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The Type II Exponentiated Half Logistic-Gompertz-G Power Series Class of Distributions: Properties and Applications

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Abstract We propose and study a new generalized class of distributions called the Type II Exponentiated Half Logistic-Gompertz-G Power Series (TIIEHL-Gom-GPS) distribution. Some structural properties including expansion of density, ordinary and conditional moments, generating function, order statistics and entropy are derived. We present some special cases of the proposed distribution. The maximum likelihood method is used for estimating the model parameters. The usefulness and importance of the new class of distributions are illustrated by means of two applications to real data sets.

Keywords Gompertz-G, Exponentiated-G Distribution, Maximum Likelihood Estimation, Power Series Distribution, Simulations.

AMS 2010 subject classifications 62E99; 60E05; 62E15

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1. Introduction

New families of distributions are widely studied and employed to describe real lifetime data sets. These new families have found wider applicability in various areas of biology, environmental sciences, economics, physics and hydrology. The type II transformations are very important and have led to various new families of distributions. Some well-known type II transformations include type II exponentiated half logistic generated family of distributions by Al-Mofleh et al. [3], type II half logistic Kumaraswamy distribution by ZeinEldin et al. [24], type II power Topp-Leone generated family of distributions by Bantan et al. [4] and type II half logistic family of distributions by Soliman et al. [22].

Al-Mofleh et al. [3], developed the type II exponentiated half logistic family of distributions with the cumulative distribution function (cdf) given by

$$F(x;\beta,\alpha,\Psi) \quad = \quad 1-\left(\frac{1-\bar{G}^{\beta}(x;\Psi)}{1+\bar{G}^{\beta}(x;\Psi)}\right)^{\alpha}, \tag{1}$$

where $\bar{G}(x;\Psi)=1-G(x;\Psi)$ and $G(x;\Psi)$ is the baseline cdf with parameter vector Ψ and $\beta,\alpha>0$, are the shape parameters. In this work the parameter β is taken to be equal to 1.

The cdf and probability density function (pdf) of the Gompertz-G (Gom-G) family of distributions by Alizadeh et al. [1] are given by

$$F(x;\lambda,\gamma,\Psi)=1-\exp\left[\frac{\lambda}{\gamma}(1-(1-G(x;\Psi))^{-\gamma})\right] \tag{2}$$

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and

$$f(x;\lambda,\gamma,\Psi)=\lambda g(x;\Psi)(1-G(x;\Psi))^{-\gamma-1}\exp\left[\frac{\lambda}{\gamma}(1-(1-G(x;\Psi))^{-\gamma})\right], \tag{3}$$

respectively, for $\lambda,\gamma>0$ and Ψ is the parameter vector from the baseline distribution. We set the parameter $\lambda=1$ from equation (2) and (3) to avoid the problem of over parameterisation.

The pdf of the Gompertz distribution is either right or left-skewed. Work by Sanku et al. [21] studied several properties and different methods of estimation of the unknown parameters of Gompertz distribution. The Gompertz distribution has a monotonically increasing hazard rate function. Some of its recent generalization include the Topp-Leone-Gompertz-G family of distributions by Oluyede et al. [19] and the Marshall-Olkin-Gompertz-G family of distributions by Chipepa and Oluyede [11].

The study of lifetimes holds significance in numerous scientific and technological areas. Various distributions have been proposed in existing literature to model lifetime data using compounding method. Examples include the type II Exponentiated Half-Logistic-Topp-Leone-G power series class of distributions by Moakofi et al. [17], a new generalized family of lifetime distributions by Goldoust et al. [13], a new two-sided class of lifetime distributions by Kharazmi et al. [15], and the Odd Log-Logistic Transmuted-G family of distributions by Alizadeh et al. [2].

We are motivated by the applicability of the Gompertz-G family of distributions in many fields. Furthermore, the usefulness of power series distributions and the versatility of generalized power series distributions inspired the development of the TIIEHL-Gom-GPS class of distributions. The proposed distribution exhibit both monotonic and non-monotonic hazard rate functions, which is a crucial improvement to the Gompertz distribution. Also, from data modeling examples presented, the new model is a strong alternative model to reliability data. We hope the newly developed model will get attention from various researchers.

This paper is organized as follows: The structural properties of the new TIIEHL-Gom-GPS class of distributions are presented in Section 2. Section 3 contains some special cases of the TIIEHL-Gom-GPS class of distributions. Monte Carlo simulation study is presented in Section 4. Applications of the proposed model to real data are given in Section 5, followed by concluding remarks.

2. The Model and Properties

Let N be a discrete random variable following a power series distribution assumed to be truncated at zero, whose probability mass function (pmf) is given by

$$P(N=n)=\frac{a_n\theta^n}{C(\theta)},\quad n=1,2,\ldots, \tag{4}$$

where $C(\theta)=\sum_{n=1}^{\infty}a_n\theta^n$ is finite, $\theta>0$, and $\{a_n\}_{n\geq1}$ a sequence of positive real numbers. The power series family of distributions includes binomial, Poisson, geometric and logarithmic distributions.

We combine the type II exponentiated half logistic-Gompertz-G (TIIEHL-Gom-G) distribution and the power series distribution to obtain a new class of distributions, namely, TIIEHL-Gom-GPS distribution. The cdf of the TIIEHL-Gom-G family of distributions is obtained by inserting the cdf in equation (2) into the cdf in equation (1) (with $\beta=1$). Given N, let $Y_1,Y_2,...,Y_N$ be identically and independently distributed (iid) random variables following TIIEHL-Gom-G distribution. Let $X=Y_{(1)}=\min(Y_1,\ldots,Y_N)$, then the conditional distribution of X

given $N = n$ is given by

$$\begin{aligned} G_{X|N=n}(x) &= 1 - \prod_{i=1}^{n}(1 - G_{TIIEHL-Gom-G}(x;\gamma,\alpha,\Psi)) = 1 - S^n_{TIIEHL-Gom-G}(x;\gamma,\alpha,\Psi) \\ &= 1 - \left(\left[\frac{\exp\left[\frac{1}{\gamma}(1-[1-G(x;\Psi)]^{-\gamma})\right]}{1+\left\{1-\exp\left[\frac{1}{\gamma}(1-[1-G(x;\Psi)]^{-\gamma})\right]\right\}}\right]^{\alpha}\right)^{n}. \end{aligned}$$

Thus, the cdf of the life length of the whole system, X, that is, the cdf of the TIIEHL-Gom-GPS distribution which is the marginal distribution of $X = Y_{(1)}$, is given by

$$\begin{aligned} F_X(x;\alpha,\gamma,\theta,\Psi) &= \sum_{n=1}^{\infty}\frac{a_n\theta^n}{C(\theta)}\left(1 - \left(\left[\frac{\exp\left[\frac{1}{\gamma}(1-[1-G(x;\Psi)]^{-\gamma})\right]}{1+\left\{1-\exp\left[\frac{1}{\gamma}(1-[1-G(x;\Psi)]^{-\gamma})\right]\right\}}\right]^{\alpha}\right)^{n}\right) \\ &= 1 - \frac{C\left(\theta\left[\frac{\exp\left[\frac{1}{\gamma}(1-[1-G(x;\Psi)]^{-\gamma})\right]}{1+\left\{1-\exp\left[\frac{1}{\gamma}(1-[1-G(x;\Psi)]^{-\gamma})\right]\right\}}\right]^{\alpha}\right)}{C(\theta)}. \end{aligned} \quad (5)$$

The corresponding pdf and hazard rate function (hrf) are given by

$$\begin{aligned} f_X(x;\alpha,\gamma,\theta,\Psi) &= 2\alpha\theta g(x;\Psi)[1-G(x;\Psi)]^{-\gamma-1}\exp\left(\frac{1}{\gamma}(1-[1-G(x;\Psi)]^{-\gamma})\right) \\ &\times \left(1+\left(1-\exp\left(\frac{1}{\gamma}(1-[1-G(x;\Psi)]^{-\gamma})\right)\right)\right)^{-\alpha-1} \\ &\times \frac{C'\left(\theta\left[\frac{\exp\left[\frac{1}{\gamma}(1-[1-G(x;\Psi)]^{-\gamma})\right]}{1+\left\{1-\exp\left[\frac{1}{\gamma}(1-[1-G(x;\Psi)]^{-\gamma})\right]\right\}}\right]^{\alpha}\right)}{C(\theta)} \end{aligned} \quad (6)$$

and

$$\begin{aligned} h_F(x) &= 2\alpha\theta g(x;\Psi)[1-G(x;\Psi)]^{-\gamma-1}\exp\left(\frac{1}{\gamma}(1-[1-G(x;\Psi)]^{-\gamma})\right) \\ &\times \left(1+\left(1-\exp\left(\frac{1}{\gamma}(1-[1-G(x;\Psi)]^{-\gamma})\right)\right)\right)^{-\alpha-1} \\ &\times \frac{C'\left(\theta\left[\frac{\exp\left[\frac{1}{\gamma}(1-[1-G(x;\Psi)]^{-\gamma})\right]}{1+\left\{1-\exp\left[\frac{1}{\gamma}(1-[1-G(x;\Psi)]^{-\gamma})\right]\right\}}\right]^{\alpha}\right)}{C\left(\theta\left[\frac{\exp\left[\frac{1}{\gamma}(1-[1-G(x;\Psi)]^{-\gamma})\right]}{1+\left\{1-\exp\left[\frac{1}{\gamma}(1-[1-G(x;\Psi)]^{-\gamma})\right]\right\}}\right]^{\alpha}\right)}, \end{aligned} \quad (7)$$

respectively, where $\alpha,\gamma,\theta > 0$ and Ψ is the parameter vector from the baseline distribution.

Table 1 shows some useful quantities including $a_n, C(\theta)$ and cdf for the type II exponentiated half logistic-Gompertz-G Poisson (TIIEHL-Gom-GP), type II exponentiated half logistic-Gompertz-G geometric (TIIEHL-Gom-GG), type II exponentiated half logistic-Gompertz-G binomial (TIIEHL-Gom-GB) and type II exponentiated half logistic-Gompertz-G logarithmic (TIIEHL-Gom-GL) distributions.

Table 1. Special Cases of the TIIEHL-Gom-GPS Distribution

Distribution	a_n	$C(\theta)$	cdf
TIIEHL-Gom-G Poisson	$(n!)^{-1}$	$e^{\theta}-1$	$1-\frac{\exp\left(\theta\left[\frac{\exp\left[\frac{1}{\gamma}(1-[1-G(x;\Psi)]^{-\gamma})\right]}{1+\left\{1-\exp\left[\frac{1}{\gamma}(1-[1-G(x;\Psi)]^{-\gamma})\right]\right\}}\right]^{\alpha}\right)-1}{e^{\theta}-1}$
TIIEHL-Gom-G Geometric	1	$\theta(1-\theta)^{-1}$	$1-\frac{(1-\theta)\left[\frac{\exp\left[\frac{1}{\gamma}(1-[1-G(x;\Psi)]^{-\gamma})\right]}{1+\left\{1-\exp\left[\frac{1}{\gamma}(1-[1-G(x;\Psi)]^{-\gamma})\right]\right\}}\right]^{\alpha}}{1-\theta\left[\frac{\exp\left[\frac{1}{\gamma}(1-[1-G(x;\Psi)]^{-\gamma})\right]}{1+\left\{1-\exp\left[\frac{1}{\gamma}(1-[1-G(x;\Psi)]^{-\gamma})\right]\right\}}\right]^{\alpha}}$
TIIEHL-Gom-G Logarithmic	n^{-1}	$-\log(1-\theta)$	$1-\frac{\log\left(1-\theta\left[\frac{\exp\left[\frac{1}{\gamma}(1-[1-G(x;\Psi)]^{-\gamma})\right]}{1+\left\{1-\exp\left[\frac{1}{\gamma}(1-[1-G(x;\Psi)]^{-\gamma})\right]\right\}}\right]^{\alpha}\right)}{\log(1-\theta)}$
TIIEHL-Gom-G Binomial	$\binom{m}{n}$	$(1+\theta)^{m}-1$	$1-\frac{\left(1+\theta\left[\frac{\exp\left[\frac{1}{\gamma}(1-[1-G(x;\Psi)]^{-\gamma})\right]}{1+\left\{1-\exp\left[\frac{1}{\gamma}(1-[1-G(x;\Psi)]^{-\gamma})\right]\right\}}\right]^{\alpha}\right)^{m}-1}{(1+\theta)^{m}-1}$

2.1. Quantile Function

The quantile function of the TIIEHL-Gom-GPS class of distributions is obtained by inverting $F(x;\alpha,\gamma,\theta,\Psi)=u$, $0\leq u\leq 1$. This is equivalent to solving the equation

$$\frac{C\left(\theta\left[\frac{\exp\left[\frac{1}{\gamma}(1-[1-G(x;\Psi)]^{-\gamma})\right]}{1+\left\{1-\exp\left[\frac{1}{\gamma}(1-[1-G(x;\Psi)]^{-\gamma})\right]\right\}}\right]^{\alpha}\right)}{C(\theta)}=1-u, \quad (8)$$

$0\leq u\leq 1$, which can be expressed as

$$Q_X(u)=G^{-1}\left[1-\left(1-\gamma\ln\left[\frac{2\left[\frac{C^{-1}(C(\theta)(1-u))}{\theta}\right]^{\frac{1}{\alpha}}}{1+\left[\frac{C^{-1}(C(\theta)(1-u))}{\theta}\right]^{\frac{1}{\alpha}}}\right]^{-\frac{1}{\gamma}}\right)\right]. \quad (9)$$

The solution of the non-linear equation (9) gives the quantiles of the TIIEHL-Gom-GPS class of distributions. Quantiles for selected parameter values for the type II exponentiated half logistic-Gompertz-Weibull Poisson (TIIEHL-Gom-WP) distribution are shown in Table 2.

Table 2. Table of Quantiles for Selected Parameters of the TIIEHL-Gom-WP Distribution

u	(1.5,1.5,1.5,1.5)	(1.5,1,1.5,1.5)	(1.5,0.5,1.5,1)	(1.5,1.5,1,1.5)	(1,1.5,1,0.5)
0.1	0.0697	0.0184	0.0005	0.0699	0.1205
0.2	0.1158	0.0394	0.0023	0.1166	0.1999
0.3	0.1597	0.0638	0.0060	0.1614	0.2749
0.4	0.2050	0.0928	0.0124	0.2081	0.3513
0.5	0.2540	0.1280	0.0234	0.2593	0.4327
0.6	0.3099	0.1725	0.0418	0.3185	0.5230
0.7	0.3773	0.2317	0.0736	0.3911	0.6280
0.8	0.4656	0.3177	0.1336	0.4885	0.7585
0.9	0.6017	0.4668	0.2722	0.6437	0.9431

2.2. Expansion of Density

In this sub-section, we present the series expansion of the TIIEHL-Gom-GPS class of distributions. The pdf in equation (6) can be written as

$$f_X(x;\alpha,\gamma,\theta,\Psi) = \sum_{s=0}^{\infty} v_{s+1} g_{s+1}(x;\Psi),$$

where $g_{s+1}(x;\Psi) = (s+1)\left(G(x;\Psi)\right)^s g(x;\Psi)$ is the exponentiated-G (Exp-G) distribution with power parameter $(s+1)$ and

$$v_{s+1} = \sum_{q,k,w,p=0}^{\infty}\sum_{n=1}^{\infty} 2\alpha\theta \frac{(\alpha n+k)^w n a_n \theta^n (-1)^{q+k+p+s}}{C(\theta)\gamma^w w!(s+1)} \binom{\alpha n+q}{q}\binom{q}{k} \times \binom{w}{p}\binom{-\gamma(p+1)-1}{s}. \quad (10)$$

Thus, the pdf of the TIIEHL-Gom-GPS class of distributions can be expressed as an infinite linear combination of Exp-G distributions. For derivations visit the appendix.

2.3. Moments and Generating Function

If X follows the TIIEHL-Gom-GPS distribution and $Y \sim$ Exp-G$(s+1)$, then the r^{th} moment, μ_r' of the TIIEHL-Gom-GPS class of distributions is obtained as

$$\mu_r' = E(X^r) = \int_{-\infty}^{\infty} x^r f(x)dx = \sum_{s=0}^{\infty} v_{s+1} E(Y^r),$$

where v_{s+1} is given by equation (10). The moment generating function (MGF) $M_x(t) = E(e^{tX})$ is given by:

$$M_X(t) = \sum_{s=0}^{\infty} v_{s+1} M_Y(t),$$

where $M_Y(t)$ is the mgf of Y and v_{s+1} is given by equation (10).
Furthermore, we can obtain the characteristic function given by $\phi_x(t) = E(e^{itX})$, where $i = \sqrt{-1}$ as

$$\phi_x(t) = \sum_{s=0}^{\infty} v_{s+1}\phi_{s+1}(t),$$

Table 3. Moments of the TIIEHL-Gom-WP distribution for some parameter values

	(1.5,0.5,0.5,0.02)	(0.5,1.5,1,1.5)	(0.5,1.5,1,1.5)	(1,1.5,1,0.5)	(1.3,1.5,1.5,0.5)
E(X)	0.1340	0.4072	0.4072	0.3974	0.3712
E(X^2)	0.0599	0.2499	0.2499	0.2302	0.1941
E(X^3)	0.0371	0.1757	0.1757	0.1553	0.1205
E(X^4)	0.0266	0.1339	0.1339	0.1148	0.0832
E(X^5)	0.0207	0.1075	0.1075	0.0901	0.0617
SD	0.2047	0.2900	0.2900	0.2688	0.2373
CV	1.5274	0.7122	0.7122	0.6763	0.6393
CS	2.0829	0.2239	0.2239	0.3319	0.4958
CK	6.9454	1.9566	1.9566	2.1462	2.4647

where $\phi_{s+1}(t)$ is the characteristic function of Exp-G distribution with power parameter $(s+1)$ and v_{s+1} is as defined in equation (10).
The coefficients of variation (CV), skewness (CS) and kurtosis (CK) can be readily obtained. The variance (σ^2), standard deviation (SD=σ), CV, CS and CK are given by

$$\sigma^2 = \mu_2' - \mu^2, \quad CV = \frac{\sigma}{\mu} = \frac{\sqrt{\mu_2' - \mu^2}}{\mu} = \sqrt{\frac{\mu_2'}{\mu^2} - 1},$$

$$CS = \frac{E\left[(X-\mu)^3\right]}{\left[E(X-\mu)^2\right]^{3/2}} = \frac{\mu_3' - 3\mu\mu_2' + 2\mu^3}{(\mu_2' - \mu^2)^{3/2}},$$

and

$$CK = \frac{E\left[(X-\mu)^4\right]}{\left[E(X-\mu)^2\right]^2} = \frac{\mu_4' - 4\mu\mu_3' + 6\mu^2\mu_2' - 3\mu^4}{(\mu_2' - \mu^2)^2},$$

respectively.

Note that the r^{th} cumulant of the random variable X can be readily obtained from the recursive relationship: $\kappa_r = \mu_r' - \sum_{s=1}^{r-1}\binom{r-1}{s-1}\mu_{r-s}'\kappa_s$, where $\mu_r' = E(X-\mu_1')^r$, so that the CS and CK are given by $\tau_1 = \frac{\kappa_3}{\kappa_2^{3/2}}$ and $\tau_2 = \frac{\kappa_4}{\kappa_2^2}$. A table of moments, SD, CV, CS, and CK for selected parameter values of the special case of the Type II Exponentiated Half Logistic-Gompertz-Weibull Poisson (TIIEHL-Gom-WP) distribution are given in Table 3.

2.4. Conditional Moments

The r^{th} conditional moment of the TIIEHL-Gom-GPS class of distributions is given by

$$\begin{aligned} E(X^r|X\geq t) &= \frac{1}{\overline{F}(t;\alpha,\gamma,\theta,\Psi)}\int_t^\infty x^r f(x;\alpha,\gamma,\theta,\Psi)dx \\ &= \frac{1}{\overline{F}(t;\alpha,\gamma,\theta,\Psi)}\sum_{s=0}^{\infty} v_{s+1}E\left(Y^r I_{\{Y^r\geq t\}}\right), \end{aligned} \tag{11}$$

where

$$E\left(Y^r I_{\{Y^r\geq t\}}\right) = \int_t^\infty y^r g_{s+1}(y;\Psi)dy = s\int_{G(u;\Psi)}^1 [Q_G(u;\Psi)]^r u^s du, \tag{12}$$

for $\alpha,\gamma,\theta > 0$, and parameter vector Ψ.

2.5. Order Statistics and Rényi Entropy

In this section, we present the distribution of the k^{th} order statistic and Rényi entropy for the TIIEHL-Gom-GPS class of distributions.

2.5.1. Order Statistics

Order statistics are very useful in probability and statistics. Let $X_1, X_2, ..., X_n$ be a random sample from TIIEHL-Gom-GPS class of distributions and suppose $X_{1:n} < X_{2:n}, ... < X_{n:n}$ denote the corresponding order statistics. The pdf of the k^{th} order statistic is given by

$$f_{k:n}(x) = \sum_{l=0}^{\infty} v^*_{l+1} g_{l+1}(x;\Psi),$$

where $g_{l+1}(x;\Psi) = (l+1)g(x;\Psi)G^l(x;\Psi)$ is an Exp-G pdf with power parameter $(l+1)$ and the linear component v^*_{l+1} is given by

$$\begin{aligned} v^*_{l+1} &= \frac{n!(-1)^{p+m+q}}{(k-1)!(n-k)!} \sum_{p,z,s,q,m,j,l=0}^{\infty} \sum_{n=1}^{\infty} \sum_{i=0}^{n-k} \binom{n-k}{i} \frac{2na_n\theta^{n+z}\alpha\theta g(x;\xi)d_{z,p}}{C^{z+1}(\theta)} \binom{k+i-1}{p} \\ &\times \binom{\alpha(z+n)+s}{s} \frac{(\alpha(n+z)+k)^m}{\gamma^m m!} \binom{\delta(w+1)+l}{l} \binom{s}{q} \binom{m}{j} \left(\frac{1}{l+1}\right). \end{aligned} \quad (13)$$

The t^{th} moment of the distribution of the k^{th} order statistic from TIIEHL-Gom-GPS class of distributions can be readily obtained from equation (13). Visit the appendix for derivations of the pdf of the k^{th} order statistic.

2.5.2. Rényi Entropy

An entropy is a measure of uncertainty or variation of a random variable. Rényi entropy [20] is a generalization of Shannon entropy [23]. Rényi entropy for the TIIEHL-Gom-GPS distribution after some simplifications can be written as

$$I_R(v) = \frac{1}{1-v} \log\left(\sum_{k=0}^{\infty} w^*_{k+1} e^{(1-\nu)I_{REG}}\right), \quad (14)$$

where $I_{REG} = \int_0^\infty [(1+k/\nu)g(x;\xi)G^{k/\nu}]^\nu dx$ is Rényi entropy for an Exp-G distribution with power parameter $(k/\nu+1)$ and

$$w^*_{k+1} = \sum_{z,m,q,s,p=0}^{\infty} \frac{d_{z,\nu}\theta^{z+\nu}(-1)^{m+q+p+k}(\alpha(z+v)+q)^s(2\alpha\theta)^\nu}{\gamma^s s!} \quad (15)$$

$$\times \binom{-\alpha(z+\nu)-\nu}{m}\binom{m}{q}\binom{s}{p}\binom{-\gamma(p+\nu)-\nu}{k}\frac{1}{(1+k/\nu)^\nu}. \quad (16)$$

Consequently, Rényi entropy of the TIIEHL-Gom-GPS class of distributions can be readily derived from Rényi entropy of the Exp-G. See the appendix for the derivations.

2.6. Parameter Estimation

Let $X_i \sim TIIEHL-Gom-GPS(\alpha,\gamma,\theta,\Psi)$ and $\Delta=(\alpha,\gamma,\theta,\Psi)^T$ be the parameter vector. The log-likelihood $\ell=\ell(\Delta)$ based on a random sample of size n is given by

$$\begin{aligned}
\ell &= \ell(\Delta)=n\ln(2\alpha\theta)+\sum_{i=1}^{n}\ln(g(x_i;\xi))-(\gamma+1)\sum_{i=1}^{n}\ln[1-G(x;\Psi)]-n\ln C(\theta)\\
&+ \left(\frac{\alpha}{\gamma}(1-[1-G(x;\Psi)]^{-\gamma})\right)-(\alpha+1)\\
&\times \sum_{i=1}^{n}\ln(1+(1-\exp(\frac{1}{\gamma}(1-[1-G(x;\Psi)]^{-\gamma}))))\\
&+ \sum_{i=1}^{n}\ln\left(C'\left(\theta\left[\frac{\exp\left[\frac{1}{\gamma}(1-[1-G(x;\Psi)]^{-\gamma})\right]}{1+\left\{1-\exp\left[\frac{1}{\gamma}(1-[1-G(x;\Psi)]^{-\gamma})\right]\right\}}\right]^{\alpha}\right)\right).
\end{aligned}$$

The maximum likelihood estimates of the parameters, denoted by $\hat{\boldsymbol{\Delta}}$ is obtained by solving the nonlinear equation $(\frac{\partial\ell}{\partial\alpha},\frac{\partial\ell}{\partial\gamma},\frac{\partial\ell}{\partial\theta},\frac{\partial\ell}{\partial\Psi_k})^T=\mathbf{0}$, using a numerical method such as Newton-Raphson procedure. The multivariate normal distribution $N_{q+3}(\underline{0},J(\hat{\boldsymbol{\Delta}})^{-1})$, where the mean vector $\underline{\mathbf{0}}=(0,0,0,\underline{0})^T$ and $J(\hat{\boldsymbol{\Delta}})^{-1}$ is the observed Fisher information matrix evaluated at $\hat{\boldsymbol{\Delta}}$, can be used to construct confidence intervals and confidence regions for the individual model parameters.

3. Special Sub-Models of the TIIEHL-Gom-GPS Distribution

In this section, we look at some special cases of the TIIEHL-Gom-GPS class of distributions. These special cases are the type II exponentiated half logistic-Gom-Weibull Poisson (TIIEHL-Gom-WP), type II exponentiated half logistic-Gom-Weibull Geometric (TIIEHL-Gom-WG), type II exponentiated half logistic-Gom-Log logistic Poisson (TIIEHL-Gom-LLoGP) and type II exponentiated half logistic-Gom-Log logistic Geometric (TIIEHL-Gom-LLoGG) distributions. The cdf and pdf of the Weibull distribution are given by $G(x;b)=1-\exp(-x^b)$ and $g(x;b)=bx^{b-1}\exp(-x^b)$, for $b>0$ and for the log-logistic distribution are given by $G(x;c)=1-(1+x^c)^{-1}$ and $g(x;c)=cx^{c-1}(1+x^c)^{-2}$, for $c>0$.

3.1. Type II Exponentiated Half Logistic-Gompertz-Weibull Poisson (TIIEHL-Gom-WP) Distribution

The cdf and pdf of the TIIEHL-Gom-WP distribution are given by

$$F_{TIIEHL-Gom-WP}(x) = 1-\frac{\exp\left(\theta\left[\frac{\exp(\frac{1}{\gamma}(1-[1-(1-e^{-x^b})]^{-\gamma}))}{1+(1-\exp(\frac{1}{\gamma}(1-[1-(1-e^{-x^b})]^{-\gamma})))}\right]^{\alpha}\right)-1}{e^{\theta}-1}$$

and

$$\begin{aligned}
f_{TIIEHL-Gom-WP}(x) &= 2\alpha\theta bx^{b-1}e^{-x^b}[1-(1-e^{-x^b}))]^{-\gamma-1}\\
&\times \exp\left(\frac{\alpha}{\gamma}(1-[1-(1-e^{-x^b})]^{-\gamma})\right)\\
&\times \left(1+\left(1-\exp\left(\frac{1}{\gamma}(1-[1-(1-e^{-x^b})]^{-\gamma})\right)\right)\right)^{-\alpha-1}\\
&\times \frac{\exp\left(\theta\left[\frac{\exp(\frac{1}{\gamma}(1-[1-(1-e^{-x^b})]^{-\gamma}))}{1+(1-\exp(\frac{1}{\gamma}(1-[1-(1-e^{-x^b})]^{-\gamma})))}\right]^{\alpha}\right)}{\exp(\theta)-1}
\end{aligned}$$

for α,γ,θ and $b>0$.

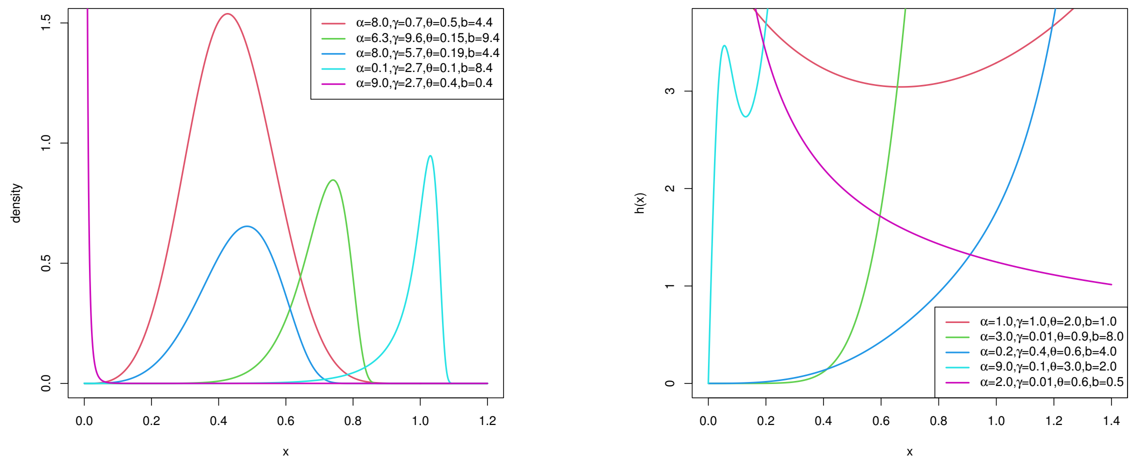


Figure 1. Plots of the pdf and hrf for the TIIEHL-Gom-WP distribution

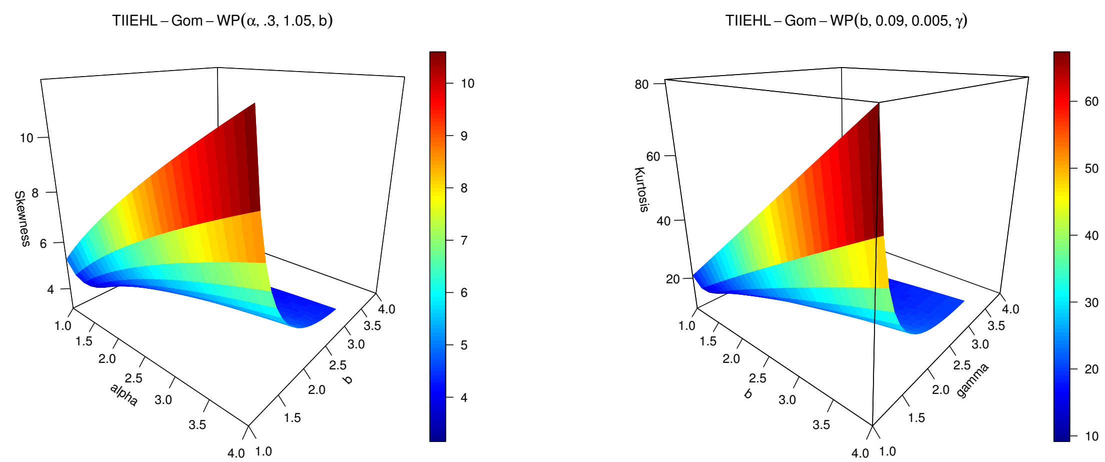


Figure 2. Plots of skewness and kurtosis for the TIIEHL-Gom-WP distribution

The plots of pdf and hrf for the TIIEHL-Gom-WP distribution are shown in Figure 1. The pdf is almost symmetric, right or left-skewed. The hazard rate function exhibits both increasing, decreasing, upside bathtub followed by bathtub and bathtub shapes. 3D plots of skewness and kurtosis of TIIEHL-Gom-WP distribution for some fixed values of parameters are shown in Figure 2.

3.2. Type II Exponentiated Half Logistic-Gompertz-Weibull Geometric (TIIEHL-Gom-WG) Distribution

The cdf and pdf of the TIIEHL-Gom-WG distribution are given by

$$F_{TIIEHL-Gom-WG}(x) = 1 - \frac{(1-\theta)\left[\frac{\exp(\frac{1}{\gamma}(1-[1-(1-e^{-x^b})]^{-\gamma}))}{1+(1-\exp(\frac{1}{\gamma}(1-[1-(1-e^{-x^b})]^{-\gamma})))}\right]^{\alpha}}{\left(1-\theta\left[\frac{\exp(\frac{1}{\gamma}(1-[1-(1-e^{-x^b})]^{-\gamma}))}{1+(1-\exp(\frac{1}{\gamma}(1-[1-(1-e^{-x^b})]^{-\gamma})))}\right]^{\alpha}\right)}$$

and

$$\begin{aligned} f_{TIIEHL-Gom-WG}(x) &= 2\alpha\theta b x^{b-1} e^{-x^b}[1-(1-e^{-x^b}))]^{-\gamma-1} \\ &\times \exp\left(\frac{\alpha}{\gamma}(1-[1-(1-e^{-x^b})]^{-\gamma})\right) \\ &\times \left(1+\left(1-\exp\left(\frac{1}{\gamma}(1-[1-(1-e^{-x^b})]^{-\gamma})\right)\right)\right)^{-\alpha-1} \\ &\times \frac{\left(1-\left(\theta\left[\frac{\exp(\frac{1}{\gamma}(1-[1-(1-e^{-x^b})]^{-\gamma}))}{1+(1-\exp(\frac{1}{\gamma}(1-[1-(1-e^{-x^b})]^{-\gamma})))}\right]^{\alpha}\right)\right)^{-2}}{(1-\theta)^{-1}} \end{aligned}$$

for $\alpha, \gamma, b > 0$ and $0 < \theta < 1$.

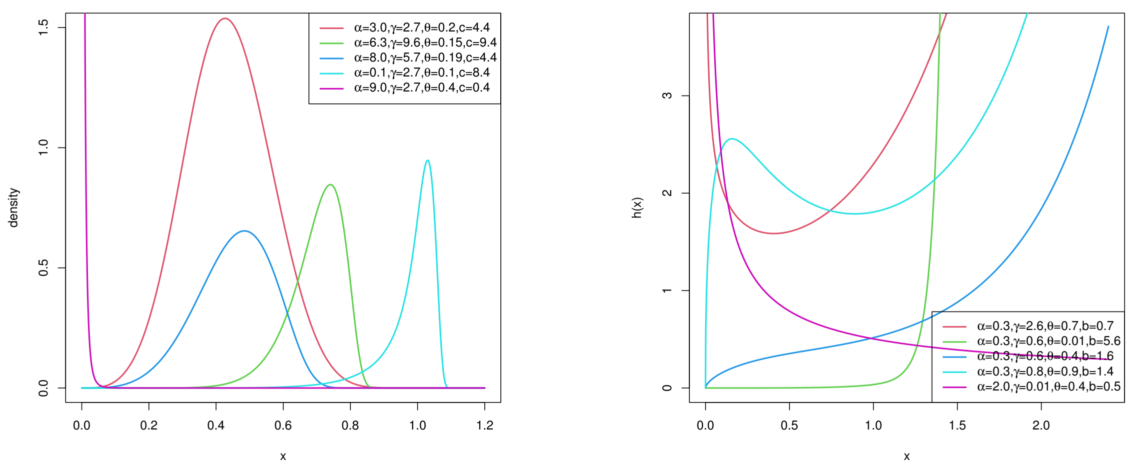


Figure 3. Plots of the pdf and hrf for the TIIEHL-Gom-WG distribution

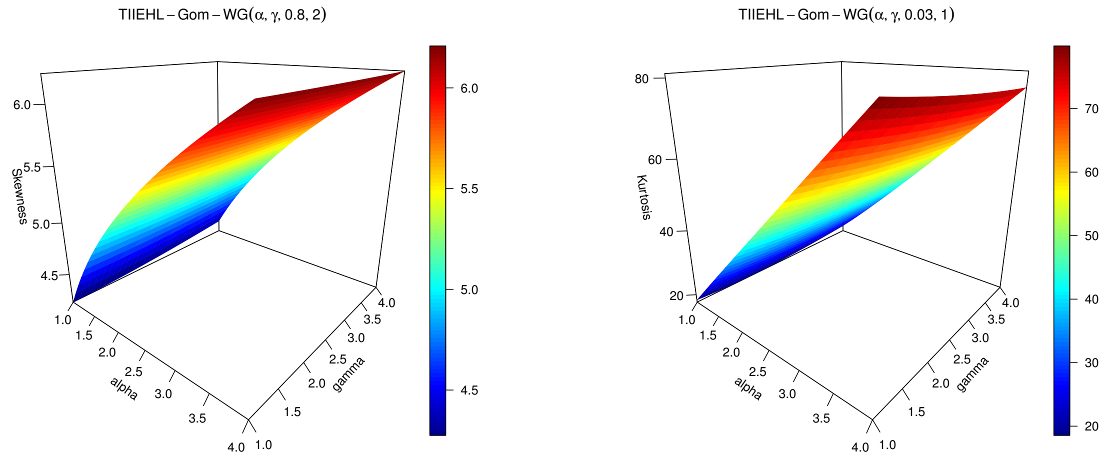


Figure 4. Plots of skewness and kurtosis for the TIIEHL-Gom-WG distribution

Figure 3 shows the pdf and hrf plots for selected parameter values for the TIIEHL-Gom-WG distribution. The distribution is almost symmetric, right or left-skewed. The hazard rate function exhibits bathtub, upside bathtub followed by bathtub, increasing, decreasing and J shapes. Figure 4 shows 3D plots of skewness and kurtosis of TIIEHL-Gom-WG distribution for some fixed values of parameters. As we fix θ and b we can see that the skewness and kurtosis are increasing.

3.3. Type II Exponentiated Half Logistic-Gompertz-Log Logistic Poisson (TIIEHL-Gom-LLoGP) Distribution

The cdf and pdf of the TIIEHL-Gom-LLoGP distribution are given by

$$F_{TIIEHL-Gom-LLoGP}(x) = 1 - \frac{\exp\left(\theta\left[\frac{\exp(\frac{1}{\gamma}(1-[(1+x^c)^{-1}]^{-\gamma}))}{1+(1-\exp(\frac{1}{\gamma}(1-[(1+x^c)^{-1}]^{-\gamma})))}\right]^{\alpha}\right)-1}{e^{\theta}-1}$$

and

$$\begin{aligned} f_{TIIEHL-Gom-LLoGP}(x) &= \alpha\theta c x^{c-1}(1+x^c)^{-2}[(1+x^c)^{-1}]^{-\gamma-1} \\ &\times \exp\left(\frac{\alpha}{\gamma}(1-[(1+x^c)^{-1}]^{-\gamma})\right) \\ &\times \left(1+\left(1-\exp\left(\frac{1}{\gamma}(1-[(1+x^c)^{-1}]^{-\gamma})\right)\right)\right)^{-\alpha-1} \\ &\times \frac{\exp\left(\theta\left[\frac{\exp(\frac{1}{\gamma}(1-[(1+x^c)^{-1}]^{-\gamma}))}{1+(1-\exp(\frac{1}{\gamma}(1-[(1+x^c)^{-1}]^{-\gamma})))}\right]^{\alpha}\right)}{\exp(\theta)-1} \end{aligned}$$

for $\alpha, \gamma, c > 0$ and $\theta > 0$.

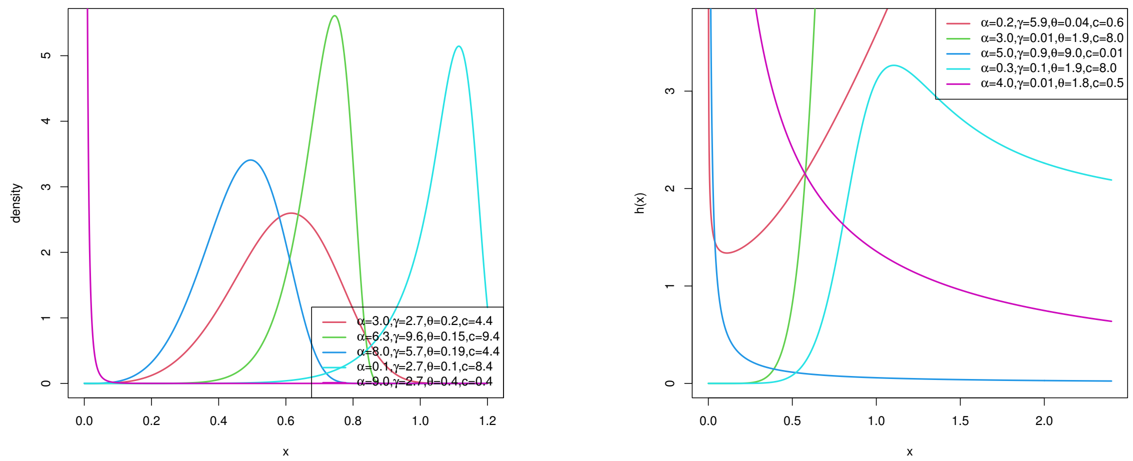


Figure 5. Plots of the pdf and hrf for the TIIEHL-Gom-LLoGP distribution

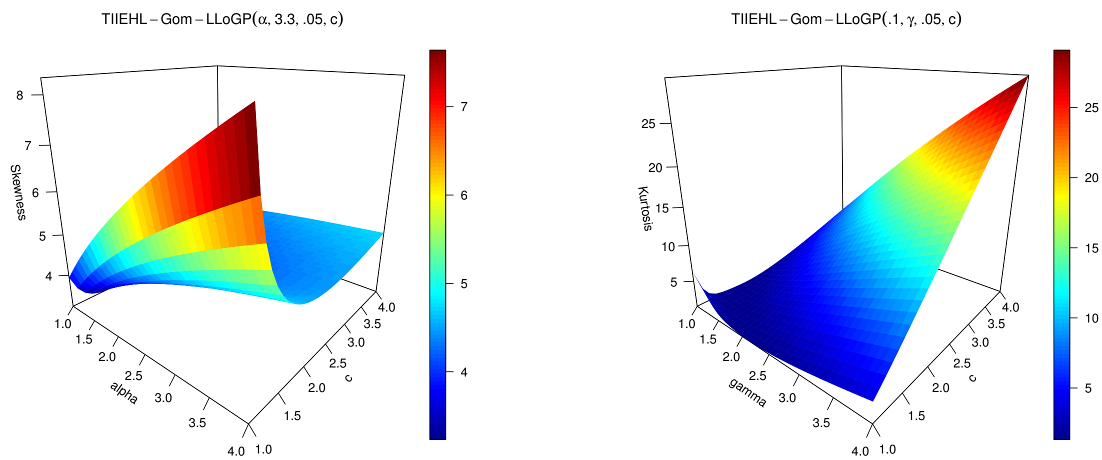


Figure 6. Plots of skewness and kurtosis for the TIIEHL-Gom-LLoGP distribution

The TIIEHL-Gom-LLoGP distribution applies to data sets that are almost symmetric, reverse-J left-skewed as shown in Figure 5. The hrf function exhibits both monotonic and non-monotonic shapes including decreasing, increasing, bathtub and upside down bathtub shapes. 3D plots of skewness and kurtosis of TIIEHL-Gom-LLoGP distribution for some fixed values of parameters are shown in Figure 6. It can be seen that the TIIEHL-Gom-LLoP distribution is capable of modelling various data sets with different levels of skewness and kurtosis.

3.4. Exponentiated Half Logistic-Gompertz-Log Logistic Geometric (TIIEHL-Gom-LLoGG) Distribution

The cdf and pdf of the TIIEHL-Gom-LLoGG distribution are given by

$$F_{TIIEHL-Gom-LLoGG}(x) = 1 - \frac{(1-\theta)\left[\frac{\exp(\frac{1}{\gamma}(1-[(1+x^c)^{-1}]^{-\gamma}))}{1+(1-\exp(\frac{1}{\gamma}(1-[(1+x^c)^{-1}]^{-\gamma})))}\right]^{\alpha}}{\left(1-\theta\left[\frac{\exp(\frac{1}{\gamma}(1-[(1+x^c)^{-1}]^{-\gamma}))}{1+(1-\exp(\frac{1}{\gamma}(1-[(1+x^c)^{-1}]^{-\gamma})))}\right]^{\alpha}\right)}$$

and

$$\begin{aligned} f_{TIIEHL-Gom-LLoGG}(x) &= \alpha\theta c x^{c-1}(1+x^c)^{-2}[(1+x^c)^{-1}]^{-\gamma-1} \\ &\times \exp\left(\frac{\alpha}{\gamma}(1-[(1+x^c)^{-1}]^{-\gamma})\right) \\ &\times \left(1+\left(1-\exp\left(\frac{1}{\gamma}(1-[(1+x^c)^{-1}]^{-\gamma})\right)\right)\right)^{-\alpha-1} \\ &\times \frac{\left(1-\left(\theta\left[\frac{\exp(\frac{1}{\gamma}(1-[(1+x^c)^{-1}]^{-\gamma}))}{1+(1-\exp(\frac{1}{\gamma}(1-[(1+x^c)^{-1}]^{-\gamma})))}\right]^{\alpha}\right)\right)^{-2}}{(1-\theta)^{-1}}, \end{aligned}$$

for $\alpha, \gamma, c > 0$ and $0 < \theta < 1$.

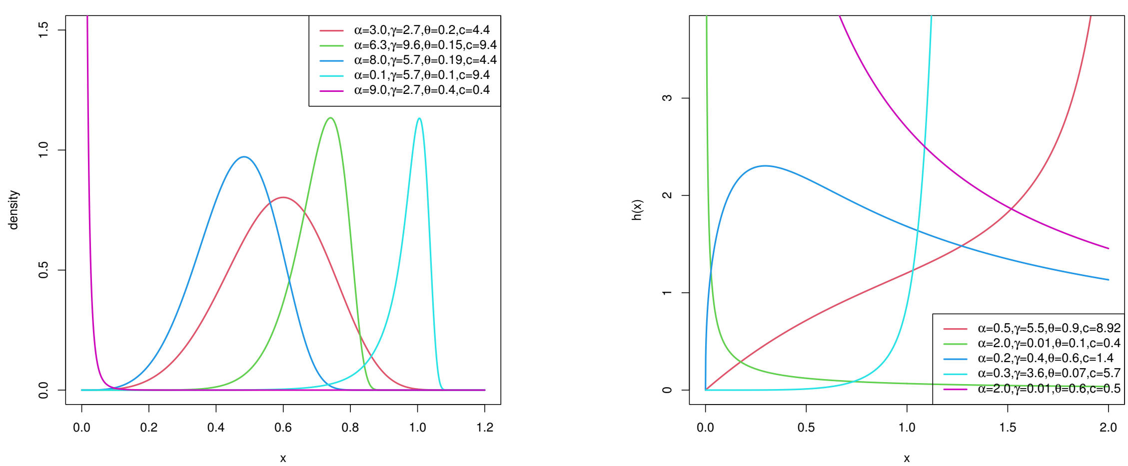


Figure 7. Plots of the pdf and hrf for the TIIEHL-Gom-LLoGG distribution

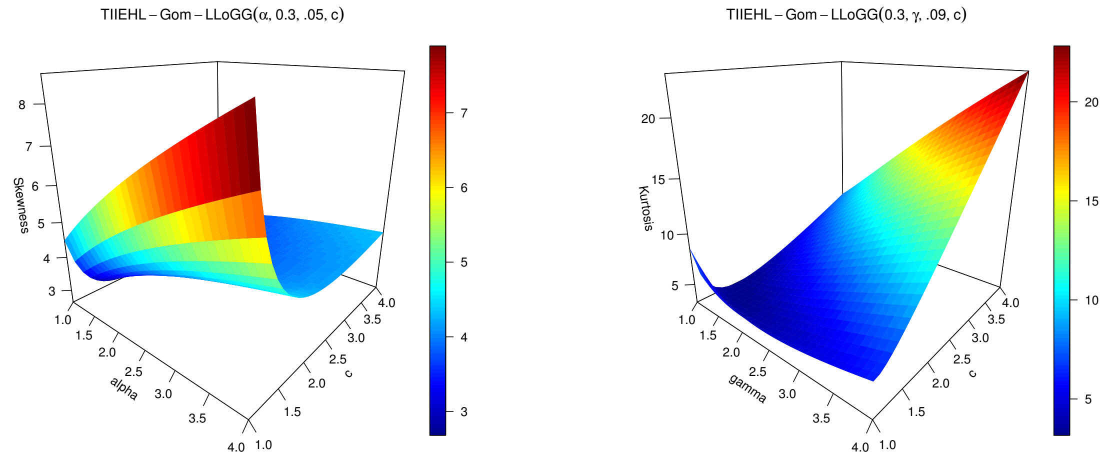


Figure 8. Plots of skewness and kurtosis for the TIIEHL-Gom-LLoGG distribution

The TIIEHL-Gom-LLoGG distribution pdf exhibits extreme tails, almost symmetric and reverse-J shapes. The shapes of the hrf can be monotonic and non-monotonic as shown in Figure 7. Figure 8 shows 3D plots of skewness and kurtosis of TIIEHL-Gom-LLoGG distribution for some fixed values of parameters.

4. Simulation Study

In this section, we examine the performance of the TIIEHL-Gom-WP distribution by conducting various simulations for different sample sizes (n= 25, 50, 100, 200 and 400) via the R package. We simulate 1000 samples for the true parameter values given in Tables 4 and 5. The average bias (ABIAS) and root mean square error (RMSE) are given by: $ABIAS(\hat{\theta}) = \frac{\sum_{i=1}^{N}\hat{\theta}_i}{N} - \theta$, and $RMSE(\hat{\theta}) = \sqrt{\frac{\sum_{i=1}^{N}(\hat{\theta}_i - \theta)^2}{N}}$, respectively.

Table 4. Monte Carlo Simulation Results for TIIEHL-Gom-WP Distribution: Mean, RMSE and Average Bias

		α=0.01,γ=1.2,θ=0.01,b=1.5			α=0.01,γ=0.8,θ=0.01,b=1.5		
Parameter	n	Mean	RMSE	Average Bias	Mean	RMSE	Average Bias
	25	0.010822	0.004842	0.000822	0.010796	0.004735	0.000796
	50	0.010930	0.003877	0.000930	0.010702	0.003798	0.000702
α	100	0.010635	0.002903	0.000635	0.010488	0.003034	0.000488
	200	0.010177	0.002058	0.000177	0.010256	0.002018	0.000256
	400	0.010172	0.001512	0.000172	0.010057	0.001373	0.000057
	25	1.210573	0.234757	0.010573	0.809479	0.160512	0.009479
	50	1.198392	0.170154	-0.001608	0.796206	0.147584	-0.003794
γ	100	1.195340	0.096566	-0.004660	0.797243	0.066258	-0.002757
	200	1.189819	0.090303	-0.010181	0.798395	0.039011	-0.001605
	400	1.190100	0.074019	-0.009900	0.794101	0.033666	-0.005899
	25	0.094061	0.327488	0.084061	0.086575	0.362279	0.076575
	50	0.056291	0.214830	0.046291	0.052126	0.205742	0.042126
θ	100	0.026892	0.119394	0.016892	0.027746	0.102226	0.017746
	200	0.032370	0.177735	0.022370	0.020659	0.065037	0.010659
	400	0.020478	0.049292	0.010478	0.022446	0.076965	0.012446
	25	1.515869	0.219857	0.015869	1.507290	0.169216	0.007290
	50	1.506467	0.179714	0.006467	1.511052	0.194726	0.011052
b	100	1.496844	0.108543	-0.003156	1.495076	0.106088	-0.004924
	200	1.503533	0.117583	0.003533	1.491970	0.033871	-0.008030
	400	1.501284	0.070689	0.001284	1.499246	0.032021	-0.000754

Table 5. Monte Carlo Simulation Results for TIIEHL-Gom-WP Distribution: Mean, RMSE and Average Bias

		α=0.01,γ=1.2,θ=0.8,b=1.5			α=0.01,γ=0.8,θ=0.8,b=1.5		
Parameter	n	Mean	RMSE	Average Bias	Mean	RMSE	Average Bias
	25	0.011965	0.006708	0.001965	0.012026	0.006175	0.002026
	50	0.011463	0.004396	0.001463	0.011388	0.004119	0.001388
α	100	0.010927	0.003370	0.000927	0.010702	0.003070	0.000702
	200	0.010650	0.002240	0.000650	0.010501	0.002339	0.000501
	400	0.010476	0.001854	0.000476	0.010455	0.001598	0.000455
	25	1.155489	0.317692	-0.044511	0.749518	0.274033	-0.050482
	50	1.157539	0.232309	-0.042461	0.789786	0.175535	-0.010214
γ	100	1.171611	0.157030	-0.028389	0.782921	0.132250	-0.017079
	200	1.180602	0.113626	-0.019398	0.775298	0.124071	-0.024702
	400	1.180315	0.101410	-0.019685	0.775047	0.112987	-0.024953
	25	0.886980	0.900470	0.086980	0.828553	0.332643	0.028553
	50	0.825923	0.279713	0.025923	0.839576	0.254904	0.039576
θ	100	0.819588	0.218065	0.019588	0.813814	0.131566	0.013814
	200	0.821583	0.203216	0.021583	0.821756	0.184034	0.021756
	400	0.799501	0.053815	-0.000499	0.818072	0.096017	0.018072
	25	1.571533	0.331907	0.071533	1.594972	0.348022	0.094972
	50	1.541545	0.238883	0.041545	1.513078	0.198906	0.013078
b	100	1.518828	0.152648	0.018828	1.516933	0.177904	0.016933
	200	1.505426	0.109504	0.005426	1.521366	0.163856	0.021366
	400	1.504392	0.086049	0.004392	1.520081	0.151288	0.020081

From the results, we can observe that as the sample size n increases, the mean estimates of the parameters tend to be closer to the true parameter values, since RMSEs decay toward zero. Clearly, the model produces consistent estimates.

5. Applications

In this section, we present applications and empirically establish the flexibility of the TIIEHL-Gom-WP distribution by means of two real data sets. We compared the TIIEHL-Gom-WP distribution to various non-nested models. We use R software to estimate model parameters and standard errors. We assessed model performance using the following goodness-of-fit statistics: -2loglikelihood (-2 log L), Akaike Information Criterion (AIC), Consistent Akaike Information Criterion (AICC), Bayesian Information Criterion (BIC), Cramér-von Mises (W^*), Andersen-Darling (A^*) (see Chen and Balakrishnan [7], for details), and Kolmogorov-Smirnov (K-S) statistic (and its p-value). Tables 6, and 7 shows model parameters estimates (standard errors in parentheses) and several goodness-of-fit statistics. We also provide fitted densities, fitted cdf, Kaplan-Meier (K-M) survival curves, Total Time on Test (TTT) plots and probability plots (as described by Chambers et al. [6]) to demonstrate how well our model fits the selected data sets.

The non-nested models considered in this paper are the Kumaraswamy-Weibull (KwW) distribution by Cordeiro et al. [12], odd Weibull-Topp-Leone-Log logistic logarithmic (OW-TL-LLoGL) distribution by Oluyede et al. [18], Topp-Leone- Gompertz-Weibull (TLGW) distribution by Oluyede et al. [19], Exponentiated Half Logistic Odd Weibull-Topp-Leone-Burr XII distribution by Chipepa et al. [8] and beta odd Lindley-Uniform (BOL-U) distribution by Chipepa et al. [9]. See the appendix for the pdfs of the non-nested models.

5.1. *Glass Fibre Data*

The first data set is on strengths of 1.5 cm glass fibres. The data set was also analyzed by Makubate et al. [16]. The data are: 0.55, 0.93, 1.25, 1.36, 1.49, 1.52, 1.58, 1.61, 1.64, 1.68, 1.73, 1.81, 2.00, 0.74, 1.04, 1.27, 1.39, 1.49, 1.53, 1.59, 1.61, 1.66, 1.68, 1.76, 1.82, 2.01, 0.77, 1.11, 1.28, 1.42, 1.50, 1.54, 1.60, 1.62, 1.66, 1.69, 1.76, 1.84, 2.24, 0.81, 1.13, 1.29, 1.48, 1.50, 1.55, 1.61, 1.62, 1.66, 1.70, 1.77, 1.84, 0.84, 1.24, 1.30, 1.48, 1.51, 1.55, 1.61, 1.63, 1.67, 1.70, 1.78, 1.89.

Table 6. Parameter estimates and goodness-of-fit statistics for various models fitted for glass fibre data set

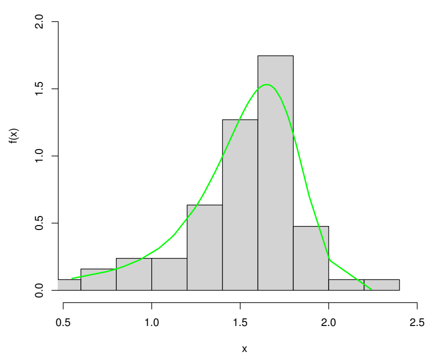
Model	Estimates				Statistics							
	α	γ	θ	b	$-2\log(L)$	AIC	$AICC$	BIC	W^*	A^*	K-S	p-value
TIIEHL-Gom-WP	0.0055	1.5505	5.8958	1.8765	24.1	32.1	32.8	40.7	0.0900	0.5524	0.1171	0.3534
	(0.0021)	(0.0564)	(2.0707)	(0.0604)								
	θ	γ	b	λ								
TL-Gom-LLoG	0.0325	2.2164	1.7307	1.1523	28.3	36.3	37.0	44.8	0.1621	0.9130	0.1312	0.2286
	(0.1033)	(3.5108)	(0.9681)	(0.9531)								
	α	λ	γ	θ								
OW-TL-LLoGL	1.4468	1.1576	3.7647	1.0770×10^{-8}	73.8	81.8	82.5	90.4	0.4950	2.7121	0.4135	8.7840×10^{-10}
	(0.3635)	(0.1865)	(0.2744)	(0.0147)								
	b	β	δ	c								
EHL-OW-TL-LLoG	1.1293	0.1464	4.3716	7.8796	34.9	42.9	43.6	51.3	0.3372	1.8409	0.1868	0.0246
	(0.7335)	(0.0736)	(1.0252)	(3.8532)								
	a	b	α	β								
KWW	7.3919	4.7793×10^{4}	0.1359	0.8776	31.2	39.2	39.9	47.7	0.2563	1.4056	0.1634	0.0693
	(2.6561)	(3.3847×10^{-5})	(0.0446)	(0.2414)								
	a	b	λ	θ								
BOL-U	3.7867	67.3210	0.2030	2.9970	30.0	38.0	38.6	46.5	0.2026	1.1312	0.1427	0.1536
	(1.1993)	(0.0003)	(0.0669)	(0.2907)								

The estimated variance-covariance matrix is given by

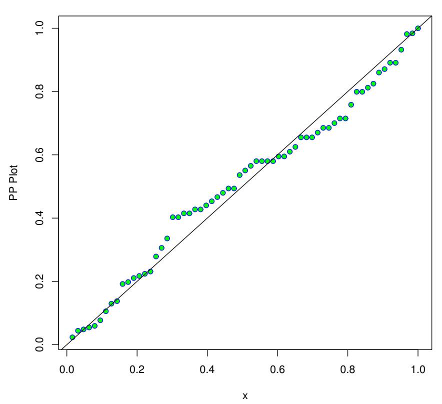
$$\begin{bmatrix} 0.000004 & -0.000010 & -0.003494 & -0.000026 \\ -0.000010 & 0.003184 & -0.029950 & 0.000458 \\ -0.003494 & -0.029951 & 4.287808 & -0.023831 \\ -0.000026 & 0.000458 & -0.023831 & 0.003651 \end{bmatrix}$$

and the 95% confidence intervals for the model parameters are given by
$\alpha \in [0.0055 \pm 0.0041]$, $\gamma \in [1.5505 \pm 0.1106]$, $\theta \in [5.8958 \pm 4.0586]$ and $b \in [1.8765 \pm 0.1184]$.

From the results shown in Table 6, we can conclude that the TIIEHL-Gom-WP model performs better on glass fibre data compared to the several competing non-nested models included in this paper. Based on Figures 9(a) and 9(b), we observe that TIIEHL-Gom-WP offers more flexibility on fitting the glass fibre data set. From Figure 10, the closeness of the fitted Kaplan-Meier survival curve and fitted cdf using TIIEHL-Gom-WP to the empirical Kaplan-Meier and empirical cdf is clear. The TTT plot indicates an increasing hazard rate function which means that the TIIEHL-Gom-WP distribution can be used to model this data.



(a)



(b)

Figure 9. Fitted densities and probability plots for glass fibre data

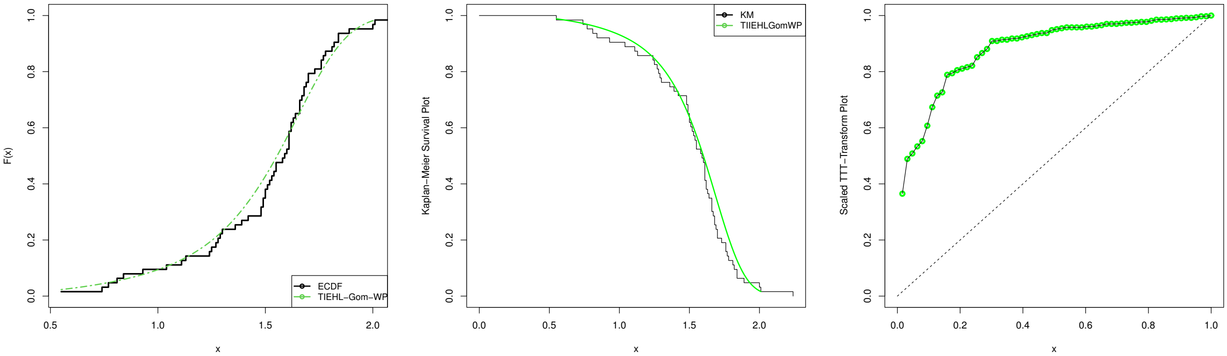


Figure 10. Kaplan-Meier (K-M) survival, estimated cdf (ECDF) plots and the total time on test (TTT) plot of the TIIEHL-Gom-WP distribution for the glass fibre data set

5.2. Carbon Fibre Data

The second data set was analyzed by Chipepa et al. [10]. The observations represents breaking stress of carbon fibres of 50 mm length (GPa). The data are: 0.39, 0.85, 1.08, 1.25, 1.47, 1.57, 1.61, 1.61, 1.69, 1.80, 1.84, 1.87, 1.89, 2.03, 2.03, 2.05, 2.12, 2.35, 2.41, 2.43, 2.48, 2.50, 2.53, 2.55, 2.55, 2.56, 2.59, 2.67, 2.73, 2.74, 2.79, 2.81, 2.82, 2.85, 2.87, 2.88, 2.93, 2.95, 2.96, 2.97, 3.09 ,3.11, 3.11, 3.15, 3.15, 3.19, 3.22, 3.22, 3.27, 3.28, 3.31, 3.31, 3.33, 3.39, 3.39, 3.56, 3.60, 3.65, 3.68, 3.70, 3.75, 4.20, 4.38, 4.42, 4.70, 4.90.

Table 7. Parameter estimates and goodness-of-fit statistics for various models fitted for carbon fibre data set

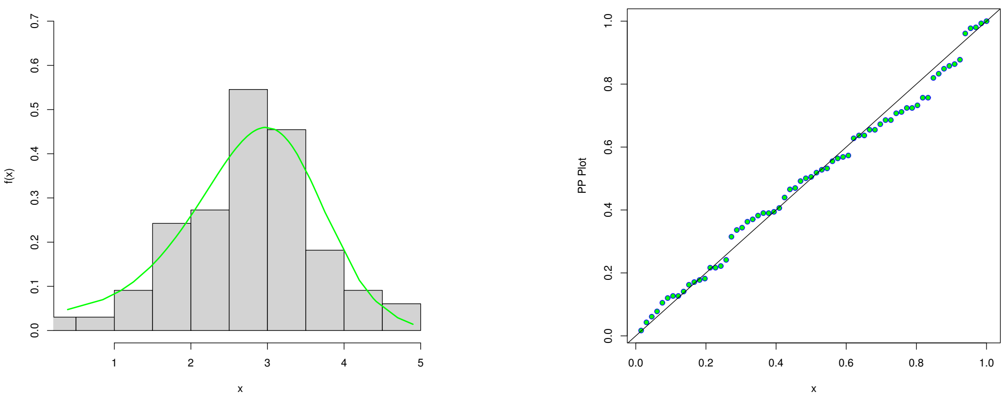
	Estimates				Statistics							
Model	α	γ	θ	b	$-2\log(L)$	AIC	$AICC$	BIC	W^*	A^*	K-S	p-value
TIIEHL-Gom-WP	0.0047	1.5781	3.2868	0.9437	170.7	178.7	179.3	187.4	0.6062	0.8358	0.0765	0.8344
	(0.0020)	(0.0525)	(1.6600)	(0.0275)								
	θ	γ	b	λ								
TL-Gom-LLoG	0.0218	2.3710	1.9934	0.5625	171.0	179.0	179.7	187.8	0.0709	0.4292	0.0794	0.8002
	(0.1038)	(5.0027)	(1.5507)	(0.6062)								
	α	λ	γ	θ								
OW-TL-LLoGL	0.5102	0.8758	4.0200	1.3525×10^{-8}	306.5	314.5	315.1	323.2	0.1841	0.9905	0.5805	2.2×10^{-16}
	(0.0898)	(0.1077)	(0.6913)	(0.0197)								
	b	β	δ	c								
EHL-OW-TL-LLoG	0.8420	0.1469	7.0243	3.7621	179.6	187.6	188.3	196.3	0.2288	1.2080	0.1309	0.2078
	(0.8155)	(0.0925)	(1.8991)	(2.2797)								
	a	b	α	β								
KWW	1.2155	2.5866×10^{3}	0.0333	2.8328	172.1	180.1	180.8	188.9	0.0930	0.52626	0.0823	0.7622
	(1.7552)	(6.6212×10^{-5})	(0.0071)	(4.0695)								
	a	b	λ	θ								
BOL-U	3.1503	71.3211	0.2964	8.4350	172.4	180.4	181.1	189.2	0.0944	0.5629	0.0876	0.691
	(0.9477)	(0.0038)	(0.1109)	(1.6137)								

The estimated variance-covariance matrix is given by

$$\begin{bmatrix} 0.000004 & -0.000008 & -0.002678 & -0.000012 \\ -0.000008 & 0.002756 & -0.023301 & 0.000240 \\ -0.002678 & -0.023301 & 2.755515 & -0.008631 \\ -0.000012 & 0.000240 & -0.008631 & 0.000756 \end{bmatrix}$$

and the 95% confidence intervals for the model parameters are given by
$\alpha \in [0.0047 \pm 0.0039]$, $\gamma \in [1.5781 \pm 0.1029]$, $\theta \in [3.2868 \pm 3.2535]$ and $b \in [0.9437 \pm 0.0539]$.

Table 7 presents model parameters estimates (standard errors in parentheses) and several goodness-of-fit statistics of the TIIEHL-Gom-WP distribution and the non-nested distributions. The K-S value is smallest and p-value corresponding to the K-S test statistic is largest for TIIEHL-Gom-WP model compared to the other non-nested models considered in this paper. Hence, it is the "best" model. Moreso, from Figures 11(a) and 11(b) it can be observed that TIIEHL-Gom-WP displays the flexibility enjoyed by fitting the carbon fibre data set. From Figure 12, the Kaplan-Meier survival and ECDF plots give enough information about the closest fit of the TIIEHL-Gom-WP to the current data set. The TTT plot indicates an increasing hazard rate function.



(a) (b)

Figure 11. Fitted densities and probability plots for carbon fibre data

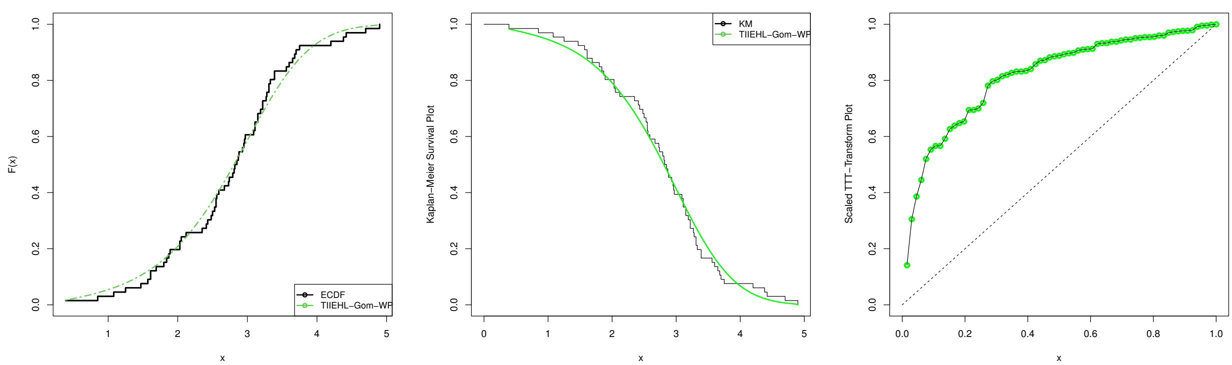


Figure 12. Kaplan-Meier (K-M) survival, estimated cdf (ECDF) plots and the total time on test (TTT) plot of the TIIEHL-Gom-WP distribution for the carbon fibre data set.

6. Concluding Remarks

A new class of distributions referred to as the Type II Exponentiated Half Logistic-Gompertz-G power series distribution is proposed. The new distribution can handle monotone as well as non-monotone hazard rate functions. The proposed distribution can be expressed as an infinite linear combination of the Exp-G distributions. We applied a special case of the new class of distributions to two real data sets and our model perform better than the competing non-nested models as shown in Tables 6 and 7.

Appendix

The following URL contains derivations of statistical properties and elements of the score vector.
https://drive.google.com/file/d/107HLSubp0ilDbA6Gf6KYtXKyzbku1rCL/view?usp=sharing

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