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Complexity Analysis of an Interior-point Algorithm for CQP Based on a New Parametric Kernel Function

Randa Chalekh, El Amir Djeffal*

LEDPA Laboratory, Mathematics Department, University of Batna 2, Batna, Algeria

Abstract In this paper, we present a primal-dual interior-point algorithm for convex quadratic programming problem based on a new parametric kernel function with a hyperbolic-logarithmic barrier term. Using the proposed kernel function we show some basic properties that are essential to study the complexity analysis of the correspondent algorithm which we find coincides with the best know iteration bounds for the large-update method, namely, $O\left(\sqrt{n}\log n \log \frac{n}{\varepsilon}\right)$ by a special choice of the parameter $p > 1$.

Keywords Convex quadratic programming, kernel function, interior-point methods, iteration bound.

AMS 2010 subject classifications 90C05, 90C51

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1. Introduction

In 1984, *N. Karmarkar* [1] was proposed a very effective technique for solving the linear programming problems is called the interior-point methods. This algorithm was essentially different from all the method before by the fact that, in the latter, one moves along the boundary of the realizable domain, whereas for the Karmarkar method, one progresses while remaining strictly inside the realizable domain. In that period, studies have been launched and went in depth on this subject such that we record more than three thousand publications in a few years and the most important result is the most widely used in practice called the primal-dual methods which are efficient and can easily be extended to other types of problems (quadratic problems, convex problems, ...). In 1997, *Roos et al* [2] was introduced the primal-dual *IPMs* which have based on logarithmic barrier function. In 2004, *Bai et al* [3] was developed some analytic tools for the analysis complexity of primal-dual *IPMs* based on a new class of so-called the kernel functions where from this study we obtained the best known complexity results for small- and large-update methods, namely, $O(\sqrt{n}\log \frac{n}{\varepsilon})$ and $O\left(\sqrt{n}\log n \log \frac{n}{\varepsilon}\right)$, respectively.

Recently, all the articles developed by the researchers follows the approach used in [3,4,5,6,7] to solve many problems based on various kernel functions, we can mention among them. In [8], *Zhang* proposed an algorithm for convex quadratic semidefinite optimization based on a new kernel function. In [9,10], *Peyghami et al* proposed two new kernel function for linear optimization and convex quadratic semidefinite optimization problems, respectively. In [11], *Achache* study the complexity analysis of an interior point algorithm for the semidefinite optimization based on a new kernel function with a double barrier term. In [12], *Djeffal et al* proposed a new kernel function for primal-dual IPMs for linear complementarity problem. In [11], *Achache et al* generalized the study for semidefinite linear complementarity problem based on a new kernel function with a double barrier term. In [13], *El Ghami et al* introduced the first kernel function with a trigonometric barrier term for linear optimization. Then, in [14] *Bouafia*

*Correspondence to: El Amir Djeffal (Email: l.djeffal@univ-batna2.dz).

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et al proposed a new kernel function with a trigonometric barrier term for linear programming problems. Most of these algorithms yields the best well-known complexity iteration bounds with the special choice of its parameters.

Our objective in this work is present a primal-dual interior-point algorithm for convex quadratic programming problem based on a new parametric kernel function with an hiperbolic– logarithmic barrier term of the form

$$\psi_h(t) = t^2 - 1 + \frac{1}{p}\left(\frac{\cosh^p\left(t^{-1}\right) - \cosh^p(1)}{at^p} - \log(t^p)\right) \tag{1}$$

where $a = \tanh(1)\cosh^p(1)$ and $p \geq 4$.

The paper is organized as follows. In Section 2, we presenting the convex quadratic programming and recall some basic concepts for central-path, search directions, the proximity functions and the generic algorithm. In Section 3, we investigate some properties of the new kernel function (1) for primal–dual interior point methods. Then, we present the complexity results for our algorithm which coincide with the best know iteration bounds for small- and large-update methods. Finally, we summarize the work with some conclusions and remarks.

Among the most important notations used in this paper are shown as follows. $\mathbb{R}^n$, $\mathbb{R}^n_+$ and $\mathbb{R}^n_{++}$ denote the set of n-dimensional real space, nonnegative space and positive space, respectively. For $x, s \in \mathbb{R}^n$, xs and $x^T s$ denote the componentwise *Hadamard* product and the euclidean scalar product of two vectors x and s, respectively. We denote by $\|x\|$ the 2-norm of x. For any $f, g : \mathbb{R}^n_{++} \to \mathbb{R}^n_{++}$ for some positive constants c_1, c_2 and c_3 we have $f(x) = O(g(x))$ and $f(x) = \Theta(g(x))$ if $f(x) \leq c_1 g(x)$ and $c_2 g(x) \leq f(x) \leq c_3 g(x)$, respectively.

2. Preliminaries

In this section, first we present the statement of the problem. Then, we will discuss about the central-path for convex quadratic programming, the new search directions and the proximity functions. Finally, we state the algorithm generic for primal-dual IPMs.

We consider the convex quadratic programming (*CQP*) problem in the standard format as follows

$$\min\left\{c^T x + \frac{1}{2}x^T Qx : Ax = b, x \geq 0\right\} \tag{P}$$

where $Q \in \mathbb{R}^{n\times n}$ is a given positive semidefinite matrix, $A \in \mathbb{R}^{m\times n}$ is a given matrix, $c \in \mathbb{R}^n$, $x \in \mathbb{R}^n$ and $b \in \mathbb{R}^m$.

The dual problem of (P) is given by

$$\max\left\{b^T y - \frac{1}{2}x^T Qx : A^T y + s - Qx = c, s \geq 0\right\} \tag{D}$$

where $s \in \mathbb{R}^n$ and $y \in \mathbb{R}^m$.

Throughout this paper we suppose that (P) and (D) satisfy the following conditions

Assum 1
Interior-point-condition (*IPC*), i.e., there exist a point (x_0, y_0, s_0) such that

$$\left\{Ax_0 = b, A^T y_0 + s_0 - Qx_0 = c, x_0 \geq 0, s_0 \geq 0\right\} \tag{2}$$

Assum 2
$rank(A) = m < n$, i.e., the matrix A has full row rank.

It is well know that the Karush-Kuhn-Tucker (*KKT*) conditions for (P) and (D) is given by the following system

$$\begin{cases} Ax = b, \quad x > 0 \\ A^T y + s - Qx = c, \quad s > 0 \\ xs = 0 \end{cases} \tag{3}$$

The main idea of primal-dual interior-point methods (*IPMs*) is to replace the complementary condition define in the system (3) by the parameterized equation $xs = \mu e$. Hence, we obtain the following system

$$\begin{cases} Ax = b, \quad x > 0 \\ A^T y + s - Qx = c, \quad s > 0 \\ xs = \mu e \end{cases} \tag{4}$$

where $\mu > 0$ and $e = (1, \ldots, 1)^T$ is the all-one vector. It is well-know that under our assumptions there exists a unique solution denoted by $(x(\mu), y(\mu), s(\mu))$ for each $\mu > 0$ where $x(\mu)$ is called the μ-center of (P) and $(y(\mu), s(\mu))$ called the μ-center of (D). Then, the limit of μ-centres when μ goes to zero exists and converges to x and (y, s) which are the solutions of (P) and (D), respectively. The set of μ-centers is called the central path of (P) and (D).

Due to *Roos et al* in [2] we can simplify the theoretical contributions by assume that $\mu_0 = 1$ and $x_0 = s_0 = e$ this means that the *IPC* can be assumed without loss of generality.

Now, to get the new feasible iterates we apply Newton's method in (4) we obtain the Newton's direction. Due to Assumption 2, then $(\Delta x, \Delta y, \Delta s)$ is the unique solution of the following linear system

$$\begin{cases} A\Delta x = 0 \\ A^T \Delta y + \Delta s - Q\Delta x = 0 \\ s\Delta x + x\Delta s = \mu \mathrm{e} - xs \end{cases} \tag{5}$$

At this stage, to simplify the analysis we introduce the scaling directions (d_x, d_s) and the scaled vector v as follows

$$d_x = \frac{v\Delta x}{x}, \quad d_s = \frac{v\Delta s}{s} \quad and \quad v = \sqrt{\frac{xs}{\mu}}. \tag{6}$$

Hence, by using the notation (6), we can write the system (5) as the following equivalent system

$$\begin{cases} \overline{A} d_x = 0 \\ \overline{A}^T \Delta y + d_s - \overline{Q} d_x = 0 \\ d_x + d_s = v^{-1} - v \end{cases} \tag{7}$$

where $\overline{A} = AV^{-1}X$, $\overline{Q} = S^{-1}VQV^{-1}X$, $V = diag(v)$, $X = diag(x)$ and $S = diag(s)$.

By taking a step size α (which defined by some line search rules) along the search direction, we construct a new iterates (x_+, y_+, s_+) according to

$$x_+ = x + \alpha\Delta x, \quad y_+ = y + \alpha\Delta y, \quad s_+ = s + \alpha\Delta s. \tag{8}$$

This process is repeated until we find iterates (x, y, s) that are close enough to $(x(\mu), y(\mu), s(\mu))$ and μ is small enough, we say that we have found an optimal solution of (P) and (D). One easily checks that the right-hand side of last equation in (7) equals the negative gradien of the logarithmic barrier function $\Psi_l(v) : \mathbb{R}^n_{++} \to \mathbb{R}_+$ which given by

$$\Psi_l(x, s; \mu) = \Psi_l(v) = \sum_{i=1}^{n} \psi_l(v_i) = \sum_{i=1}^{n} \left(\frac{v_i^2 - 1}{2} - \log v_i \right) \tag{9}$$

where $\psi_l(t)$ is called the kernel function of the classical logarithmic barrier function $\Psi_l(v)$. Hence, we obtain the following system

$$\begin{cases} \overline{A} d_x = 0 \\ \overline{A}^T \Delta y + d_s - \overline{Q} d_x = 0 \\ d_x + d_s = -\nabla\Psi_l(v) \end{cases} \tag{10}$$

So that the logarithmic barrier function $\Psi_l(v)$ consider as a proximity function to measure the distance between any iteration and the μ-center, in addition to proximity measure $\delta(v) : \mathbb{R}^n_{++} \to \mathbb{R}_+$ which defined as follows

$$\delta(x, s; \mu) = \delta(v) = \frac{1}{2}\|d_x + d_s\| = \frac{1}{2}\|\nabla\Psi_l(v)\| \tag{11}$$

Since d_x and d_s are orthogonal, due to (10) we get

$$d_x = d_s = 0 \Leftrightarrow \nabla\Psi_l(v) = 0 \Leftrightarrow v = e \Leftrightarrow \Psi_l(v) = 0$$

this means the desired result $(x, s) = (x(\mu), s(\mu))$.

In this paper, our goal is present the following generic algorithm to solve *CQP* which based on a new hiperbolic-logarithmic kernel function $\psi_h(v)$ befined by (1) instaed of the logarithmic kernel function $\psi_l(v)$ defined in (9).

INPUT :
A proximity function $\Psi_h(v)$; an accuracy parameter $\varepsilon > 0$;
a threshold parameter $\tau \geq 1$; a fixed barrier update parameter
$0 < \theta < 1$; initial point (x_0, y_0, s_0) and the parameter $\mu_0 > 0$.
ITERATION :
BEGIN
  $x = x_0;\ y = y_0;\ s = s_0;\ \mu = \mu_0;\ v = \sqrt{\frac{xs}{\mu}}$.
  WHILE $(n\mu \geq \varepsilon)$ **DO**
  BEGIN
    update of μ and v : $\mu = (1-\theta)\mu;\ v = \frac{v}{\sqrt{1-\theta}}$.
    WHILE $(\Psi_h(v) > \tau)$ **DO**
    BEGIN
      solve (7) via (5) to find search directions;
      determine a step size $\alpha > 0$;
      $(x, y, s) = (x, y, s) + \alpha(\Delta x, \Delta y, \Delta s)$;
      $v = \sqrt{\frac{xs}{\mu}}$.
    END.
  END.
END.

Figure 1. GENERIC INTERIOR-POINT ALGORITHM FOR *CQP*.

From all the previous explanation, it is clear that the above algorithm works as follows. It starts with the parameters $\epsilon > 0$, $\tau \leq 1$, $0 < \theta < 1$ (should be chosen in such a way that the number of iterations required by the algorithm is as small as possible), the parameter $\mu > 0$, an initial point (x, y, s) and the scaled vector v. Therefore, in each outer iteration we apdate the values of μ and v by the factor $1-\theta$ then if the condition $\Psi_h(v) > \tau$ is met we enter the inner iteration that aims to find the search directions $(\Delta x, \Delta y, \Delta s)$ using the two systems (5) and (7), compute the step size α and we apply (8) to get new iterates. This process is repeated until we get μ is small enough (i.e., $n\mu < \epsilon$) and v satisfy $\Psi_h(v) \leq \tau$, say we have found an optimal solution of *CQP*.

3. The new kernel function and its properties

In this section, we investigate some basic properties of the new kernel function with a new hiperbolic–logarithmic barrier term (1) which using in the complexity analysis of our algorithm. First, we recall the following definition.

Definition 1
$\psi(t) : \mathbb{R}_{++} \rightarrow \mathbb{R}_+$ is a kernel function if ψ is twice differentiable and satisfies the following conditions

$$\psi(1) = \psi^{'}(1) = 0.$$
$$\psi^{''}(t) > 0, \quad \forall t > 0.$$
$$\lim_{t\rightarrow 0^+} \psi(t) = \lim_{t\rightarrow +\infty} \psi(t) = +\infty.$$

And we give the first three derivatives of (1) with respect to t as follows.

$$\psi_h^{'}(t) = 2t - \frac{(\tanh(1)-1)\cosh^p(1)+\cosh^p(t^{-1})}{at^{p+1}} - \frac{\tanh(t^{-1})\cosh^p(t^{-1})}{at^{p+2}}. \quad (12)$$

$$\psi_h^{''}(t) = 2 + \frac{(p+1)}{a}\left(\frac{(\tanh(1)-1)\cosh^p(1)+\cosh^p(t^{-1})}{t^{p+2}} + \frac{2\tanh(t^{-1})\cosh^p(t^{-1})}{t^{p+3}}\right) + \frac{p\tanh^2(t^{-1})\cosh^p(t^{-1})+\cosh^{p-2}(t^{-1})}{at^{p+4}}. \quad (13)$$

$$\psi_h^{'''}(t) = -\left(\frac{(p+2)}{a}\left((p+1)\left(\frac{(\tanh(1)-1)\cosh^p(1)+\cosh^p(t^{-1})}{t^{p+3}} + \frac{3\tanh(t^{-1})\cosh^p(t^{-1})}{t^{p+4}}\right) + \frac{3(p\tanh^2(t^{-1})\cosh^p(t^{-1})+\cosh^{p-2}(t^{-1}))}{t^{p+5}}\right) + \frac{p^2\tanh^3(t^{-1})\cosh^p(t^{-1})+(3p+2)\tanh(t^{-1})\cosh^{p-2}(t^{-1})}{at^{p+6}}\right). \quad (14)$$

To facilitate the study, we use the following notations.

$$\begin{array}{c} f(t) = (\tanh(1)-1)\cosh^p(1)+\cosh^p(t^{-1}) \\ g(t) = \tanh(t^{-1})\cosh^p(t^{-1}) \\ h(t) = p\tanh^2(t^{-1})\cosh^p(t^{-1})+\cosh^{p-2}(t^{-1}) \\ q(t) = p^2\tanh^3(t^{-1})\cosh^p(t^{-1})+(3p+2)\tanh(t^{-1})\cosh^{p-2}(t^{-1}) \end{array} \quad (15)$$

Lemma 1
For all $p \geq 4$ and $t > 0$, we have the following properties

$$\cosh(t^{-1}) > 0 \quad \text{and} \quad 0 < \tanh(t^{-1}) < 1. \quad (16)$$

$$f(t) > 0, \quad g(t) > 0, \quad h(t) > 0 \quad \text{and} \quad q(t) > 0. \quad (17)$$

$$\cosh^p(t^{-1})\left(\frac{\tanh(t^{-1})}{at^{p+2}} - \frac{1}{(at)^p}\right) > 0, \quad 0 < t < 1. \quad (18)$$

Proof
(16) is true since for all $t > 0$ the two functions $\tanh(t^{-1})$, $\cosh(t^{-1})$ are decreasing in $]0,+\infty[$ and

$$\lim_{t\to+\infty}\cosh(t^{-1}) = 1 > 0, \ \lim_{t\to 0^+}\tanh(t^{-1}) = 1 \text{ and } \lim_{t\to+\infty}\tanh(t^{-1}) = 0.$$

From (15) and (16), we note that the four function f, g, h and q are decreasing in $]0,+\infty[$ because for all $t > 0$ we have

$$f^{'}(t) = -\frac{p}{t^2}\tanh(t^{-1})\cosh^p(t^{-1}) < 0$$

$$g^{'}(t) = -\frac{1}{t^2}\left(\cosh^{p-2}(t^{-1}) + p\tanh^2(t^{-1})\cosh^p(t^{-1})\right) < 0$$

$$h^{'}(t) = -\frac{1}{t^2}\left((3p-2)\tanh(t^{-1})\cosh^{p-2}(t^{-1}) + p^2\tanh^3(t^{-1})\cosh^p(t^{-1})\right) < 0$$

$$q^{'}(t) = -\frac{1}{t^2}\left((3p+2)\cosh^{p-4}(t^{-1}) + p(6p+2)\tanh^2(t^{-1})\cosh^{p-2}(t^{-1}) + p^3\tanh^4(t^{-1})\cosh^p(t^{-1})\right) < 0$$

with $\lim_{t\to+\infty} f(t) = a + 1 - \cosh^p(1) > 0$, $\lim_{t\to+\infty} g(t) = 0$, $\lim_{t\to+\infty} h(t) = 1 > 0$ and $\lim_{t\to+\infty} q(t) = 0$. Then, we obtain the desired result (17).

To prove (18), we put $\alpha(t) = \cosh^p\left(t^{-1}\right)\left(\frac{\tanh\left(t^{-1}\right)}{at^{p+2}} - \frac{1}{(at)^p}\right)$ and we calculate

$$\begin{aligned}\alpha^{'}(t) &= p\cosh^p\left(t^{-1}\right)\left(\frac{1}{a^p t^{p+1}} + \frac{\tanh\left(t^{-1}\right)}{a^p t^{p+2}}\right) - \cosh^p\left(t^{-1}\right)\left(\frac{(p+2)\tanh\left(t^{-1}\right)}{at^{p+3}}\right.\\ &\left. + \frac{p\tanh^2\left(t^{-1}\right)+\cosh^{-2}\left(t^{-1}\right)}{at^{p+4}}\right)\\ &\leq -\cosh^p\left(t^{-1}\right)\left(\frac{2\left(\tanh^2\left(t^{-1}\right)+\tanh\left(t^{-1}\right)-1\right)+\cosh^{-2}\left(t^{-1}\right)}{a^p t^{p+2}}\right) \leq 0\end{aligned}$$

where the two inequalities follows from the fact that $0 < t < 1$, $a > 0$, $p \geq 4$, (17) and $\tanh^2\left(t^{-1}\right) + \tanh\left(t^{-1}\right) > 1$. Hence, we conclude that the function $\alpha(t)$ is decreasing in $]0,1[$ and $\lim_{t\to 1}\alpha(t) = 1 - a^{-p} > 0$. □

By using (1), (12), (13), (15) and (17), it is easy to verify the three conditions of Definition 1 and conclude that the function defined by (1) is a kernel function. From the second condition it is clear that $\psi_h(t)$ is coercive and has the barrier property. Moreover, (13) and (17) imply that for all $t > 0$ we have

$$\psi_h^{''}(t) \geq 2 > 0. \tag{19}$$

Now, we are ready to present the essential properties of (1).

Lemma 2
For all $p \geq 4$ and $t > 0$, $\psi_h(t)$ check the following

$$\psi_h(t) \text{ is exponentially convex.} \tag{20}$$
$$\psi_h^{''}(t) \text{ is monotonically decreasing.} \tag{21}$$
$$t\psi_h^{''}(t) - \psi_h^{'}(t) > 0, \quad t > 1. \tag{22}$$
$$2\psi_h^{''}(t)^2 - \psi_h^{'}(t)\psi_h^{'''}(t) > 0, \quad t < 1. \tag{23}$$

Proof
For (20), we know that by Lemma 2.1.2 in [15] it is sufficient to prove that $t\psi_h^{''}(t) + \psi_h^{'}(t) > 0$, so by using (12) and (13) we get

$$t\psi_h^{''}(t) + \psi_h^{'}(t) = 4t + \frac{pf(t)}{at^{p+1}} + \frac{(2p+1)g(t)}{at^{p+2}} + \frac{h(t)}{at^{p+3}} > 0, \quad \forall t > 0.$$

From (15), (17), the positivity of t and p, we obtain $\psi_h^{'''}(t) < 0$. Thus, the desired result (21).
The two last inequalities holds due to (12), (13) and (14).

$$t\psi_h^{''}(t) - \psi_h^{'}(t) = \frac{(p+2)f(t)}{at^{p+1}}\frac{(2p+3)g(t)}{at^{p+2}} + \frac{h(t)}{at^{p+3}} > 0, \quad \text{for all } t > 1$$

and

$$\begin{aligned}2\psi_h^{''}(t)^2 - \psi_h^{'}(t)\psi_h^{'''}(t) = \; & 8 + (p+1)\left(\frac{2(p+6)f(t)}{at^{p+2}} + \frac{2(3p+14)g(t)}{at^{p+3}} + \right.\\ & \left.\frac{pf^2(t)}{a^2t^{2p+4}} + \frac{4pf(t)g(t)}{a^2t^{2p+5}} + \frac{(5p+2)g^2(t)}{a^2t^{2p+6}}\right) + \frac{2q(t)}{at^{p+5}}\\ & + \frac{2(p+6)h(t)}{at^{p+4}} + \frac{(p-2)f(t)g(t)}{a^2t^{2p+6}} + \frac{2h^2(t)-q(t)g(t)}{a^2t^{2p+8}}\\ & + \frac{(5p+2)g(t)h(t)-q(t)f(t)}{a^2t^{2p+7}} > 0, \quad \text{for all } t < 1.\end{aligned}$$

Which completes the proof. □

Therefore, from the above lemma we conclude that our kernel function $\psi_h(t)$ is an eligible kernel function.

Lemma 3
For all $p \geq 4$, the kernel function $\psi_h(t)$ hold the following result

$$(t-1)^2 \leq \quad \psi_h(t) \quad \leq \frac{1}{4}\left(\psi_h^{'}(t)\right)^2, \qquad t>0. \tag{24}$$

$$\psi_h(t) \quad \leq \frac{\kappa p+\iota}{2}(t-1)^2, \quad t \geq 1. \tag{25}$$

With $\kappa = 3 + \dfrac{\tanh^2(1)\cosh^p(1)}{a}, \quad \iota = 5 + \dfrac{\cosh^{p-2}(1)}{a}$.

Proof
For (24). From the first condition of Definition 1 and (19), we get

$$\psi_h(t) = \int_1^t \int_1^\xi \psi_h^{''}(z)\,\mathrm{d}z\,\mathrm{d}\xi, \quad \forall t>0. \tag{26}$$

Using (26), we obtain

$$\psi_h(t) = \int_1^t \int_1^\xi \psi_h^{''}(z)\,\mathrm{d}z\,\mathrm{d}\xi \geq \int_1^t \int_1^\xi 2\,\mathrm{d}z\,\mathrm{d}\xi = (t-1)^2, \quad \forall t>0$$

and

$$\begin{aligned}\psi_h(t) = \int_1^t \int_1^\xi \psi_h^{''}(z)\,\mathrm{d}z\,\mathrm{d}\xi \quad &\leq \quad \frac{1}{2}\int_1^t \int_1^\xi \psi_h^{''}(\xi)\psi_h^{''}(z)\,\mathrm{d}z\,\mathrm{d}\xi \\ &= \quad \frac{1}{2}\int_1^t \psi_h^{''}(\xi)\psi_h^{'}(\xi)\,\mathrm{d}\xi = \frac{1}{4}\left(\psi_h^{'}(t)\right)^2.\end{aligned}$$

For (25), using Taylor's theorem, (21) and (13) for some ξ, $1 \leq \xi \leq t$ we get

$$\begin{aligned}\psi_h(t) \quad &= \quad \psi_h(1) + \psi_h^{'}(1)(t-1) + \frac{1}{2}\psi_h^{''}(1)(t-1)^2 + \frac{1}{6}\psi_h^{'''}(\xi)(\xi-1)^3 \\ &\leq \quad \frac{1}{2}\psi_h^{''}(1)(t-1)^2 \\ &= \quad \frac{1}{2}\left(\left(3+\frac{\tanh^2(1)\cosh^p(1)}{a}\right)p + 5 + \frac{\cosh^{p-2}(1)}{a}\right)(t-1)^2,\end{aligned}$$

to completes the proof we put $\kappa = 3 + \dfrac{\tanh^2(1)\cosh^p(1)}{a}$ and $\iota = 5 + \dfrac{\cosh^{p-2}(1)}{a}$. □

Let $\gamma(s) : [0,+\infty[\to [1,+\infty[$ be the inverse function of $\psi_h(t)$ for all $t \geq 1$ and $\rho(s) : [0,+\infty[\to]0,1]$ be the inverse function of $-\frac{1}{2}\psi_h^{'}(t)$ for all $t \in]0,1]$. Then, we have the following lemma.

Lemma 4
For $p \geq 4$, we have

$$1 + \sqrt{\frac{2s}{\kappa p + \iota}} \leq \gamma(s) \leq 1 + \sqrt{s}. \tag{27}$$

$$\frac{\cosh\left(t^{-1}\right)}{at} \leq (2s+2)^{\frac{1}{p}}. \tag{28}$$

Proof
For (27), let $\psi_h(t) = s, t \geq 1$ i.e., $t = \gamma(s), s \geq 0$. By using the left part of the inequality (24), we have

$$s = \psi_h(t) \geq (t-1)^2 \text{ this implies that } t = \gamma(s) \leq 1 + \sqrt{s}$$

and from (25), we get

$$s = \psi_h(t) \leq \frac{\kappa p + \iota}{2}(t-1)^2 \text{ this implies that } t = \gamma(s) \geq 1 + \sqrt{\frac{2s}{\kappa p + \iota}}.$$

For (28), let $s = -\frac{1}{2}\psi_h'(t)$ for $0 < t \leq 1$. Due to the definition of ρ and (12), we get

$$\begin{aligned} s = -\frac{1}{2}\psi_h'(t) \quad &\Leftrightarrow \quad s = -\frac{1}{2}\left(2t - \frac{f(t)}{at^{p+1}} - \frac{g(t)}{at^{p+2}}\right) \\ &\Leftrightarrow \quad \frac{g(t)}{at^{p+2}} = 2s + 2t - \frac{f(t)}{at^{p+1}} \leq 2s + 1 \qquad (29) \\ &\Leftrightarrow \quad \left(\frac{\cosh\left(t^{-1}\right)}{at}\right)^p \leq 2s + 2 \qquad (30) \\ &\Leftrightarrow \quad \frac{\cosh\left(t^{-1}\right)}{at} \leq (2s+2)^{\frac{1}{p}} \qquad (31) \end{aligned}$$

where in the third equivalence the inequality holds since $t \leq 1$ and (17). Then, we use (18) to obtain the desired result. □

In the following lemma, we give a relationship between (11) and $\Psi_h(v)$.

Lemma 5
Let $\delta(v)$ be defined as in (11). Then we have

$$\delta(v) \geq \sqrt{\Psi_h(v)}.$$

Proof
Due to (9), (11) and (24) we have

$$\Psi_h(v) = \sum_{i=1}^{n} \psi_h(v_i) \leq \sum_{i=1}^{n} \frac{1}{4}\left(\psi_h'(v_i)\right)^2 = \frac{1}{4}\|\nabla\Psi_h(v)\|^2 = \delta^2(v)$$

this implie the desired result. □

Now, we present the theorem which gives an estimate of the effect of a μ-update on the value of $\Psi(v)$.

Theorem 1
(Theorem 3.2, [3]*)* Let $\gamma(s) : [0, +\infty[\to [1, +\infty[$ be the inverse function of $\psi(t)$ defined in (1) for all $t \geq 1$. Then we have for any positive vector v and any $\beta \geq 1$ that

$$\Psi(\beta v) \leq n\psi\left(\beta\gamma\left(\frac{\Psi(v)}{n}\right)\right). \qquad (32)$$

In the following lemma we compute two upper bounds for the effect of a μ-update on the value of $\Psi_h(v)$.

Lemma 6
Let $0 < \theta < 1$ and $v_+ = \frac{v}{\sqrt{1-\theta}}$. If $\Psi_h(v) \leq \tau$, then

$$\Psi_h(v_+) \leq \frac{\kappa p + \iota}{2(1-\theta)}\left(\theta\sqrt{n} + \sqrt{\tau}\right)^2. \qquad (33)$$

$$\Psi_h(v_+) \leq \frac{2\tau + n\theta + 2\sqrt{n\tau}}{1-\theta}. \qquad (34)$$

Proof

For (33), since $\frac{1}{\sqrt{1-\theta}} \geq 1$ and $\gamma\left(\frac{\Psi_h(v)}{n}\right) \geq 1$, we have $\frac{\gamma\left(\frac{\Psi_h(v)}{n}\right)}{\sqrt{1-\theta}} \geq 1$. Using Theorem 1 with $\beta = \frac{1}{\sqrt{1-\theta}}$, $\Psi_h(v) \leq \tau$, (25) and the second inequality in (27), we obtain

$$\begin{aligned}
\Psi_h(v_+) \leq n\psi_h\left(\frac{1}{\sqrt{1-\theta}}\gamma\left(\frac{\Psi_h(v)}{n}\right)\right) &\leq n\frac{\kappa p+\iota}{2}\left(\frac{1}{\sqrt{1-\theta}}\gamma\left(\frac{\Psi_h(v)}{n}\right)-1\right)^2 \\
&\leq n\frac{\kappa p+\iota}{2}\left(\frac{1+\sqrt{\frac{\Psi_h(v)}{n}}}{\sqrt{1-\theta}}-1\right)^2 \\
&\leq n\frac{\kappa p+\iota}{2}\left(\frac{1+\sqrt{\frac{\tau}{n}}-\sqrt{1-\theta}}{\sqrt{1-\theta}}\right)^2 \\
&\leq n\frac{\kappa p+\iota}{2}\left(\frac{\theta+\sqrt{\frac{\tau}{n}}}{\sqrt{1-\theta}}\right)^2 \\
&= \frac{\kappa p+\iota}{2(1-\theta)}\left(\theta\sqrt{n}+\sqrt{\tau}\right)^2
\end{aligned}$$

where the last inequality holds because $1-\sqrt{1-\theta} = \frac{\theta}{1+\sqrt{1-\theta}} \leq \theta$.

For (34), by Theorem 1 with $\beta = \frac{1}{\sqrt{1-\theta}}$, (27) and the fact $\psi_h(t) \leq t^2-1$ for all $t \in [1,+\infty[$, we get

$$\begin{aligned}
\Psi_h(v_+) \leq n\psi_h\left(\frac{1}{\sqrt{1-\theta}}\gamma\left(\frac{\Psi_h(v)}{n}\right)\right) &\leq n\left(\left(\frac{1}{\sqrt{1-\theta}}\gamma\left(\frac{\Psi_h(v)}{n}\right)\right)^2-1\right) \\
&= \frac{n}{1-\theta}\left(\gamma\left(\frac{\Psi_h(v)}{n}\right)+\theta-1\right) \\
&\leq \frac{n}{1-\theta}\left(\left(1+\sqrt{\frac{\Psi_h(v)}{n}}\right)^2+\theta-1\right) \\
&= \frac{2\tau+n\theta+2\sqrt{n\tau}}{1-\theta}
\end{aligned}$$

this completes the proof. □

We denote for all $p \geq 4$

$$\overline{\Psi}_{h_0} = \frac{\kappa p+\iota}{2(1-\theta)}\left(\theta\sqrt{n}+\sqrt{\tau}\right)^2 \tag{35}$$

$$\widehat{\Psi}_{h_0} = \frac{2\tau+n\theta+2\sqrt{n\tau}}{1-\theta}. \tag{36}$$

where $\overline{\Psi}_{h_0}$ and $\widehat{\Psi}_{h_0}$ an upper bounds of $\Psi_h(v)$ for small-update and large-update, respectively.

Remark 1
For small-update methods with $\tau = O(1)$ and $\theta = \Theta(\frac{1}{\sqrt{n}})$, we have $\overline{\Psi}_{h_0} = O(p)$ and for large-update methods with $\tau = O(n)$ and $\theta = \Theta(1)$, we have $\widehat{\Psi}_{h_0} = O(n)$.

3.1. Complexity analysis

In this subsection, we compute the iteration bounds for small- and large-update methods based on our kernel function (1). It must first compute a default step size α and express the decrease of the proximity function during an inner iteration. It is well know that during an inner iteration the value of μ is fixed and after each step of the algorithm we obtain a new iterates (x_+, y_+, s_+), then by using (6) and (8) we get

$$x_+ = x + \alpha\Delta x = x\left(e + \alpha\frac{\Delta x}{x}\right) = x\left(e + \alpha\frac{d_x}{v}\right) = \frac{x}{v}\left(v + \alpha d_x\right)$$
$$s_+ = s + \alpha\Delta s = s\left(e + \alpha\frac{\Delta s}{s}\right) = s\left(e + \alpha\frac{d_s}{v}\right) = \frac{s}{v}\left(v + \alpha d_s\right)$$

thus we have

$$v_+ = \sqrt{\frac{x_+ s_+}{\mu}} = \sqrt{(v + \alpha d_x)(v + \alpha d_s)} \tag{37}$$

from (20) and (37) it is clear that

$$\begin{aligned}\Psi_h(v_+) &= \Psi_h\left(\sqrt{(v + \alpha d_x)(v + \alpha d_s)}\right) \\ &\leq \frac{1}{2}\left(\Psi_h(v + \alpha d_x) + \Psi_h(v + \alpha d_s)\right).\end{aligned}$$

For $\alpha > 0$, we define

$$f(\alpha) = \Psi_h(v_+) - \Psi_h(v) \tag{38}$$
$$f_1(\alpha) = \frac{1}{2}\left(\Psi_h(v + \alpha d_x) + \Psi_h(v + \alpha d_s)\right) - \Psi_h(v). \tag{39}$$

taking the two successive derivatives of $f_1(\alpha)$ with respect to α, we have

$$f_1^{'}(\alpha) = \frac{1}{2}\sum_{i=1}^{n}\left(\psi_h^{'}(v_i + \alpha d_{x_i})d_{x_i} + \psi_h^{'}(v_i + \alpha d_{s_i})d_{s_i}\right) \tag{40}$$

and

$$f_1^{''}(\alpha) = \frac{1}{2}\sum_{i=1}^{n}\left(\psi_h^{''}(v_i + \alpha d_{x_i})d_{x_i}^2 + \psi_h^{''}(v_i + \alpha d_{s_i})d_{s_i}^2\right). \tag{41}$$

Therefore

$$f(\alpha) \leq f_1(\alpha), \quad f(0) = f_1(0) = 0$$

and by using the second equation in (10) and (11), we have

$$f_1^{'}(0) = \frac{1}{2}\left\langle \nabla\Psi_h(v), (d_x + d_s)\right\rangle = -\frac{1}{2}\|\nabla\Psi_h(v)\|^2 = -2\delta(v)^2. \tag{42}$$

We denote

$$v_{min} = \min_{i\in\{1,\ldots,n\}} v_i, \quad \delta = \delta(v) \quad \text{and} \quad \Psi_h = \Psi_h(v).$$

For the rest of this paper, we assume that $\Psi_h \geq \tau \geq 1$. Using Lemma 5 we get $\delta \geq \sqrt{\tau} \geq 1$. So that before presenting our results we will remind four lemmas that we will use (for proof, it is possible to resort to [3]).

Lemma 7
(Lemma 4.1, [3]*)* Let $f_1(\alpha)$ be defined as in (39) and δ as in (11). Then

$$f_1^{''}(\alpha) \leq 2\delta^2\psi_h^{''}\left(v_{min} - 2\alpha\delta\right). \tag{43}$$

Lemma 8
(Lemma 4.2, [3]*)* If α satisfies the inequality

$$-\psi_h^{'}(v_{min} - 2\alpha\delta) + \psi_h^{'}(v_{min}) \leq 2\delta, \tag{44}$$

then

$$f_1^{'}(\alpha) \leq 0$$

Lemma 9
(Lemma 4.3, [3]*)* Let $\rho(s) : [0, +\infty[\rightarrow]0, 1]$ denote the inverse function of $-\frac{1}{2}\psi_h^{'}(t)$. The largest step size α that satisfies (44) is given by

$$\overline{\alpha} := \frac{\rho(\delta) - \rho(2\delta)}{2\delta}.$$

Lemma 10
(Lemma 4.4, [3]*)* Let $\overline{\alpha}$ be as defined in Lemma 9. Then

$$\overline{\alpha} \geq \frac{1}{\psi_h^{''}(\rho(2\delta))}. \tag{45}$$

Lemma 11
Let ρ and $\overline{\alpha}$ be as defined in Lemma 9 and Lemma 10, respectively. Then

$$\overline{\alpha} \quad \geq \quad \frac{1}{2 + 4a^{p+3}(p+1)(4\delta + 2)^{\frac{p+4}{p}}}, \qquad p \geq 4 \tag{46}$$

where $\overline{\alpha}$ is the largest step size of the worst case associated with $\psi_h(t)$.

Proof
For (46), according to (13) with $t = \rho(2\delta) \in]0, 1]$, then we have

$$\begin{aligned} \psi_h^{''}(\rho(2\delta)) &= 2 + \frac{p+1}{a}\left(\frac{f(\rho(2\delta))}{\rho(2\delta)^{p+2}} + \frac{2g(\rho(2\delta))}{\rho(2\delta)^{p+3}}\right) + \frac{h(\rho(2\delta))}{a\rho(2\delta)^{p+4}} \\ &\leq 2 + \frac{(p+1)\left(f(\rho(2\delta)) + 2g(\rho(2\delta))\right) + h(\rho(2\delta))}{a\rho(2\delta)^{p+4}} \end{aligned} \tag{47}$$

due to (15) and (16), we obtain

$$\begin{aligned} f(\rho(2\delta)) &= a + \cosh^p\left(\rho^{-1}(2\delta)\right) - \cosh^p(1) \leq \cosh^p\left(\rho^{-1}(2\delta)\right) \\ g(\rho(2\delta)) &= \tanh\left(\rho^{-1}(2\delta)\right)\cosh^p\left(\rho^{-1}(2\delta)\right) \leq \cosh^p\left(\rho^{-1}(2\delta)\right) \end{aligned}$$

$$\begin{aligned} h(\rho(2\delta)) &= p\tanh^2\left(\rho^{-1}(2\delta)\right)\cosh^p\left(\rho^{-1}(2\delta)\right) + \cosh^{p-2}\left(\rho^{-1}(2\delta)\right) \\ &\leq (p+1)\cosh^p\left(\rho^{-1}(2\delta)\right). \end{aligned}$$

From (47) and (28) we have

$$\begin{aligned} \psi_h^{''}(\rho(2\delta)) &\leq 2 + \frac{4(p+1)}{a}\left(\frac{\cosh\left(\rho^{-1}(2\delta)\right)}{\rho(2\delta)}\right)^{p+4} \\ &\leq 2 + 4a^{p+3}(p+1)(4\delta + 2)^{\frac{p+4}{p}} \end{aligned}$$

and by using Lemma 10 we get

$$\overline{\alpha} \geq \frac{1}{\psi_h^{''}(\rho(2\delta))} \geq \frac{1}{2 + 4a^{p+3}(p+1)(4\delta + 1)^{\frac{p+4}{p}}}.$$

this proves the lemma. □

For our algorithm we define the default step size $\alpha^\star$ with $\alpha^\star \leq \overline{\alpha}$ as follows

$$\alpha^\star = \frac{1}{2 + 4a^{p+3}(p+1)(4\delta+2)^{\frac{p+4}{p}}}. \tag{48}$$

Now, we express the decrease of the proximity function Ψ_h during an inner iteration with the default step size $\alpha^\star$ which given by (48). To this end, we established the following results.

Lemma 12
(Lemma 12, [15]*)* Let $h(t)$ be a twice differentiable convex function with $h(0) = 0$, $h^{'}(0) < 0$ and let $h(t)$ attain its global minimum at its stationary point $t^* > 0$. If $h^{''}(t)$ is increasing with respect to t, then one has for any $t \in [0, t^*]$

$$h(t) \leq \frac{th^{'}(0)}{2}.$$

Lemma 13
(Lemma 4.5, [3]*)* If the step size α satisfies $\alpha \leq \overline{\alpha}$, then

$$f(\alpha) \leq -\alpha\delta^2. \tag{49}$$

Using Lemma 13 and (48), we have the following theorem.

Theorem 2
(Theorem 4.6, [3]*)* With $\alpha^\star$ being the default step size, as given by (48), one has

$$f(\alpha^\star) \leq -\frac{\delta^2}{\psi_h^{''}(\rho(2\delta))}. \tag{50}$$

Lemma 14
Let $\alpha^\star$ defined as in (48). Then we have

$$f(\alpha^\star) \quad \leq \quad -\frac{\Psi_h^{\frac{p-4}{2p}}}{2 + 144a^{p+3}(p+1)}. \tag{51}$$

Proof
To prove this lemma, we use Theorem 2 with (48) and $\delta \geq \sqrt{\Psi_h}$. We obtain

$$\begin{aligned} f(\alpha^\star) \leq \frac{-\delta^2}{\psi_h^{''}(\rho(2\delta))} \quad &\leq \quad \frac{-\delta^2}{2 + 4a^{p+3}(p+1)(4\delta+2)^{\frac{p+4}{p}}} \\ &\leq \quad \frac{-\delta^2}{2\delta^{\frac{p+4}{p}} + 4a^{p+3}(p+1)(4\delta+2\delta)^{\frac{p+4}{p}}} \\ &\leq \quad \frac{-\delta^{2-\frac{p+4}{p}}}{2 + 144a^{p+3}(p+1)} \\ &\leq \quad \frac{-\Psi_h^{\frac{p-4}{2p}}}{2 + 144a^{p+3}(p+1)}. \end{aligned}$$

□

At this stage, we count how many inner iterations required by the algorithm to obtain the situation $\Psi_h \leq \tau$. Let Ψ_{h_0} denote the value of Ψ_h after μ-update and the subsequent values in the same outer iteration are denoted as Ψ_{h_k}, $k = 1, \ldots, K$, where K denotes the total number of inner iterations in the outer iteration and for this we invoke the following lemma.

Lemma 15
(Proposition 1.3.2, [15]*)* Let a sequence $t_k > 0$, $k = 0, \ldots, K$ that verifies

$$t_{k+1} \leq t_k - \kappa t_k^{1-\nu} \quad \text{with} \quad \kappa > 0, \quad 0 < \nu \leq 1 \quad \text{and} \quad k = 0, \ldots, K,$$

then

$$K \leq \frac{t_0^\nu}{\kappa \nu}.$$

Using the Lemma 15 for $t_k = \Psi_{h_k}$, we can get the following lemma.

Lemma 16
Let K be the total number of inner iterations in the outer iteration. Then we have

$$K \leq \left(4 + 288a^{p+3}\right) p \Psi_{h_0}^{\frac{p+4}{2p}}.$$

Proof
Using the definition of $f(\alpha)$ (38) and Lemma 13 with $\alpha \leq \overline{\alpha}$, we have

$$f(\alpha) = \Psi_{h_{k+1}} - \Psi_{h_k} \leq -\kappa \delta^2$$

then we suppose that they exist $\kappa > 0$ and $0 < \nu \leq 1$, such that

$$\Psi_{h_{k+1}} - \Psi_{h_k} \leq -\kappa \Psi_{h_k}^{1-\nu}, \tag{52}$$

according to Lemma 15 with $t_k = \Psi_{h_k}$, we obtain

$$K \leq \frac{\Psi_{h_0}^\nu}{\kappa \nu}.$$

From (52) and (51), we conclude that

$$\nu = \frac{p+4}{2p} \quad \text{and} \quad \kappa = \frac{1}{2 + 144a^{p+3}(p+1)}.$$

then one can easily deduce the desired result. □

In the following theorem, we estimate the total number of iterations of the algorithm.

Theorem 3
The total number of iterations required to obtain the optimal solution of *CQP* is bounded by

$$\left(4 + 288a^{p+3}\right) p \Psi_{h_0}^{\frac{p+4}{2p}} \frac{\log \frac{n}{\varepsilon}}{\theta}.$$

Proof
The number of outer iterations is bounded above by $\frac{1}{\theta} \log \frac{n}{\varepsilon}$ (see [16]). By multiplying the number of outer iterations by the number of inner iterations, we obtain

$$K \frac{\log \frac{n}{\varepsilon}}{\theta} \leq \left(4 + 288a^{p+3}\right) p \Psi_{h_0}^{\frac{p+4}{2p}} \frac{\log \frac{n}{\varepsilon}}{\theta}$$

which completes the proof. □

Using Remark 1, we obtain the complexity result of small- and large-update methods which we summarize in the following table.

From the results of Table 1, we get the best known iteration bounds for small- and large-update, namely, $O(\sqrt{n} \log \frac{n}{\varepsilon})$ and $O\left(\sqrt{n} \log n \log \frac{n}{\varepsilon}\right)$ if p take any constant value and take $p = \frac{\log n}{2} - 1$, respectively.

The kernel function	Small-update method	Large-update method
$\psi_h(t)$ with $p \geq 4$	$O\left(p^{\frac{3p+4}{2p}}\sqrt{n}\log\frac{n}{\varepsilon}\right)$	$O\left(pn^{\frac{p+4}{2p}}\log\frac{n}{\varepsilon}\right)$

Table 1. COMPLEXITY OF LARGE- AND SMALL-UPDATE METHODS

3.2. Concluding remarks and future research

In this work, we were able to deal with the complexity analysis of primal-dual interior-point algorithm for *CQP* problem based on a new kernel function with hiperbolic–logarithmic barrier term (1). Therefore, we have analyzed the algorithm and we obtain the best iteration complexity bound $O\left(\sqrt{n}\log n\log\frac{n}{\varepsilon}\right)$ for large-update method with a specific choice of the parameter p, namely, $p = \frac{\log n}{2} - 1$.

One of the most important areas of research that attracts our attention is generalize the complexity bound based on a new kernel function with hiperbolic barrier term for primal-dual interior-point methods in the convex quadratic semidefinte optimization.

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