STATISTICS, OPTIMIZATION AND INFORMATION COMPUTING
Stat., Optim. Inf. Comput., Vol. **11**, January 2023, pp 22–28.
Published online in International Academic Press (www.IAPress.org)

On nonsmooth multiobjective semi-infinite programming with switching constraints using tangential subdifferentials

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Abstract We investigate optimality conditions for a nonsmooth multiobjective semi-infinite programming problem subject to switching constraints. In particular, we employ a surrogate problem and a suitable constraint qualification to state necessary M-stationary conditions in terms of tangential subdifferentials. An example is given at the end to illustrate our main result.

Keywords Nonsmooth multiobjective optimization, Switching constraints, M-stationarity conditions, Constraint qualifications, Tangential subdifferentials

AMS 2010 subject classifications 90C30, 90C33

DOI: 10.19139/soic-2310-5070-1704

1. Introduction

We take up the following nonsmooth multiobjective semi-infinite programming problem subject to switching constraints, NMPSC for short

$$\begin{cases} \min & f(x) = (f_1(x), ..., f_m(x)), \\ s.t. & g_s(x) \leq 0,\ \forall s \in S, \\ & h_k(x) = 0,\ \forall k \in K = \{1, ..., q\}, \\ & G_i(x)H_i(x) = 0,\ \forall i \in I = \{1, ..., l\}, \end{cases} \tag{1}$$

where the index set S is an arbitrary nonempty set, not necessary finite. The real-valued functions f_j, $j \in J = \{1, ..., m\}$, g_s, $s \in S$, h_k, $k \in K$, G_i and H_i, $i \in I$ are defined on $\mathbb{R}^n$ and not necessary convex nor differentiable. The feasible region of (1) is given by

$$\Pi := \{x \in \mathbb{R}^n : g_s(x) \leq 0, s \in S, h_k(x) = 0, k \in K, G_i(x)H_i(x) = 0, i \in I\}.$$

The terminology "switching constraints" originates from the fact that if the product of two functions is equal to zero, then at least one of them must be equal to zero. Problems under the form (1) were recently introduced to investigate the discretization of optimal control problems with switching constraints [1, 2, 3], and to study mathematical programs with either-or-constraints [4, 5, 6]. Moreover, NMPSC can be seen as an extension of another class of optimization problems, namely mathematical programming with equilibrium constraints (MPEC) [7, 8, 9], which has the same form as NMPSC subject to an additional condition "$G_i(x) \geq 0$ and $H_i(x) \geq 0$

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ISSN 2310-5070 (online) ISSN 2311-004X (print)

for all $i \in I$". Although the latter condition does not appear in the problems studied in many papers of optimal control and related fields, we find that the published results on NMPSC are very few where compared to the MPEQ, which motivated us to deal with this type of problems as they provide a more general setting.

The major difficulty in solving (1) is that it typically violates the majority of classical constraint qualifications (such as Mangasarian-Fromovitz constraint qualification, linear independence constraint qualification), and hence the standard KKT conditions are not relevant in the context of mathematical programming with switching constraints (MPSC). This led to introduce various stationarity concepts (weak, Mordukhovich, and strong stationarity) for MPSC and to derive some associated constraint qualifications [4]. Kanzow et al. [10] proposed several relaxation methods from the numerical treatment of MPEC to MPSC. Li and Guo extended some weak and verifiable constraint qualifications for nonlinear programs to MPSC in [11]. Very recently, Mehlitz investigated a second-order optimality conditions for MPSC in [12].

In this paper, we are concerned with a nonsmooth, multiobjective and semi-infinite version of MPSC, and introduce a constraint qualification of a surrogate problem, which will guarantee an optimality condition, called M-stationarity, to hold at a local minimum. This will be performed using the concept of tangential subdifferential which includes many types of subdifferentials like Gâteaux derivatives, convex subdifferentials or those of Clarke and Michel-Penot. We point out that this concept has been efficiently employed in [13] to establish necessary optimality conditions but for a nonsmooth multiobjective bilevel programming problem without assuming neither convexity nor locally Lipschitzity of the upper level objectives and constraint functions.

The organization of the paper is as follows: In the next section, we present needed notations and recall some definitions. In Section 3, we propose M-stationary conditions for local efficient solutions of (1) involving an appropriate constraint qualification of a surrogate problem and we give an example that illustrates the main result. Finally, a conclusion is given in Section 4.

2. Preliminaries

From now on, we take the following order in the Euclidean space: $a, b \in \mathbb{R}^m$ satisfies

- $a \leq b$ if and only if $a_i \leq b_i$ for all $i = 1, 2, \dots, m$ with strict inequality for at least one i.
- $a < b$ if and only if $a_i < b_i$ for all $i = 1, 2, \dots, m$.

Given a nonempty subset $\mathcal{S}$ of $\mathbb{R}^n$, $co\mathcal{S}$ and $cl\mathcal{S}$ denote the convex hull, and closure of $\mathcal{S}$, respectively. Also, the polar cone, the strictly negative polar cone and the orthogonal complement of $\mathcal{S}$ are respectively defined by

$$\begin{array}{rcl} \mathcal{S}^\circ & = & \{x \in \mathbb{R}^n : \langle x, d\rangle \leq 0, \forall d \in \mathcal{S}\}, \\ \mathcal{S}^s & = & \{x \in \mathbb{R}^n : \langle x, d\rangle < 0, \forall d \in \mathcal{S} \setminus \{0\}\}, \\ \mathcal{S}^\perp & = & \{x \in \mathbb{R}^n : \langle x, d\rangle = 0, \forall d \in \mathcal{S}\}. \end{array}$$

It can easily be shown that $\mathcal{S}^\perp = \mathcal{S}^\circ \cap (-\mathcal{S})^\circ$. Moreover, at $\overline{x} \in cl\mathcal{S}$, the tangent cone, the convex cone generated by $\mathcal{S}$ and the linear hull of $\mathcal{S}$ are respectively given by

$$T(\mathcal{S}, \overline{x}) = \Big\{ v \in \mathbb{R}^n : \exists t_n \downarrow 0,\ \exists v_n \to v,\ \overline{x} + t_n v_n \in \mathcal{S} \Big\},$$

$$cone(\mathcal{S}) = \Big\{ y = \sum_{i=1}^{k} \lambda_i y_i : k \in \mathbb{N}, \lambda_i \geq 0,\ y_i \in \mathcal{S}, i = 1, 2, ..., k \Big\},$$

$$lin(\mathcal{S}) = \Big\{ y = \sum_{i=1}^{k} \lambda_i y_i : k \in \mathbb{N}, \lambda_i \in \mathbb{R},\ y_i \in \mathcal{S}, i = 1, 2, ..., k \Big\}.$$

Recall that for any two sets $\mathcal{S}_1$ and $\mathcal{S}_2$ in $\mathbb{R}^n$ one has $lin(\mathcal{S}_1 \cup \mathcal{S}_2) = lin(\mathcal{S}_1) + lin(\mathcal{S}_2)$.

A function $\varphi : \mathbb{R}^n \to \mathbb{R}$ will be called tangentially convex at $\overline{x} \in \mathbb{R}^n$ [14] if its directional derivative at $\overline{x}$,

$$\varphi'(\overline{x}, d) = \lim_{t \downarrow 0} \frac{\varphi(\overline{x} + td) - \varphi(\overline{x})}{t},$$

is finite for any direction $d \in \mathbb{R}^n$ and convex in this argument. Observe that the directional derivative of a tangentially convex function is sublinear as a function of the direction because it is positive homogeneous. Moreover, φ will be called Hadamard directionally differentiable at $\overline{x} \in \mathbb{R}^n$, if its Hadamard directional derivative

$$\varphi^H(\overline{x}, d) = \lim_{t \downarrow 0, d' \to d} \frac{\varphi(\overline{x} + td') - \varphi(\overline{x})}{t},$$

is defined for all directions d. In this case one has $\varphi^H(\overline{x}, d) = \varphi'(\overline{x}, d)$. For the converse, φ is Hadamard directionally differentiable at $\overline{x}$ in d if φ is locally Lipschitz at $\overline{x}$ and directionally differentiable. On the other hand, the tangential subdifferential of $\varphi : \mathbb{R}^n \to \mathbb{R}$ at $\overline{x} \in \mathbb{R}^n$ is given by $\partial_T \varphi(\overline{x}) = \Big\{ y^* \in \mathbb{R}^n : \langle y^*, d \rangle \leq \varphi'(\overline{x}, d)\ \forall d \in \mathbb{R}^n \Big\}$ [14, 15]. For a tangentially convex function, this subdifferential is nonempty, compact, convex and its support functional coincides with the directional derivative. Furthermore, tangentially convex functions constitute a large class that contains convex functions on open domains where the tangential subdifferential falls into the classical Fenchel subdifferential, Gâteaux differentiable functions on open domains with a tangential subdifferential reduced to the gradient. This class also includes locally Lipschitz functions that are either Clarke regular [16] or Michel-Penot regular [17], and their tangential subdifferential coincides with the Clarke subdifferential in the first case and the Michel-Penot subdifferential in the second.

Hereafter, we assume that $\overline{x} \in \Pi$, $f_j,\ j \in J$ is Hadamard directionally differentiable at $\overline{x}$, and $g_s,\ s \in S,\ h_k,\ k \in K,\ G_i$ and $H_i,\ i \in I$ are tangentially convex at $\overline{x}$. We say that $\overline{x}$ is a local (weak) efficient solution to (1) if there is a neighbourhood V of $\overline{x}$ such that for each $y \in V \cap \Pi$ the inequality $f(y) \leq (<) f(\overline{x})$ does not hold It is straightforward to check that every local efficient solution for (1) is local weak efficient. When $V = \mathbb{R}^n$, the word local will be omitted.

We denote by $\mathbb{R}_+^{|S|}$ the collection of all functions $\lambda : S \to \mathbb{R}$ taking positive values λ_s only at finitely many points of S, and zero otherwise. For $\overline{x} \in \Pi$, we let $S(\overline{x}) := \{s \in S \mid g_s(\overline{x}) = 0\}$ be the index set of all active constraints at $\overline{x}$ and $A(\overline{x}) := \{\lambda \in \mathbb{R}_+^{|S|} \mid \lambda_s g_s(\overline{x}) = 0, \forall s \in S\}$ be that of active constraint multipliers at $\overline{x}$. Notice that $\lambda \in A(\overline{x})$ if there exists a finite index set $R \subset S(\overline{x})$ such that $\lambda_s > 0$ for all $s \in R$ and $\lambda_s = 0$ for all $s \in S \setminus R$. Let us also define

$$I_G = I_G(\overline{x}) := \{i \in I \mid G_i(\overline{x}) = 0, H_i(\overline{x}) \neq 0\},$$

$$I_H = I_H(\overline{x}) := \{i \in I \mid G_i(\overline{x}) \neq 0, H_i(\overline{x}) = 0\}$$

and

$$I_{GH} = I_{GH}(\overline{x}) := \{i \in I \mid G_i(\overline{x}) = 0, H_i(\overline{x}) = 0\}.$$

We suppose that I_{GH} is a nonempty set and denote by $\mathcal{P}(I_{GH})$ the set of all disjoint bipartitions of I_{GH}; i.e., $\mathcal{P}(I_{GH}) = \{(B_1, B_2) :\ B_1 \cup B_2 = I_{GH},\ B_1 \cap B_2 = \emptyset\}$. The point $\overline{x}$ is called weakly stationary, W-stationary for short, if there exist multipliers solving the system

$$\begin{aligned} 0 \in \sum_{j \in J} \lambda_j \partial_T f_j(\overline{x}) + \sum_{s \in S(\overline{x})} \lambda_s^g \partial_T g_s(\overline{x}) + \sum_{k \in K} \lambda_k^h \partial_T h_k(\overline{x}) \\ + \sum_{i \in I} \lambda_i^G \partial_T G_i(\overline{x}) + \sum_{i \in I} \lambda_i^H \partial_T H_i(\overline{x}), \\ \forall s \in S(\overline{x}) : \lambda_s^g \geq 0, \quad \forall i \in I_H(\overline{x}) : \lambda_i^G = 0, \quad \forall i \in I_G(\overline{x}) : \lambda_i^H = 0. \end{aligned} \tag{2}$$

It is called Mordukhovich-stationary, M-stationary for short, if in addition to (2), $\lambda_i^G \lambda_i^H = 0$ for all $i \in I_{GH}(\overline{x})$. Finally, it is strongly stationary, S-stationary for short, if in addition to (2), $\lambda_i^G = 0$ *and* $\lambda_i^H = 0$ for all $i \in I_{GH}(\overline{x})$. Clearly, S-stationarity yields M-stationarity, which yields W-stationarity.

Now, we present two useful lemmas which we need to prove our main result.

Lemma 2.1 ([18]). *Let $\{\mathcal{S}_j \mid j \in J\}$ be a family of nonempty convex sets in $\mathbb{R}^n$. Then, every nonzero vector of $\mathcal{C} = cone(\bigcup_{j\in J} \mathcal{S}_j)$ can be written as a non-negative linear combination of at most n linear independent vectors, each belonging to a different $\mathcal{S}_j$.*

Lemma 2.2 ([19]). *Let S, T and P be three arbitrary nonempty index sets (possibly infinite). Consider the maps $\varphi : S \to \mathbb{R}^n$, $\phi : T \to \mathbb{R}^n$ and $\psi : P \to \mathbb{R}^n$. If the set $co\{\varphi(s), s \in S\} + cone\{\phi(t), t \in T\} + lin\{\psi(p), p \in P\}$ is closed, the following two assertions are equivalent:*

(i) $\begin{cases} \langle \varphi(s), d\rangle < 0,\ s \in S, \\ \langle \phi(t), d\rangle \leq 0,\ t \in T, \\ \langle \psi(p), d\rangle = 0,\ p \in P, \end{cases}$ *has no solution* $d \in \mathbb{R}^n$;

(ii) $0 \in co\{\varphi(s), s \in S\} + cone\{\phi(t), t \in T\} + lin\{\psi(p), p \in P\}$.

3. M-stationary conditions for local efficient solutions

In this section, we derive M-stationary conditions for local efficient solutions of (1). To proceed, we consider the following nonlinear programming problem with respect to a partition (B_1, B_2) of I_{GH}.

$$\begin{cases} \min & f(x) = (f_1(x), ..., f_m(x)), \\ s.t. & g_s(x) \leq 0,\ \forall s \in S, \\ & h_k(x) = 0,\ \forall k \in K, \\ & G_i(x) = 0,\ \forall i \in I_G \cup B_1, \\ & H_i(x) = 0,\ \forall i \in I_H \cup B_2. \end{cases} \tag{3}$$

The feasible set of (3) is given by

$$\Upsilon_{B_1,B_2} := \{x \in \mathbb{R}^n : g_s(x) \leq 0, s \in S, h_k(x) = 0, k \in K, G_i(x) = 0, i \in I_G \cup B_1, H_i(x) = 0, i \in I_H \cup B_2\}.$$

It is easy to show that $\Upsilon_{B_1,B_2} \subset \Pi$.

Let us define the following Abadie type constraint qualifications:

$$\partial_T\text{-ACQ}(B_1, B_2)\ :\ \Lambda_{(B_1,B_2)}(\overline{x}) \subseteq T(\Upsilon_{B_1,B_2}, \overline{x}),$$

where

$$\Lambda_{(B_1,B_2)}(\overline{x}) = (\bigcup_{s\in S} \partial_T g_s(\overline{x}))^- \cap (\bigcup_{k\in K} \partial_T h_k(\overline{x}))^\perp \cap (\bigcup_{i\in I_G\cup B_1} \partial_T G_i(\overline{x}))^\perp \cap (\bigcup_{i\in I_H\cup B_2} \partial_T H_i(\overline{x}))^\perp.$$

We are now in position to give necessary optimality conditions for local efficient solutions of (1).

Theorem 3.1. *Let $\overline{x}$ be a local efficient solution of (1). Assume that there exists a partition $(B_1, B_2) \in \mathcal{P}(I_{GH})$ such that ∂_T-ACQ(B_1, B_2) holds for $\overline{x}$ and*

$$D = cone\left(\bigcup_{s\in S} \partial_T g_s(\overline{x})\right) + lin\left(\bigcup_{k\in K} \partial_T h_k(\overline{x}) \cup \bigcup_{i\in I_G\cup I_{B_1}} \partial_T G_i(\overline{x}) \cup \bigcup_{i\in I_H\cup I_{B_2}} \partial_T H_i(\overline{x})\right) \tag{4}$$

is closed, then $\overline{x}$ is an M-stationary point of (1).

Proof
We claim that

$$\Big(\bigcup_{j\in J} \partial_T f_j(\overline{x})\Big)^s \cap T(\Pi,\overline{x}) = \emptyset. \tag{5}$$

Indeed, suppose that there exists $y^* \in \big(\bigcup_{j\in J} \partial_T f_j(\overline{x})\big)^s \cap T(\Pi,\overline{x})$. Then, from $y^* \in \big(\bigcup_{j\in J} \partial_T f_j(\overline{x})\big)^s$, it follows that

$$\langle x^*, y^*\rangle < 0, \quad \forall x^* \in \partial_T f_j(\overline{x}) \setminus \{0\},\ \forall j \in J. \tag{6}$$

For each $j \in J$, define $\varphi_j : \partial_T f_j(\overline{x}) \subset \mathbb{R}^n \to \mathbb{R}$ as $\varphi_j(x^*) = \langle x^*, y^*\rangle$ for all $x^* \in \partial_T f_j(\overline{x})$. The continuity of φ_j on $\partial_T f_j(\overline{x})$, which is compact, implies the existence of $\overline{x}^*_j \in \partial_T f_j(\overline{x})$ with $\varphi_j(\overline{x}^*_j) = \max\limits_{x^*\in\partial_T f_j(\overline{x})} \langle x^*, y^*\rangle$. Hence, according to (6), we obtain for every $j \in J$

$$f'_j(\overline{x}, y^*) = \max_{x^*\in\partial_T f_j(\overline{x})} \langle x^*, y^*\rangle = \langle \overline{x}^*_j, y^*\rangle < 0. \tag{7}$$

Now, since $y^* \in T(\Pi,\overline{x})$, there is $t_k \downarrow 0$ and $y^*_k \to y^*$ satisfying $\overline{x} + t_k y^*_k \in \Pi$ for all k. Because $\overline{x}$ is local efficient solution of (1), there is $\overline{x} + t_k y^*_k \in B(\overline{x}, r)$, for some $r > 0$ and for k high enough, such that there is $j_0 \in J$ verifying $f_{j_0}(\overline{x} + t_k y^*_k) \geq f_{j_0}(\overline{x})$. In combining this with the fact that

$$f'_{j_0}(\overline{x}, y^*) = f^H_{j_0}(\overline{x}, y^*) = \lim_{k\to\infty} \frac{f_{j_0}(\overline{x} + t_k y^*_k) - f_{j_0}(\overline{x})}{t_k},$$

we obtain $f'_{j_0}(\overline{x}, y^*) \geq 0$, which contradicts (7), and consequently, (5) is fulfilled.
On the basis of ∂_T-ACQ(B_1, B_2) and taking into account that $T(\Upsilon_{B_1,B_2}, \overline{x}) \subset T(\Pi, \overline{x})$, we have

$$\Big(\bigcup_{j\in J} (\partial_T f_j(\overline{x})\Big)^s \cap \Big(\bigcup_{s\in S(\overline{x})} \partial_T g_s(\overline{x})\Big)^\circ \cap \Big(\bigcup_{k\in K} \partial_T h_k(\overline{x})\Big)^\perp$$
$$\cap \Big(\bigcup_{i\in I_G\cup B_1} \partial_T G_i(\overline{x})\Big)^\perp \cap \Big(\bigcup_{i\in I_H\cup B_2} \partial_T H_i(\overline{x})\Big)^\perp = \emptyset.$$

Then, we see that the system

$$\begin{cases} \langle \zeta_j, y^*\rangle < 0, & \forall j \in J,\ \forall \zeta_j \in \partial_T f_j(\overline{x}),\\ \langle \vartheta_t, y^*\rangle \leq 0, & \forall s \in S,\ \forall \vartheta_s \in \partial_T g_s(\overline{x}),\\ \langle \eta_k, y^*\rangle = 0, & \forall k \in K,\ \forall \eta_k \in \partial_T h_k(\overline{x})\\ \langle \theta_i, y^*\rangle = 0, & \forall i \in I_G \cup B_1,\ \forall \theta_i \in \partial_T G_i(\overline{x}),\\ \langle \xi_i, y^*\rangle = 0, & \forall i \in I_H \cup B_2,\ \forall \xi_i \in \partial_T H_i(\overline{x}), \end{cases}$$

has no solution $y^* \in \mathbb{R}^n$. On the other hand, since $\partial_T f_j(\overline{x})$ is compact for all $j \in J$, the set $\bigcup\limits_{j=1}^{m} \partial_T f_j(\overline{x})$ is also compact, and hence $\bigcup\limits_{j=1}^{m} \partial_T f_j(\overline{x}) + D$ is closed because so is D. Thus, by virtue of Lemma 2.2, we are led to

$$0 \in co\Big(\bigcup_{j\in J} \partial_T f_j(\overline{x})\Big) + cone\Big(\bigcup_{s\in S(\overline{x})} \partial_T g_s(\overline{x})\Big) + lin\Big(\bigcup_{k\in K} \partial_T h_k(\overline{x})\Big)$$
$$+ lin\Big(\bigcup_{i\in I_G\cup B_1} \partial_T G_i(\overline{x})\Big) + lin\Big(\bigcup_{i\in I_H\cup B_2} \partial_T H_i(\overline{x})\Big).$$

On the basis of Lemma 2.1, we deduce that there exist $\lambda = (\lambda_1, ..., \lambda_m) \in \mathbb{R}^m_+$ with $\sum_{j=1}^{m} \lambda_j = 1$, $\lambda^g \in A(\overline{x})$, $\lambda^h = (\lambda^h_1, ..., \lambda^h_q) \in \mathbb{R}^q$, $\rho = (\rho_1, ..., \rho_l) \in \mathbb{R}^l$ and $\sigma = (\sigma_1, ..., \sigma_l) \in \mathbb{R}^l$ such that

$$0 \in \sum_{j\in J} \lambda_j \partial_T f_j(\overline{x}) + \sum_{s\in S(\overline{x})} \lambda^g_s \partial_T g_s(\overline{x}) + \sum_{k\in K} \lambda^h_k \partial_T h_k(\overline{x}) + \sum_{i\in I_G\cup B_1} \rho_i \partial_T G_i(\overline{x}) + \sum_{i\in I_H\cup B_2} \sigma_i \partial_T H_i(\overline{x}).$$

By taking

$$\lambda^G_i = \begin{cases} \rho_i, & i \in I_G(\overline{x}) \cup B_1, \\ 0, & i \in I_H(\overline{x}) \cup B_2, \end{cases} \qquad \lambda^H_i = \begin{cases} 0, & i \in I_G(\overline{x}) \cup B_1, \\ \sigma_i, & i \in I_H(\overline{x}) \cup B_2, \end{cases}$$

and using the fact that for all $s \in S(\overline{x}) : \lambda^g_s \geq 0$, we deduce that $\overline{x}$ is an M-stationary point of (1). □

To illustrate Theorem 3.1, we present the following example of (1).

Example 3.2. Consider the functions $f = (f_1, f_2) : \mathbb{R}^2 \to \mathbb{R}^2$, $g_s : \mathbb{R}^2 \to \mathbb{R}$, $\forall s \in S = [0, +\infty)$, $h : \mathbb{R}^2 \to \mathbb{R}$, $G = (G_1, G_2, G_3) : \mathbb{R}^2 \to \mathbb{R}^3$, $H = (H_1, H_2, H_3) : \mathbb{R}^2 \to \mathbb{R}^3$ defined by

$$f_1(x, y) = \begin{cases} \frac{x^3}{y} - x, & y \neq 0, \\ -x, & y = 0, \end{cases} \qquad f_2(x, y) = |x| + y,$$

$$g_s(x, y) = -x - s \ (s \in S), \quad h(x, y) = \begin{cases} 0, & y \geq 0, \\ -y, & y < 0, \end{cases}$$

$$G_1(x, y) = \begin{cases} y, & y \geq 0, \\ 0, & y < 0, \end{cases}, \quad H_1(x, y) = \begin{cases} y, & y \geq 0, \\ 0, & y < 0, \end{cases},$$
$$G_2(x, y) = \begin{cases} y, & y \geq 0, \\ 0, & y < 0, \end{cases}, \quad H_2(x, y) = \begin{cases} 1, & y \geq 0, \\ 1 - y, & y < 0, \end{cases},$$
$$G_3(x, y) = \begin{cases} 1, & y \geq 0, \\ 1 - y, & y < 0, \end{cases}, \quad H_3(x, y) = \begin{cases} y, & y \geq 0, \\ 0, & y < 0, \end{cases}.$$

We have $\Pi = \mathbb{R}_+ \times \{0\}$ and $\overline{x} = (0, 0) \in \Pi$ is a local efficient solution of (1). It is easily seen that

$$\partial_T f_1(\overline{x}) = \{(-1, 0)\},\ \partial_T f_2(\overline{x}) = [-1, 1] \times \{0\},\ \partial_T g_s(\overline{x}) = \{(-1, 0)\}\ \forall s \in S,$$
$$\partial_T h(\overline{x}) = \partial_T G_3(\overline{x}) = \partial_T H_2(\overline{x}) = \{0\} \times [-1, 0],$$
$$\partial_T G_1(\overline{x}) = \partial_T G_2(\overline{x}) = \partial_T H_1(\overline{x}) = \partial_T H_3(\overline{x}) = \{0\} \times [0, 1],$$
$$T(\Pi, \overline{x}) = \mathbb{R}_+ \times \{0\},\ I_{GH}(\overline{x}) = \{1\},\ I_G(\overline{x}) = \{2\},\ I_H(\overline{x}) = \{3\}.$$

In choosing $B_1 = \emptyset$ and $B_2 = I_{GH}(\overline{x})$, we can easily check that the constraint qualification ∂_T-ACQ(B_1, B_2) holds at $\overline{x}$ and that D, defined by (4), is closed. Consequently, $\overline{x}$ satisfies the assumptions of Theorem 3.1. In taking $\lambda_1 = \lambda^g = \frac{1}{2}$, $\lambda_2 = \lambda^h = 1, \lambda^G_1 = \lambda^G_2 = \lambda^H_3 = \frac{1}{3}$ and $\lambda^G_3 = \lambda^H_1 = \lambda^H_2 = 0$, the condition (2) is verified with $\lambda^G_1 \lambda^H_1 = 0$, which means that $\overline{x}$ is an M-stationary point of (1).

Remark 1. *The use of tangential subdifferentials instead of other subdifferentials such as Clarke subdifferentials presents some advantages. Indeed, for our problem, the functions are not necessarily locally Lipschitz at the local efficient solution of (1), as is the case with the function f_1 in the above example.*

4. Conclusion

In this work, we have established necessary M-stationary conditions for a nonsmooth multiobjective semi-infinite programming with switching constraints by using a surrogate problem and tangential subdifferentials. Moreover, we have employed Abadie-type constraint qualifications that are weaker than most of known nonsmooth constraint qualifications like those of Slater, Cottle, Zangwill, etc. To the best of our knowledge, this is the first work that treats the nonsmooth and semi-infinite case for multiobjective programming with switching constraints. For future research, we can derive optimality conditions for the same problem we studied using weaker subdifferentials such as convexificators.

Acknowledgement

We would like to express our sincere gratitude to the anonymous reviewer for his helpful comments and suggestions.

REFERENCES

1. A. Rund, and K. Kunisch,, *Nonconvex penalization of switching control of partial differential equations*, Systems & Control Letters, vol. 106, pp. 1–8, 2017.
2. M. Gugat, *Optimal switching boundary control of a string to rest in finite time*, ZAMM-Journal of Applied Mathematics and Mechanics, vol. 88, no. 4, pp. 283–305, 2010.
3. S. Sager, *Reformulations and algorithms for the optimization of switching decisions in nonlinear optimal control*, Journal of Process Control, vol. 19, no. 8, pp 1238–1247, 2009.
4. P. Mehlitz, *Stationarity conditions and constraint qualifications for mathematical programs with switching constraints*, Mathematical Programming, vol. 181, no. 1, pp 149–186, 2020.
5. S. Dempe, and H. Schreier, *Operations research: deterministische Modelle und Methoden*, Springer-Verlag, 2007.
6. R. A. Sarker, and C. S. Newton, *Optimization Modelling*, CRC Press, Boca Raton, 2008.
7. Z. Q. Luo, J. S. Pang, and D. Ralph, *Mathematical Programs with Equilibrium Constraints*, Cambridge University Press, Cambridge, 1996.
8. J. V. Outrata, M. Kocvara, and J. Zowe, *Nonsmooth approach to optimization problems with equilibrium constraints*, Kluwer Academic, Dordrecht, 1998.
9. G. H. Lin, and M. Fukushima, *A modified relaxation scheme for mathematical programs with complementarity constraints*, Annals of Operations Research, vol. 133, no. 1, pp. 63–84, 2005.
10. C. Kanzow, P. Mehlitz, and D. Steck, *Relaxation schemes for mathematical programmes with switching constraints*, Optimization Methods and Software, vol. 36, no. 6, pp. 1223–1258, 2021.
11. G. Li, and L. Guo, *Mordukhovich stationarity for mathematical programs with switching constraints under weak constraint qualifications*, Optimization, pp. 1–22, 2022.
12. P. Mehlitz, *On the linear independence constraint qualification in disjunctive programming*, Optimization, pp. 1–37, 2019.
13. M. Jennane, E. M. Kalmoun, and L. El Fadil, *Optimality conditions for nonsmooth multiobjective bilevel optimization using tangential subdifferentials*, RAIRO–Operations Research, vol. 55, no. 5, pp. 3041–3048, 2021.
14. B. N. Pshenichnyi, *Necessary Conditions for an Extremum*, New York, NY: Marcel Dekker Inc, 1971.
15. C. Lemaréchal, *An introduction to the theory of nonsmooth optimization*, Optimization. vol. 17, no. 6, pp. 827–858, 1986.
16. F. H. Clarke, *Optimization and Nonsmooth Analysis*, Wiley, New York, 1983.
17. P. Michel and J. P. Penot, *A generalized derivative for calm and stable functions*, Differential and Integral Equations, vol. 5, no. 2, pp. 433–454, 1992.
18. R. T. Rockafellar, *Convex Analysis*, in: Princeton Mathematical Series, vol. 28. Princeton, NJ: Princeton University Press, 1970.
19. M. A. Goberna, and M. A López, *Linear Semi-Infinite Optimization*, Chichester: Wiley, 1998.