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A Comparative Study between Partial Bayes and Empirical Bayes Method in Gamma Distribution

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Abstract Though the name Partial Bayes was used earlier in a different context, but in statistics this was started from 2021, [20]. Also, we know that empirical Bayes method was studied extensively for several decades. In this paper, these two methods are compared in two parameter gamma distribution having the shape and scale parameters. As expected, it is found that empirical Bayes method is good in some cases. However, partial Bayes method performs even better in some cases where the shape parameter is sufficiently small, i.e. variation in the data is small. Even, overall performances of these two methods do not differ too much. But whenever we have information that shape parameter is small, then partial Bayes method in this case performs well. These results are also found by extensive simulation technique. The performance of these two estimators are also compared using two real datasets.

Keywords Empirical Bayes Method, Partial Bayes Estimation, Integrated Risk, Efficiency

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1. Introduction

Modeling positive (right skewed) data set, particularly when data are not censored in reliability and survival analysis the two parameter gamma distribution has been used extensively. Also, in several applied field such as life testing experiments, insurance, meteorology, climatology and many other physical situations, it offers better fit. A random variable $\boldsymbol{X}$ is said to follow gamma distribution, denoted as $G(\alpha,\beta)$, if the probability density function can be written in the following form

$$f(x \mid \alpha,\beta) = \frac{\beta^{\alpha}}{\Gamma\alpha} x^{\alpha-1} e^{-\beta x}; \quad x > 0, \tag{1}$$

where $\alpha > 0$ and $\beta > 0$ are the shape and scale parameters respectively. This is a flexible distribution that has increasing, decreasing or constant failure rate functions depending on the value of shape parameter $\alpha > 0$, $\alpha < 0$ or $\alpha = 0$, respectively [4]. Also, it exhibits some nice relationship with other popular distributions, including exponential and other distributions.

In literature, several attempts have been made towards estimating the two parameters of gamma distribution both in classical and Bayesian ways. The estimates based on maximum likelihood methods of the parameters of gamma distribution have been discussed in [1], [14], [27] etc. A class of moment based estimators were introduced in [21] and some of them were shown to be efficient. Saulo et al. [12] proposed two new alternative estimation methods for two parameter gamma distribution, known as modified moment estimators (MMEs) and new MMEs. It can be observed that the involvement of the gamma function in the shape parameter in (1), can led to complicated analyses

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both in MLE and Bayesian methods. Earlier, the Bayesian analysis for the gamma distribution was discussed in [7] and [23]. Son and Oh [29] obtained the Bayes estimators of the two parameter gamma distribution under the assumption of non-informative prior and using Gibbs sampling technique. Apolloni and Bassis [2] provided an estimation procedure of the two parameter gamma distribution based on algorithmic inference approach without assuming any prior of these parameters. Moala et al. [9] discussed the Bayes estimates and credible intervals for the unknown parameters of gamma distribution assuming different non-informative priors. A new two-parameter biased estimator has been proposed in [28] for gamma regression models.

Apart from the Bayesian method, empirical and hierarchical Bayesian approaches are also available. Robbins [11] introduced the empirical Bayes (EB) methods and applied in the problems where the researchers wish to make inferential statements about the unknown hyper-parameters. Some works on empirical Bayes estimation procedure have been worked out in [26], [19], [3], [13], [6], [30] etc. Hierarchical Bayesian method is more robust than ordinary Bayes method as it involves multiple stages for estimating the parameters involved in the prior distribution. Actually these two methods are step by step understanding of the prior belief, i.e. sometimes the parameters of the prior distribution are partially or completely unknown and those parameters are estimated from available data. Recently, in Bayesian setup a new estimation approach, known as Partial Bayes (PB) estimation has been proposed by Banerjee and Seal [20]. According to them, when the prior information regarding the joint parameters of a multi-parameter model is not available and one of the parameter is estimated in presence of another unknown parameter, this method is useful. The prior regarding the parameter of interest is to be realized and is estimated in Bayesian way. This is not fully known due to the presence of another unknown model parameter and this unknown part is estimated by using some classical estimation techniques, such as maximum likelihood estimates or method of moments etc. The fundamental difference between these two types of method is: in EB approach, the hyper-parameters involved in the Bayes estimate is unknown, whereas in PB approach hyper-parameters are known but at least one of the model parameters is unknown.

So, clearly the traditional EB method and recent PB method are two distinct approaches in Bayesian inference. Now, our motivation behind this study is to make a comparison between these two methods. For this purpose we consider gamma distribution as the baseline model and assume conjugate prior for the scale parameter β. The Bayes estimate is obtained under weighted squared error loss function (WSELF). Integrated risks for both the estimators, i.e. empirical and partial Bayes for the scale parameter β are obtained through an extensive simulation. Furthermore, the performances are also compared in terms of their risk efficiencies for different parameter combinations and varying sample sizes.

The article is organized in the following manner. In Section 2, Bayes estimate for the scale parameter β under conjugate gamma prior is obtained. Partial Bayes and empirical Bayes estimation methods are discussed in Sections 3 and 4 respectively. In Section 5, numerical results are shown to compare the performance of these two methods in terms of their risk values. In Section 6, two real datasets have been utilized to understand the behaviour of the estimators. The study concludes in Section 7 with a brief discussion.

2. Estimation of scale parameter

Let $x_1, x_2, \cdots, x_n$ be a random sample drawn from two parameter gamma distribution (1) and the scale parameter β is only the parameter of interest to us. The likelihood function of the observed data is

$$L = L(x \mid \alpha, \beta) = \left(\frac{\beta^\alpha}{\Gamma\alpha}\right)^n \prod_{i=1}^{n} x_i^{\alpha-1} e^{-\beta \sum_{i=1}^n x_i}. \tag{2}$$

The posterior distribution summarizes the sample information in presence of the available probabilistic information of the parameters. Here, it is obtained by considering conjugate gamma prior for the unknown scale parameter β of the baseline distribution. It is well known that the joint conjugate prior does not exist when both the parameters are unknown [15]. Also a conjugate prior is often used in limited information scenario. The prior distribution of β is

$$\pi(\beta) = \frac{b^a}{\Gamma a} \beta^{a-1} e^{-b\beta}, \tag{3}$$

where $a > 0$ and $b > 0$ are the shape and scale hyper-parameters respectively. Now, by combining the likelihood function (2) and the prior distribution (3), we have the following posterior distribution of β.

$$\begin{aligned}\Pi(\beta \mid x) &= \frac{f(x \mid \alpha, \beta)\,\pi(\beta)}{\int_0^\infty f(x \mid \alpha, \beta)\,\pi(\beta)\,d\beta} \\ &= \frac{\beta^{(\alpha\, n+a)-1} e^{-\left(\sum_{i=1}^n x_i + b\right)\beta}}{\int_0^\infty \beta^{(\alpha\, n+a)-1} e^{-\left(\sum_{i=1}^n x_i + b\right)\beta} d\beta} \\ &= \frac{\left(b + \sum_{i=1}^n x_i\right)^{\alpha\, n+a}}{\Gamma\left(\alpha\, n + a\right)} \beta^{(\alpha\, n+a)-1} e^{-\left(b + \sum_{i=1}^n x_i\right)\beta}. \end{aligned} \tag{4}$$

Therefore,

$$\beta \mid X \sim gamma\left(\alpha\, n + a, b + \sum_{i=1}^n x_i\right).$$

To obtain the Bayes estimate, specification of a loss function is a crucial part as it measures the loss which is incurred while making a decision $\delta(x)$ about the true parameter value β. For the estimation of the scale parameter, a modified form of squared error loss function i.e. the weighted squared error loss function (WSELF) is used [5] and it is defined as

$$l(\beta, \delta(x)) = \left(\frac{\delta(x)}{\beta} - 1\right)^2,$$

$$\text{i.e.}\quad l(\beta, \delta(x)) = (\delta(x) - \beta)^2\, \omega(\beta);\quad \text{with}\quad \omega(\beta) = \frac{1}{\beta^2}.$$

Therefore, the Bayes estimate under WSELF is obtained as

$$\begin{aligned}\hat{\beta}_{BE} &= \frac{E\left\{\beta\omega(\beta) | X = x\right\}}{E\left\{\omega(\beta) | X = x\right\}} \\ &= \frac{\alpha\, n + a - 2}{b + \sum_{i=1}^n x_i}. \end{aligned} \tag{5}$$

3. Partial Bayes (PB) Estimation

Sometimes when there does not exist any proper belief regarding the joint parameters of the model and we want to estimate the parameter of interest in presence of some other nuisance parameters, then the Partial Bayes (PB) method is beneficial. In this method, the Bayes estimate of the concerned parameter involves any classical estimate of the nuisance parameter. So, the estimate is obtained by mixing both the Bayesian and classical idea and also different from empirical Bayes method.

Now, to obtain the PB estimate of the scale parameter β in presence of the nuisance parameter α, let us make the following assumption that the model parameter α is unknown and it is to be estimated by some classical approach. Here, we use the maximum likelihood estimation (MLE) method due to its various desirable properties like consistency, asymptotic efficiency, and invariance [25]. So, we estimate α through MLE and plug that value in the Bayes estimate expression (5). Generally, after differentiating the log likelihood function and setting the resultant derivative to zero, the maximum likelihood estimators are obtained. But, in gamma distribution when both the model parameters are unknown it is difficult to obtain the MLE in closed from.

The log-likelihood function becomes

$$log\, L = \alpha\, n\, log\, \beta - n\, log\, \Gamma\alpha + (\alpha - 1) \sum_{i=1}^n log\, x_i - \beta \sum_{i=1}^n x_i. \tag{6}$$

Differentiating (6) with respect to α and β, we may apply the approximation of the digamma function involved in the log-derivative equation. Here, we use the result from [20], to get the MLE of the shape parameter α and final expression becomes,

$$\hat{\alpha}_{MLE} = \frac{1}{2\left[log\,\bar{x} - \overline{logx}\right]}. \tag{7}$$

After substituting the $\hat{\alpha}_{MLE}$ into the Bayes estimate expression (5), we have the following PB estimator.

$$\hat{\beta}_{PB} = \frac{\hat{\alpha}_{MLE}\, n + a - 2}{b + \sum_{i=1}^{n} x_i} = \frac{\frac{n}{2\left[log\,\bar{x} - \overline{logx}\right]} + a - 2}{b + \sum_{i=1}^{n} x_i}, \tag{8}$$

provided the hyper-parameters a and b are known.

4. Empirical Bayes (EB) Estimation

Empirical Bayes method is used to estimate the hyper-parameters directly from the data. To calculate the EB estimate, the first step is to assume the parameters of the prior to be unknown. Here for simplicity, we consider one of the hyper-parameters is unknown. i.e. the scale parameter b of the gamma prior is unknown and it is to be estimated by empirical Bayes approach. The posterior marginal density of X is expressed as,

$$\begin{aligned} m(X) &= \int_0^{\infty} f\left(X \mid \alpha, \beta\right) \pi(\beta)\, d\beta \\ &= \left(\frac{1}{\Gamma\alpha}\right)^n \prod_{i=1}^{n} x_i^{\alpha-1} \frac{b^a}{\Gamma a} \int_0^{\infty} \beta^{(\alpha\, n + a) - 1} e^{-\left(b + \sum_{i=1}^{n} x_i\right)^{\beta}} d\beta \\ &= \left(\frac{1}{\Gamma\alpha}\right)^n \prod_{i=1}^{n} x_i^{\alpha-1} \frac{b^a}{\Gamma a} \frac{\Gamma\left(\alpha\, n + a\right)}{\left(b + \sum_{i=1}^{n} x_i\right)^{\alpha\, n + a}}. \end{aligned} \tag{9}$$

Taking logarithm on the both sides of the above equation (9), the expression becomes,

$$log\, m(X) = -n\, log\, \Gamma\alpha + (\alpha - 1) \sum_{i=1}^{n} log\, x_i + a\, log\, b - log\, \Gamma a + log\, \Gamma(\alpha\, n + a) - (\alpha\, n + a)\; log\left(b + \sum_{i=1}^{n} x_i\right).$$

Differentiating the above log-likelihood with respect to b and equating to zero, we get

$$\begin{aligned} & \frac{\partial}{\partial b} log\, m(X) = 0 \\ \Rightarrow \quad & a \sum_{i=1}^{n} x_i - \alpha\, b\, n = 0 \\ \Rightarrow \quad & \hat{b}_M = \frac{a \sum_{i=1}^{n} x_i}{\alpha\, n} = \frac{a\, \bar{x}}{\alpha}. \end{aligned} \tag{10}$$

Therefore, by plugging $\hat{b}_M$ into the Bayes estimate (5), the EB estimator of the scale parameter β becomes,

$$\hat{\beta}_{EB} = \frac{\alpha\, n + a - 2}{\hat{b}_M + \sum_{i=1}^{n} x_i} = \frac{\alpha\, n + a - 2}{\frac{a\, \bar{x}}{\alpha} + \sum_{i=1}^{n} x_i}, \tag{11}$$

provided α and a are known.

5. Risk function for EB and PB estimators

If $\delta(x)$ is an estimator of θ, then the risk function is defined as

$$R\left(\theta, \delta(x)\right) = E_{X|\theta}\left[L\left(\theta, \delta(x)\right)\right].$$

Here, we study the performance of the PB and EB estimators on the basis of risk values. Under WSELF, the risk function of PB estimator is:

$$E_{X|\beta}\left[\frac{\hat{\beta}_{PB}}{\beta} - 1\right]^2 = E_{X|\beta}\left[\frac{\frac{n}{2\left[\overline{log\, \bar{x}} - \overline{log\, x}\right]} + a - 2}{\beta\left(b + \sum_{i=1}^{n} x_i\right)} - 1\right]^2, \tag{12}$$

and the risk function of EB estimator is

$$E_{X|\beta}\left[\frac{\hat{\beta}_{EB}}{\beta} - 1\right]^2 = E_{X|\beta}\left[\frac{\alpha\, n + a - 2}{\beta\left(\frac{a\bar{x}}{\alpha} + \sum_{i=1}^{n} x_i\right)} - 1\right]^2. \tag{13}$$

Table 1. Bayes estimates and corresponding integrated risk (IR) of the scale parameter β when α = 0.07.

Parameter choices	Sample sizes (n)	Partial Bayes		Empirical Bayes		Risk efficiency
		Estimate	IR	Estimate	IR	
$a = 3.5, b = 0.1$	25	17.48871	0.2756426	48.73681	7.4821070	27.144230
	50	18.35458	0.2295992	34.50786	0.5664766	2.4672410
	75	18.84280	0.2098712	33.15751	0.2709554	1.2910560
	100	19.25284	0.1995069	33.40518	0.1749845	0.8770851
	300	19.80639	0.1828095	33.44517	0.0505348	0.2764340
$a = 5.0, b = 0.3$	25	10.07065	0.2048069	25.71582	9.8135470	47.916090
	50	9.978588	0.1739358	17.14818	0.6536450	3.7579680
	75	9.915531	0.1615695	16.05809	0.2929509	1.8131570
	100	9.908242	0.1574777	15.96175	0.1841226	1.1691980
	300	9.613898	0.1612054	15.60387	0.0507229	0.3146476
$a = 2.5, b = 5.0$	25	0.1928994	0.3566304	0.5970489	5.3924700	15.120610
	50	0.2247639	0.2910540	0.4613580	0.4984108	1.7124340
	75	0.2414587	0.2577186	0.4568100	0.2552579	0.9904519
	100	0.2532897	0.2377006	0.4666648	0.1687698	0.7100098
	300	0.2752531	0.1981348	0.4772773	0.0504291	0.2545190
$a = 4.0, b = 0.25$	25	8.582835	0.2415147	23.02760	8.3530050	34.585900
	50	8.738991	0.2045716	15.88911	0.5975816	2.9211370
	75	8.829574	0.1889019	15.10896	0.2785609	1.4746330
	100	8.931564	0.1818174	15.14210	0.1780866	0.9794806
	300	8.982629	0.1744432	15.02647	0.0505941	0.2900322
$a = 3.0, b = 0.75$	25	1.773787	0.3218410	5.095942	6.4990880	20.193470
	50	1.929007	0.2557290	3.739363	0.5333403	2.0855680
	75	2.013674	0.2290756	3.640503	0.2631510	1.1487510
	100	2.078629	0.2146777	3.690851	0.1718638	0.8005663
	300	2.185897	0.1889360	3.732335	0.0504795	0.2671778

Due, to the complexity in the above expressions, we proceed for comparison of risks through simulation study.

In this comparison study, the natures of the parameters and hyper-parameters play important role to keep the same characteristics of both the estimation process. Keeping this in mind, the problem is two fold: for PB estimation, the hyper-parameters a, b are supposed to be known, say a_0, b_0 and an initial parameter value is chosen for α, say α_0. In contrast, for EB estimation, the model parameter α is some known value α_0 and b_0 is considered as an initial value for the hyper-parameter b. We choose the same α_0 and b_0 to maintain the similar characteristics having in the former estimation process. Also, the shape parameter a is assumed to be known throughout the study. The algorithm for calculating the integrated risk of both the estimators is as follows.

Table 2. Bayes estimates and corresponding integrated risk (IR) of the scale parameter β when α = 0.1.

Parameter choices	Sample sizes (n)	Partial Bayes		Empirical Bayes		Risk efficiency
		Estimate	IR	Estimate	IR	
$a = 3.5, b = 0.1$	25	18.58185	0.2472902	38.68820	1.8658080	7.5450120
	50	19.58792	0.2035477	33.63540	0.3259699	1.6014420
	75	20.12938	0.1850623	33.57426	0.1878741	1.0151930
	100	20.43152	0.1744383	33.50212	0.1155488	0.6624048
	300	20.75827	0.1620532	33.67480	0.0348106	0.2148097
$a = 5.0, b = 0.3$	25	10.27036	0.1884662	19.75371	2.2932830	12.1681400
	50	10.23523	0.1593288	16.33314	0.3560626	2.2347650
	75	10.22439	0.1485426	15.99976	0.1970423	1.3265040
	100	10.20261	0.1431134	15.82114	0.1187352	0.8296583
	300	9.927780	0.1474491	15.67338	0.0348545	0.2363830
$a = 2.5, b = 5.0$	25	0.220847	0.3130839	0.498775	1.5003130	4.7920490
	50	0.252886	0.2475859	0.462048	0.3037544	1.2268640
	75	0.268441	0.2174946	0.470251	0.1814571	0.8343060
	100	0.277122	0.2002071	0.473228	0.1134567	0.5666969
	300	0.291701	0.1718851	0.481437	0.0347880	0.2023912
$a = 4.0, b = 0.25$	25	8.932833	0.2206569	18.018600	2.0228860	9.1675640
	50	9.156107	0.1847893	15.342990	0.3364857	1.8209160
	75	9.289066	0.1700373	15.203180	0.1910109	1.1233470
	100	9.361238	0.1617502	15.118670	0.1166139	0.7209507
	300	9.371085	0.1564860	15.116890	0.0348240	0.2225376
$a = 3.0, b = 0.75$	25	1.934432	0.2818258	4.128834	1.6919900	6.0036730
	50	2.100920	0.2220109	3.688211	0.3150251	1.4189630
	75	2.186023	0.1981228	3.713971	0.1846797	0.9321476
	100	2.233580	0.1846135	3.720573	0.1144923	0.6201729
	300	2.302003	0.1659307	3.761312	0.0347986	0.2097174

- *Step 1 :* Initialize a and b_0.
- *Step 2 :* Generate $\beta_1, \beta_2, \cdots, \beta_m$ from *gamma(a,* b_0*)*.
- *Step 3 :* Initialize α_0 with a fixed value.
- *Step 4 :* Generate $X_1, X_2, \cdots, X_K$ each of size n from *gamma(*α_0, β_1*)*.
- *Step 5 :* Calculate $\hat{\alpha}_{MLE}$ and $\hat{b}_M$ for *K* times.
- *Step 6 :* Calculate $\hat{\beta}_{PB}$ and $\hat{\beta}_{EB}$ for *K* times.
- *Step 7 :* Calculate the loss $\left(\frac{\hat{\beta}_{PB}}{\beta_1} - 1\right)^2$ and $\left(\frac{\hat{\beta}_{EB}}{\beta_1} - 1\right)^2$ for *K* times.

Table 3. Bayes estimates and corresponding integrated risk (IR) of the scale parameter β when α = 0.2.

Parameter choices	Sample sizes (n)	Partial Bayes Estimate	Partial Bayes IR	Empirical Bayes Estimate	Empirical Bayes IR	Risk efficiency
$a = 3.5, b = 0.1$	25	21.61015	0.1940002	33.43734	0.4876370	2.5135900
	50	22.35440	0.1483680	32.73770	0.1105453	0.7450753
	75	22.94670	0.1327892	33.59245	0.0736583	0.5547005
	100	23.07599	0.1246019	33.83282	0.0516122	0.4142166
	300	23.16312	0.1134408	34.17726	0.0163301	0.1439524
$a = 5.0, b = 0.3$	25	11.03357	0.1545381	16.23696	0.5324451	3.4453970
	50	10.99762	0.1238630	15.46015	0.1127755	0.9104854
	75	11.09719	0.1145925	15.73437	0.0745274	0.6503694
	100	11.06346	0.1101029	15.79185	0.0519571	0.4718955
	300	10.88412	0.1076161	15.87598	0.0163348	0.1517874
$a = 2.5, b = 5.0$	25	0.2861933	0.2341391	0.459327	0.4528784	1.9342280
	50	0.3073097	0.1685755	0.4624306	0.1091969	0.6477624
	75	0.3199714	0.1464087	0.4778662	0.0731084	0.4993445
	100	0.3238546	0.1347884	0.4826388	0.0513970	0.3813166
	300	0.3294386	0.1170489	0.4893217	0.0163275	0.1394927
$a = 4.0, b = 0.25$	25	10.00892	0.1788340	15.25264	0.5035284	2.8156180
	50	10.18536	0.1389073	14.77370	0.1112723	0.8010547
	75	10.38914	0.1258683	15.11391	0.0739441	0.5874725
	100	10.41866	0.1190641	15.20307	0.0517248	0.4344284
	300	10.40716	0.1112433	15.33203	0.0163315	0.1468091
$a = 3.0, b = 0.75$	25	2.344042	0.2143747	3.666493	0.4707497	2.1959200
	50	2.459032	0.1565669	3.635681	0.1098487	0.7016087
	75	2.538447	0.1381430	3.743081	0.0733788	0.5311803
	100	2.559117	0.1284752	3.774952	0.0515027	0.4008760
	300	2.582230	0.1147722	3.820124	0.0163287	0.1422706

- *Step 8 :* Take an average with respect to density to obtain the risk function $E_{X|\beta_1}\left[\frac{\hat{\beta}_{PB}}{\beta_1}-1\right]^2$ and $E_{X|\beta_1}\left[\frac{\hat{\beta}_{EB}}{\beta_1}-1\right]^2$.
- *Step 9 :* Repeat *Step 4 - Step 10* for remaining $\beta_i's, \quad i = 2, 3, \cdots, m$.
- *Step 10 :* Take an average with respect to $\beta_i's$ to obtain the integrated risk of $\hat{\beta}_{PB}$ and $\hat{\beta}_{EB}$ respectively.
- *Step 11 :* Calculate the integrated risk efficiency of PB estimate with respect to EB estimate by using

$$R_E\left(\hat{\beta}_{PB}, \hat{\beta}_{EB}\right) = \frac{IR\left(\hat{\beta}_{EB}\right)}{IR\left(\hat{\beta}_{PB}\right)}.$$

Therefore, if $R_E < 1$, then EB estimator performs better than the PB estimator, whereas if $R_E > 1$, then PB estimators outperform the EB estimator. In order to compare the performances of both the estimators, we carry out an extensive simulation study by using [24] (version 3.6.1). In particular, the choices of hyper-parameters are considered as (3.5,0.1), (5,0.3), (2.5,5), (4,0.25) and (3,0.75). For all combinations of hyper-parameters, we generate $m = 50$ times β from the prior distribution. The initial shape parameter of the model is chosen as $\alpha = 0.07, 0.1, 0.2, 0.5, 1$ and for every combinations of (α, β), we generate random samples of sizes

$n = 25, 50, 75, 100, 300$ from the baseline distribution. The average estimators, integrated risk values and the corresponding efficiency are obtained over K=1000 simulated samples. In order to study the performance of the estimators, the comparison has been evaluated on the basis of the integrated risk efficiency criteria.

Table 4. Bayes estimates and corresponding integrated risk (IR) of the scale parameter β when α = 0.5.

Parameter choices	Sample sizes (n)	Partial Bayes		Empirical Bayes		Risk efficiency
		Estimate	IR	Estimate	IR	
$a = 3.5, b = 0.1$	25	27.17094	0.1264698	32.93048	0.0841465	0.6653485
	50	27.27450	0.0843969	33.59052	0.0427108	0.5060700
	75	27.54539	0.0697373	34.30609	0.0276786	0.3968980
	100	27.43628	0.0618230	34.31149	0.0200651	0.3245575
	300	27.10038	0.0536785	34.47978	0.0069873	0.1301699
$a = 5.0, b = 0.3$	25	12.95606	0.1112644	15.47253	0.0851411	0.7652144
	50	12.85426	0.0770174	15.65056	0.0428168	0.5559370
	75	12.92026	0.0649391	15.95270	0.0277464	0.4272684
	100	12.84006	0.0582931	15.94325	0.0200882	0.3446065
	300	12.61750	0.0524162	16.00654	0.0069879	0.1333155
$a = 2.5, b = 5$	25	0.3833854	0.1394703	0.4672243	0.0835771	0.5992466
	50	0.3876364	0.0895982	0.4798510	0.0426512	0.4760276
	75	0.3925298	0.0728563	0.4907919	0.0276346	0.3793022
	100	0.3914215	0.0639875	0.4911357	0.0200504	0.3133477
	300	0.3876249	0.0543831	0.4938671	0.0069870	0.1284767
$a = 4.0, b = 0.25$	25	12.23853	0.1221711	14.83291	0.0844644	0.6913616
	50	12.24548	0.0819855	15.08459	0.0427442	0.5213623
	75	12.35689	0.0680832	15.39542	0.0277010	0.4068698
	100	12.30578	0.0605505	15.39387	0.0200727	0.3315038
	300	12.15350	0.0531949	15.46448	0.0069875	0.1313567
$a = 3.0, b = 0.75$	25	3.024071	0.1328925	3.664751	0.0838487	0.6309510
	50	3.042164	0.0865509	3.750449	0.0426797	0.4931166
	75	3.074928	0.0709130	3.833086	0.0276564	0.3900052
	100	3.063684	0.0625654	3.834712	0.0200577	0.3205871
	300	3.028455	0.0538888	3.854766	0.0069871	0.1296585

The simulation results have been reported in Table 1-5 and the following observations can be drawn from the mentioned tables. It has been observed that when $\alpha < 0.5$, the integrated risk efficiency is greater than 1 for small and moderate sizes of generated samples. In that case, the PB estimator works well compared to EB estimator. However, for large sample, $R_E < 1$, i.e. EB estimator performs well as compared to PB estimator. When $\alpha \geq 0.5$, it is evident from Table 3 and 4 that the EB estimator is more preferable than PB estimator as integrated risk efficiency is less than 1 for all choices of sample. The integrated risk efficiencies are very sensitive with the variation in α. This comparative study mostly depends on the choices of the model's shape parameter. If it is small, then there is not much variation in the data set and if the size of data set is small or moderate, then PB estimator is preferable. Otherwise, for greater variation in large data set, EB estimator is found to be effective than PB estimator. As the sample size increases, the integrated risks of both the estimators decrease. It shows the consistency of both the estimators. These are the important findings of this study.

Table 5. Bayes estimates and corresponding integrated risk (IR) of the scale parameter β when α = 1.0.

Parameter choices	Sample sizes (n)	Partial Bayes		Empirical Bayes		Risk efficiency
		Estimate	IR	Estimate	IR	
$a = 3.5, b = 0.1$	25	31.29433	0.1051508	33.60700	0.0414585	0.3942768
	50	30.81490	0.0577802	34.24492	0.0194550	0.3367068
	75	30.52105	0.0430690	34.48706	0.0135894	0.3155249
	100	30.46864	0.0367719	34.55111	0.0104971	0.2854659
	300	29.97220	0.0248034	34.55662	0.0034520	0.1391725
$a = 5.0, b = 0.3$	25	14.63176	0.0972879	15.65825	0.0415587	0.4271721
	50	14.36623	0.0550255	15.91232	0.0194727	0.3538849
	75	14.21274	0.0414922	16.01572	0.0135984	0.3277330
	100	14.17772	0.0356702	16.04212	0.0105015	0.2944066
	300	13.92535	0.0244786	16.04068	0.0034520	0.1410215
$a = 2.5, b = 5.0$	25	0.4473831	0.1115936	0.4800865	0.0414032	0.3710178
	50	0.4408087	0.0596985	0.4901827	0.0194440	0.3257025
	75	0.4367315	0.0440638	0.4938475	0.0135834	0.3082675
	100	0.4360889	0.0374323	0.4948365	0.0104942	0.2803508
	300	0.4291916	0.0249694	0.4949997	0.0034519	0.1382461
$a = 4.0, b = 0.25$	25	14.01156	0.1033161	15.09200	0.0414899	0.4015821
	50	13.80321	0.0569953	15.36400	0.0194608	0.3414448
	75	13.67615	0.0425642	15.46964	0.0135923	0.3193372
	100	13.65501	0.0363991	15.49727	0.0104986	0.2884294
	300	13.43886	0.0246758	15.49844	0.0034519	0.1398930
$a = 3.0, b = 0.75$	25	3.500543	0.1086916	3.752290	0.0414295	0.3811654
	50	3.445541	0.0586724	3.827272	0.0194494	0.3314911
	75	3.412310	0.0434622	3.855101	0.0135864	0.3126019
	100	3.406425	0.0370079	3.862538	0.0104957	0.2836058
	300	3.350842	0.0248390	3.863481	0.0034519	0.1389741

6. Real data analysis

In this section, we apply the previously mentioned Bayesian methodologies on two real datasets in order to assess the applicability of the proposed study. We consider one large data with large variance as well as one small data with small variance to demonstrate the significance of the partial and empirical Bayes approaches. Data set I represents the survival times of 121 patients with Breast cancer whereas Data set II represents the Relief time of 20 patients receiving analgesic respectively. To check whether the considered datasets follow the gamma distribution or not is the initial step. We use Kolmogorov-Smirnov (KS) test to determine the models' goodness-of-fit. The KS statistic and the corresponding P-value for Data set I is (0.0761, 0.4847) and for Data set II is (0.1734, 0.5845). Apart from the theoretical fitting, we further display fitted density with the histogram and empirical CDF plot in Figures 1 and 2 respectively for both the datasets. These graphical representations indicate that the baseline distribution provides satisfactory fitting to both the datasets.

In this comparative study, the performance of partial and empirical Bayes estimators are evaluated on the basis of posterior risk values and it has been found that

$$E_{\beta|X} L(\beta, \delta(x)) = \frac{1}{\alpha n + a - 1}. \tag{14}$$

When using the PB estimation method, we substitute $\hat{\alpha}_{MLE}$ in (14) whereas for EB approach, we consider some known values of α which is chosen close to $\hat{\alpha}_{MLE}$ for comparison purpose. Similarly, to determine the PB estimator, the hyper-parameter b is some known quantity and it is considered to be closed to $\hat{b}_M$ which is obtained by the EB estimation process. As we want to establish a comparison study between both the estimation methodologies, we have taken the known parameter values nearest to the estimated ones.

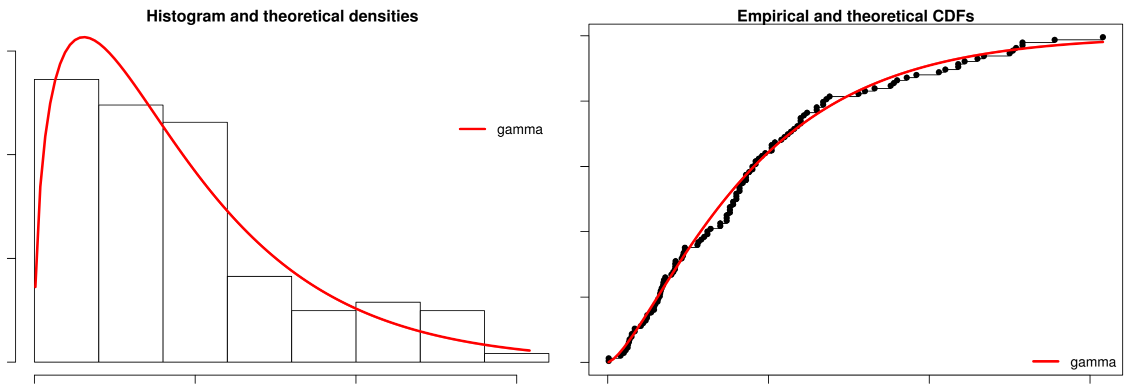

Figure 1. Histogram (left) and empirical vs theoretical cdf (right) of the Breast cancer data.

6.1. *Data set I*

Our first data set is related with the survival times of 121 patients with Breast cancer. This data set has been originally reported in [8] and given in Table 6. Recently, this data have been also used in [17] and [10].

Table 6. The survival times of 121 patients with Breast cancer.

0.3	0.3	4.0	5.0	5.6	6.2	6.3	6.6	6.8	7.4	7.5	8.4	8.4
10.3	11.0	11.8	12.2	12.3	13.5	14.4	14.4	14.8	15.5	15.7	16.2	16.3
16.5	16.8	17.2	17.3	17.5	17.9	19.8	20.4	20.9	21.0	21.0	21.1	23.0
23.4	23.6	24.0	24.0	27.9	28.2	29.1	30.0	31.0	31.0	32.0	35.0	35.0
37.0	37.0	37.0	38.0	38.0	38.0	39.0	39.0	40.0	40.0	40.0	41.0	41.0
41.0	42.0	43.0	43.0	43.0	44.0	45.0	45.0	46.0	46.0	47.0	48.0	49.0
51.0	51.0	51.0	52.0	54.0	55.0	56.0	57.0	58.0	59.0	60.0	60.0	60.0
61.0	62.0	65.0	65.0	67.0	67.0	68.0	69.0	78.0	80.0	83.0	88.0	89.0
90.0	93.0	96.0	103.0	105.0	109.0	109.0	111.0	115.0	117.0	125.0	126.0	127.0
129.0	129.0	139.0	154.0									

The length of the given data set is $n = 121$ and variance is 1244.46, which indicates that the data are widely spread out from the mean. The estimated MLEs of the parameters α and β are obtained as $\hat{\alpha}_{MLE} = 1.4963$ and $\hat{\beta}_{MLE} = 0.0323$. We use this $\hat{\alpha}_{MLE}$ to derive the PB estimator and for EB approach, the choices of α are taken as 1.5, 2.5, 3.5, 4.5 and 5.5. We compute the partial and empirical Bayes estimators with the corresponding posterior risk values for different choices of known hyper-parameter a . Table 7 provides the PB and EB estimators with their 95% credible interval. Table 8 also includes the posterior risk values of both the estimators obtained in Table 7 to observe their performances. From Table 7, it is seen that the PB and EB estimators belong to their corresponding credible interval. Also from Table 8, it is observed that the posterior risk of EB estimator is comparatively small than that of PB estimator. So, we may conclude that the EB estimator seems to provide a better performance than the PB estimator for highly dispersed large data set.

Table 7. Partial and empirical Bayes estimates and 95% credible interval in Breast cancer data.

Choices of hyper-parameter	PBE	95% Credible Interval Lower	95% Credible Interval Upper	EBE	95% Credible Interval Lower	95% Credible Interval Upper
$a = 0.5$	0.028820	0.024882	0.033807	0.032021	0.027844	0.037247
	0.028851	0.024909	0.033843	0.053606	0.048056	0.060205
	0.028867	0.024922	0.033861	0.075190	0.068526	0.082905
	0.028872	0.024926	0.033867	0.096775	0.089147	0.105454
	0.028877	0.024931	0.033874	0.118360	0.109870	0.127900
$a = 1.0$	0.028832	0.024898	0.033813	0.032022	0.027850	0.037240
	0.028894	0.024951	0.033885	0.053606	0.048061	0.060200
	0.028920	0.024973	0.033915	0.075191	0.068530	0.082900
	0.028935	0.024986	0.033933	0.096775	0.089150	0.105450
	0.028945	0.024995	0.033945	0.118360	0.109873	0.127897
$a = 2.5$	0.028858	0.024936	0.033817	0.032025	0.027868	0.037220
	0.029016	0.025072	0.034003	0.053608	0.048075	0.060184
	0.029083	0.025130	0.034081	0.075192	0.068542	0.082887
	0.029124	0.025166	0.034129	0.096776	0.089161	0.105438
	0.029145	0.025184	0.034154	0.118361	0.109883	0.127886
$a = 4.0$	0.028888	0.024977	0.033827	0.032028	0.027885	0.037199
	0.029137	0.025193	0.034119	0.053610	0.048089	0.060168
	0.029250	0.025291	0.034252	0.075193	0.068554	0.082874
	0.029307	0.025340	0.034319	0.096777	0.089172	0.105427
	0.029349	0.025376	0.034367	0.118362	0.109893	0.127876

Table 8. Posterior risk (PR) of the partial and empirical Bayes estimates for Breast cancer data.

Choices of hyper-parameter	PR of PBE putting $\hat{\alpha}_{MLE}$	PR of EBE $\alpha = 1.5$	PR of EBE $\alpha = 2.5$	PR of EBE $\alpha = 3.5$	PR of EBE $\alpha = 4.5$	PR of EBE $\alpha = 5.5$
$a = 0.5$	0.00613	0.00552	0.00331	0.00236	0.00184	0.0015
$a = 1.0$	0.00612	0.00551	0.00331	0.00236	0.00184	0.0015
$a = 2.5$	0.00606	0.00546	0.00329	0.00235	0.00183	0.0015
$a = 4.0$	0.00601	0.00542	0.00327	0.00234	0.00183	0.0015

6.2. Data set II

The second data set contains the lifetimes data relating to Relief times (in minutes) of 20 patients receiving an analgesic and this data set has been reported in [1]. Recently, several works including [16], [22], [18] etc. also used this data in their study and it is listed as follows.

1.1, 1.4, 1.3, 1.7, 1.9, 1.8, 1.6, 2.2, 1.7, 2.7, 4.1, 1.8, 1.5, 1.2, 1.4, 3, 1.7, 2.3, 1.6, 2.

The length of the data set is $n = 20$ and variance is 0.4958. So, we consider a small data set with less variability where all the data are concentrated to the mean. The MLE of the parameters α and β are $\hat{\alpha}_{MLE} = 9.6701$ and $\hat{\beta}_{MLE} = 5.0895$. In order to obtain the PB estimator, we use the $\hat{\alpha}_{MLE}$ and in case of EB approach the choices of α are taken as 5, 5.75, 6.5, 7.25 and 8.0. As the hyper-parameter a is known, we take it as $a = 0.5, 1.0, 2.5, 4.0$. So, both the estimators with their 95% credible interval and the associated posterior risk values are given in Tables 9 and 10 respectively for different choices of hyper-parameters.

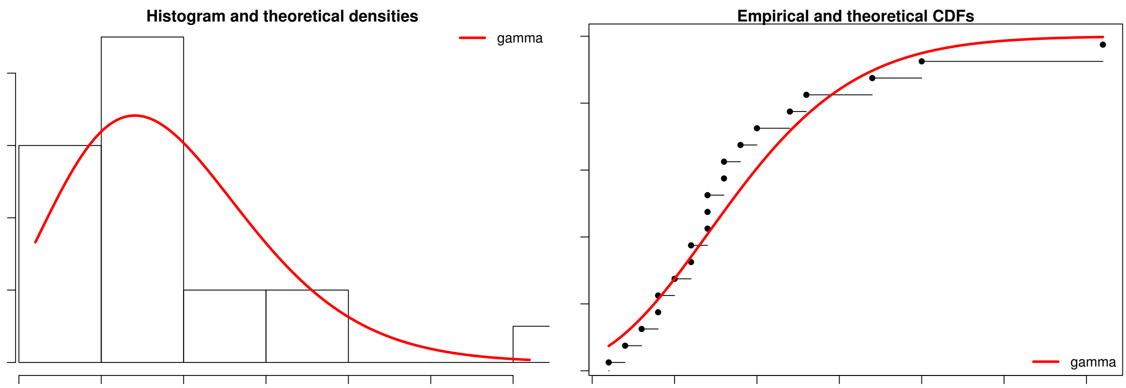

Figure 2. Histogram (left) and empirical vs theoretical cdf (right) of the Relief times data.

Table 9. Partial and empirical Bayes estimates and 95% credible interval in Relief times data.

Choices of hyper-parameter	PBE	95% Credible Interval Lower	95% Credible Interval Upper	EBE	95% Credible Interval Lower	95% Credible Interval Upper
	4.938890	4.307814	5.724288	2.579209	2.142315	3.170409
	4.941478	4.310071	5.727287	2.973912	2.499622	3.602613
$a = 0.5$	4.944068	4.312331	5.730290	3.368623	2.859310	4.032426
	4.946661	4.314593	5.733296	3.763339	3.220977	4.460257
	4.947959	4.315725	5.734800	4.158059	3.584319	4.886406
	4.927467	4.298591	5.709902	2.579468	2.143464	3.169015
	4.933895	4.304199	5.717351	2.974138	2.500703	3.601318
$a = 1.0$	4.939049	4.308695	5.723323	3.368823	2.860335	4.031213
	4.942922	4.312074	5.727811	3.763518	3.221952	4.459111
	4.945507	4.314329	5.730807	4.158222	3.585253	4.885318
	4.893869	4.271470	5.667580	2.580231	2.146863	3.164895
	4.908993	4.284671	5.685095	2.974804	2.503907	3.597484
$a = 2.5$	4.921668	4.295734	5.699773	3.369414	2.863374	4.027614
	4.930579	4.303512	5.710094	3.764050	3.224850	4.455710
	4.939523	4.311318	5.720451	4.158704	3.588028	4.882086
	4.861240	4.245142	5.626467	2.580972	2.150191	3.160867
	4.885966	4.266735	5.655086	2.975453	2.507052	3.593725
$a = 4.0$	4.904677	4.283074	5.676742	3.369992	2.866363	4.024078
	4.919749	4.296236	5.694187	3.764571	3.227706	4.452361
	4.932380	4.307266	5.708806	4.159178	3.590767	4.878899

It may be noticed from Table 9 that all the credible intervals contain their corresponding PB and EB estimators. From Table 10, it is evident that the PB estimator has smaller posterior risk as compared to the EB estimator. So, in case of data set having small variation in it, PB estimator is appeared to be better than EB estimator.

Table 10. Posterior risk (PR) of the partial and empirical Bayes estimates for Relief times data.

Choices of	PR of PBE	PR of EBE				
hyper-parameter	putting $\hat{\alpha}_{MLE}$	$\alpha = 5.0$	$\alpha = 5.75$	$\alpha = 6.5$	$\alpha = 7.25$	$\alpha = 8.0$
$a = 0.5$	0.00527	0.01005	0.00873	0.00772	0.00692	0.00627
$a = 1.0$	0.00526	0.01000	0.00870	0.00769	0.00690	0.00625
$a = 2.5$	0.00522	0.00985	0.00858	0.00760	0.00683	0.00619
$a = 4.0$	0.00518	0.00971	0.00847	0.00752	0.00676	0.00613

7. Conclusion

In this article, an attempt has been made to establish a comparative study between two Bayesian estimation methodologies i.e. empirical Bayes approach and partial Bayes estimation. The former is one of the methods, traditionally used when prior parameters are unknown whereas the later one is used when there is no proper information regarding the joint parameters of the model and we want to estimate only one of them in presence of the other nuisance parameter. Two parameter gamma distribution is considered as a baseline model and scale parameter being the parameter of interest. Both the partial and empirical Bayes estimators have been derived by considering gamma conjugate prior under the weighted squared error loss function. As the expression of risk is not obtained in an explicit form, it is not feasible to differentiate these methods theoretically. Thus, an extensive simulation study is carried out to investigate the behaviour of both the estimators and it is evaluated with respect to integrated risk values. From the simulation study, it is remarked that, when we do not have variation in the sample and also the sample size is small or moderate, then PB estimator works well in comparison to EB estimator. Otherwise, for greater variability in a large sample, the EB estimator seems to be better than the PB estimator. Moreover, two real datasets have been considered to illustrate the applicability of both the estimation procedures. So, it is evident that both the estimation methods have their own significance. A researcher should decide whether to choose between these two estimation methods based on the available data information.

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