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Filtering Problem for Sequences with Periodically Stationary Multi-seasonal Increments with Spectral Densities Allowing Canonical Factorizations

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Abstract We propose solution of the problem of the mean square optimal estimation of linear functionals which depend on the unobserved values of a stochastic sequence (signal) with periodically stationary generalized multiple increments of fractional order. These stochastic sequences combine cyclostationary, multi-seasonal, integrated and fractionally integrated patterns. Estimates are based on observations of the signal sequence with additive periodically stationary noise sequence. In cases where the spectral densities of the signal and the noise sequences are known and allow the canonical factorizations, we provide formulas for calculation the mean square error and the spectral characteristic of the optimal estimate of the functionals in terms of the coefficients of such factorizations. In the case where the spectral densities are not exactly known while certain sets of admissible spectral densities are available we apply the minimax (robust) method of estimation. Formulas are proposed that determine the least favourable spectral densities and the minimax (robust) spectral characteristics of the optimal linear estimate of the functional.

Keywords Periodically Stationary Sequence, SARIMA, Fractional Integration, Filtering, Optimal Linear Estimate, Mean Square Error, Least Favourable Spectral Density Matrix, Minimax Spectral Characteristics

AMS 2010 subject classifications. Primary: 60G10, 60G25, 60G35. Secondary: 62M20, 62P20, 93E10, 93E11

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1. Introduction

Non-stationary time series models have found wide-ranging applications in economics, finance, climatology, air pollution, signal processing. A fundamental example is a general multiplicative model, known as $SARIMA(p,d,q)\times(P,D,Q)_s$ (Seasonal Autoregressive Integrated Moving Average), introduced in the book by Box and Jenkins et al. [4]. This model incorporates both integrated and seasonal factors. It is described by the equation

$$\Psi(B^s)\psi(B)(1-B)^d(1-B^s)^D x_t=\Theta(B^s)\theta(B)\varepsilon_t, \tag{1}$$

where ε_t is a sequence of independent and identically distributed (i.i.d.) random variables, $\psi(z)$, $\theta(z)$ are polynomials of p and q degrees, respectively, with roots outside the unit circle, and where $\Psi(z)$ and $\Theta(z)$ are two polynomials of degrees P and Q, respectively, with roots outside the unit circle. The parameters d and D can take fractional values, resulting to what is known as seasonal ARFIMA, or SARFIMA model. The process described by equation (1) is stationary and invertible when $|d+D|<1/2$ and $|D|<1/2$. One application of seasonal ARFIMA models to the analysis of monetary aggregates used by the U.S. Federal Reserve is demonstrated in the work of Porter-Hudak [32].

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In the field of statistical inference for seasonal long-memory sequences, recent research has yielded several notable results. One such contribution is the work by Tsai, Rachinger, and Lin [36], who developed methods for estimating model parameters when measurement errors are present. In another study, Baillie, Kongcharoen, and Kapetanios [2] compared two commonly used estimation procedures for prediction problem based on ARFIMA models. Specifically, they compared the performance of maximum likelihood estimation (MLE) to a two-step local Whittle estimator. Through a simulation study, they found that the MLE estimator outperformed the two-step local Whittle estimator. In addition, Hassler and Pohle [13] evaluated the predictive performance of various forecasting methods for inflation and return volatility time series. Their analysis provided compelling evidence in support of models with a fractional integration component.

Another class of non-stationary processes is the periodically correlated or cyclostationary processes, introduced by Gladyshev [9]. These processes belong to the class of time-dependent spectrum processes and are widely used in signal processing and communications. For recent works on cyclostationarity and its applications, see the review by Napolitano [30]. Periodic time series can be viewed as an extension of seasonal models [1, 3, 20, 31].

The methods used for parameter estimation and filtering of time series data often fail to account for real-world challenges such as outliers, measurement errors, incomplete information about spectral structure. As a result, there is a growing interest in robust estimation methods that can effectively handle such issues. For example, Reisen et al. [33] and Solci et al. [35] have proposed robust estimates for SARIMA and PAR models. Other researchers, including Grenander [11], Hosoya [14], Franke [7], Vastola and Poor [37], Moklyachuk [26, 27], and Luz and Moklyachuk [21], Liu et al. [19], have also investigated various aspects of minimax extrapolation, interpolation, and filtering problems for stationary sequences and processes.

In this article, we extend our investigation of robust filtering for stochastic sequences with periodically stationary long memory multiple seasonal increments (or sequences with periodically stationary general multiplicative (GM) increments) by focusing on spectral densities that allow canonical factorizations, whereas in [21], the results were obtained using Fourier transformations of the spectral densities.

The mentioned sequences were introduced by Luz and Moklyachuk in the paper [23], motivated by an increasing interest in models with multiple seasonal and periodic patterns (see the works of Dudek [6], Gould et al. [10], Hurd and Piparas [15]). This research continues previous works on minimax filtering of stationary vector-valued processes, periodically correlated processes, and processes with stationary increments. Specifically, Moklyachuk and Masyutka [28], Moklyachuk and Golichenko (Dubovetska) [5], Luz and Moklyachuk [22] have performed research in these areas. Additionally, we mention the works by Moklyachuk, Masyutka, and Sidei [29], which derive minimax estimates of stationary processes from observations with missing values.

The article is structured as follows. In Section 2, we provide a brief review of the GM increment sequence $\chi^{(d)}_{\overline{\mu},\overline{s}}(\vec{\xi}(m))$ and the stochastic sequence $\xi(m)$ with periodically stationary (periodically correlated, cyclostationary) GM increments, as well as the spectral theory of vector-valued GM increment sequences. In Section 3, we address the classical filtering problem for linear functionals $A\xi$ and $A_N\xi$ that are constructed from unobserved values of the sequence $\xi(m)$. We assume that the spectral densities of the sequence $\xi(m)$ and a noise sequence $\eta(m)$ are known and allow canonical factorizations. The estimates are derived in terms of coefficients of canonical factorizations of the spectral densities, making use of results obtained in [24] by using the Fourier transformations of the spectral densities. Section 4 focuses on the minimax (robust) estimation for cases where the spectral densities of sequences are not precisely known while some sets of admissible spectral densities are specified. For illustration, we propose particular types of admissible spectral density sets, which are generalizations of the sets described in a survey article by Kassam and Poor [17] for stationary stochastic processes.

2. Stochastic sequences with periodically stationary generalized multiple increments

2.1. Preliminary notations and definitions

Consider a stochastic sequence $\xi(m)$, $m\in\mathbb{Z}$, and a backward shift operator B_μ with the step $\mu\in\mathbb{Z}$, such that $B_\mu\xi(m)=\xi(m-\mu)$; $B:=B_1$. Then $B^s_\mu=B_\mu B_\mu\cdot\ldots\cdot B_\mu$. Define a *multiplicative incremental operator*

$$\chi^{(d)}_{\overline{\mu},\overline{s}}(B)=\prod_{i=1}^{r}(1-B^{s_i}_{\mu_i})^{d_i}=\sum_{k=0}^{n(\gamma)}e_\gamma(k)B^k,$$

where $d:=d_1+d_2+\ldots+d_r$, $\overline{d}=(d_1,d_2,\ldots,d_r)\in(\mathbb{N}^*)^r$, $\overline{s}=(s_1,s_2,\ldots,s_r)\in(\mathbb{N}^*)^r$ and $\overline{\mu}=(\mu_1,\mu_2,\ldots,\mu_r)\in(\mathbb{N}^*)^r$ or $\in(\mathbb{Z}\setminus\mathbb{N})^r$; $n(\gamma):=\sum_{i=1}^{r}\mu_is_id_i$. Here $\mathbb{N}^*=\mathbb{N}\setminus\{0\}$. The explicit representation of the coefficients $e_\gamma(k)$ is given in [23]. Within the article, δ_{lp} denotes Kronecker symbols, $\binom{n}{l}=\frac{n!}{l!(n-l)!}$.

Definition 1 ([23])
For a stochastic sequence $\xi(m)$, $m\in\mathbb{Z}$, the sequence

$$\begin{aligned}\chi^{(d)}_{\overline{\mu},\overline{s}}(\xi(m)) &:= \chi^{(d)}_{\overline{\mu},\overline{s}}(B)\xi(m)=(1-B^{s_1}_{\mu_1})^{d_1}(1-B^{s_2}_{\mu_2})^{d_2}\cdot\ldots\cdot(1-B^{s_r}_{\mu_r})^{d_r}\xi(m)\\ &= \sum_{l_1=0}^{d_1}\ldots\sum_{l_r=0}^{d_r}(-1)^{l_1+\ldots+l_r}\binom{d_1}{l_1}\cdot\ldots\cdot\binom{d_r}{l_r}\xi(m-\mu_1s_1l_1-\cdots-\mu_rs_rl_r) \qquad (2)\end{aligned}$$

is called *a stochastic generalized multiple (GM) increment sequence* of differentiation order d with a fixed seasonal vector $\overline{s}\in(\mathbb{N}^*)^r$ and a varying step $\overline{\mu}\in(\mathbb{N}^*)^r$ or $\overline{\mu}\in(\mathbb{Z}\setminus\mathbb{N})^r$.

The theory of (wide sense) stationary stochastic sequences describes second order random variables $\eta(m)$, $m\in\mathbb{Z}$, such that the mean value $a=\mathsf{E}\eta(m_0)$ and the covariance function $\gamma(h)=\mathsf{Cov}(\eta(m_0),\eta(m_0+h))$ are finite and do not depend on m_0. The following definition describes the stationarity of the increment sequence $\chi^{(d)}_{\overline{\mu},\overline{s}}(\xi(m))$, $m\in\mathbb{Z}$, with a generalization of the mean value and covariance function, arising from the presence of the increment step $\overline{\mu}$.

Definition 2 ([23])
A stochastic GM increment sequence $\chi^{(d)}_{\overline{\mu},\overline{s}}(\xi(m))$ is called a wide sense stationary if the mathematical expectations

$$\begin{aligned}\mathsf{E}\chi^{(d)}_{\overline{\mu},\overline{s}}(\xi(m_0)) &= c^{(d)}_{\overline{s}}(\overline{\mu}),\\ \mathsf{E}\chi^{(d)}_{\overline{\mu}_1,\overline{s}}(\xi(m_0+m))\chi^{(d)}_{\overline{\mu}_2,\overline{s}}(\xi(m_0)) &= D^{(d)}_{\overline{s}}(m;\overline{\mu}_1,\overline{\mu}_2)\end{aligned}$$

exist for all $m_0,m,\overline{\mu},\overline{\mu}_1,\overline{\mu}_2$ and do not depend on m_0. The function $c^{(d)}_{\overline{s}}(\overline{\mu})$ is called a mean value and the function $D^{(d)}_{\overline{s}}(m;\overline{\mu}_1,\overline{\mu}_2)$ is called a structural function of the stationary GM increment sequence (of a stochastic sequence with stationary GM increments).
The stochastic sequence $\xi(m)$, $m\in\mathbb{Z}$ determining the stationary GM increment sequence $\chi^{(d)}_{\overline{\mu},\overline{s}}(\xi(m))$ by (2) is called a stochastic sequence with stationary GM increments (or GM increment sequence of order d).

Remark 1
Spectral properties of one-pattern increment sequence $\chi^{(n)}_{\mu,1}(\xi(m)):=\xi^{(n)}(m,\mu)=(1-B_{\overline{\mu}})^n\xi(m)$ and the continuous time increment process $\xi^{(n)}(t,\tau)=(1-B_\tau)^n\xi(t)$ are described in [38], [39].

Example 1
Consider an increment operator $\chi^{(d+D)}_{\overline{\mu},(1,s)}(B)=(1-B_{\mu_0})^d(1-B^s_{\mu_1})^D$. In this case the SARIMA time series (1) can be modeled by a GM increment sequence $x_m=\chi^{(d+D)}_{(1,1),(1,s)}(\xi(m))$ with the step $\overline{\mu}=(1,1)$, which is defined as an ARMA model

$$\Psi(B^s)\psi(B)x_m=\Theta(B^s)\theta(B)\varepsilon_m,$$

where ε_m is a sequence of i.i.d. random variables, and $\Psi(z^s)\psi(z)$ and $\Theta(z^s)\theta(z)$ are two polynomials with roots outside the unit circle.

2.2. Definition and spectral representation of stochastic sequences with periodically stationary GM increment

In this subsection, we present definition, justification and a brief review of the spectral theory of stochastic sequences with periodically stationary multiple seasonal increments, introduced in [23].

Definition 3
A stochastic sequence $\xi(m)$, $m\in\mathbb{Z}$ is called a *stochastic sequence with periodically stationary (periodically correlated) GM increments* with period T if the mathematical expectations

$$\begin{aligned}\mathsf{E}\chi^{(d)}_{\overline{\mu},T\overline{s}}(\xi(m+T)) &= \mathsf{E}\chi^{(d)}_{\overline{\mu},T\overline{s}}(\xi(m))=c^{(d)}_{T\overline{s}}(m,\overline{\mu}),\\ \mathsf{E}\chi^{(d)}_{\overline{\mu}_1,T\overline{s}}(\xi(m+T))\chi^{(d)}_{\overline{\mu}_2,T\overline{s}}(\xi(k+T)) &= D^{(d)}_{T\overline{s}}(m+T,k+T;\overline{\mu}_1,\overline{\mu}_2)=D^{(d)}_{T\overline{s}}(m,k;\overline{\mu}_1,\overline{\mu}_2)\end{aligned}$$

exist for every $m,k,\overline{\mu}_1,\overline{\mu}_2$ and $T>0$ is the least integer for which these equalities hold.

It follows from Definition 3 that the sequence

$$\xi_p(m)=\xi(mT+p-1),\quad p=1,2,\dots,T;\quad m\in\mathbb{Z} \tag{3}$$

forms a vector-valued sequence $\vec{\xi}(m)=\{\xi_p(m)\}_{p=1,2,\dots,T}$, $m\in\mathbb{Z}$ with stationary GM increments as follows:

$$\chi^{(d)}_{\overline{\mu},\overline{s}}(\xi_p(m))=\chi^{(d)}_{\overline{\mu},T\overline{s}}(\xi(mT+p-1)),\quad p=1,2,\dots,T,$$

where $\chi^{(d)}_{\overline{\mu},\overline{s}}(\xi_p(m))$ is the GM increment of the p-th component of the vector-valued sequence $\vec{\xi}(m)$.
The following theorem describes the spectral structure of the vector-valued GM increment [16], [23].

Theorem 1
1. The mean value and the structural function of the vector-valued stochastic stationary GM increment sequence $\chi^{(d)}_{\overline{\mu},\overline{s}}(\vec{\xi}(m))$ can be represented in the form

$$c^{(d)}_{\overline{s}}(\overline{\mu}) = c\prod_{i=1}^{r}\mu_i^{d_i}, \tag{4}$$

$$D^{(d)}_{\overline{s}}(m;\overline{\mu}_1,\overline{\mu}_2) = \int_{-\pi}^{\pi}e^{i\lambda m}\chi^{(d)}_{\overline{\mu}_1}(e^{-i\lambda})\chi^{(d)}_{\overline{\mu}_2}(e^{i\lambda})\frac{1}{|\beta^{(d)}(i\lambda)|^2}dF(\lambda), \tag{5}$$

where

$$\chi^{(d)}_{\overline{\mu}}(e^{-i\lambda})=\prod_{j=1}^{r}(1-e^{-i\lambda\mu_j s_j})^{d_j},\quad \beta^{(d)}(i\lambda)=\prod_{j=1}^{r}\prod_{k_j=-[s_j/2]}^{[s_j/2]}(i\lambda-2\pi i k_j/s_j)^{d_j},$$

c is a vector, $F(\lambda)$ is the matrix-valued spectral function of the stationary stochastic sequence $\chi^{(d)}_{\overline{\mu},\overline{s}}(\vec{\xi}(m))$. The vector c and the matrix-valued function $F(\lambda)$ are determined uniquely by the GM increment sequence $\chi^{(d)}_{\overline{\mu},\overline{s}}(\vec{\xi}(m))$.

2. The stationary vector-valued GM increment sequence $\chi^{(d)}_{\overline{\mu},\overline{s}}(\vec{\xi}(m))$ admits the spectral representation

$$\chi^{(d)}_{\overline{\mu},\overline{s}}(\vec{\xi}(m))=\int_{-\pi}^{\pi}e^{im\lambda}\chi^{(d)}_{\overline{\mu}}(e^{-i\lambda})\frac{1}{\beta^{(d)}(i\lambda)}d\vec{Z}_{\xi^{(d)}}(\lambda), \tag{6}$$

where $\vec{Z}_{\xi^{(d)}}(\lambda)=\{Z_p(\lambda)\}_{p=1}^{T}$ is a (vector-valued) stochastic process with uncorrelated increments on $[-\pi,\pi)$ connected with the spectral function $F(\lambda)$ by the relation

$$\mathsf{E}(Z_p(\lambda_2)-Z_p(\lambda_1))(\overline{Z_q(\lambda_2)-Z_q(\lambda_1)})=F_{pq}(\lambda_2)-F_{pq}(\lambda_1),$$

$$-\pi\leq\lambda_1<\lambda_2<\pi,\quad p,q=1,2,\dots,T.$$

Consider another vector-valued stochastic sequence with the stationary GM increments $\vec{\zeta}(m)=\vec{\xi}(m)+\vec{\eta}(m)$, where $\vec{\eta}(m)$ is a vector-valued stationary stochastic sequence, uncorrelated with $\vec{\xi}(m)$, with the spectral representation

$$\vec{\eta}(m)=\int_{-\pi}^{\pi}e^{i\lambda m}d\vec{Z}_{\eta}(\lambda),$$

where $\vec{Z}_{\eta}(\lambda)=\{Z_{\eta,p}(\lambda)\}_{p=1}^{T}$, $\lambda\in[-\pi,\pi)$, is a stochastic process with uncorrelated increments, that corresponds to the spectral function $G(\lambda)$ [12]. The stochastic stationary GM increment $\chi_{\overline{\mu},\overline{s}}^{(d)}(\vec{\zeta}(m))$ allows the spectral representation

$$\chi_{\overline{\mu},\overline{s}}^{(d)}(\vec{\zeta}(m)) = \int_{-\pi}^{\pi}e^{i\lambda m}\frac{\chi_{\overline{\mu}}^{(d)}(e^{-i\lambda})}{\beta^{(d)}(i\lambda)}d\vec{Z}_{\xi^{(d)}}(\lambda)+\int_{-\pi}^{\pi}e^{i\lambda m}\chi_{\overline{\mu}}^{(d)}(e^{-i\lambda})d\vec{Z}_{\eta}(\lambda),$$

while $d\vec{Z}_{\eta}(\lambda)=(\beta^{(d)}(i\lambda))^{-1}d\vec{Z}_{\eta^{(d)}}(\lambda)$, $\lambda\in[-\pi,\pi)$. Therefore, in the case where the spectral functions $F(\lambda)$ and $G(\lambda)$ have the spectral densities $f(\lambda)$ and $g(\lambda)$, the spectral density $p(\lambda)=\{p_{ij}(\lambda)\}_{i,j=1}^{T}$ of the stochastic sequence $\vec{\zeta}(m)$ is determined by the formula

$$p(\lambda)=f(\lambda)+|\beta^{(d)}(i\lambda)|^2g(\lambda).$$

For a regular stationary GM increment sequence $\chi_{\overline{\mu},\overline{s}}^{(d)}(\vec{\xi}(m))$ [24], there exists an innovation sequence $\vec{\varepsilon}(u)=\{\varepsilon_k(u)\}_{k=1}^{q}$, $u\in\mathbb{Z}$ and a sequence of matrix-valued functions $\varphi^{(d)}(k,\overline{\mu})=\{\varphi_{ij}^{(d)}(k,\overline{\mu})\}_{i=\overline{1,T}}^{j=\overline{1,q}}$, $k\geq 0$, such that

$$\sum_{k=0}^{\infty}\sum_{i=1}^{T}\sum_{j=1}^{q}|\varphi_{ij}^{(d)}(k,\overline{\mu})|^2<\infty,\quad \chi_{\overline{\mu},\overline{s}}^{(d)}(\vec{\xi}(m))=\sum_{k=0}^{\infty}\varphi^{(d)}(k,\overline{\mu})\vec{\varepsilon}(m-k). \tag{7}$$

Representation (7) is called *a canonical moving average representation* of the stochastic stationary GM increment sequence $\chi_{\overline{\mu},\overline{s}}^{(d)}(\vec{\xi}(m))$. The spectral function $F(\lambda)$ of this sequence has the spectral density $f(\lambda)=\{f_{ij}(\lambda)\}_{i,j=1}^{T}$ admitting the canonical factorization

$$f(\lambda)=\varphi(e^{-i\lambda})\varphi^*(e^{-i\lambda}),$$

where the function $\varphi(z)=\sum_{k=0}^{\infty}\varphi(k)z^k$ has analytic in the unit circle $\{z:|z|\leq 1\}$ components $\varphi_{ij}(z)=\sum_{k=0}^{\infty}\varphi_{ij}(k)z^k; i=1,\dots,T; j=1,\dots,q$. Based on moving average representation (7) define

$$\varphi_{\overline{\mu}}(z)=\sum_{k=0}^{\infty}\varphi^{(d)}(k,\overline{\mu})z^k=\sum_{k=0}^{\infty}\varphi_{\overline{\mu}}(k)z^k.$$

Then the following relation holds true:

$$\varphi_{\overline{\mu}}(e^{-i\lambda})\varphi_{\overline{\mu}}^*(e^{-i\lambda})=\frac{|\chi_{\overline{\mu}}^{(d)}(e^{-i\lambda})|^2}{|\beta^{(d)}(i\lambda)|^2}f(\lambda)=\prod_{j=1}^{r}\frac{\left|1-e^{-i\lambda\mu_js_j}\right|^{2d_j}}{\prod_{k_j=-[s_j/2]}^{[s_j/2]}|\lambda-2\pi k_j/s_j|^{2d_j}}f(\lambda). \tag{8}$$

In the following the one-sided moving average representation (7) and relation (8) are used for finding the mean square optimal estimates of unobserved values of vector-valued sequences with stationary GM increments.

3. Hilbert space projection method of filtering

3.1. *Filtering of vector-valued stochastic sequence with stationary GM increments*

Consider a vector-valued stochastic sequence $\vec{\xi}(m)$ with stationary GM increments constructed from transformation (3) and a vector-valued stationary stochastic sequence $\vec{\eta}(m)$ uncorrelated with the sequence $\vec{\xi}(m)$. Let the stationary GM increment sequence $\chi_{\overline{\mu},\overline{s}}^{(d)}(\vec{\xi}(m))=\{\chi_{\overline{\mu},\overline{s}}^{(d)}(\xi_p(m))\}_{p=1}^{T}$ and the stationary sequence $\vec{\eta}(m)$

have absolutely continuous spectral functions $F(\lambda)$ and $G(\lambda)$ with the spectral densities $f(\lambda)=\{f_{ij}(\lambda)\}_{i,j=1}^{T}$ and $g(\lambda)=\{g_{ij}(\lambda)\}_{i,j=1}^{T}$ respectively. Without loss of generality assume that $\mathsf{E}\chi_{\overline{\mu},\overline{s}}^{(d)}(\vec{\xi}(m))=0$, $\mathsf{E}\vec{\eta}(m)=0$ and $\overline{\mu}>\overline{0}$.

Filtering problem. Consider the problem of mean square optimal linear estimation of the functional

$$A\vec{\xi}=\sum_{k=0}^{\infty}(\vec{a}(k))^{\top}\vec{\xi}(-k), \tag{9}$$

which depends on unobserved values of a stochastic sequence $\vec{\xi}(k)=\{\xi_p(k)\}_{p=1}^{T}$ with stationary GM increments. Estimates are based on observations of the sequence $\vec{\zeta}(k)=\vec{\xi}(k)+\vec{\eta}(k)$ at points $k=0,-1,-2,\ldots$.

We suppose that the conditions on coefficients $\vec{a}(k)=\{a_p(k)\}_{p=1}^{T}$, $k\geq 0$

$$\sum_{k=0}^{\infty}\|\vec{a}(k)\|<\infty,\quad \sum_{k=0}^{\infty}(k+1)\|\vec{a}(k)\|^2<\infty, \tag{10}$$

and *the minimality condition* on the spectral densities $f(\lambda)$ and $g(\lambda)$

$$\int_{-\pi}^{\pi}\mathrm{Tr}\left[\frac{|\beta^{(d)}(i\lambda)|^2}{|\chi_{\overline{\mu}}^{(d)}(e^{-i\lambda})|^2}\Big(f(\lambda)+|\beta^{(d)}(i\lambda)|^2g(\lambda)\Big)^{-1}\right]d\lambda<\infty. \tag{11}$$

are satisfied. The second condition (11) is the necessary and sufficient one under which the mean square error of the optimal estimate of functional $A\vec{\xi}$ is not equal to 0.

Any linear estimate $\widehat{A}\vec{\xi}$ of the functional $A\vec{\xi}$ allows the representation [24]

$$\widehat{A}\vec{\xi}=\sum_{k=0}^{\infty}(\vec{a}(k))^{\top}(\vec{\xi}(-k)+\vec{\eta}(-k))-\int_{-\pi}^{\pi}(\vec{h}_{\overline{\mu}}(\lambda))^{\top}d\vec{Z}_{\xi^{(d)}+\eta^{(d)}}(\lambda), \tag{12}$$

where $\vec{h}_{\overline{\mu}}(\lambda)=\{h_p(\lambda)\}_{p=1}^{T}$ is the spectral characteristic of the estimate $\widehat{A}\vec{\eta}$.

In the Hilbert space $L_2(p)$, define a subspace

$$L_2^0(p)=\overline{span}\{e^{i\lambda k}\chi_{\overline{\mu}}^{(d)}(e^{-i\lambda})(\beta^{(d)}(i\lambda))^{-1}\vec{\delta}_l:\ \vec{\delta}_l=\{\delta_{lp}\}_{p=1}^{T},\ l=1,\ldots,T,\ k\leq 0\}.$$

Define the following matrix-valued Fourier coefficients:

$$\begin{aligned}
S_{\overline{\mu}}(k) &= \frac{1}{2\pi}\int_{-\pi}^{\pi}e^{-i\lambda k}\frac{|\beta^{(d)}(i\lambda)|^2}{|\chi_{\overline{\mu}}^{(d)}(e^{-i\lambda})|^2}\Big(g(\lambda)(f(\lambda)+|\beta^{(d)}(i\lambda)|^2g(\lambda))^{-1}\Big)^{\top}d\lambda,\quad k\in\mathbb{Z},\\
P_{\overline{\mu}}(k) &= \frac{1}{2\pi}\int_{-\pi}^{\pi}e^{-i\lambda k}\frac{|\beta^{(d)}(i\lambda)|^2}{|\chi_{\overline{\mu}}^{(d)}(e^{-i\lambda})|^2}\Big((f(\lambda)+|\beta^{(d)}(i\lambda)|^2g(\lambda))^{-1}\Big)^{\top}d\lambda,\quad k\in\mathbb{Z},\\
Q(k) &= \frac{1}{2\pi}\int_{-\pi}^{\pi}e^{-i\lambda k}\Big(f(\lambda)(f(\lambda)+|\beta^{(d)}(i\lambda)|^2g(\lambda))^{-1}g(\lambda)\Big)^{\top}d\lambda,\quad k\in\mathbb{Z}.
\end{aligned}$$

Define the vectors $\mathbf{a}=((\vec{a}(0))^{\top},(\vec{a}(1))^{\top},(\vec{a}(2))^{\top},\ldots)^{\top}$ and $\mathbf{a}_{\overline{\mu}}=((\vec{a}_{\overline{\mu}}(0))^{\top},(\vec{a}_{\overline{\mu}}(1))^{\top},(\vec{a}_{\overline{\mu}}(2))^{\top},\ldots)^{\top}$, where the coefficients $\vec{a}_{\overline{\mu}}(k)=\vec{a}_{-\overline{\mu}}(k-n(\gamma))$, $k\geq 0$,

$$\vec{a}_{-\overline{\mu}}(m)=\sum_{l=\max\{m,0\}}^{m+n(\gamma)}e_{\gamma}(l-m)\vec{a}(l),\quad m\geq -n(\gamma). \tag{13}$$

Define the matrices $\mathbf{S}_{\overline{\mu}}$, $\mathbf{P}_{\overline{\mu}}$ and $\mathbf{Q}$ by the matrix-valued entries $(\mathbf{S}_{\overline{\mu}})_{l,k}=S_{\overline{\mu}}(l+1+k-n(\gamma))$, $(\mathbf{P}_{\overline{\mu}})_{l,k}=P_{\overline{\mu}}(l-k)$ and $(\mathbf{Q})_{l,k}=Q(l-k)$, $l,k\geq 0$.

The solution to the filtering problem is described by the following theorem in terms of Fourier coefficients $\{S_{\overline{\mu}}(k),P_{\overline{\mu}}(k),Q(k):k\in\mathbb{Z}\}$.

Theorem 2 ([24])
A solution $\widehat{A}\vec{\xi}$ to the filtering problem for the linear functional $A\vec{\xi}$ of the values of a vector-valued stochastic sequence $\vec{\xi}(m)$ with stationary GM increments under conditions (10) and (11) is calculated by formula (12). The spectral characteristic $\vec{h}_{\overline{\mu}}(\lambda)$ and the value of the mean square error $\Delta(f,g;\widehat{A}\vec{\xi})$ are calculated by the formulas

$$(\vec{h}_{\overline{\mu}}(\lambda))^{\top} = \left(\chi_{\overline{\mu}}^{(d)}(e^{i\lambda})(A(e^{-i\lambda}))^{\top}g(\lambda) - (C_{\overline{\mu}}(e^{i\lambda}))^{\top}\right)p^{-1}(\lambda)\frac{\overline{\beta^{(d)}(i\lambda)}}{\chi_{\overline{\mu}}^{(d)}(e^{i\lambda})}, \tag{14}$$

where

$$A(e^{-i\lambda}) = \sum_{k=0}^{\infty}\vec{a}(k)e^{-i\lambda k}, \quad C_{\overline{\mu}}(e^{i\lambda}) = \sum_{k=0}^{\infty}\left(\mathbf{P}_{\overline{\mu}}^{-1}\mathbf{S}_{\overline{\mu}}\mathbf{a}_{\overline{\mu}}\right)_k e^{i\lambda(k+1)},$$

and

$$\Delta\left(f,g;\widehat{A}\vec{\xi}\right) = \mathsf{E}\left|A\vec{\xi} - \widehat{A}\vec{\xi}\right|^2 = \left\langle\mathbf{S}_{\overline{\mu}}\mathbf{a}_{\overline{\mu}}, \mathbf{P}_{\overline{\mu}}^{-1}\mathbf{S}_{\overline{\mu}}\mathbf{a}_{\overline{\mu}}\right\rangle + \langle\mathbf{Q}\mathbf{a},\mathbf{a}\rangle. \tag{15}$$

Remark 2
The filtering problem in the presence of fractional integration is considered in [24].

3.2. Filtering based on factorizations of the spectral densities

The main goal of the article is to derive the classical and minimax estimates of the functional $A\vec{\xi}$ in terms of the coefficients of the canonical factorizations of the spectral densities $f(\lambda)$, $g(\lambda)$ and $f(\lambda)+|\beta^{(d)}(i\lambda)|^2 g(\lambda)$.

Let the following canonical factorizations take place

$$\frac{|\beta^{(d)}(i\lambda)|^2}{|\chi_{\overline{\mu}}^{(d)}(e^{-i\lambda})|^2}(f(\lambda)+|\beta^{(d)}(i\lambda)|^2 g(\lambda)) = \Theta_{\overline{\mu}}(e^{-i\lambda})\Theta_{\overline{\mu}}^{*}(e^{-i\lambda}), \quad \Theta_{\overline{\mu}}(e^{-i\lambda}) = \sum_{k=0}^{\infty}\theta_{\overline{\mu}}(k)e^{-i\lambda k}, \tag{16}$$

$$g(\lambda) = \sum_{k=-\infty}^{\infty}g(k)e^{i\lambda k} = \Phi(e^{-i\lambda})\Phi^{*}(e^{-i\lambda}), \quad \Phi(e^{-i\lambda}) = \sum_{k=0}^{\infty}\phi(k)e^{-i\lambda k}. \tag{17}$$

Define the matrix-valued function $\Psi_{\overline{\mu}}(e^{-i\lambda}) = \{\Psi_{\overline{\mu},ij}(e^{-i\lambda})\}_{i=\overline{1,q}}^{j=\overline{1,T}}$ by the equation

$$\Psi_{\overline{\mu}}(e^{-i\lambda})\Theta_{\overline{\mu}}(e^{-i\lambda}) = E_q,$$

where E_q is an identity $q\times q$ matrix. One can check that the following factorization takes place

$$\frac{|\beta^{(d)}(i\lambda)|^2}{|\chi_{\overline{\mu}}^{(d)}(e^{-i\lambda})|^2}(f(\lambda)+|\beta^{(d)}(i\lambda)|^2 g(\lambda))^{-1} = \Psi_{\overline{\mu}}^{*}(e^{-i\lambda})\Psi_{\overline{\mu}}(e^{-i\lambda}), \quad \Psi_{\overline{\mu}}(e^{-i\lambda}) = \sum_{k=0}^{\infty}\psi_{\overline{\mu}}(k)e^{-i\lambda k}, \tag{18}$$

Remark 3
Any spectral density matrix $f(\lambda)$ is self-adjoint: $f(\lambda)=f^{*}(\lambda)$. Thus, $(f(\lambda))^{\top}=\overline{f(\lambda)}$. One can check that the inverse spectral density $f^{-1}(\lambda)$ is also self-adjoint: $f^{-1}(\lambda)=(f^{-1}(\lambda))^{*}$ and $(f^{-1}(\lambda))^{\top}=\overline{f^{-1}(\lambda)}$.

The following Lemmas provide representations of $\mathbf{P}_{\overline{\mu}}$ and $\mathbf{P}_{\overline{\mu}}^{-1}\mathbf{S}_{\overline{\mu}}\mathbf{a}_{\overline{\mu}}$, which contain coefficients of factorizations (16) – (18).

Lemma 1
Let factorization (16) takes place and let $q\times T$ matrix function $\Psi_{\overline{\mu}}(e^{-i\lambda})$ satisfy the equation $\Psi_{\overline{\mu}}(e^{-i\lambda})\Phi_{\overline{\mu}}(e^{-i\lambda}) = E_q$. Define the linear operators $\Psi_{\overline{\mu}}$ and $\Theta_{\overline{\mu}}$ in the space ℓ_2 by the matrices with the matrix entries $(\Psi_{\overline{\mu}})_{k,j} = \psi_{\overline{\mu}}(k-j)$, $(\Theta_{\overline{\mu}})_{k,j} = \theta_{\overline{\mu}}(k-j)$ for $0\le j\le k$, $(\Psi_{\overline{\mu}})_{k,j}=0$, $(\Theta_{\overline{\mu}})_{k,j}=0$ for $0\le k<j$. Then:

a) the linear operator $\mathbf{P}_{\overline{\mu}}$ admits a factorization

$$\mathbf{P}_{\overline{\mu}} = (\Psi_{\overline{\mu}})^{\top}\overline{\Psi}_{\overline{\mu}};$$

b) the inverse operator $(\mathbf{P}_{\overline{\mu}})^{-1}$ admits a factorization

$$(\mathbf{P}_{\overline{\mu}})^{-1} = \overline{\Theta}_{\overline{\mu}}(\Theta_{\overline{\mu}})^{\top}.$$

Proof
See [25]. □

Lemma 2
Let factorizations (16) and (17) take place. Define by $\widetilde{e}_{\overline{\mu}}(m) = \left(\Theta_{\overline{\mu}}^{\top}\mathbf{S}_{\overline{\mu}}\widetilde{\mathbf{a}}_{\overline{\mu}}\right)_m$, $m \geq 0$, the mth element of the vector $\widetilde{\mathbf{e}}_{\overline{\mu}} = \Theta_{\overline{\mu}}^{\top}\mathbf{S}_{\overline{\mu}}\widetilde{\mathbf{a}}_{\overline{\mu}}$. Then

$$\widetilde{e}_{\overline{\mu}}(m) = \sum_{j=-n(\gamma)}^{\infty} Z_{\overline{\mu}}(m+j+1)\vec{a}_{-\mu}(j),$$

where $Z_{\overline{\mu}}(j)$, $j \in \mathbb{Z}$, are defined as

$$Z_{\overline{\mu}}(j) = \sum_{l=0}^{\infty}\overline{\psi}_{\overline{\mu}}(l)\overline{g}(l-j),\ j \in \mathbb{Z}, \quad g(k) = \sum_{m=\max\{0,-k\}}^{\infty}\phi(m)\phi^{*}(k+m),\ k \in \mathbb{Z}.$$

Proof
See Appendix. □

Define the linear operators $\mathbf{G}^{-}$, $\mathbf{G}^{+}$,$\widetilde{\Phi}^{+}$ in the space ℓ_2 by matrices with the matrix entries $(\mathbf{G}^{-})_{l,k} = \overline{g}(l-k)$, $(\mathbf{G}^{+})_{l,k} = \overline{g}(l+k)$, $(\widetilde{\Phi}^{+})_{l,k} = \phi^{\top}(k+j)$, $l,k \geq 0$. And the linear operator $\widetilde{\Phi}$ in the space ℓ_2 determined by a matrix with the matrix entries $(\widetilde{\Phi})_{k,j} = \phi^{\top}(k-j)$ for $0 \leq j \leq k$, $(\widetilde{\Phi})_{k,j} = 0$ for $0 \leq k < j$.

Define also the coefficients $\{\vec{b}_{-\overline{\mu}}(k) : k \geq 0\}$ as follows: $\vec{b}_{-\overline{\mu}}(0) = 0$, $\vec{b}_{-\overline{\mu}}(k) = \vec{a}_{-\overline{\mu}}(-k)$ for $1 \leq k \leq n(\gamma)$, $\vec{b}_{-\overline{\mu}}(k) = 0$ for $k > n(\gamma)$, where coefficients $\vec{a}_{-\overline{\mu}}(k)$ are calculated by formula (13), and the vectors

$$\mathbf{a}_{-\overline{\mu}} = ((\vec{a}_{-\overline{\mu}}(0))^{\top}, (\vec{a}_{-\overline{\mu}}(1))^{\top}, (\vec{a}_{-\overline{\mu}}(2))^{\top}, \ldots)^{\top}, \quad \mathbf{b}_{-\overline{\mu}} = ((\vec{b}_{-\overline{\mu}}(0))^{\top}, (\vec{b}_{-\overline{\mu}}(1))^{\top}, (\vec{b}_{-\overline{\mu}}(2))^{\top}, \ldots)^{\top}.$$

The following theorem describes a solution to the filtering problem in the case when the spectral densities $f(\lambda)$ and $g(\lambda)$ admit canonical factorizations (16) – (18) .

Theorem 3
Suppose that condition (10) is fulfilled and the spectral functions $F(\lambda)$ and $G(\lambda)$ of the stochastic sequences $\vec{\xi}(m)$ and $\vec{\eta}(m)$ have the spectral densities $f(\lambda)$ and $g(\lambda)$ admitting canonical factorizations (16) – (18). A solution $\widehat{A}\vec{\xi}$ to the filtering problem for the linear functional $A\vec{\xi}$ of the values of a vector-valued stochastic sequence $\vec{\xi}(m)$ with stationary GM increments is calculated by formula (12). The spectral characteristic $\vec{h}_{\overline{\mu}}(\lambda)$ is calculated by the formulas

$$\begin{aligned}\vec{h}_{\overline{\mu}}(\lambda) &= \frac{\chi_{\overline{\mu}}^{(d)}(e^{-i\lambda})}{\beta^{(d)}(i\lambda)}\left(\sum_{k=0}^{\infty}\psi_{\overline{\mu}}^{\top}(k)e^{-i\lambda k}\right)\sum_{m=0}^{\infty}((\widetilde{\Psi}_{\overline{\mu}})^{*}\mathbf{G}^{-}\mathbf{a}_{-\overline{\mu}} + (\widetilde{\Psi}_{\overline{\mu}})^{*}\mathbf{G}^{+}\mathbf{b}_{-\overline{\mu}})_m e^{-i\lambda m}\\ &= \frac{\chi_{\overline{\mu}}^{(d)}(e^{-i\lambda})}{\beta^{(d)}(i\lambda)}\left(\sum_{k=0}^{\infty}\psi_{\overline{\mu}}^{\top}(k)e^{-i\lambda k}\right)\sum_{m=0}^{\infty}(\overline{\psi}_{\overline{\mu}}\mathbf{C}_{\overline{\mu},g}^{-} + \overline{\psi}_{\overline{\mu}}\mathbf{C}_{\overline{\mu},g}^{+})_m e^{-i\lambda m}, \end{aligned} \tag{19}$$

where $\overline{\psi}_{\overline{\mu}} = (\overline{\psi}_{\overline{\mu}}(0), \overline{\psi}_{\overline{\mu}}(1), \overline{\psi}_{\overline{\mu}}(2), \ldots)$,

$$(\overline{\psi}_{\overline{\mu}}\mathbf{C}_{\overline{\mu},g}^{\pm})_m = \sum_{k=0}^{\infty}\overline{\psi}_{\overline{\mu}}(k)\mathbf{c}_{\overline{\mu},g}^{\pm}(k+m),$$

$$\mathbf{c}^{-}_{\overline{\mu},g}(m)=\sum_{k=0}^{\infty}\overline{g}(m-k)a_{-\overline{\mu}}(k)=\sum_{l=0}^{\infty}\overline{\phi}(l)\sum_{k=0}^{l+m}\phi^{\top}(l+m-k)a_{-\overline{\mu}}(k)=\sum_{l=0}^{\infty}\overline{\phi}(l)(\widetilde{\Phi}\mathbf{a}_{-\overline{\mu}})_{l+m}=(\widetilde{\Phi}^{*}\widetilde{\Phi}\mathbf{a}_{-\overline{\mu}})_{m},$$

$$\mathbf{c}^{+}_{\overline{\mu},g}(m)=\sum_{k=0}^{\infty}\overline{g}(m+k)b_{-\overline{\mu}}(k)=\sum_{l=0}^{\infty}\overline{\phi}(l)\sum_{k=0}^{\infty}\phi^{\top}(l+m+k)b_{-\overline{\mu}}(k)=\sum_{l=0}^{\infty}\overline{\phi}(l)(\widetilde{\Phi}^{+}\mathbf{b}_{-\overline{\mu}})_{l+m}=(\widetilde{\Phi}^{*}\widetilde{\Phi}^{+}\mathbf{b}_{-\overline{\mu}})_{m}.$$

The the value of the mean square error $\Delta(f,g;\widehat{A}\vec{\xi})$ is calculated by the formulas

$$\Delta\left(f,g;\widehat{A}\xi\right)=\|\widetilde{\Phi}\mathbf{a}\|^{2}-\|\overline{\psi}_{\overline{\mu}}(\mathbf{C}^{-}_{\overline{\mu},g}+\mathbf{C}^{+}_{\overline{\mu},g})\|^{2}. \tag{20}$$

Proof
See Appendix. □

Remark 4
The following factorizations hold true:

$$\mathbf{G}^{-}=\widetilde{\Phi}^{*}\widetilde{\Phi},\quad \mathbf{G}^{+}=\widetilde{\Phi}^{*}\widetilde{\Phi}^{+}.$$

The filtering problem for the functional $A_N\vec{\xi}$ is solved directly by Theorem 3 by putting $\vec{a}(k)=\vec{0}$ for $k>N$. To solve the filtering problem for the pth coordinate of the single vector $\vec{\xi}(-N)$, we put $\vec{a}(N)=\vec{\delta}_p$, $\vec{a}(k)=\vec{0}$ for $k\neq N$.

The following corollaries take place.

Corollary 1
Under conditions of Theorem 3, a solution $\widehat{A}_N\vec{\xi}$ to the filtering problem for the linear functional $A_N\vec{\xi}$ of the values of a vector-valued stochastic sequence $\vec{\xi}(m)$ with stationary GM increments is calculated by the formula

$$\widehat{A}_N\vec{\xi}=\sum_{k=0}^{N}(\vec{a}(k))^{\top}(\vec{\xi}(-k)+\vec{\eta}(-k))-\int_{-\pi}^{\pi}(\vec{h}_{\overline{\mu},N}(\lambda))^{\top}d\vec{Z}_{\xi^{(n)}+\eta^{(n)}}(\lambda). \tag{21}$$

The spectral characteristic $\vec{h}_{\overline{\mu},N}(\lambda)$ and the value of the mean square error $\Delta(f,g;\widehat{A}_N\vec{\xi})$ of the optimal estimate $\widehat{A}_N\vec{\xi}$ are calculated by formulas (19) and (20) for the vectors $\mathbf{a}$, $\mathbf{a}_{-\overline{\mu}}$, $\mathbf{b}_{-\overline{\mu}}$ calculated as

$$\mathbf{a}_N=((\vec{a}(0))^{\top},(\vec{a}(1))^{\top},(\vec{a}(2))^{\top},\dots,(\vec{a}(N))^{\top},0,\dots)^{\top},$$

$$\mathbf{a}_{-\overline{\mu},N}=((\vec{a}_{-\overline{\mu},N}(0))^{\top},(\vec{a}_{-\overline{\mu},N}(1))^{\top},...,(\vec{a}_{-\overline{\mu},N}(N))^{\top},0,\dots)^{\top},$$

$$\mathbf{b}_{-\overline{\mu},N}=(0,(\vec{a}_{-\overline{\mu},N}(-1))^{\top},(\vec{a}_{-\overline{\mu},N}(-2))^{\top},...,(\vec{a}_{-\overline{\mu},N}(-n(\gamma)))^{\top},0\dots)^{\top}$$

where

$$\vec{a}_{-\overline{\mu},N}(m)=\sum_{l=\max\{m,0\}}^{\min\{m+n(\gamma),N\}}e_{\gamma}(l-m)\vec{a}(l),\quad -n(\gamma)\leq m\leq N. \tag{22}$$

Corollary 2
Under conditions of Theorem 3, the optimal linear estimate $\widehat{\xi}_p(-N)$ of an unobserved value $\xi_p(-N)$, $N\geq 0$, of the stochastic vector sequence $\vec{\xi}(m)$ with GM stationary increments based on observations of the sequence $\vec{\xi}(m)+\vec{\eta}(m)$ at points $m=0,-1,-2,\dots$, where the noise sequence $\vec{\eta}(m)$ is uncorrelated with $\vec{\xi}(m)$, is calculated by the formula

$$\widehat{\xi}_p(-N)=(\xi_p(-N)+\eta_p(-N))-\int_{-\pi}^{\pi}(\vec{h}_{\overline{\mu},N,p}(\lambda))^{\top}d\vec{Z}_{\xi^{(d)}+\eta^{(d)}}(\lambda). \tag{23}$$

Put

$$\vec{a}_{-\overline{\mu},N,p}(m)=e_{\gamma}(N-m)\vec{\delta}_p, \quad N-n(\gamma)\leq m\leq N.$$

If $N<n(\gamma)$, the spectral characteristic $\vec{h}_{\overline{\mu},N,p}(\lambda)$ and the value of the mean square error $\Delta(f,g;\widehat{\xi}_p(-N))$ of the optimal estimate $\widehat{\xi}_p(-N)$ are calculated by formulas (19) and (20) for the vectors $\mathbf{a}$, $\mathbf{a}_{-\overline{\mu}}$, $\mathbf{b}_{-\overline{\mu}}$ calculated as

$$\mathbf{a}_{N,p}=(0,0,\dots,(\delta_p)^{\top},0,\dots)^{\top},$$

$$\mathbf{a}_{-\overline{\mu},N,p}=((\vec{a}_{-\overline{\mu},N,p}(0))^{\top},(\vec{a}_{-\overline{\mu},N,p}(1))^{\top},\dots,(\vec{a}_{-\overline{\mu},N,p}(N))^{\top},0,\dots)^{\top},$$

$$\mathbf{b}_{-\overline{\mu},N,p}=(0,(\vec{a}_{-\overline{\mu},N,p}(-1))^{\top},(\vec{a}_{-\overline{\mu},N,p}(-2))^{\top},\dots,(\vec{a}_{-\overline{\mu},N,p}(N-n(\gamma)))^{\top},0\dots)^{\top}.$$

If $N\geq n(\gamma)$, the spectral characteristic $\vec{h}_{\overline{\mu},N,p}(\lambda)$ and the value of the mean square error $\Delta(f,g;\widehat{\xi}_p(-N))$ of the optimal estimate $\widehat{\xi}_p(-N)$ are calculated by formulas

$$\vec{h}_{\overline{\mu},N,p}(\lambda)=\frac{\chi_{\overline{\mu}}^{(d)}(e^{-i\lambda})}{\beta^{(d)}(i\lambda)}\left(\sum_{k=0}^{\infty}\psi_{\overline{\mu}}^{\top}(k)e^{-i\lambda k}\right)\sum_{m=0}^{\infty}((\widetilde{\Psi}_{\overline{\mu}})^{*}\mathbf{G}^{-}\mathbf{a}_{-\overline{\mu},N,p})_m e^{-i\lambda m} \tag{24}$$

and

$$\Delta\left(f,g;\widehat{\xi}_p(-N)\right)=\left\langle\overline{g}(0)\vec{\delta}_p,\vec{\delta}_p\right\rangle-\|(\widetilde{\Psi}_{\overline{\mu}})^{*}\mathbf{G}^{-}\mathbf{a}_{-\overline{\mu},N,p}\|^2,$$

where

$$\mathbf{a}_{-\overline{\mu},N,p}=(0,\dots,0,(\vec{a}_{-\overline{\mu},N,p}(N-n(\gamma)))^{\top},\dots,(\vec{a}_{-\overline{\mu},N,p}(N))^{\top},0,\dots)^{\top}.$$

3.3. Filtering of stochastic sequences with periodically stationary GM increment

Consider the filtering problem for the functionals

$$A\xi=\sum_{k=0}^{\infty}a^{(\xi)}(k)\xi(-k), \quad A_M\xi=\sum_{k=0}^{N}a^{(\xi)}(k)\xi(-k) \tag{25}$$

which depend on the unobserved values of a stochastic sequence $\xi(m)$ with periodically stationary GM increments. Estimates are based on observations of the sequence $\xi(m)+\eta(m)$ at points $m=0,-1,-2,\dots$, where the periodically stationary noise sequence $\eta(m)$ is uncorrelated with $\xi(m)$.

The functional $A\xi$ can be represented in the form

$$\begin{aligned}A\xi &= \sum_{k=0}^{\infty}a^{(\xi)}(k)\xi(-k)=\sum_{m=0}^{\infty}\sum_{p=1}^{T}a^{(\xi)}(mT+p-1)\xi(-mT-p+1)\\ &= \sum_{m=0}^{\infty}\sum_{p=1}^{T}a_p(m)\xi_p(-m)=\sum_{m=0}^{\infty}(\vec{a}(m))^{\top}\vec{\xi}(-m)=A\vec{\xi},\end{aligned}$$

where

$$\vec{\xi}(m)=(\xi_1(m),\xi_2(m),\dots,\xi_T(m))^{\top},\ \xi_p(m)=\xi(mT+p-1);\ p=1,2,\dots,T; \tag{26}$$

and

$$\vec{a}(m)=(a_1(m),a_2(m),\dots,a_T(m))^{\top},\ a_p(m)=a^{(\xi)}(mT+p-1);\ p=1,2,\dots,T. \tag{27}$$

In the same way, the functional $A\eta$ is represented as

$$A\eta=\sum_{k=0}^{\infty}a^{(\xi)}(k)\eta(-k)=\sum_{m=0}^{\infty}(\vec{a}(m))^{\top}\vec{\eta}(-m)=A\vec{\eta},$$

where

$$\vec{\eta}(m) = (\eta_1(m), \eta_2(m), \dots, \eta_T(m))^{\top}, \; \eta_p(m) = \eta(mT+p-1); \; p = 1, 2, \dots, T. \quad (28)$$

Making use of the introduced notations and statements of Theorem 2 we can claim that the following theorem holds true.

Theorem 4
Let a stochastic sequence $\xi(m)$ with periodically stationary GM increments and a stochastic periodically stationary sequence $\eta(m)$ generate by formulas (26) and (28) vector-valued stochastic sequences $\vec{\xi}(m)$ and $\vec{\eta}(m)$ with the spectral functions $F(\lambda)$ and $G(\lambda)$, which has the spectral densities matrices $f(\lambda) = \{f_{ij}(\lambda)\}_{i,j=1}^{T}$ and $g(\lambda) = \{g_{ij}(\lambda)\}_{i,j=1}^{T}$ admitting the canonical factorizations (16) – (18). A solution $\widehat{A}\xi$ of the filtering problem for the functional $A\xi = A\vec{\xi}$ under conditions (10) is calculated by formula (12) for the coefficients $\vec{a}(m)$, $m \geq 0$, defined in (27). The spectral characteristic $\vec{h}_{\overline{\mu}}(\lambda) = \{h_p(\lambda)\}_{p=1}^{T}$ and the value of the mean square error $\Delta(f, g; \widehat{A}\xi)$ of the estimate $\widehat{A}\xi$ are calculated by formulas (19) and (20) respectively.

The functional $A_M\xi$ can be represented in the form

$$\begin{aligned} A_M\xi &= \sum_{k=0}^{M} a^{(\xi)}(k)\zeta(-k) = \sum_{m=0}^{N}\sum_{p=1}^{T} a^{(\xi)}(mT+p-1)\xi(-mT-p+1) \\ &= \sum_{m=0}^{N}\sum_{p=1}^{T} a_p(m)\xi_p(-m) = \sum_{m=0}^{N} (\vec{a}(m))^{\top}\vec{\xi}(-m) = A_N\vec{\xi}, \end{aligned}$$

where $N = [\frac{M}{T}]$, the sequence $\vec{\xi}(m)$ is determined by formula (26),

$$\begin{aligned} (\vec{a}(m))^{\top} &= (a_1(m), a_2(m), \dots, a_T(m))^{\top}, \\ a_p(m) &= a^{\zeta}(mT+p-1); \; 0 \leq m \leq N; 1 \leq p \leq T; \; mT+p-1 \leq M; \\ a_p(N) &= 0; \quad M+1 \leq NT+p-1 \leq (N+1)T-1; 1 \leq p \leq T. \end{aligned} \quad (29)$$

An estimate of a single unobserved value $\xi(-M)$, $M \geq 0$ of a stochastic sequence $\xi(m)$ with periodically stationary GM increments is obtained by making use of the notations $\xi(-M) = \xi_p(-N) = (\vec{\delta}_p)^{\top}\vec{\xi}(N)$, $N = [\frac{M}{T}]$, $p = M+1-NT$. We can conclude that the following corollaries hold true.

Corollary 3
Let a stochastic sequence $\xi(m)$ with periodically stationary GM increments and a stochastic periodically stationary sequence $\eta(m)$ generate by formulas (26) and (28) vector-valued stochastic sequences $\vec{\xi}(m)$ and $\vec{\eta}(m)$. A solution $\widehat{A}_M\xi$ to the filtering problem for the functional $A_M\xi = A_N\vec{\xi}$ under condition (11) is calculated by formula (21) for the coefficients $\vec{a}(m)$, $0 \leq m \leq N$, defined in (29). The spectral characteristic and the value of the mean square error of the estimate $\widehat{A}_M\xi$ are calculated by formulas (19) and (20) respectively.

Corollary 4
Let a stochastic sequence $\xi(m)$ with periodically stationary GM increments and a stochastic periodically stationary sequence $\eta(m)$ generate by formulas (26) and (28) vector-valued stochastic sequences $\vec{\xi}(m)$ and $\vec{\eta}(m)$. A solution $\widehat{\xi}(-M)$ to the filtering problem for an unobserved value $\xi(-M) = \xi_p(-N) = (\vec{\delta}_p)^{\top}\vec{\xi}(-N)$, $N = [\frac{M}{T}]$, $p = M+1-NT$, under condition (11) is calculated by formula (23). The spectral characteristic and the value of the mean square error of the estimate $\widehat{\xi}(-M)$ are calculated by formulas (19) and (20) or (24) and (25) respectively.

4. Minimax (robust) method of filtering

Solutions of the problem of estimating the functionals $A\vec{\xi}$ and $A_N\vec{\xi}$ constructed from unobserved values of the stochastic sequence $\vec{\xi}(m)$ with stationary GM increments $\chi_{\overline{\mu},\overline{s}}^{(d)}(\vec{\xi}(m))$ having the spectral density matrix $f(\lambda)$ based on its observations with stationary noise $\vec{\xi}(m)+\vec{\eta}(m)$ at points $m=0,-1,-2,\dots$ are proposed in Theorem 3 and Corollary 1 in the case where the spectral density matrices $f(\lambda)$ and $g(\lambda)$ of the basic sequence and the noise are exactly known.

In this section, we study the case where the complete information about the spectral density matrices is not available, while some sets of admissible spectral densities $\mathcal{D}=\mathcal{D}_f\times\mathcal{D}_g$ is known. The minimax approach of estimation of the functionals from unobserved values of stochastic sequences is considered, which consists in finding an estimate that minimizes the maximal values of the mean square errors for all spectral densities from a class $\mathcal{D}$ simultaneously. This method will be applied for the concrete classes of spectral densities.

The proceed with the stated problem, we recall the following definitions [26].

Definition 4
For a given class of spectral densities $\mathcal{D}=\mathcal{D}_f\times\mathcal{D}_g$, the spectral densities $f^0(\lambda)\in\mathcal{D}_f$, $g^0(\lambda)\in\mathcal{D}_g$ are called the least favourable densities in the class $\mathcal{D}$ for optimal linear filtering of the functional $A\xi$ if the following relation holds true

$$\Delta(f^0,g^0)=\Delta(h(f^0,g^0);f^0,g^0)=\max_{(f,g)\in\mathcal{D}_f\times\mathcal{D}_g}\Delta(h(f,g);f,g).$$

Definition 5
For a given class of spectral densities $\mathcal{D}=\mathcal{D}_f\times\mathcal{D}_g$ the spectral characteristic $\vec{h}^0(\lambda)$ of the optimal estimate of the functional $A\xi$ is called minimax (robust) if the following relations hold true

$$\begin{aligned}\vec{h}^0(\lambda)\in H_{\mathcal{D}} &= \bigcap_{(f,g)\in\mathcal{D}_f\times\mathcal{D}_g} L_2^0(p),\\ \min_{\vec{h}\in H_{\mathcal{D}}}\max_{(f,g)\in\mathcal{D}_f\times\mathcal{D}_g}\Delta(\vec{h};f,g) &= \max_{(f,g)\in\mathcal{D}_f\times\mathcal{D}_g}\Delta(\vec{h}^0;f,g).\end{aligned}$$

Taking into account the introduced definitions and the relations derived in the previous sections we can verify that the following lemmas hold true.

Lemma 3
The spectral densities $f^0\in\mathcal{D}_f$, $g^0\in\mathcal{D}_g$ which admit the canonical factorizations (8), (16) and (17) are least favourable densities in the class $\mathcal{D}$ for the optimal linear filtering of the functional $A\vec{\xi}$ based on observations of the sequence $\vec{\xi}(m)+\vec{\eta}(m)$ at points $m=0,-1,-2,\dots$ if the matrix coefficients of the canonical factorizations (16) and (17) determine a solution of the constrained optimization problem

$$\|\widetilde{\Phi}\mathbf{a}\|^2-\|\overline{\psi}_{\overline{\mu}}(\mathbf{C}^-_{\overline{\mu},g}+\mathbf{C}^+_{\overline{\mu},g})\|^2\to\sup, \tag{30}$$

$$\begin{aligned}f(\lambda) &= \frac{|\beta^{(d)}(i\lambda)|^2}{|\chi_{\overline{\mu}}^{(d)}(e^{-i\lambda})|^2}\Theta_{\overline{\mu}}(e^{-i\lambda})\Theta^*_{\overline{\mu}}(e^{-i\lambda})-|\beta^{(d)}(i\lambda)|^2\Phi(e^{-i\lambda})\Phi^*(e^{-i\lambda})\in\mathcal{D}_f,\\ g(\lambda) &= \Phi(e^{-i\lambda})\Phi^*(e^{-i\lambda})\in\mathcal{D}_g.\end{aligned}$$

The minimax spectral characteristic $\vec{h}^0=\vec{h}_{\overline{\mu}}(f^0,g^0)$ is calculated by formula (19) if $\vec{h}_{\overline{\mu}}(f^0,g^0)\in H_{\mathcal{D}}$.

Lemma 4
The spectral density $g^0\in\mathcal{D}_g$ which admits the canonical factorizations (16), (17) with the known spectral density $f(\lambda)$ is the least favourable in the class $\mathcal{D}_g$ for the optimal linear filtering of the functional $A\xi$ based on observations of the sequence $\vec{\xi}(m)+\vec{\eta}(m)$ at points $m=0,-1,-2,\dots$ if the matrix coefficients of the canonical factorizations

$$\begin{aligned}f(\lambda)+|\beta^{(d)}(i\lambda)|^2g^0(\lambda) &= \frac{|\beta^{(d)}(i\lambda)|^2}{|\chi_{\overline{\mu}}^{(d)}(e^{-i\lambda})|^2}\left(\sum_{k=0}^{\infty}\theta^0_{\overline{\mu}}(k)e^{-i\lambda k}\right)\left(\sum_{k=0}^{\infty}\theta^0_{\overline{\mu}}(k)e^{-i\lambda k}\right)^*,\\ g^0(\lambda) &= \left(\sum_{k=0}^{\infty}\phi^0(k)e^{-i\lambda k}\right)\left(\sum_{k=0}^{\infty}\phi^0(k)e^{-i\lambda k}\right)^*\end{aligned}$$

are determined by the equation $\Psi^0_{\overline{\mu}}(e^{-i\lambda})\Theta^0_{\overline{\mu}}(e^{-i\lambda})=E_q$ and a solution $\{\psi^0_{\overline{\mu}}(k),\phi^0(k):k\geq 0\}$ of the constrained optimization problem

$$\|\widetilde{\Phi}\mathbf{a}\|^2-\|\overline{\psi}_{\overline{\mu}}(\mathbf{C}^-_{\overline{\mu},g}+\mathbf{C}^+_{\overline{\mu},g})\|^2\to\sup, \tag{31}$$
$$g(\lambda)=\Phi(e^{-i\lambda})\Phi^*(e^{-i\lambda})\in\mathcal{D}_g.$$

The minimax spectral characteristic $\vec{h}^0=\vec{h}_{\overline{\mu}}(f,g^0)$ is calculated by formula (19) if $\vec{h}_{\overline{\mu}}(f,g^0)\in H_{\mathcal{D}}$.

Lemma 5
The spectral density $f^0\in\mathcal{D}_f$ which admits the canonical factorizations (8), (16) with the known spectral density $g(\lambda)$ is the least favourable spectral density in the class $\mathcal{D}_f$ for the optimal linear filtering of the functional $A\vec{\xi}$ based on observations of the sequence $\vec{\xi}(m)+\vec{\eta}(m)$ at points $m=0,-1,-2,\dots$ if matrix coefficients of the canonical factorization

$$f^0(\lambda)+|\beta^{(d)}(i\lambda)|^2g(\lambda)=\frac{|\beta^{(d)}(i\lambda)|^2}{|\chi^{(d)}_{\overline{\mu}}(e^{-i\lambda})|^2}\left(\sum_{k=0}^{\infty}\theta^0_{\overline{\mu}}(k)e^{-i\lambda k}\right)\left(\sum_{k=0}^{\infty}\theta^0_{\overline{\mu}}(k)e^{-i\lambda k}\right)^*$$

are determined by the equation $\Psi^0_{\overline{\mu}}(e^{-i\lambda})\Theta^0_{\overline{\mu}}(e^{-i\lambda})=E_q$ and a solution $\{\psi^0_{\overline{\mu}}(k):k\geq 0\}$ of the constrained optimization problem

$$\|\overline{\psi}_{\overline{\mu}}(\mathbf{C}^-_{\overline{\mu},g}+\mathbf{C}^+_{\overline{\mu},g})\|^2\to\inf, \tag{32}$$

$$f(\lambda)\;=\;\frac{|\beta^{(d)}(i\lambda)|^2}{|\chi^{(d)}_{\overline{\mu}}(e^{-i\lambda})|^2}\Theta_{\overline{\mu}}(e^{-i\lambda})\Theta^*_{\overline{\mu}}(e^{-i\lambda})-|\beta^{(d)}(i\lambda)|^2\Phi(e^{-i\lambda})\Phi^*(e^{-i\lambda})\in\mathcal{D}_f$$

for the fixed matrix coefficients $\{\phi(k):k\geq 0\}$. The minimax spectral characteristic $\vec{h}^0=\vec{h}_{\overline{\mu}}(f^0,g)$ is calculated by formula (19) if $\vec{h}_{\overline{\mu}}(f^0,g)\in H_{\mathcal{D}}$.

The more detailed analysis of properties of the least favorable spectral densities and the minimax-robust spectral characteristics shows that the minimax spectral characteristic h^0 and the least favourable spectral densities f^0 and g^0 form a saddle point of the function $\Delta(h;f,g)$ on the set $H_{\mathcal{D}}\times\mathcal{D}$. The saddle point inequalities

$$\Delta(h;f^0,g^0)\geq\Delta(h^0;f^0,g^0)\geq\Delta(h^0;f,g)\quad\forall(f,g)\in\mathcal{D},\forall h\in H_{\mathcal{D}}$$

hold true if $h^0=\vec{h}_{\overline{\mu}}(f^0,g^0)$, $\vec{h}_{\overline{\mu}}(f^0,g^0)\in H_{\mathcal{D}}$ and (f^0,g^0) is a solution of the constrained optimization problem

$$\widetilde{\Delta}(f,g)=-\Delta(\vec{h}_{\overline{\mu}}(f^0,g^0);f,g)\to\inf,\quad(f,g)\in\mathcal{D}, \tag{33}$$

where the functional $\Delta(\vec{h}_{\overline{\mu}}(f^0,g^0);f,g)$ is calculated by the formula

$$\Delta\left(\vec{h}_{\overline{\mu}}(f^0,g^0);f,g\right)\;=\;\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{|\chi^{(d)}_{\overline{\mu}}(e^{-i\lambda})|^2}{|\beta^{(d)}(i\lambda)|^2}(\mathbf{h}^0_{\overline{\mu},f}(e^{-i\lambda}))^{\top}\Psi^0_{\overline{\mu}}(e^{-i\lambda})f(\lambda)(\Psi^0_{\overline{\mu}}(e^{-i\lambda}))^*\overline{\mathbf{h}^0_{\overline{\mu},f}(e^{-i\lambda})}d\lambda$$
$$+\frac{1}{2\pi}\int_{-\pi}^{\pi}(\mathbf{h}^0_{\overline{\mu},g}(e^{-i\lambda}))^{\top}\Psi^0_{\overline{\mu}}(e^{-i\lambda})g(\lambda)(\Psi^0_{\overline{\mu}}(e^{-i\lambda}))^*\overline{\mathbf{h}^0_{\overline{\mu},g}(e^{-i\lambda})}d\lambda,$$

where

$$\mathbf{h}^0_{\overline{\mu},f}(e^{-i\lambda})\;=\;\sum_{m=0}^{\infty}(\overline{\psi}^0_{\overline{\mu}}(\mathbf{C}^-_{\overline{\mu},g})^0+\overline{\psi}^0_{\overline{\mu}}(\mathbf{C}^+_{\overline{\mu},g})^0)_me^{-i\lambda m},$$
$$\mathbf{h}^0_{\overline{\mu},g}(e^{-i\lambda})\;=\;\left(\sum_{k=0}^{\infty}(\theta^0_{\overline{\mu}}(k))^{\top}e^{-i\lambda k}\right)\left(\sum_{k=0}^{\infty}a(k)e^{-i\lambda k}\right)$$
$$-\chi^{(d)}_{\overline{\mu}}(e^{-i\lambda})\left(\sum_{m=0}^{\infty}(\overline{\psi}^0_{\overline{\mu}}(\mathbf{C}^-_{\overline{\mu},g})^0+\overline{\psi}^0_{\overline{\mu}}(\mathbf{C}^+_{\overline{\mu},g})^0)_me^{-i\lambda m}\right).$$

The constrained optimization problem (33) is equivalent to the unconstrained optimization problem

$$\Delta_{\mathcal{D}}(f,g)=\widetilde{\Delta}(f,g)+\delta(f,g|\mathcal{D})\to\inf, \tag{34}$$

where $\delta(f,g|\mathcal{D})$ is the indicator function of the set $\mathcal{D}$, namely $\delta(f,g|\mathcal{D})=0$ if $(f,g)\in\mathcal{D}$ and $\delta(f|\mathcal{D})=+\infty$ if $(f,g)\notin\mathcal{D}$. The condition $0\in\partial\Delta_{\mathcal{D}}(f^0,g^0)$ characterizes a solution (f^0,g^0) of the stated unconstrained optimization problem. This condition is the necessary and sufficient condition that the point (f^0,g^0) belongs to the set of minimums of the convex functional $\Delta_{\mathcal{D}}(f,g)$ [27, 34]. Thus, it allows us to find the equalities for the least favourable spectral densities in some special classes of spectral densities $\mathcal{D}$.

The form of the functional $\Delta(\vec{h}_{\overline{\mu}}(f^0,g^0);f,g)$ is suitable for application of the Lagrange method of indefinite multipliers to the constrained optimization problem (33). Thus, the complexity of the problem is reduced to finding the subdifferential of the indicator function of the set of admissible spectral densities. We illustrate the solving of the problem (34) for concrete sets admissible spectral densities in the following subsections. A semi-uncertain filtering problem, when the spectral density $f(\lambda)$ is known and the spectral density $g(\lambda)$ belongs to in class $\mathcal{D}_g$, is considered as well.

4.1. Least favorable spectral density in classes $\mathcal{D}_0\times\mathcal{D}_{1\delta}$

Consider the minimax filtering problem for the functional $A\vec{\xi}$ for sets of admissible spectral densities $\mathcal{D}_0^k$, $k=1,2,3,4$ of the sequence with GM increments $\vec{\xi}(m)$

$$\begin{aligned}
\mathcal{D}_{f0}^1 &= \left\{f(\lambda)\left|\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{|\chi_{\overline{\mu}}^{(d)}(e^{-i\lambda})|^2}{|\beta^{(d)}(i\lambda)|^2}f(\lambda)d\lambda=P\right.\right\},\\
\mathcal{D}_{f0}^2 &= \left\{f(\lambda)\left|\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{|\chi_{\overline{\mu}}^{(d)}(e^{-i\lambda})|^2}{|\beta^{(d)}(i\lambda)|^2}\mathrm{Tr}\,[f(\lambda)]d\lambda=p\right.\right\},\\
\mathcal{D}_{f0}^3 &= \left\{f(\lambda)\left|\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{|\chi_{\overline{\mu}}^{(d)}(e^{-i\lambda})|^2}{|\beta^{(d)}(i\lambda)|^2}f_{kk}(\lambda)d\lambda=p_k,k=\overline{1,T}\right.\right\},\\
\mathcal{D}_{f0}^4 &= \left\{f(\lambda)\left|\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{|\chi_{\overline{\mu}}^{(d)}(e^{-i\lambda})|^2}{|\beta^{(d)}(i\lambda)|^2}\left\langle B_1,f(\lambda)\right\rangle d\lambda=p\right.\right\},
\end{aligned}$$

where $p,p_k,k=\overline{1,T}$ are given numbers, P,B_1 are given positive-definite Hermitian matrices, and sets of admissible spectral densities $\mathcal{D}_V^U$, $k=1,2,3,4$ for the stationary noise sequence $\vec{\eta}(m)$

$$\begin{aligned}
\mathcal{D}_{g1\delta}^1 &= \left\{g(\lambda)\left|\frac{1}{2\pi}\int_{-\pi}^{\pi}\left|g_{ij}(\lambda)-g_{ij}^1(\lambda)\right|d\lambda\le\delta_i^j,i,j=\overline{1,T}\right.\right\},\\
\mathcal{D}_{g1\delta}^2 &= \left\{g(\lambda)\left|\frac{1}{2\pi}\int_{-\pi}^{\pi}\left|\mathrm{Tr}(g(\lambda)-g_1(\lambda))\right|d\lambda\le\delta\right.\right\},\\
\mathcal{D}_{g1\delta}^3 &= \left\{g(\lambda)\left|\frac{1}{2\pi}\int_{-\pi}^{\pi}\left|g_{kk}(\lambda)-g_{kk}^1(\lambda)\right|d\lambda\le\delta_k,k=\overline{1,T}\right.\right\},\\
\mathcal{D}_{g1\delta}^4 &= \left\{g(\lambda)\left|\frac{1}{2\pi}\int_{-\pi}^{\pi}\left|\langle B_2,g(\lambda)-g_1(\lambda)\rangle\right|d\lambda\le\delta\right.\right\},
\end{aligned}$$

where $g_1(\lambda)=\{g_{ij}^1(\lambda)\}_{i,j=1}^T$ is a fixed spectral density, B_2 is a given positive-definite Hermitian matrix, $\delta,\delta_k,k=\overline{1,T}$, $\delta_i^j,i,j=\overline{1,T}$, are given numbers.

The condition $0\in\partial\Delta_{\mathcal{D}}(f^0,g^0)$ implies the following equations which determine the least favourable spectral densities for these given sets of admissible spectral densities.

For the first set of admissible spectral densities $\mathcal{D}_{f0}^1 \times \mathcal{D}_{g1\delta}^1$:

$$\left(\mathbf{h}_{\overline{\mu},f}^0(e^{i\lambda})\right)\left(\mathbf{h}_{\overline{\mu},f}^0(e^{i\lambda})\right)^* = (\Theta_{\overline{\mu}}^0(e^{-i\lambda}))^\top \vec{\alpha}_f \cdot \vec{\alpha}_f^* \overline{\Theta_{\overline{\mu}}^0(e^{-i\lambda})}, \tag{35}$$

$$\left(\mathbf{h}_{\overline{\mu},g}^0(e^{i\lambda})\right)\left(\mathbf{h}_{\overline{\mu},g}^0(e^{i\lambda})\right)^* = (\Theta_{\overline{\mu}}^0(e^{-i\lambda}))^\top \left\{\beta_{ij}\gamma_{ij}(\lambda)\right\}_{i,j=1}^T \overline{\Theta_{\overline{\mu}}^0(e^{-i\lambda})}, \tag{36}$$

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\left|g_{ij}^0(\lambda)-g_{ij}^1(\lambda)\right| d\lambda = \delta_i^j, \tag{37}$$

where $\vec{\alpha}_f$, β_{ij} are Lagrange multipliers, functions $|\gamma_{ij}(\lambda)| \leq 1$ and

$$\gamma_{ij}(\lambda) = \frac{g_{ij}^0(\lambda)-g_{ij}^1(\lambda)}{\left|g_{ij}^0(\lambda)-g_{ij}^1(\lambda)\right|} : \; g_{ij}^0(\lambda)-g_{ij}^1(\lambda) \neq 0, \; i,j=\overline{1,T}.$$

For the second set of admissible spectral densities $\mathcal{D}_{f0}^2 \times \mathcal{D}_{g1\delta}^2$ we have equations

$$\left(\mathbf{h}_{\overline{\mu},f}^0(e^{i\lambda})\right)\left(\mathbf{h}_{\overline{\mu},f}^0(e^{i\lambda})\right)^* = \alpha_f^2 (\Theta_{\overline{\mu}}^0(e^{-i\lambda}))^\top \overline{\Theta_{\overline{\mu}}^0(e^{-i\lambda})}, \tag{38}$$

$$\left(\mathbf{h}_{\overline{\mu},g}^0(e^{i\lambda})\right)\left(\mathbf{h}_{\overline{\mu},g}^0(e^{i\lambda})\right)^* = \beta^2\gamma_2(\lambda) (\Theta_{\overline{\mu}}^0(e^{-i\lambda}))^\top \overline{\Theta_{\overline{\mu}}^0(e^{-i\lambda})}, \tag{39}$$

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\left|\mathrm{Tr}\,(g^0(\lambda)-g_1(\lambda))\right| d\lambda = \delta, \tag{40}$$

where α_f^2, β^2 are Lagrange multipliers, function $|\gamma_2(\lambda)| \leq 1$ and

$$\gamma_2(\lambda) = \mathrm{sign}\,\left(\mathrm{Tr}\,(g^0(\lambda)-g_1(\lambda))\right) : \; \mathrm{Tr}\,(g^0(\lambda)-g_1(\lambda)) \neq 0.$$

For the third set of admissible spectral densities $\mathcal{D}_{f0}^3 \times \mathcal{D}_{g1\delta}^3$ we have equations

$$\left(\mathbf{h}_{\overline{\mu},f}^0(e^{i\lambda})\right)\left(\mathbf{h}_{\overline{\mu},f}^0(e^{i\lambda})\right)^* = (\Theta_{\overline{\mu}}^0(e^{-i\lambda}))^\top \left\{\alpha_{fk}^2\delta_{kl}\right\}_{k,l=1}^T \overline{\Theta_{\overline{\mu}}^0(e^{-i\lambda})}, \tag{41}$$

$$\left(\mathbf{h}_{\overline{\mu},g}^0(e^{i\lambda})\right)\left(\mathbf{h}_{\overline{\mu},g}^0(e^{i\lambda})\right)^* = (\Theta_{\overline{\mu}}^0(e^{-i\lambda}))^\top \left\{\beta_k^2\gamma_k^2(\lambda)\delta_{kl}\right\}_{k,l=1}^T \overline{\Theta_{\overline{\mu}}^0(e^{-i\lambda})}, \tag{42}$$

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\left|g_{kk}^0(\lambda)-g_{kk}^1(\lambda)\right| d\lambda = \delta_k, \tag{43}$$

where α_{fk}^2, β_k^2 are Lagrange multipliers, δ_{kl} are Kronecker symbols, functions $\left|\gamma_k^2(\lambda)\right| \leq 1$ and

$$\gamma_k^2(\lambda) = \mathrm{sign}\,(g_{kk}^0(\lambda)-g_{kk}^1(\lambda)) : \; g_{kk}^0(\lambda)-g_{kk}^1(\lambda) \neq 0, \; k=\overline{1,T}.$$

For the fourth set of admissible spectral densities $\mathcal{D}_{f0}^4 \times \mathcal{D}_{g1\delta}^4$ we have equations

$$\left(\mathbf{h}_{\overline{\mu},f}^0(e^{i\lambda})\right)\left(\mathbf{h}_{\overline{\mu},f}^0(e^{i\lambda})\right)^* = \alpha_f^2 (\Theta_{\overline{\mu}}^0(e^{-i\lambda}))^\top B_1 \overline{\Theta_{\overline{\mu}}^0(e^{-i\lambda})}, \tag{44}$$

$$\left(\mathbf{h}_{\overline{\mu},g}^0(e^{i\lambda})\right)\left(\mathbf{h}_{\overline{\mu},g}^0(e^{i\lambda})\right)^* = \beta^2\gamma_2'(\lambda) (\Theta_{\overline{\mu}}^0(e^{-i\lambda}))^\top B_2 \overline{\Theta_{\overline{\mu}}^0(e^{-i\lambda})}, \tag{45}$$

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\left|\left\langle B_2, g^0(\lambda)-g_1(\lambda)\right\rangle\right| d\lambda = \delta, \tag{46}$$

where α_f^2, β^2 are Lagrange multipliers, function $|\gamma_2'(\lambda)| \leq 1$ and

$$\gamma_2'(\lambda) = \mathrm{sign}\,\left\langle B_2, g^0(\lambda)-g_1(\lambda)\right\rangle : \; \left\langle B_2, g^0(\lambda)-g_1(\lambda)\right\rangle \neq 0.$$

The following theorem holds true.

Theorem 5
The least favorable spectral densities $f^0(\lambda)$, $g^0(\lambda)$ in the classes $\mathcal{D}_{f0}^k \times \mathcal{D}_{g1\delta}^k$, $k=1,2,3,4$ for the optimal linear filtering of the functional $A\vec{\xi}$ from observations of the sequence $\vec{\xi}(m)+\vec{\eta}(m)$ at points $m=0,-1,-2,\dots$ are determined by canonical factorizations (8), (16) and (17), equations (35)–(37), (38)–(40), (41)–(43), (44)–(46), respectively, constrained optimization problem (30) and restrictions on densities from the corresponding classes $\mathcal{D}_{f0}^k, \mathcal{D}_{g1\delta}^k, k=1,2,3,4$. The minimax-robust spectral characteristic of the optimal estimate of the functional $A\vec{\xi}$ is determined by the formula (19).

4.2. *Semi-uncertain filtering problem in classes $\mathcal{D}_\varepsilon$ of least favorable noise spectral density*

Consider a semi-uncertain filtering problem for the functional $A\vec{\xi}$, where the spectral density $g(\lambda)$ of the stationary noise sequence $\vec{\eta}(m)$ is known and the spectral density $f(\lambda)$ of the sequence with GM increments $\vec{\xi}(m)$ belongs to the sets of admissible spectral densities $\mathcal{D}_\varepsilon^k, k=1,2,3,4$

$$\mathcal{D}_\varepsilon^1 = \left\{ f(\lambda) \middle| f(\lambda) = (1-\varepsilon)f_1(\lambda)+\varepsilon W(\lambda), \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{|\chi_{\overline{\mu}}^{(d)}(e^{-i\lambda})|^2}{|\beta^{(d)}(i\lambda)|^2} f(\lambda)d\lambda = P \right\},$$

$$\mathcal{D}_\varepsilon^2 = \left\{ f(\lambda) \middle| \mathrm{Tr}\,[f(\lambda)] = (1-\varepsilon)\mathrm{Tr}\,[f_1(\lambda)]+\varepsilon \mathrm{Tr}\,[W(\lambda)], \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{|\chi_{\overline{\mu}}^{(d)}(e^{-i\lambda})|^2}{|\beta^{(d)}(i\lambda)|^2} \mathrm{Tr}\,[f(\lambda)]d\lambda = p \right\},$$

$$\mathcal{D}_\varepsilon^3 = \left\{ f(\lambda) \middle| f_{kk}(\lambda) = (1-\varepsilon)f_{kk}^1(\lambda)+\varepsilon w_{kk}(\lambda), \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{|\chi_{\overline{\mu}}^{(d)}(e^{-i\lambda})|^2}{|\beta^{(d)}(i\lambda)|^2} f_{kk}(\lambda)d\lambda = p_k, k=\overline{1,T} \right\},$$

$$\mathcal{D}_\varepsilon^4 = \left\{ f(\lambda) \middle| \langle B_1, f(\lambda)\rangle = (1-\varepsilon)\langle B_2, f_1(\lambda)\rangle+\varepsilon \langle B_2, W(\lambda)\rangle, \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{|\chi_{\overline{\mu}}^{(d)}(e^{-i\lambda})|^2}{|\beta^{(d)}(i\lambda)|^2} \langle B_2, f(\lambda)\rangle d\lambda = p \right\},$$

where $f_1(\lambda)$ is a fixed spectral density, $W(\lambda)$ is an unknown spectral density, $p, p_k, k=\overline{1,T}$, are given numbers, P, B_2 are given positive-definite Hermitian matrices.

The condition $0 \in \partial\Delta_{\mathcal{D}}(f^0, g)$ implies the equations which determine the least favourable spectral densities of the noise sequence $\vec{\xi}(m)$. Note that the elements $\mathbf{C}_{\overline{\mu},g}^-$ and $\mathbf{C}_{\overline{\mu},g}^+$ are known and determined by the coefficients $\{\phi(k), k\geq 0\}$ of the canonical factorization of the spectral density matrix $g(\lambda)$.

For the first set of admissible spectral density $\mathcal{D}_\varepsilon^1$ we have an equation

$$\left(\sum_{m=0}^{\infty}\left(\overline{\psi}_{\overline{\mu}}^0(\mathbf{C}_{\overline{\mu},g}^-+\mathbf{C}_{\overline{\mu},g}^+)\right)_m e^{-i\lambda m}\right)\left(\sum_{m=0}^{\infty}\left(\overline{\psi}_{\overline{\mu}}^0(\mathbf{C}_{\overline{\mu},g}^-+\mathbf{C}_{\overline{\mu},g}^+)\right)_m e^{-i\lambda m}\right)^*$$
$$=\left(\sum_{k=0}^{\infty}\theta_{\overline{\mu}}^0(k)e^{-i\lambda k}\right)^\top(\vec{\alpha}_f\cdot\vec{\alpha}_f^*+\Gamma(\lambda))\overline{\left(\sum_{k=0}^{\infty}\theta_{\overline{\mu}}^0(k)e^{-i\lambda k}\right)}, \quad (47)$$

where $\vec{\alpha}_f$ ia a vector of Lagrange multipliers, matrix $\Gamma(\lambda)\leq 0$ and $\Gamma(\lambda)=0$ if $f^0(\lambda)>(1-\varepsilon)f_1(\lambda)$.

For the second set of admissible spectral densities $\mathcal{D}_\varepsilon^2$ we have an equation

$$\left(\sum_{m=0}^{\infty}\left(\overline{\psi}_{\overline{\mu}}^0(\mathbf{C}_{\overline{\mu},g}^-+\mathbf{C}_{\overline{\mu},g}^+)\right)_m e^{-i\lambda m}\right)\left(\sum_{m=0}^{\infty}\left(\overline{\psi}_{\overline{\mu}}^0(\mathbf{C}_{\overline{\mu},g}^-+\mathbf{C}_{\overline{\mu},g}^+)\right)_m e^{-i\lambda m}\right)^*$$
$$=(\alpha_f^2+\gamma(\lambda))\left(\sum_{k=0}^{\infty}\theta_{\overline{\mu}}^0(k)e^{-i\lambda k}\right)^\top\overline{\left(\sum_{k=0}^{\infty}\theta_{\overline{\mu}}^0(k)e^{-i\lambda k}\right)}, \quad (48)$$

where α_f^2 is a Lagrange multiplier, function $\gamma(\lambda)\leq 0$ and $\gamma(\lambda)=0$ if $\mathrm{Tr}\,[f^0(\lambda)]>(1-\varepsilon)\mathrm{Tr}\,[f_1(\lambda)]$.

For the third set of admissible spectral densities $\mathcal{D}_{\varepsilon}^{3}$, we have an equation

$$\left(\sum_{m=0}^{\infty}\left(\overline{\psi}_{\overline{\mu}}^{0}(\mathbf{C}_{\overline{\mu},g}^{-}+\mathbf{C}_{\overline{\mu},g}^{+})\right)_{m}e^{-i\lambda m}\right)\left(\sum_{m=0}^{\infty}\left(\overline{\psi}_{\overline{\mu}}^{0}(\mathbf{C}_{\overline{\mu},g}^{-}+\mathbf{C}_{\overline{\mu},g}^{+})\right)_{m}e^{-i\lambda m}\right)^{*}$$
$$=\left(\sum_{k=0}^{\infty}\theta_{\overline{\mu}}^{0}(k)e^{-i\lambda k}\right)^{\top}\left\{(\alpha_{fk}^{2}+\gamma_{k}(\lambda))\delta_{kl}\right\}_{k,l=1}^{T}\overline{\left(\sum_{k=0}^{\infty}\theta_{\overline{\mu}}^{0}(k)e^{-i\lambda k}\right)}, \quad (49)$$

where α_{fk}^{2} are Lagrange multipliers, δ_{kl} are Kronecker symbols, functions $\gamma_k(\lambda)\leq 0$ and $\gamma_k(\lambda)=0$ if $f_{kk}^{0}(\lambda)>(1-\varepsilon)f_{kk}^{1}(\lambda)$.

For the fourth set of admissible spectral densities $\mathcal{D}_{\varepsilon}^{4}$, we have AN equation

$$\left(\sum_{m=0}^{\infty}\left(\overline{\psi}_{\overline{\mu}}^{0}(\mathbf{C}_{\overline{\mu},g}^{-}+\mathbf{C}_{\overline{\mu},g}^{+})\right)_{m}e^{-i\lambda m}\right)\left(\sum_{m=0}^{\infty}\left(\overline{\psi}_{\overline{\mu}}^{0}(\mathbf{C}_{\overline{\mu},g}^{-}+\mathbf{C}_{\overline{\mu},g}^{+})\right)_{m}e^{-i\lambda m}\right)^{*}$$
$$=(\alpha_{f}^{2}+\gamma'(\lambda))\left(\sum_{k=0}^{\infty}\theta_{\overline{\mu}}^{0}(k)e^{-i\lambda k}\right)^{\top}B_1\overline{\left(\sum_{k=0}^{\infty}\theta_{\overline{\mu}}^{0}(k)e^{-i\lambda k}\right)}, \quad (50)$$

where α_{f}^{2} is a Lagrange multiplier, function $\gamma'(\lambda)\leq 0$ and $\gamma'(\lambda)=0$ if $\langle B_1, f^0(\lambda)\rangle>(1-\varepsilon)\langle B_1, f_1(\lambda)\rangle$.

The following theorem holds true.

Theorem 6
Let the spectral density $g(\lambda)$ be known. The least favorable spectral density $f^0(\lambda)$ in the classes $\mathcal{D}_{\varepsilon}^{k}$, $k=1,2,3,4$ for the optimal linear foltering of the functional $A\vec{\xi}$ from observations of the sequence $\vec{\xi}(m)+\vec{\eta}(m)$ at points $m=0,-1,-2,\dots$ is determined by canonical factorizations (8) and (16), equations (47), (48), (49), (50), respectively, constrained optimization problem (32) and restrictions on density from the corresponding classes $\mathcal{D}_{\varepsilon}^{k}$, $k=1,2,3,4$. The minimax-robust spectral characteristic of the optimal estimate of the functional $A\vec{\xi}$ is determined by the formula (19).

5. Conclusions

In this article, we presented a solution of the filtering problem for stochastic sequences with periodically stationary GM increments, introduced in the article by Luz and Moklyachuk [23]. We proposed a solution of the filtering problem in terms of coefficients of canonical factorizations of the spectral densities of the involved stochastic sequences. The results obtained in [24] are based on the Fourier transformations of the spectral densities.

In the case where the spectral densities of sequences are not exactly known while the sets of admissible spectral densities are specified (spectral uncertainty), the minimax-robust approach to filtering problem was applied. We described the minimax (robust) estimates of the functionals and relations determining the least favourable spectral densities and the minimax spectral characteristics of the optimal estimates of linear functionals for a list of specific classes of admissible spectral densities.

Appendix

Proof of Lemma 2

Factorizations (17), (18) and Remark 3 imply

$$\begin{aligned}\sum_{k\in\mathbb{Z}}\vec{s}_{\overline{\mu}}(k)e^{i\lambda k} &= \frac{|\beta^{(d)}(i\lambda)|^2}{|\chi_{\overline{\mu}}^{(d)}(e^{-i\lambda})|^2}\left[g(\lambda)(f(\lambda)+|\beta^{(d)}(i\lambda)|^2g(\lambda))^{-1}\right]^{\top}=(\Psi_{\overline{\mu}}(e^{-i\lambda}))^{\top}\overline{\Psi}_{\overline{\mu}}(e^{-i\lambda})\overline{g}(\lambda)\\ &= \sum_{l=0}^{\infty}\psi_{\overline{\mu}}^{\top}(l)e^{-i\lambda l}\sum_{j\in\mathbb{Z}}Z_{\overline{\mu}}(j)e^{i\lambda j}=\sum_{k\in\mathbb{Z}}\sum_{l=0}^{\infty}\psi_{\overline{\mu}}^{\top}(l)Z_{\overline{\mu}}(l+k)e^{i\lambda k}.\end{aligned}$$

Then

$$\begin{aligned}(\Theta_{\overline{\mu}}^{\top}\mathbf{S}_{\overline{\mu}}\widetilde{\mathbf{a}}_{\overline{\mu}})_m &= \sum_{j=-n(\gamma)}^{\infty}\sum_{p=m}^{\infty}\theta_{\overline{\mu}}^{\top}(p-m)S_{\overline{\mu}}(p+j+1)\vec{a}_{-\overline{\mu}}(j)\\ &= \sum_{j=-n(\gamma)}^{\infty}\sum_{p=m}^{\infty}\sum_{l=0}^{\infty}\theta_{\overline{\mu}}^{\top}(p-m)\psi_{\overline{\mu}}^{\top}(l)Z_{\overline{\mu}}(l+p+j+1)\vec{a}_{-\overline{\mu}}(j)\\ &= \sum_{j=-n(\gamma)}^{\infty}\sum_{p=m}^{\infty}\sum_{k=p}^{\infty}(\psi_{\overline{\mu}}(k-p)\theta_{\overline{\mu}}(p-m))^{\top}Z_{\overline{\mu}}(k+j+1)\vec{a}_{-\overline{\mu}}(j)\\ &= \sum_{j=-n(\gamma)}^{\infty}\sum_{k=m}^{\infty}\mathrm{diag}_q(\delta_{k,m})Z_{\overline{\mu}}(k+j+1)\vec{a}_{-\overline{\mu}}(j)\\ &= \sum_{j=-n(\gamma)}^{\infty}Z_{\overline{\mu}}(m+j+1)\vec{a}_{-\overline{\mu}}(j).\end{aligned}$$

The representation for $Z_{\overline{\mu}}(j)$ follows from

$$\sum_{j\in\mathbb{Z}}Z_{\overline{\mu}}(j)e^{i\lambda j}=\overline{\Psi}_{\overline{\mu}}(e^{-i\lambda})\overline{g}(\lambda)=\sum_{k\in\mathbb{Z}}\sum_{l=0}^{\infty}\overline{\Psi}_{\overline{\mu}}(l)\overline{g}_{\overline{\mu}}(l-k)e^{i\lambda k}. \quad \square$$

Proof of Theorem 3

Under the conditions of Lemmas 1 and 2 on the spectral densities $f(\lambda)$ and $g(\lambda)$, formulas (14) and (15) can be rewritten as follows. We make the following transformations

$$\begin{aligned}&\frac{|\beta^{(d)}(i\lambda)|^2}{|\chi_{\overline{\mu}}^{(d)}(e^{-i\lambda})|^2}\left[(f(\lambda)+|\beta^{(d)}(i\lambda)|^2g(\lambda))^{-1}\right]^{\top}\left(\sum_{k=0}^{\infty}\left(\mathbf{P}_{\overline{\mu}}^{-1}\mathbf{S}_{\overline{\mu}}\mathbf{a}_{\overline{\mu}}\right)_k e^{i\lambda(k+1)}\right)\\ =&\left(\sum_{k=0}^{\infty}\psi_{\overline{\mu}}^{\top}(k)e^{-i\lambda k}\right)\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\overline{\psi}_{\overline{\mu}}(j)(\overline{\Theta}_{\overline{\mu}}\widetilde{\mathbf{e}}_{\overline{\mu}})_k e^{i\lambda(k+j+1)}\\ =&\left(\sum_{k=0}^{\infty}\psi_{\overline{\mu}}^{\top}(k)e^{-i\lambda k}\right)\sum_{m=0}^{\infty}\sum_{p=0}^{m}\sum_{k=p}^{m}\overline{\psi}_{\overline{\mu}}(m-k)\overline{\theta}_{\overline{\mu}}(k-p)\widetilde{e}_{\overline{\mu}}(p)e^{i\lambda(m+1)}\\ =&\left(\sum_{k=0}^{\infty}\psi_{\overline{\mu}}^{\top}(k)e^{-i\lambda k}\right)\sum_{m=0}^{\infty}\sum_{p=0}^{m}\mathrm{diag}_q(\delta_{m,p})\widetilde{e}_{\overline{\mu}}(m)e^{i\lambda(m+1)}\\ =&\left(\sum_{k=0}^{\infty}\psi_{\overline{\mu}}^{\top}(k)e^{-i\lambda k}\right)\sum_{m=0}^{\infty}\widetilde{e}_{\overline{\mu}}(m)e^{i\lambda(m+1)},\end{aligned}$$

and

$$\begin{aligned}&\frac{|\beta^{(d)}(i\lambda)|^2}{|\chi_{\overline{\mu}}^{(d)}(e^{-i\lambda})|^2}\left[(f(\lambda)+|\beta^{(d)}(i\lambda)|^2g(\lambda))^{-1}\right]^{\top}(g(\lambda))^{\top}A(e^{-i\lambda})\chi_{\overline{\mu}}^{(d)}(e^{i\lambda})\\
=\;&\Psi_{\overline{\mu}}^{\top}(e^{-i\lambda})\overline{\Psi_{\overline{\mu}}(e^{-i\lambda})g(\lambda)}A(e^{-i\lambda})\chi_{\overline{\mu}}^{(d)}(e^{i\lambda})\\
=\;&\left(\sum_{k=0}^{\infty}\psi_{\overline{\mu}}^{\top}(k)e^{-i\lambda k}\right)\sum_{m\in\mathbb{Z}}^{\infty}\sum_{j=-n(\gamma)}^{\infty}Z_{\overline{\mu}}(m+j)\vec{a}_{-\overline{\mu}}(j)e^{i\lambda m}.\end{aligned}$$

Then obtain:

$$\begin{aligned}\vec{h}_{\overline{\mu}}(\lambda)&=\frac{\chi_{\overline{\mu}}^{(d)}(e^{-i\lambda})}{\beta^{(d)}(i\lambda)}\left(\sum_{k=0}^{\infty}\psi_{\overline{\mu}}^{\top}(k)e^{-i\lambda k}\right)\sum_{m=0}^{\infty}\sum_{j=-\mu n}^{\infty}\theta_{\overline{\mu}}^{\top}(l)Z_{\overline{\mu}}(j-m)\vec{a}_{-\overline{\mu}}(j)e^{-i\lambda m}\\
&=\frac{\chi_{\overline{\mu}}^{(d)}(e^{-i\lambda})}{\beta^{(d)}(i\lambda)}\left(\sum_{k=0}^{\infty}\psi_{\overline{\mu}}^{\top}(k)e^{-i\lambda k}\right)\\
&\quad\times\left(\sum_{m=0}^{\infty}\sum_{j=0}^{\infty}Z_{\overline{\mu}}(j-m)\vec{a}_{-\overline{\mu}}(j)e^{-i\lambda m}+\sum_{m=0}^{\infty}\sum_{j=1}^{n(\gamma)}Z_{\overline{\mu}}(-j-m)\vec{b}_{-\overline{\mu}}(j)e^{-i\lambda m}\right)\\
&=\frac{\chi_{\overline{\mu}}^{(d)}(e^{-i\lambda})}{\beta^{(d)}(i\lambda)}\left(\sum_{k=0}^{\infty}\psi_{\overline{\mu}}^{\top}(k)e^{-i\lambda k}\right)\\
&\quad\times\left(\sum_{m=0}^{\infty}\sum_{j=0}^{\infty}\sum_{p=m}^{\infty}\overline{\psi}_{\overline{\mu}}(p-m)\overline{g}(p-j)\vec{a}_{-\overline{\mu}}(j)e^{-i\lambda m}+\sum_{m=0}^{\infty}\sum_{j=1}^{n(\gamma)}\sum_{p=m}^{\infty}\overline{\psi}_{\overline{\mu}}(p-m)\overline{g}(p+j)\vec{b}_{-\overline{\mu}}(j)e^{-i\lambda m}\right)\\
&=\frac{\chi_{\overline{\mu}}^{(d)}(e^{-i\lambda})}{\beta^{(d)}(i\lambda)}\left(\sum_{k=0}^{\infty}\psi_{\overline{\mu}}^{\top}(k)e^{-i\lambda k}\right)\sum_{m=0}^{\infty}((\widetilde{\Psi}_{\overline{\mu}})^{*}\mathbf{G}^{-}\mathbf{a}_{-\overline{\mu}}+(\widetilde{\Psi}_{\overline{\mu}})^{*}\mathbf{G}^{+}\mathbf{b}_{-\overline{\mu}})_{m}e^{-i\lambda m}\\
&=\frac{\chi_{\overline{\mu}}^{(d)}(e^{-i\lambda})}{\beta^{(d)}(i\lambda)}\left(\sum_{k=0}^{\infty}\psi_{\overline{\mu}}^{\top}(k)e^{-i\lambda k}\right)\sum_{m=0}^{\infty}(\overline{\psi}_{\overline{\mu}}\mathbf{C}_{\overline{\mu},g}^{-}+\overline{\psi}_{\overline{\mu}}\mathbf{C}_{\overline{\mu},g}^{+})_{m}e^{-i\lambda m}.\end{aligned}$$

The value of the mean square error $\Delta(f,g;\widehat{A}\xi)$ is calculated by the formula

$$\begin{aligned}\Delta\left(f,g;\widehat{A}\xi\right)&=\Delta\left(f,g;\widehat{A}\eta\right)=\mathsf{E}\left|A\eta-\widehat{A}\eta\right|^2\\
&=\frac{1}{2\pi}\int_{-\pi}^{\pi}(\vec{A}(e^{i\lambda}))^{\top}g(\lambda)\overline{\vec{A}(e^{i\lambda})}d\lambda+\frac{1}{2\pi}\int_{-\pi}^{\pi}(\vec{h}_{\overline{\mu}}(\lambda))^{\top}(f(\lambda)+|\beta^{(d)}(i\lambda)|^2g(\lambda))\overline{\vec{h}_{\overline{\mu}}(\lambda)}d\lambda\\
&\quad-\frac{1}{2\pi}\int_{-\pi}^{\pi}(\vec{h}_{\overline{\mu}}(\lambda))^{\top}\beta^{(d)}(i\lambda)g(\lambda)\overline{A(e^{-i\lambda})}d\lambda-\frac{1}{2\pi}\int_{-\pi}^{\pi}(A(e^{-i\lambda}))^{\top}\overline{\beta^{(d)}(i\lambda)}g(\lambda)\overline{\vec{h}_{\overline{\mu}}(\lambda)}d\lambda\\
&=\|\widetilde{\Phi}\mathbf{a}\|^2-\|\overline{\psi}_{\overline{\mu}}(\mathbf{C}_{\overline{\mu},g}^{-}+\mathbf{C}_{\overline{\mu},g}^{+})\|^2.\quad\square\end{aligned}$$

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