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Rao-Robson-Nikulin Goodness-of-fit Test Statistic for Censored and Uncensored Real Data with Classical and Bayesian Estimation

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Abstract In this work, we provide a new Pareto type-II extension for censored and uncensored real-life data. With an emphasis on the applied elements of the model, some mathematical properties of the new distribution are deduced without excess. A variety of traditional methods, including the Bayes method, are used to estimate the parameters of the new distribution. The censored case maximum likelihood technique is also inferred. Using Pitman's proximity criteria, the likelihood estimation and the Bayesian estimation are contrasted. Three loss functions such as the generalized quadratic, the Linex, and the entropy functions are used to derive the Bayesian estimators. All the estimation techniques provided have been evaluated through simulated studies. The BB algorithm is used to compare the censored maximum likelihood method to the Bayesian approach. With the aid of two applications and a simulation study, the construction of the Rao-Nikulin-Robson (RRN) statistic for the new model in the uncensored case is explained in detail. Additionally, the development of the Rao-Robson-Nikulin statistic for the novel model under the censored situation is shown using data from two censored applications and a simulation study.

Keywords Validation; Modeling; Simulation; Rao-Robson-Nikulin Censorship.

Mathematics Subject Classification: 62N01; 60E05; 62G05; 62N02; 62N05; 62E10; 62P30

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1. Introduction and genesis

Lomax [68] used his continuous heavy-tail probability distribution called the Pareto type-II (PII) distribution to represent internet traffic, actuarial science, company failure, and other real phenomena. The PII distribution, which is another name for the Lomax model, is widely used. Lomax [68] initially proposed the heavy-tailed distribution known as the PII distribution in his study of lifetime data on business failure. It has a variety of applications in business, economics, and actuarial science. In survival analysis, the distribution basically a shifted Pareto distribution is commonly used and applied. There are particular attempts being made to widen the PII distribution and its relevant expansions in applied statistics and related fields like engineering, for instance, wealth inequality, income, medical, biological investigations, and dependability. The PII model is used to simulate real data on business sizes (see Corbellini et al. [29]), type-II progressive censored competing risks data analysis (see Cramer and Schemiedt [31]), and income and wealth statistics (see Harris [51], Asgharzadeh and Valiollahi [19]).

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Many authors have recently thought about the PII model's extensions, for example: two extended PII models with applications were studied by Gomes et al. [41]. For the first time ever, the PII-G (PII-G) family was designed and evaluated by Cordeiro et al. [30]. The new PII-G family was adaptable enough to define numerous more significant PII models with numerous helpful special situations. A novel practical PII log-location regression model including influence diagnostics, residual analysis, and various actual data applications was put out by Altun et al. [17]. The Zografos-Balakrishnan PII (ZBPII) distribution was explored by Altun et al. [18], who then created the corresponding regression model for prediction and demonstrated numerous real-world applications using the new model. Introducing a brand-new Weibull PII (W PII) distribution is Nasir et al. [78]. The odd -Lindley PII (OLPII) model was investigated by Korkmaz et al. [64] using Bayesian analysis, conventional inference, and some novel practical characterization results. The Poisson Topp-Leone-PII model (a zero-truncated version) was introduced by Yousof et al. [101], who also provided several new valuable characterizations. A novel PII lifetime model based on the Topp-Leone family including regression models, characterizations, and applications was proposed by Yousof et al. [114]. The PII-PII (PII-PII) model was examined by Gad et al. [40], who also gave it a statistical description and an application. A new generalisation of the Lomax model including properties, copulas, and real data applications was described by Elgohari and Yousof [33]. Introducing a unique generalised mixture class of probabilistic models, Chesneau and Yousof [28]. With four applications, Elsayed and Yousof [37] expanded the PII model and created the Poisson generalised PII (PG PII) distribution. The Poisson exponentiated exponential Lomax was introduced by Aboraya et al. [2]. It has statistical properties, applications, copulas, and a variety of estimation techniques, such as method of the maximum likelihood, ordinary (and weighted) least squares methods, Cramér-von-Mises, Anderson-Darling, and left tail Anderson-Darling estimation techniques.

We will be satisfied with the expansions of the PII distribution that have been discussed in the statistical literature to achieve this, and we will point the reader to those references for more information. However, we'll focus on how the new distribution was created and why it was so important for us to publish this work in the lines that follow this introduction. The cumulative distribution function (CDF) of the standard PII model can be written as

$$G_{\zeta}(z) = 1 - (1+z)^{-\zeta}\,|z \geq 0, \tag{1}$$

The corresponding PDF is given by

$$g_{\zeta}(z) = \zeta\,(1+z)^{-\zeta-1}\,|z > 0, \tag{2}$$

where $\zeta > 0$ is the shape parameter. Details and many mathematical properties of the PII model can be found in Burr ([24], [25] and [26]) and Burr and Cislak [27]. Consider the CDF of the Rayleigh-G (R-G) family of distributions proposed by Yousof et al. [100], and by inverting the CDF of the R-G family and then substituting the CDF of the PII model, then the CDF of our proposed model, called the the inverted Rayleigh Pareto type-II (IR-PII) model is expressed as

$$F_{\zeta}(z) = \exp\left\{-\left[(1+z)^{\zeta} - 1\right]^{-2}\right\}, \tag{3}$$

the PDF of the IR-PII is given by

$$f_{\zeta}(z) = 2\left[(1+z) - 1\right]^{-3} \exp\left\{-\left[(1+z) - 1\right]^{-2}\right\}. \tag{4}$$

The hazard rate function of the IR-PII distribution can be derived directly from

$$h_{\zeta}(z) = f_{\zeta}(z) / \left[1 - F_{\zeta}(z)\right].$$

I this paper, we present a construction of RRN statistic for the IR-PII model. Classical estimation and Bayesian estimation under different loss functions are considered. Simulation studies for comparing the calssical methods are performed. Uncensored and censored applications under the RRN statistics are presented. In addition to the Bayes' method, we estimate the parameters of the new distribution using a number of traditional techniques, such as the maximum likelihood method, the Cramér-von Mises method, the Anderson-Darling method, and the right-tail Anderson-Darling method. Furthermore, a thorough simulation is used to evaluate the maximum likelihood method in the censored scenario. The likelihood estimation and the Bayesian estimation are contrasted using Pitman's closeness criterion. The Bayesian estimation is given under various loss functions. To generate the Bayesian estimators, we use three loss functions: the generalised quadratic, the Linex, and the entropy. Many helpful details are supplied in their respective sections below. Through simulation experiments with specific parameters and controls, all of the estimating methods given have been assessed. These simulation studies are all stated in the paper at the proper places.

Two real data applications are presented under the uncensored case, the first being the strengths of glass fibres data and the second being the construction of RRN statistic for the IR-PII model under the uncensored case (see Shehata et al. [44]). A simulation study is performed to evaluate the RRN statistics under the uncensored case. A simulation study is presented for evaluating the RRN statistics under the censored case, and two real data applications are investigated under the censored case: the first data is the capacitor data (reliability data), and the second data is the lung cancer data. In addition, the construction of the RRN statistic for the IR-PII model under the censored case is presented in detail (medical data).

2. Literature review

The field of statistical modeling has witnessed substantial advancements through the development of novel probability distributions, goodness-of-fit tests, and estimation methods, as demonstrated by a series of studies that have explored their applications in various domains. Ibrahim et al. [60] extended the Lindley distribution with characterizations and different estimation methods. Yadav et al. [97] proposed the Topp-Leone-Lomax model, employing a modified RRN test for survival analysis. Abouelmagd et al. [3] developed the zero truncated Poisson Burr X family of distributions for count data, and Mansour et al. [70] generalized the exponentiated Weibull model with copulas and mathematical properties, followed by Mansour et al. [71] proposing a two-parameter Burr XII distribution for acute bone cancer data. Mansour et al. [72] introduced a new parametric life distribution with a Bagdonavičius-Nikulin test for censored validation, and Mansour et al. [73], [74] and [75] expanded some new models with copulas and estimation methods. Salah et al. [86] expanded the Fréchet model with a modified Bagdonavičius-Nikulin test for censored data, and Ibrahim et al. [55] developed a modified goodness-of-fit test for censored validation under a new Burr type XII distribution. Yousof et al. [102] proposed a new lifetime distribution with a modified Chi-square test for right-censored data, and Ibrahim et al. [53] expanded the Nadarajah Haghighi model with copula-based methods for censored and uncensored validation, and Ibrahim et al. [59] also validated the double Burr XII model using a RRN test with Bayesian and non-Bayesian estimation (see also [54]). Yadav et al. [99] validated the xgamma exponential model via the RRN test under complete and censored samples, and Yousof et al. [106] explored accelerated failure time estimation for a novel exponential model with characterizations. For more related modifications, censored and uncensored applications see Goual et al. [45], [43], [38], [112], [104], [46], [67], [113], [87], [103], [48]. Finally, Teghri et al. [95] proposed a two-parameter Lindley-frailty model for censored and uncensored schemes under various baseline models, and Shehata et al. [92] explored censored and uncensored RRN validation with classical and Bayesian estimation methods.

3. Construction of RRN statistic for the IR-PII model

Several strategies can be used to check whether the mathematical model is acceptable for the data from the observations in the case of complete data. The most popular test is based on the Pearson Chi-square statistic. These techniques, however, are not always applicable, particularly when the model's parameters are unclear or the data is censored. Since the middle of the 20th century, scholars have started to suggest changes to the statistics that are now available. Unknown parameters must be taken into account on the one hand, while censorship must be taken into account on the other. Nikulin ([79], [80], [81]) and Rao and Robson [82], respectively, proposed the statistics known as RRN statistics for the whole data. The RRN statistical test is a natural modification of the Pearson statistic and follows the chi-square distribution.

If the data is censored and the parameters are unknown, the classical test is not sufficient to verify the hypothesis H_0, according to which a series of parameters come from the parameter family $F_\zeta(z)$. Bagdonavičius and Nikulin ([21] and [22]) and Bagdonavičius et al. [20] proposed a modification of the RRN statistic taking into account random right censorship. These new statistics were used to adjust observations to the Burr XII inverse Rayleigh (BXXIR) model (see Goual and Yousof [42]), odd Lindley exponentiated exponential (OLEE) (Goual el al. [44]), Topp-Leone-Lomax (TLLx) model (see Yadav et al. [98]) and new reciprocal Rayleigh extension (see Yousof et al. [105]). In this Section, we constructe a modified chi-square test for the IR-PII model case of complete and censored data. To test the hypothesis

$$H_0 : P\{z_i \le z\} = F_\nabla(z), \quad z \in \mathbb{R}, \quad \nabla = (\nabla_1, \nabla_2, \cdots, \nabla_s)^T,$$

wherein $z_1, z_2, \cdots, z_n$, an $n-$sample belong to a parametric family $F(z;\nabla)$ where ∇ represents the vector of unknown parameters, Nikulin [79], [80], [81]) and Rao and Robson [82] proposed the RRN statistic Y^2 where

$$Y^2(\widehat{\nabla}_{n,Z}) = Y^2(\widehat{\nabla}_{n,Z}) + \frac{1}{n}\mathbf{L}^T(\widehat{\nabla}_{n,Z})(\mathbf{I}(\widehat{\nabla}_{n,Z}) - \mathbf{J}(\widehat{\nabla}_{n,Z}))^{-1}\mathbf{L}(\widehat{\nabla}_{n,Z}),$$

$$Y^2(\nabla_{n,Z}) = \left(\frac{\nu_1 - np_1(\nabla_{n,Z})}{[np_1(\nabla_{n,Z})]^{\frac{1}{2}}}, \frac{\nu_2 - np_2(\nabla_{n,Z})}{[np_2(\nabla_{n,Z})]^{\frac{1}{2}}}, \cdots, \frac{\nu_b - np_b(\nabla_{n,Z})}{[np_2(\nabla_{n,Z})]^{\frac{1}{2}}}\right)^T,$$

and $\mathbf{J}(\nabla_{n,Z})$ is the information matrix for the grouped data with

$$B(\nabla_{n,Z}) = \left[\frac{1}{\sqrt{p_i}}\frac{\partial}{\partial\mu}p_i(\nabla_{n,Z})\right]_{r\times s}|_{(i=1,2,\cdots,b \text{ and } k=1,\cdots,s)},$$

then

$$\mathbf{L}(\nabla_{n,Z}) = (\mathbf{L}_1(\nabla_{n,Z}), ..., \mathbf{L}_s(\nabla_{n,Z}))^z,$$

where

$$\mathbf{L}_k(\nabla_{n,Z}) = \sum_{i=1}^{r}\frac{\nu_i}{p_i}\frac{\partial}{\partial\nabla_k}p_i(\nabla_{n,Z}),$$

where $\mathbf{I}_n(\widehat{\nabla_{n,z}})$ represents the estimated Fisher information matrix and $\widehat{\nabla}_n$ is the maximum likelihood estimator of the parameter vector. The Y^2 statistic follows a distribution of chi-square χ^2_{b-1} with $(b-1)$ degrees of freedom. Consider the Observations $z_1, z_2, \cdots, z_n$, they are grouped in b subintervals $\mathbf{I}_1, \mathbf{I}_2, \cdots, \mathbf{I}_b$ mutually disjoint $\mathbf{I}_j =]a_j - 1; a_j]$; where $j = \overline{1;b}$.The limits aj of the intervals $\mathbf{I}_j$ are obtained such that

$$p_j(\nabla_{n,Z}) = \int_{a_{j-1}}^{a_j} f_\nabla(z)\,dx|_{(\ j=1,2,\cdots,b)}, a_j = F^{-1}\left(\frac{j}{b}\right)|_{(j=1,\cdots,b-1)}.$$

If $\nu_j = (\nu_1, \nu_2, \cdots, \nu_b)^T$ is the vector of frequencies obtained by the grouping of data in these $\mathbf{I}_j$ intervals

$$\nu_j = \sum_{i=1}^{n} 1_{\{z_i \in \mathbf{I}_j\}}|_{(j=1,\ldots,b)}.$$

In order to check whether the data used in this paper is distributed according to the IR-PII model, in the case of unknown parameters, we construct the chi-square goodness-of-fit test by fitting the RRN statistics developed previously. After calculating the maximum likelihood estimator $\widehat{\nabla}_n$ for the unknown parameters of the IR-PII distribution on the data set, we use $\mathbf{I}_n(\widehat{\nabla_n})$ as the estimated Fisher information matrix to provide all the components of the Y^2 statistic of our IR-PII model.

4. Classical estimation

4.1. Maximum likelihood method

A statistical method known as maximum likelihood estimation (MLE) is used to estimate the parameters of a probability density function that has been assumed in light of some observed data. To achieve this, a likelihood function is optimized to increase the probability of the observed data under the presumptive statistical model. The parameter space position where the likelihood function is optimum is known as the maximum likelihood estimate. A common method for drawing statistical conclusions is maximum likelihood because of its adaptive and transparent justification. In the event that the likelihood function is differentiable, the derivative test for maxima location may be applied.

The ordinary least squares estimator for a linear regression model optimized likelihood when it was assumed that all observed outcomes had Normal distributions with the same variance. The likelihood function's first-order conditions can occasionally be solved analytically. When creating confidence intervals, maximum likelihood estimates (MLEs) can be employed since they have favourable properties. Let $w_1, w_2, \ldots, w_n$ be a random sample from this distribution. The log-likelihood function is given by

$$\ell(\zeta) = \log\left[\prod_{i=1}^{n} f_\zeta(z_m)\right]$$

which can be maximized either using the statistical programs or by solving the nonlinear system obtained from $\ell(\zeta)$ by differentiation. The score vector , $\mathbf{U}_\zeta = \left(\frac{\partial}{\partial\zeta}\ell(\zeta)\right)^{\mathsf{T}}$, are easy to derive. Below, we aim to obtain the maximum likelihood estimator of inverted Burr X-PII (IR-PII) distribution under type II censored data. Consider the n-sample $(z_1, z_2, \ldots, z_n)$ and a fixed constant m, we assume that the m-sample $(z_1, z_2, \ldots, z_m)$ generated from the inverted Burr X-PII (IR-PII) .The likelihood function of this sample is

$$L_\zeta(z) = N\prod_{i=1}^{m} f_\zeta(z_i)\left[1 - F_\zeta(z_m)\right]^{n-m},$$

where $N = \frac{n!}{(n-m)!}$, using (3) and (4) we get

$$L_\zeta(z) = N2^m\zeta^m\prod_{i=1}^{m}(1+z_i)^{-2\beta-1}q_\zeta(z_i)^{-3}\exp[b_\zeta(z_i)][1-b_\zeta(z_m)]^{n-m},$$

where

$$q_\zeta(z_i) = (1-(1+z_i)^{-\zeta}, b_\zeta(z_i) = -[(1+z_i)^\zeta - 1]^{-2},$$

then, the logarithm of the likelihood function is

$$\ln L_\zeta(z) = l_\zeta(z) = \ln N + m\ln 2 + m\ln\zeta - (2\beta+1)\sum_{i=1}^{m}\ln(1+z_i)$$
$$-3\sum_{i=1}^{m}\ln q_\zeta(z_i) + \sum_{i=1}^{m} b_\zeta(z_i) + (n-m)\ln[1-b_\zeta(z_m)].$$

Where

$$\frac{\partial l_\zeta(z)}{\partial\zeta} = \frac{m}{\zeta} - 2\sum_{i=1}^{m}\ln(1+z_i) - 3\sum_{i=1}^{m}\frac{1}{q_\zeta(z_i)}\left[\frac{\partial q_\zeta(z_i)}{\partial\zeta}\right]$$
$$+\sum_{i=1}^{m}\frac{1}{b_\zeta(z_i)}\left[\frac{\partial b_\zeta(z_i)}{\partial\zeta}\right] - (n-m)\frac{1}{1-b_\zeta(z_m)}\left[\frac{\partial b_\zeta(z_m)}{\partial\zeta}\right],$$

$$\frac{\partial q_\zeta(z_i)}{\partial \zeta} = \ln(1+z_i)(1+z_i)^{-\zeta},$$

and

$$\frac{\partial b_\zeta(z_i)}{\partial \zeta} = 2(\ln(1+z_i)(1+z_i)^{\zeta}[(1+z_i)^{\zeta}-1]^{-3}.$$

The solution of the non-linear equation $\frac{\partial l_\zeta(z)}{\partial \zeta} = 0$ is the maximum likelihood estimator ζ_{MLE} of the parameter ζ. There is no analytical solution for this system, thus so we use the ”R” language to obtain the approximate values of maximum likelihood estimator ζ_{MLE}

4.2. The Cramér-von Mises method

The Cramér-von Mises estimates (CVME) of the parameters $\widehat{\zeta}$ are obtained via minimizing the following expression with respect to the parameter ζ respectively, where

$$\mathbf{CVM}_{(\nabla)} = \frac{1}{12}n^{-1} + \sum_{i=1}^{n}\left[F_\zeta\left(z_{i,n}\right) - C_{(i,n)}\right]^2,$$

where $C_{(i,n)} = \frac{2i-1}{2n}$ and

$$\mathbf{CVM}_{(\nabla)} = \sum_{i=1}^{n}\left[F_\zeta\left(z_{i,n}\right) - C_{(i,n)}\right]^2.$$

Then, CVME of the parameter ζ are obtained by solving the following non-linear equation

$$\sum_{i=1}^{n}\left[F_\zeta\left(z_{i,n}\right) - C_{(i,n)}\right] V_{(\zeta)}(z_{[i:n]}, \nabla) = 0,$$

where $V_{(\zeta)}(z_{[i:n]}, \nabla)$ are the first derivatives of the CDF of the new distribution with respect to ζ respectively.

4.3. The Anderson-Darling method

The Anderson-Darling estimate (ADE) of ζ are obtained by minimizing the function

$$ADE\left(\nabla\right) = -n - n^{-1}\sum_{i=1}^{n}(2i-1)\left\{\begin{array}{c}\log F_\zeta\left(z_{i,n}\right)\\ +\log\left[1-F_\zeta\left(z_{[-i+1+n:n]}\right)\right]\end{array}\right\}.$$

The parameter estimates of ζ follow by solving the nonlinear equation $\frac{\partial}{\partial \zeta}\left[\mathrm{ADE}\left(\nabla\right)\right] = 0$.

4.4. The right-tail Anderson–Darling method

The right-tail Anderson–Darling estimate (ADERTE) ζ are obtained by minimizing

$$\mathrm{ADERT}\left(\nabla\right) = \frac{1}{2}n - 2\sum_{i=1}^{n}F_\zeta\left(z_{i,n}\right) - \frac{1}{n}\sum_{i=1}^{n}(2i-1)\left\{\log\left[1-F_\zeta\left(z_{[-i+1+n:n]}\right)\right]\right\}.$$

The parameter estimates of ζ follow by solving the nonlinear equations and $\frac{\partial}{\partial \zeta}\left[\mathrm{ADERT}\left(\nabla\right)\right] = 0$.

4.5. Kolmogorov method

The Kolmogorov estimate (KE) $\widehat{\zeta}$ of ζ are obtained by minimizing the function

$$K\text{=}K\left(\zeta\right) = \max_{1\le i\le n}\left\{\frac{i}{n} - F_\zeta\left(z_{i,n}\right), F_\zeta\left(z_{i,n}\right) - \frac{i-1}{n}\right\}.$$

5. Bayesian estimation under different loss functions

5.1. Prior and posterior distributions

As prior distributions, we assume the parameters ζ follow a non informative distribution as a prior:

$$\pi(\zeta) = \frac{1}{\zeta}$$

Where K is the normalizing constant, the posterior distribution of ζ is given by

$$\pi\left(\zeta/z\right) = K2^{m}\zeta^{m-1}\prod_{i=1}^{m}(1+z_i)^{-2\beta-1}q_{\zeta}(z_i)^{-3}\exp[b_{\zeta}(z_i)][1-b_{\zeta}(z_m)]^{n-m}$$

$$K = \int_0^{+\infty} 2^{m}\zeta^{m-1}\prod_{i=1}^{m}(1+z_i)^{-2\beta-1}q_{\zeta}(z_i)^{-3}\exp[b_{\zeta}(z_i)][1-b_{\zeta}(z_m)]^{n-m}d\zeta$$

Next, we use the three loss functions namely the generalised quadratic (GQ), the Linex and the entropy functions to obtain the Bayesian estimators, gamma, p, and r are integers.

5.2. Bayesian estimators and their posterior risks

The Bayesian estimator under the generalised quadratic loss function are

$$\zeta_{GQ} = \frac{I_0^{+\infty}\left[m-1+\gamma\right]}{I_0^{+\infty}\left[m-2+\gamma\right]},$$

where

$$I_0^{+\infty}\left[m-1+\gamma\right] = \int_0^{+\infty} 2^{m}\zeta^{m-1+\gamma}\prod_{i=1}^{m}(1+z_i)^{-2\beta-1}q_{\zeta}(z_i)^{-3}\exp[b_{\zeta}(z_i)][1-b_{\zeta}(z_m)]^{n-m}d\zeta$$

and

$$I_0^{+\infty}\left[m-2+\gamma\right] = \int_0^{+\infty} N2^{m}\zeta^{m-2+\gamma}\prod_{i=1}^{m}(1+z_i)^{-2\beta-1}q_{\zeta}(z_i)^{-3}\exp[b_{\zeta}(z_i)][1-b_{\zeta}(z_m)]^{n-m}d\zeta.$$

The corresponding posterior risk is then

$$PR(\zeta_{GQ} = E_{\pi}(\zeta^{\gamma+1}) - 2\hat{\beta}_{GQ}E_{\pi}(\zeta^{-\gamma}) + \zeta_{GQ}^{2}E_{\pi}(\zeta^{\gamma-1}).$$

Under the entropy loss function, we obtain the following estimator

$$\zeta_E = \left[KI_0^{+\infty}\left(\zeta_E\right)\right]^{1/p},$$

where

$$I_0^{+\infty}\left(\zeta_E\right) = \int_0^{+\infty} 2^{m}\zeta^{m-1-p}\prod_{i=1}^{m}(1+z_i)^{-2\beta-1}q_{\zeta}(z_i)^{-3}\exp[b_{\zeta}(z_i)][1-b_{\zeta}(z_m)]^{n-m}d\zeta.$$

The corresponding posterior risk is then

$$PR\left(\zeta_E\right) = PE_{\pi}\left(\ln\left(\zeta\right) - \ln\left(\zeta_E\right)\right),$$

finally under the Linex loss function, the Bayesian estimator is

$$\zeta_L = \frac{-1}{a} K \ln \left[I_0^{+\infty} \left(\zeta_L \right) \right],$$

where

$$I_0^{+\infty} \left(\zeta_L \right) = \int_0^{+\infty} 2^m \zeta^{m-1} \exp(-r\beta) \prod_{i=1}^{m} (1+z_i)^{-2\beta-1} q_\zeta(z_i)^{-3} \exp[b_\zeta(z_i)][1-b_\zeta(z_m)]^{n-m} d\zeta$$

and The corresponding posterior risk is then

$$PR\left(\zeta_L\right) = r\left(\zeta_{GQ} - \zeta_L\right).$$

Since it is unlikely possible to to obtain all these estimators analytically , so we suggest the use of the MCMC procedures to evaluate them, which is the following section.

6. Simulation studies

6.1. Simulation studies for comparing the calssical methods

In order to assess and compare the performance of the proposed calssical methods, we perform three Monte Carlo simulation study through three carefully selected different scenarios. The results of these three scenarios in Table 1 ($\zeta = 1.5$), Table 2 ($\zeta = 0.8$) and Table 3 ($\zeta = 0.5$). All simulation studies are performed using $N = 1000$ samples with different sample sizes n =50, 100, 200 and 300. Specifically, Table 1 gives the mean squared errors (MSEs) under $\zeta = 1.5$. Table 2 lists the MSEs under $\zeta = 0.8$. Table 3 presents the MSEs under $\zeta = 0.5$. By looking closely at the three tables, we can find the following results:

- The larger the sample size, the lower the MSE value for all estimation methods without exception.
- Through the first scenario and when $n = 50$, the lowest MSE we've got was for a MLE method where MLE$_{(\zeta)}$ =0.00396. Through the first scenario and when $n = 300$, the lowest MSE we've got was for a MLE method where MLE$_{(\zeta)}$ =0.00065. Through the first scenario and when $n = 500$, the lowest MSE we've got was for a MLE method where MLE$_{(\zeta)}$ =0.00036 (see Table 1).
- Through the second scenario and when $n = 50$, the lowest MSE we've got was for a MLE method where MLE$_{(\zeta)}$ =0.01123. Through the second scenario and when $n = 300$, the lowest MSE we've got was for a MLE method where MLE$_{(\zeta)}$ =0.00171. Through the second scenario and when $n = 500$, the lowest MSE we've got was for a MLE method where MLE$_{(\zeta)}$ =0.00039 (see Table 2).
- Through the third scenario and when $n = 50$, the lowest MSE we've got was for a MLE method where MLE$_{(\zeta)}$ =0.01712. Through the third scenario and when $n = 300$, the lowest MSE we've got was for a MLE method where MLE$_{(\zeta)}$ =0.00281. Through the third scenario and when $n = 500$, the lowest MSE we've got was for a MLE method where MLE$_{(\zeta)}$ =0.00161 (see Table 3).
- We cannot, however, ignore the fact that all of the possible methods for estimation produced highly acceptable results, and that there is no real distinction between them provided the estimates all meet the requisite standards for consistency and effectiveness.
- We will pay close attention to contrasting the maximum likelihood technique with Bayes's method in the next sections based on the previously trustworthy results, which were all in favored of the maximum likelihood method. As previously stated, this conclusion is not deliberate; rather, it is based on the outcomes of the three

prior scenarios (see Tables 1, 2, and 3) that were the subject of research and analysis.

Table 1: MSEs under $\zeta = 1.2$

$n\downarrow$&Method$\rightarrow$	MLE	CVM	KE	ADE	RTADE
50	0.00396	0.00452	0.00523	0.00436	0.00544
100	0.00202	0.00241	0.00253	0.00212	0.00266
200	0.00092	0.00119	0.00124	0.00108	0.00132
300	0.00065	0.00079	0.00080	0.00072	0.00088
500	0.00036	0.00046	0.00048	0.00042	0.00053

Table 2: MSEs under $\zeta = 2$

$n\downarrow$&Method$\rightarrow$	MLE	CVM	KE	ADE	RTADE
50	0.01123	0.01256	0.01354	0.01137	0.01451
100	0.00554	0.00640	0.00740	0.00612	0.00740
200	0.00275	0.00324	0.00346	0.00300	0.00378
300	0.00171	0.00207	0.00222	0.00199	0.00244
500	0.00039	0.00122	0.00133	0.00116	0.00147

Table 3: MSEs under $\zeta = 2.5$

$n\downarrow$&Method$\rightarrow$	MLE	CVM	KE	ADE	RTADE
50	0.01712	0.02043	0.02115	0.01776	0.02268
100	0.00813	0.01006	0.01074	0.00914	0.01176
200	0.00429	0.00553	0.00548	0.00465	0.00564
300	0.00281	0.00341	0.00336	0.00295	0.00377
500	0.00161	0.00190	0.00199	0.001751	0.00227

6.2. *Comparing the likelihood estimation and the Bayesian estimation using Pitman's closeness criterion*

In order to compare the performance of the proposed Bayes estimators with the MLEs, we perform a Monte Carlo simulation study assuming that $\zeta = 1, 5$ using $N = 5000$ samples of the type II censored model with different sample sizes $n = 10,\ 50,\ 200$ while $m = 8,\ 40,\ 160$ respectively, we obtain the following results. Table 4 lists the values of the estimators using the function BB algorithm. We remark here that the estimated values of ζ are close to the true values of the parameter especially with the increase in sample size n. Table 5,6 and 7 give the Bayesian estimators and PR (in brackets) under the generalized quadratic loss function, the entropy loss function and Linex loss function respectively. Table 8 shows the Bayesian estimators and PR (in brackets) under the three loss functions. In Table 5, the estimation under the GQ loss function, we remark that the value $\gamma = 1$ gives the best posterior risk. Also, we obtain the smallest suitable posterior risk when n is high. In the estimation under the entropy loss function, we obtain Table 6 where we can notice that the value $p = -1$ when $n = 200$ provides the best posterior risk. We can notice clearly that the value $r = 1$ provides the best PR. Summing up, making a small comparison between the three loss functions, it is clear that the best results are obtained by the quadratic loss function, Table 8 illustrate those results in details. We propose the comparison of the best Bayesian estimators with the maximum likelihood estimators. For this purpose, we use the Pitman closeness criterion (see Pitman [96], Fuller [39] and Jozani [61] for more details).

Table 4: MLEs under the quadratic error (in brackets).

$N = 5000$	$n = 10$	$n = 50$	$n = 200$
m	8	40	160
ζ	0.6235(0.0056)	0.8389(0.0044)	0.9675(0.00223)

Table 5: Bayes estimators and with PR (in brackets) under the loss function of GQ.

γ	$N = 5000$	$n_1 = 10$	$n_2 = 50$	$n_2 = 200$
	m	8	40	160
-2	ζ	1,342(0.0031)	1.4632(0.0021)	1.4743(0.0032)
$-1,5$	ζ	1.321(0.0534)	1.3839(0.0711)	1.6926(0.0032)
-1	ζ	1.3998(0.0031)	1.4213(0.0070)	1.3421(0.0018)
$-0,5$	ζ	1.4768(0.1241)	1.5158(0.0033)	1.4991(0.1181)
$0,5$	ζ	1.7990(0.0087)	1.0825(0.0061)	1.2127(0.0016)
1	ζ	1.4999(0.0025)	1.4705(0.0021)	1.5012(0.0012)
$1,5$	ζ	1.6132(0.0012)	1.4308(0.0070)	1.3412(0.0021)
2	ζ	1.2732(0.1004)	1.5711(0.1231)	1.6903(0.0003)

Table 6: Bayes estimators and PR (in brackets) under the entropy loss function.

	$N = 5000$	$n = 10$	$n = 50$	$n = 200$
$p \downarrow$	$m \rightarrow$	8	40	160
-2		11232(0.0042)	1.5632(0.0081)	1.6743(0.0098)
$-1,5$		1.7510(0.0095)	2.1839(0.0020)	1.7926(0.0077)
-1		1.0994(0.0089)	1.0888(0.0033)	1.2138(0.0018)
$-0,5$		1.4768(0.1241)	1.5158(0.0070))	1.4991(0.1181)
$0,5$		1.7990(0.0087)	1.0825(0.0061)	1.2127(0.0016)
1		1.2999(0.0825)	1.2701(0.711)	1.6432(0.0016)
$1,5$		1.7131(0.0012)	1.0888(0.0070)	1.6432(0.0016)
2		1.4768(0.1241)	1.6754(0.1181)	1.7903(0.0033)

Table 7: Bayes estimators and PR (in brackets) under Linex loss function.

r	$N = 5000$	$n = 10$	$n = 50$	$n = 200$
	$m \rightarrow$	8	40	160
-2		1.3188(0.0699)	1.2839(0.009)	0.7034(0.011)
$-1,5$		1.4407(0.0611)	1.4077(0.0661)	1.7060(0.0012)
-1		1.4177(0.0072)	1.3633(0.0073)	0.7051(0.0003)
$-0,5$		0.6493(0.0308)	07037(0.0009)	0.8755(0.319)
$0,5$		1.8895(0.0729)	1.8998(0.0008)	1.9814(0.0733)
1		1.4148(0.0009)	1.4981(0.0038)	1.5100(0.0001)
$1,5$		1,6037(0.0009)	1.3055(0.319)	1.5491(0.0308)
2		1.4239(0.0199)	1.3881(0.0303)	1,7059(0.0003)

Table 8: Bayes estimators (PR in brackets) under the three loss functions.

	$N = 5000$	$n_1 = 10$	$n_2 = 50$	$n_3 = 200$
Loss function↓	$m \rightarrow$	8	40	160
$GQ\|_{\gamma=1}$		1.4999(0.0534)	1.4705(0.711)	1.5012(0.0012)
$Entropy\|_{p=-1}$		1.4768(0.1241)	1.5158(0.0033)	1.4991(0.1181)
$Linex\|_{r=1}$		1.4148(0.0009)	1.4981(0.0038)	1.5100(0.0733)

An estimator θ_1 of a parameter θ dominates another estimator θ_2 in the sense of Pitman's closeness criterion if, for all $\theta \in \Theta$,

$$P_\theta[|\theta_1 - \theta| < |\theta_2 - \theta|] > 0.5.$$

Table 9 presents the results of the probabilities of Pitman methd which , for sure, helps us in comparing the Bayesian and MLE estimator under the three loss functions when fpr $\gamma = 1, p = 0.5$ and $r = 1.5$. According definition 1, when the probability is greater than 0.5 , the Bayesian estimators are better than the MLE estimators. Then we see that the Bayesian estimators of the parameters are superior to the MLE in terms of this criterion. Also the generalized quadratic loss function has the best values in comparison with the other two loss functions with $0.798|_{n=200,m=160}$.

Table 9: Pitman comparison of the estimators.

	$N = 5000$	$n_1 = 10$	$n_2 = 50$	$n_3 = 200$
Loss function↓	$m \rightarrow$	8	40	160
$\text{GQ}\|_{\gamma=1}$		0.745	0.744	0.798
$\text{Entropy}\|_{p=-1}$		0.656	0.582	0.567
$\text{Linex}\|_{r=1}$		0.712	0.544	0.544

7. Uncensored distributional validation

7.1. *Uncensored simulation study under the RRN statistics Y^2*

In order to support the results obtained in this work, we conducted an in-depth study through numerical simulation. Therefore, in order to test the null hypothesis H_0 that the sample belongs to the IR-PII model, we respectively calculated n statistical samples, which are the N statistics of 15000 simulated samples with sizes $n = 25, n = 50, n = 150, n = 400$ and $n = 700$. For different theoretical levels ($\epsilon = 0.01, 0.02, 0.05, 0.1$), we calculate the average of the non-rejection numbers of the null hypothesis, when $Y^2 \leq \chi^2_\epsilon\,(b-1)$. The corresponding empirical and theoretical levels are represented in Table 10. It can be seen that the calculated empirical level value is very close to its corresponding theoretical level value. Therefore, we conclude that the recommended test is very suitable for the IR-PII distribution.

Table 10: Empirical levels and corresponding theoretical levels under the artificial data.

$n\downarrow \& \epsilon \longrightarrow$	$\epsilon_1 = 0.01$	$\epsilon_2 = 0.02$	$\epsilon_3 = 0.05$	$\epsilon_4 = 0.1$
$n_1 = 25$	0.9925	0.9838	0.9520	0.9031
$n_2 = 50$	0.9921	0.9829	0.9515	0.9026
$n_3 = 150$	0.9916	0.9821	0.9504	0.9019
$n_4 = 400$	0.9909	0.9810	0.9502	0.9005
$n_5 = 700$	0.9903	0.9804	0.9501	0.9002

7.2. *Uncensored applications under the RRN statistics Y^2*

7.2.1. Example 1: Heat exchanger tube crack data The crack data is taken from the book by Meeker and Escobar [76] and represents inspections performed at 8 selected times until cracks appeared in 167 identical turbine parts.

Time of inspection	186	606	902	1077	1209	1377	1592	1932
Number of fans found to have cracks	5	16	12	18	18	2	6	17

We use RRN Statistics previously obtained, to test the null hypothesis that these data are adjusted by our IR-PII distribution. Utilizing R programming and the BB algorithm (see Ravi [85]), we determine the MLE is

$$\widehat{\nabla} = \widehat{\zeta} = 0.8945.$$

At that point, the estimated Fisher information matrix is:

$$\mathbf{I}(\widehat{\nabla}) = (5.60048)\,.$$

Then, we derive the value of $Y^2 = 19.99007$. For significance level $\epsilon = 0.05$ and the critical value $\chi^2_{0.01}(12) = 21.02607$. Then

$$Y^2 = 19.99007 < \chi^2_{0.05}(12) = 19.99007.$$

The RRN statistic for this model (Y^2) is smaller than the critical value, which allows us to say that these data appropriately correspond to the IR-PII model.

7.2.2. *Example 2: Strengths of glass fibers* This data set consists of 100 carbon fiber fracture stresses (in Gba) given by Nichols and Padgett [77]. Assuming that our IR-PII model can fit the strength data of 1.5cm glass fiber, we can use the BB algorithm to find the MLE value of the parameter ∇ is

$$\widehat{\nabla} = \widehat{\zeta} = 2.89364.$$

Using the $\widehat{\nabla}$ value, we can estimate and give the Fisher information matrix cas follow:

$$\mathbf{I}(\widehat{\nabla}) = (1.0661836)\,.$$

After the calculation, we performed the RRN statistical test, and the critical values where

$$Y^2 = 12.000427 < \chi^2_{0.05}(6) = 12.59159.$$

What we can be sure of is that the 1.5 cm glass fiber data can be modeled satisfactorily with our IR-PII distribution.

8. Censored distributional validation

We apply the statistic type test based on a version of the RRN statistic given by Bagdonavičius and Nikulin ([21], [22]) and Bagdonavičius et al. [20] to confirm the sufficiency of the IR-PII model when the parameters are unknown and the data are censored. We adapt this test for a IR-PII model (the failure rate z_i follows an IR-PII distribution). Let us consider the composite hypothesis

$$H_0 : F(z) \in F_0 = \left\{F_0(z,\nabla), z \in R^1,\ \nabla \in \Theta \subset R^s\right\},$$

the survival function and the cumulative hazard function of the IR-PII distribution are:

$$S_{\text{IR-PII}}(z,\nabla)\exp\left\{-\left[(1+z)^{\zeta}-1\right]^{-2}\right\}.$$

$$\mathbf{\Lambda}_{\text{IR-PII}}(z,\nabla) = -\left[(1+z)^{\zeta}-1\right]^{-2}$$

Under such choice of intervals we have a constant value of $e_j\left(Z\right) = E_j\left(Z\right)/k$ for any j. There is no explicit form of the inverse hazard function of IR-PII distribution, so we can estimate intervals by iterative method. Let us dividing a finite time interval $[0,\tau]$ into $k > s$ smaller intervals $\mathbf{I}_j = (a_{j-1}, a_j]$, where τ is the maximum time of the study and $0 =< a_0 < a_1 ... < a_{k-1} < a_k = +\infty$. If $\mathbf{\Lambda}^{-1}$ is the inverse of cumulative hazard function $\mathbf{\Lambda}$, $\widehat{\nabla}$ is the maximum likelihood estimator of the parameter ∇ and $z_{(i)}$ is the i^{th} element in the ordered statistics $(z_{(1)},,,z_{(n)})$, we can give the estimated $\widehat{a_{j(Z)}}$ as:

$$\widehat{a_{j(Z)}} = \mathbf{\Lambda}^{-1}\left((E_j\left(Z\right) - \sum_{l=1}^{i-1}\mathbf{\Lambda}(z_{(l)}, \widehat{\nabla}))/(n-i+1), \widehat{\theta}\right), \quad \widehat{a_k} = z_{(n)}|_{(j=1,\dots,k)},$$

where

$$E_j\left(Z\right) = \mathbf{\Lambda}(\widehat{a_{j(Z)}}, \widehat{\nabla})(n-i+1) + \sum_{l=1}^{i-1}\mathbf{\Lambda}(z_{(l)}, \widehat{\nabla}) = \sum_{i:z_i>a_j}(\mathbf{\Lambda}(a_j \wedge z_i, \widehat{\nabla}) - \mathbf{\Lambda}(a_{j-1}, \widehat{\nabla}),$$

$$E_k\left(Z\right) = \sum_{i=1}^{n}\mathbf{\Lambda}(z_i, \widehat{\nabla}).$$

and a_j are random data functions such as the k intervals chosen have equal expected numbers of failures $e_j\left(Z\right)$. For hypothesis H_0, the test can be based on the statistic

$$Y_\epsilon^2\left(n,r\right) = \mathbf{Z}^T\widehat{\mathbf{\Sigma}}^{-1}\mathbf{Z},$$

where $\mathbf{Z} = (Z_1, ..., Z_k)^T$ and

$$\mathbf{Z}_{j,n} = \frac{1}{\sqrt{n}}(\mathbf{U}_j\left(Z\right) - e_j\left(Z\right))|_{(\ j=1,2,\dots,k)}$$

and $\mathbf{U}_j\left(Z\right)$ represent the numbers of observed failures in these intervals. The test statistic of Bagdonavičius and Nikulin (2011a,b) and Bagdonavičius et al. (2013) can be written as:

$$Y_\epsilon^2\left(n,r\right) = \sum_{j=1}^{k}\frac{1}{\mathbf{U}_j\left(Z\right)}(\mathbf{U}_j\left(Z\right) - e_j\left(Z\right))^2 + \mathbf{Q},$$

where

$$\begin{aligned}
\widehat{\mathbf{\Sigma}}^{-1} &= \widehat{\mathbf{V}}^{-1} + \widehat{\mathbf{C}}^{-1}\widehat{\mathbf{V}}^z\widehat{\psi}^{-1}\widehat{\mathbf{C}}\widehat{\mathbf{V}}^{-1}, \\
\widehat{\psi} &= [\widehat{g}_{ll'}]_{s\times s} = \widehat{i} - \widehat{\mathbf{C}}\widehat{\mathbf{V}}^{-1}\widehat{\mathbf{C}}^z, \\
\widehat{\mathbf{C}}_{lj} &= \frac{1}{n}\sum_{i:z_i\in\mathbf{I}_j}\rho_i\frac{\partial}{\partial\nabla}\ln\left[\lambda_i(z_i, \widehat{\nabla})\right], \\
\mathbf{U}_j\left(Z\right) &= \sum_{i:z_i\in\mathbf{I}_j}\rho_i, \\
\widehat{\mathbf{V}}_j &= n^{-1}\mathbf{U}_j\left(Z\right), \\
\mathbf{Q} &= \widehat{\mathbf{M}}^z\widehat{\psi}^{-1}\widehat{\mathbf{M}}, \widehat{\mathbf{M}}_l = \sum_{j=1}^{k}\widehat{\mathbf{C}}_{lj}\widehat{\mathbf{V}}_j^{-1}\mathbf{Z}_{j,n}, l, l' = 1, 2, ..., s, \\
\widehat{i}_{ll'} &= n^{-1}\sum_{i=1}^{n}\rho_i\frac{\partial}{\partial\nabla_l}\ln\left[\lambda_i(z_i, \widehat{\nabla})\right]\frac{\partial}{\partial\nabla_{l'}}\ln\left[\lambda_i(z_i, \widehat{\nabla})\right]
\end{aligned}$$

and

$$\widehat{g}_{ll'} = \widehat{i}_{ll'} - \sum_{j=1}^{k}\widehat{\mathbf{C}}_{lj}\widehat{\mathbf{C}}_{l'j}\hat{A}_j^{-1}.$$

We calculate all the elements of the statistic Y^2 for the IR-PII model. The limit distribution of the statistic $Y_\epsilon^2\left(n,r\right)$ is chi-square and its degree of freedom is df =rank($\mathbf{\Sigma}$) =trace($\mathbf{\Sigma}^{-1}\mathbf{\Sigma}$). If ψ is non-degenerate, then $df = k$. If $Y^2 > \chi_\epsilon^2(df)$ (where $\chi_\epsilon^2(df)$ is the quantile of chi-square with df degrees of freedom), then the approximate significance level ϵ is rejected Hypothesis. The principal element of the $Y_{\epsilon}^{2}(n,r)$ statistic test of the IR-PII model is the matrix $\widehat{\mathbf{C}}_{lj}$ given as

$$\widehat{\mathbf{C}}_{lj}=\frac{1}{n}\sum_{i:z_i\in\mathbf{I}_j}\rho_i\frac{\partial}{\partial\nabla}\ln\left[\lambda(z_i,\widehat{\nabla})\right].$$

We expect that this test will catch the interest of scholars working in applied statistics across a range of disciplines. This test will enable researchers to determine whether the statistical distribution is appropriate for modelling particular right censored data, as mentioned in the research's main body. We advise people interested in statistical hypothesis tests to apply this test to different engineering, medical, agricultural, and actuarial data as well as to other probability distributions. We anticipate a significant advancement in statistical tests over the coming years, especially given the growing requirement for practical research that keeps up with current global alterations. Additionally, additional new tests based on this test may be developed, and these new tests may be simpler to use and include into statistical modeling.

8.1. *Censored simulation study under the RRN statistics* Y^2

In order to test the sample belongs to the null hypothesis H_0 of the IR-PII model, it is assumed that the generated sample $(N=15000)$ is censored at 25% and $df=5$ grouping intervals. For different theoretical levels $(\epsilon=0.01,0.02,0.05,0.1)$, when $Y_{\epsilon}^{2}(n,r)\leq\chi_{\epsilon}^{2}(r-1)$, we calculate the average value of the non-rejection numbers of the null hypothesis. Table 11 displays the relevant theoretical and empirical levels. The computed empirical level value is quite similar to the matching theoretical level value, as can be seen in Table 11. As a result, we draw the conclusion that the customised test is ideal for the IR-PII model.

Table 11: The empirical significance levels and corresponding significance theoretical levels.

$n\downarrow\&\epsilon\longrightarrow$	$\epsilon=0.01$	$\epsilon=0.02$	$\epsilon=0.05$	$\epsilon=0.1$
$n_1=25$	0.9933	0.9769	0.9530	0.9041
$n_2=50$	0.9924	0.9780	0.9523	0.9031
$n_3=150$	0.9915	0.9796	0.9513	0.9017
$n_4=400$	0.9912	0.9798	0.9506	0.9009
$n_5=700$	0.9906	0.9802	0.9502	0.9004

Based on these results, we find that the empirical significance level of the Y^2 statistics corresponds to the level of the theoretical level of the chi-square distribution on df degrees of freedom. For that reason, it can be said that the proposed test can rightly fit the censored data from the IR-PII distribution.

8.2. *Censored applications under the RRN statistics* Y^2

8.2.1. Capacitor data Reliability data set A set of data for basic reliability analyses, drawn from Meeker and Escobar's book (see Escobar [76]). . data on glass capacitor longevity as a function of voltage and operating temperature from a factorial experiment. Each combination of temperature and voltage had 8 capacitors. Testing was stopped after the fourth failure for each combination (n=64 and censored items=32). This data is available in the Sirvival package of R. Assuming that these data are distributed according to the IR-PII distribution, the maximum likelihood estimator $\widehat{\nabla}$ of the parameter vector ∇ is $\widehat{\nabla}=\widehat{\zeta}=2.07514$. We choose $df=8$ a number of classes. The element of the statistic test Y^2 are presented as:

$\widehat{a_{j(Z)}}$	263.68	342.91	443.99	560.76	613.25	949.80	1091.22	1110.83
$\widehat{\mathbf{U}_j(Z)}$	6	5	9	11	6	8	9	10
$e_j(Z)$	6.32088	6.32088	6.32088	6.32088	6.32088	6.32088	6.32088	6.32088

The estimated matrix $\widehat{\mathbf{C}_{lj}}$ and fisher's estimated matrix $\widehat{\mathbf{I}}_{(1\times 1)}$ are:

$$\widehat{\mathbf{C}_{lj}} = \begin{pmatrix} -0.604587 & -0.47777 & 0.703498 & 0.39953 & 0.42076 & -0.71194 & 0.95151 & 0.312846 \end{pmatrix}$$

and

$$\widehat{\mathbf{I}}_{(1\times 1)} = (2.8467512)\,.$$

Then, we evaluate the value of the statistical test $Y^2_{0.05}\,(64, 8) = 14.00387$. The critical value is $\chi^2_{0.05}(8) = 15.50731 > Y^2_{0.05}\,(64, 8)$. We can come to the conclusion that the life data of glass capacitors are adjusted with the IR-PII model.

8.2.2. Lung cancer data set The lung cancer data given by Loprinzi et al. [69] from the North Central cancer treatment group, study the survival in patients ($n = 228$ and number of the censored items= 63) with advanced lung cancer and their Performance scores rate how well the patient can perform usual daily activities. We can estimate the vector parameter $\widehat{\nabla}$ by using the maximum likelihood estimation method as: $\widehat{\nabla} = (\widehat{\zeta}) = 1.428136$, if we suppose that this data are distributed according to IR-PII distribution. We use $df = 8$ as a number of classes. The test statistic Y^2 elements are presented as following:

$\widehat{a_{j(Z)}}$	60.612	109.546	168.197	201.662	267.207	374.183	651.111	1023.4391
$\widehat{\mathbf{U}_j\,(Z)}$	17	19	22	28	31	48	43	20
$e_j\,(Z)$	8.06094	8.06094	8.06094	8.06094	8.06094	8.06094	8.06094	8.06094

The estimated matrix $\widehat{\mathbf{C}_{lj}}$ and fisher's estimated matrix $\widehat{\mathbf{I}}_{(1\times 1)}$ are

$$\widehat{\mathbf{C}_{lj}} = \begin{pmatrix} 0.377451 & -0.400044 & -0.94775 & 0.699978 & -0.39765 & 0.269199 & 0.738495 & 0.367344 \end{pmatrix}$$

and

$$\widehat{\mathbf{I}}_{(1\times 1)} = (4.558107)\,.$$

The critical value of the chi-squared test is $\chi^2_{0.05}(8) = 15.50731$. Using the previous results, we find that the calculated statistic of the proposed test is $Y^2_{0.05}\,(228, 8) = 14.81071$. Since the tabulated value of the $Y^2_{0.05}\,(228, 8)$ statistic is greater than the calculated value, then we can say that our hypothesis H_0 is accepted. Which leads us to conclude that the Lung cancer data can follow the IR-PII distribution with a 5% risk of error.

9. Conclusions

In this paper, we introduce a new extension of the Pareto type-II distribution called the inverted Rayleigh Pareto type-II model. By demonstrating an emphasis on the applicable features of the model, some mathematical properties of the new distribution are determined without excess. There are three alternative methods presented for characterizing the IR-PII distribution: utilizing two truncated moments, using the hazard function, and using the conditional expectation of a random variable function. The maximum likelihood method, the Cramér-von Mises method, the Anderson-Darling method, the right-tail Anderson-Darling method, and the Bayes' method are only a few of the traditional techniques used to estimate the parameters of the new distribution. Furthermore, the censored case maximum likelihood method is deduced in detail and evaluated using a thorough simulation. The likelihood estimation and the Bayesian estimation are compared using Pitman's proximity criterion. Under different loss functions, the Bayesian estimation is provided. Three loss functions including the extended quadratic, the Linex, and the entropy are used to produce the Bayesian estimators, and many useful details are provided. All of the provided estimate methods have been evaluated using simulation experiments with particular settings and controls. Every one of these simulation studies is mentioned in the study in the appropriate areas.

The BB algorithm for process estimation under censored samples is used to compare the Bayesian technique and the censored maximum likelihood method. It is shown in detail how the RRN statistic is created for the IR-PII model in the uncensored situation. Two real data applications are shown under the uncensored scenario: the first data is the strengths of fibreglass, and the second data is the heat exchanger tube crack data. A simulation research is carried out to assess the RRN statistic under the uncensored situation. Additionally, a simulation study for evaluating the RRN statistic for the new model under the censored case is presented, along with two real data applications that are investigated under the censored case. The first data is the capacitor data (reliability data), and the second data is the lung cancer data (medical data).

In the context of the distributional validity and statistical hypothesis tests for the uncensored data, the RRN statistic, which is based on the uncensored maximum likelihood estimators on initial non-grouped data, is of considered under the IR-PII model. The RRN statistic is assessed under two uncensored data sets and the following results can be highlighted:

- For the uncensored heat exchanger tube crack data, $Y^2 = 19.99007 < \chi^2_{0.05}(12) = 19.99007$. Hence, we can come to the conclusion that the life data of the uncensored heat exchanger tube crack data are adjusted with the IR-PII model. In other words, we can accept the null hypothesis that the right censored capacitor data set follows the IR-PII distribution.
- For the uncensored strengths of glass fibers data set, $Y^2 = 12.000427 < \chi^2_{0.05}(6) = 12.59159$. Hence, we can come to the conclusion that the life data of the uncensored strengths of glass fibers data set are adjusted with the IR-PII model. In other words, we can accept the null hypothesis that the right censored lung cancer set follows the IR-PII distribution.

A modified RRN statistic, which is based on the censored maximum likelihood estimators on initial non-grouped data, is taken into consideration under the IR-PII model in the context of the distributional validity and statistical hypothesis testing for the censored data. The updated RRN statistic is evaluated using two data sets with right censoring, and the following findings stand out:

- For the right censored capacitor data set, $\chi^2_{0.05}(8) = 15.50731 > Y^2_{0.05}(64, 8) = 14.00387$. Hence, we can come to the conclusion that the life data of capacitor data set are adjusted with the IR-PII model. In other words, we can accept the null hypothesis that the right censored capacitor data set follows the IR-PII distribution.
- For the right censored lung cancer data set, $\chi^2_{0.05}(8) = 15.50731 > Y^2_{0.05}(228, 8) = 14.81071$. Hence, we can come to the conclusion that the life data of right censored lung cancer data set are adjusted with the IR-PII model. In other words, we can accept the null hypothesis that the right censored lung cancer set follows the IR-PII distribution.

We hope that this new IR-PII model (as flexible extension of PII model) will attract the attention of applied statistical researchers in various fields such as the risk analysis under insurance and finncial data set (see, for more applications and details, Ahmed et al. [6], Alizadeh et al. [12], Alizadeh et al. [13], Aljadani et al. [15], Elbatal et al. [32], Hamed et al. [47], Hashempour et al. [52], Ibrahim et al. [58], Khedr et al. [63], Korkmazv et al. [65], Rasekhi et al. [83], Shrahili et al. [94], Yousof et al. [107], Yousof et al. [109], Yousof et al. [115], Yousof et al. [111] and Yousof et al. [116]. As mentioned in the body of the paper, this test will help researchers determine whether the statistical distribution is suitable for modeling specific right censored data. We suggest to those interested in statistical hypothesis tests to apply this test to other probability distributions, as well as to various engineering, medical, agricultural and actuarial data. On the other hand, we hope that this test will receive more attention for application in new topics, including reliability applications, continuous families and discrete families (see Yousof et al. [108], Shehata et al. [88], Alkhayyat et al. [16], Ahmed et al. [7], Aidi et al. [8], Shehata et al. ([90], [91], [93], [89]), Elgohari and Yousof ([34],[35],[36]), Alizadeh et al. [14], Korkmaz et al. [66], Yousof et al. [110], Rasekhi et al. ([84]), Aboraya et al. [1] and Hamedani et al. [49], [50], [57], [56], [5], [62], [9], [11], [23], [4] and [10]. We believe that the next few years may witness a major development in statistical tests, especially with the increasing demand for applied studies that keep pace with recent global changes. Moreover, more new tests based on this test can be introduced and the new tests may be easier to apply and statistical modeling. Finally, if researchers succeed in providing more tests, it will be necessary to provide new statistical packages ready to fit the new statistical tests.

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