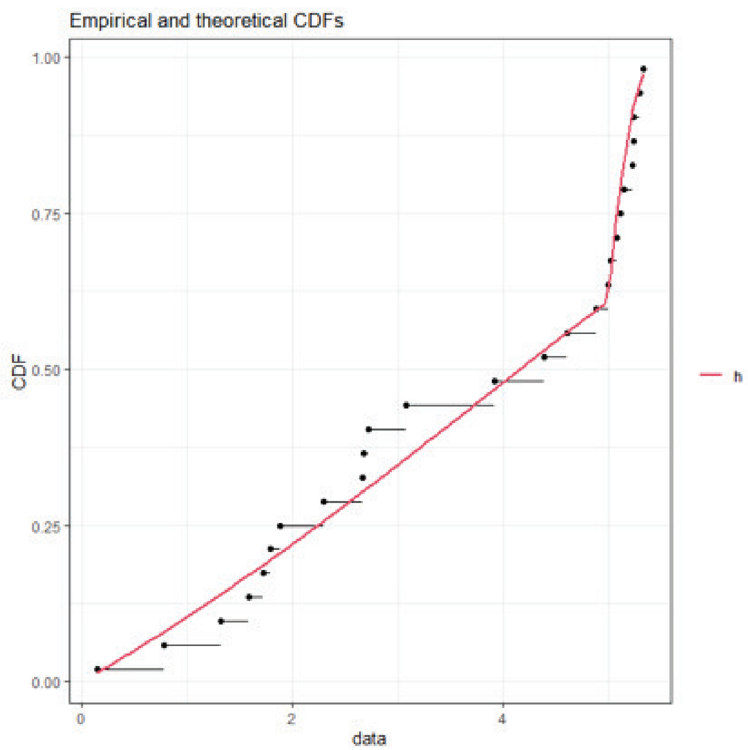

Figure 6. The empirical CDF (dots); and the estimated CDF of Gompertz CE model based on MLE (solid line).

Table 1. Biases and MSPEs of the point predictors for the censored lifetime.

Scheme 1: $\theta_1 = 3, \theta_2 = 2.5, \lambda = 0.025$ and $\tau = 0.8$.							
(n,r)	s	MLP		CMP		BUP	
		Bias	MSPE	Bias	MSPE	Bias	MSPE
(40,30)	32	-0.0244	0.0066	-0.0039	0.0064	0.0292	0.0057
	34	-0.0332	0.0086	-0.0106	0.0082	0.1215	0.0147
	36	-0.0318	0.0095	-0.0056	0.0097	0.0318	0.0073
	38	-0.0394	0.0118	-0.0070	0.0128	0.0990	0.0314
	40	-0.0584	0.0182	-0.0103	0.0205	-0.0030	0.0121
(50,40)	42	-0.0216	0.0059	-0.0047	0.0056	-0.0138	0.0049
	44	-0.0224	0.0057	-0.0039	0.0055	0.0002	0.0044
	46	-0.0276	0.0071	-0.0065	0.0070	-0.0112	0.0043
	48	-0.0220	0.0082	0.0018	0.0087	-0.0229	0.0064
	50	-0.0435	0.0133	-0.0049	0.0149	0.0402	0.0134
(60,50)	52	-0.0191	0.0044	-0.0038	0.0042	-0.0141	0.0355
	54	-0.0186	0.0052	-0.0025	0.0050	0.0135	0.0051
	56	-0.0235	0.0059	-0.0061	0.0059	0.0037	0.0043
	58	-0.0279	0.0071	-0.0083	0.0070	0.0085	0.0076
	60	-0.0401	0.0105	-0.0076	0.0111	0.0152	0.0175

6.1. Simulation Study

In this part of the study, a Monte Carlo simulations are performed for evaluating the suggested prediction techniques. For the point predictors, performance evaluation is measured by assessing biases and calculating mean square prediction errors (MSPEs). The bias and MSPE of a predictor $\widehat{Y}$ of $Y = T_{s:n}(s \geq r+1)$, are defined as follows, respectively

$$Bias(\widehat{Y}) = \frac{1}{M}\Sigma_{K=1}^{M}(\widehat{Y}_k - Y),$$

$$MSPE(\widehat{Y}) = \frac{1}{M}\Sigma_{K=1}^{M}(\widehat{Y}_k - Y)^2.$$

Scheme 2: $\theta_1 = 2, \theta_2 = 1.5, \lambda = 0.03$ and $\tau = 1.2$.							
(n,r)	s	MLP		CMP		BUP	
		Bias	MSPE	Bias	MSPE	Bias	MSPE
(40,30)	32	-0.0393	0.0188	-0.0063	0.0182	0.0051	0.0172
	34	-0.0542	0.0213	-0.0171	0.0202	-0.0075	0.0204
	36	-0.0503	0.0265	-0.0073	0.0269	0.0004	0.0274
	38	-0.0663	0.0371	-0.0140	0.0392	-0.0071	0.0399
	40	-0.0939	0.0521	-0.0121	0.0625	-0.0004	0.0649
(50,40)	42	-0.0396	0.0156	-0.0115	0.0146	-0.0011	0.0147
	44	-0.0410	0.0169	-0.0107	0.0163	-0.0017	0.0165
	46	-0.0358	0.0195	-0.0022	0.0200	0.0053	0.0203
	48	-0.0443	0.0244	-0.0043	0.0254	0.0026	0.0259
	50	-0.0814	0.0408	-0.0210	0.0425	-0.0092	0.0435
(60,50)	52	-0.0398	0.0135	-0.0145	0.0125	-0.0047	0.0125
	54	-0.0392	0.0147	-0.0128	0.0139	-0.0044	0.0140
	56	-0.0330	0.0160	-0.0044	0.0159	0.0029	0.0161
	58	-0.0403	0.0204	-0.0077	0.0204	-0.0008	0.0206
	60	-0.0650	0.0316	-0.0131	0.0320	-0.0011	0.0326

Additionally, we conduct a comparison of the PIs discussed in Section 5 by evaluating their estimated average lengths (Als) and coverage probabilities (CPs).
Hence, a comparative study is performed using various schemes of censoring and different sample sizes from the Gompertz distribution within the context of the CE model. For specific values of n, r, and s, we create Type-II censored samples following the procedure outlined in Section 3. This process is carried out according to the subsequent schemes:

Scheme 1: $\theta_1 = 0.4, \theta_2 = 0.7, \lambda = 1$ and $\tau = 0.8$.
Scheme 2: $\theta_1 = 2, \theta_2 = 1, \lambda = 0.1$ and $\tau = 1$.

In both instances, we determine the point predictor values: MLP, CMP, and BUP. Additionally, we calculate 95% PIs using pivotal quantity, HCD, and SL methods. We generate Type-II censored samples from the Gompertz model using these two schemes, repeated $M = 1000$ times in the simulation. By employing these generated samples, we calculate the MLEs of the parameters as well as the prediction biases and MSPEs for the predictors.

Table 2. ALs and CPs of 95% PIs of the censored lifetimes.

Scheme 1: $\theta_1 = 3, \theta_2 = 2.5, \lambda = 0.025$ and $\tau = 0.8$.							
(n,r)	s	Pivotal Method		HCD Method		SL Method	
		AL	CP	AL	CP	AL	CP
(40,30)	32	0.1816	0.652	0.1682	0.646	0.1682	0.645
	34	0.2532	0.795	0.2472	0.792	0.2460	0.0792
	36	0.3159	0.874	0.3187	0.880	0.3119	0.876
	38	0.3921	0.912	0.4233	0.927	0.3894	0.909
	40	0.5667	0.951	∞	0.951	0.5631	0.948
(50,40)	42	0.1590	0.642	0.1470	0.626	0.1470	0.626
	44	0.2246	0.837	0.2191	0.836	0.2178	0.839
	46	0.2856	0.896	0.2883	0.895	0.2815	0.900
	48	0.3567	0.949	0.3861	0.955	0.3538	0.938
	50	0.5426	0.976	∞	0.982	0.5385	0.976
(60,50)	52	0.1451	0.651	0.1340	0.646	0.1340	0.647
	54	0.2083	0.821	0.2030	0.811	0.2017	0.813
	56	0.2632	0.899	0.2658	0.900	0.2591	0.901
	58	0.3356	0.950	0.3641	0.967	0.3325	0.946
	60	0.5120	0.981	∞	0.984	0.5076	0.978

R software is used to accomplish these calculations. The outcomes from this process are displayed in Table 1. Furthermore, Table 2 reports the ALs and CPs of the PIs.

The results obtained in these tables lead to make the following observations:

1. As s increases, the biases and MSPEs of the point predictors increase when considering constant values of n and r. This result can be attributed to the variability in the lifetime to be predicted as s reaches higher values.
2. When considering the bias as a measure of predictive quality, it can be observed that the BUP has the best results in most of the cases. In terms of the MSPEs, it can be noted that the point predictors are very competitive. However, the MLP outperforms the CMP and the BUP when s reaches higher values. The similarity in MSPEs for the three predictors when s is close to r might be due to the similarity between MLEs and PMLEs in the analyzed cases. This leads to say that the MLP is a strong contender in terms of predictive accuracy among the methods being compared.

Scheme 2: $\theta_1 = 2, \theta_2 = 1.5, \lambda = 0.03$ and $\tau = 1.2$.							
(n,r)	s	Pivotal Method		HCD Method		SL Method	
		AL	CP	AL	CP	AL	CP
(40,30)	32	0.2947	0.637	0.2731	0.628	0.2731	0.629
	34	0.4155	0.839	0.4057	0.83	0.4037	0.831
	36	0.5245	0.880	0.5292	0.883	0.5178	0.876
	38	0.6478	0.892	0.6991	0.915	0.6435	0.890
	40	0.9545	0.943	∞	0.949	0.9482	0.937
(50,40)	42	0.2593	0.640	0.2398	0.621	0.2398	0.621
	44	0.3740	0.824	0.3647	0.816	0.3626	0.821
	46	0.4743	0.895	0.4787	0.894	0.4675	0.900
	48	0.5967	0.934	0.6462	0.939	0.5919	0.934
	50	0.8819	0.963	∞	0.969	0.8751	0.960
(60,50)	52	0.2400	0.653	0.2216	0.638	0.2216	0.639
	54	0.3434	0.831	0.3346	0.833	0.3324	0.834
	56	0.4397	0.915	0.4439	0.914	0.4328	0.912
	58	0.5548	0.942	0.6020	0.947	0.5498	0.94
	60	0.8491	0.979	∞	0.981	0.8418	0.975

Table 3. Lifetimes of prototypes of a solar lighting device on a simple step-stress test.

Temperature Level	Recorded data
$S_1 : 293K$	0.140 0.783 1.324 1.582 1.716 1.794 1.883 2.293 2.660 2.674 2.725 3.085 3.924 4.396 4.612 4.892
$S_2 : 353K$	5.002 5.022 5.082 5.112 5.147 5.238 5.244 5.247 5.305 5.337 5.407 5.408 5.445 5.483 5.717

3. The SL method appears to be more efficient than other methods based on the AL criterion, especially as s increases. The HCD PIs outperform pivotal PIs for cases where s is close to r, while pivotal PIs become more competitive as s approaches n. The SL PIs are generally superior based on both criteria. Another observation, the CPs of all PIs tend to increase as s increases. In this sense, the CP is at its worst when predicting the lifetime immediately following the last observed lifetime.

Table 4. Point predictors and PIs for future lifetimes of $Y = T_{s:n}$.

Point predictors of $Y = T_{s:n}$				
s	True value	MLP	CMP	BUP
28	5.408	5.374	5.405	5.415
30	5.483	5.457	5.497	5.506
31	5.717	5.504	5.550	5.559
33	——	5.620	5.684	5.692
35	——	5.818	5.928	5.940
95% PIs of $Y = T_{s:n}$				
s	True value	Pivotal PI	HCD PI	SL PI
28	5.408	(5.341 , 5.598)	(5.340 , 5.517)	(5.339 , 5.515)
30	5.483	(5.369 , 5.740)	(5.383 , 5.663)	(5.373 , 5.650)
31	5.717	(5.391 , 5.8169)	(5.418 , 5.746)	(5.402 , 5.723)
33	——	(5.458 , 6.010)	(5.516 , 5974)	(5.481 , 5.903)
35	——	(5.581 , 6.429)	(5.686 , ∞)	(5.617 , 6.264)

6.2. Data Analysis

To illustrate the predicting techniques proposed in this chapter, we conduct an analysis using real data. The dataset utilized is sourced from the work of Han and Kundu [23]. It encompasses 31 instances of failure times (measured in hundred hours) from a subset of 35 prototypes of a solar lighting apparatus, characterized by two primary failure modes: controller malfunction and capacitor malfunction. In this study, the stress-inducing factor is temperature, which was varied within the range of 293K to 353K during testing. The standard operational temperature is 293K, and the stress alteration occurred over a period of 500 hours. These specific data have been previously employed by Kotb and El-Din [24], as well as Amleh [7]. The recorded data are presented in Table 3.

For the purpose of illustrating the precision of our model, namely the Gompertz CE model, we have plotted the actual cdf of the lifetimes in Fig. 6. Alongside the cdf derived from the MLEs. Precisely, to examine the goodness-of- fit of the data to the Gompertz CE model, the Kolmogorov-Smirnov test is used. The test statistic for the distance between the fitted and experimental distribution function is 0.1122 and the corresponding p-value is close to 1. Therefore, it is justified to use the Gompertz distribution within the CE model as a suitable model to fit these data.

Assume that the life test ends when the 26th lifetime is observed. This means we have Type-II censored sample, with a sample size of 35 with 26 failures occurred. Our objective is to calculate the point predictors for the lifetimes that we have not observed yet: $Y = T_{s:n}$, where s takes values 28, 30, 31, 33, and 35. We also want to determine the corresponding PIs. Initially, we determine the MLEs for λ, θ_1 and θ_2 by simultaneously solving Eq.s (14), (15) and (16). The computations yield $\widehat{\lambda} = 0.5254, \widehat{\theta_1} = 0.1543$ and $\widehat{\theta_2} = 1.4748$. These estimates are then used to predict future censored lifetimes and generate both point predictions and PIs, which are presented in Table 4.

The closeness of the point predictions to the actual values is evident using all the proposed predictors. Furthermore, the acquired point predictions fall within all the considered PIs. It can be noted that all the obtained PIs include the actual values of forthcoming order statistics. Additionally, it's noticeable that the Pls become wider as the parameter s increases. This is attributed to the increased variability in the fluctuation of $Y = T_{s:n}$, particularly as Y diverges from observed failure times. Despite the close adherence of all PIs according to the AL criterion, the intervals constructed using the SL method are characterized by their minimal length.

7. Conclusions

This paper focuses on predicting the future lifetimes of a simple step stress test using the Gompertz distribution within the CE model. The study considers cases where data is Type-II censored. The paper introduces several point predictors, including MLP, CMP, and BUP. In addition, the paper discusses another aspect of prediction, where we construct PIs for these future lifetimes. Through an extensive Monte Carlo simulation, the performance of these predictors is compared, taking into account biases and MSPEs. The evaluation of PIs includes considerations of their ALs and CPs. In summary, it is observed that the MLP as a point predictor has the best performance. On the other hand, the SL based PIs outperform the pivot and HCD PI.

It is important to note that although the study primarily addresses Type-II censoring, the techniques discussed can also be adapted for other censoring schemes such as Type-I, hybrid or progressive censoring.

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