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Sine-Cosine Weighted Circular Distributions

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Abstract This paper introduces a new family of multimodal and skew-symmetric circular distributions, namely, the sine-cosine weighted circular distribution. The fundamental properties of this family are examined in the context of a general case and three specific examples. Additionally, general solutions for estimating the parameters of any sine-cosine weighted circular distribution using maximum likelihood are provided. A likelihood-ratio test is performed to check the symmetry of the data. Some simulations are run to assess the performance of the maximum-likelihood estimators. Lastly, two examples are presented that illustrate how the proposed model may be utilized to analyze two real-world case studies with asymmetric datasets.

Keywords Circular statistics, Sine-cosine weighted circular distribution, Maximum likelihood estimate, Trigonometric moment

AMS 2010 subject classifications 60E05 , 62H11

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1. Introduction

Classical models for circular data are symmetric, including the von Mises, wrapped Cauchy, and cardioid distributions [21, 14].

Wrapping is a technique that is frequently used to create skew-symmetric circular models. It involves wrapping a skew-symmetric family, defined on the natural line, around the unit circle. Wrapped stable models [21, 26], wrapped skew-normal models [25], and wrapped Laplace models [13] are all instances of skew-symmetric models that may be constructed by this method.

There are two other approaches that result in distributions capable of simulating asymmetry and, presumably, multimodality. They are both extensions to the von Mises distribution. In the first, additional terms are added to the von Mises density exponent. [8] and [20] present two examples of this approach in practice (see also, [11]). The second approach is to employ finite mixtures of von Mises distributions (see,[27]).
Alongside this, perturbation is a frequently used technique for generating skew-symmetric models. Perhaps the most well-known use of this approach is Azzalini's ([6],[7]) skew-normal distribution. [28], as well as [1], followed Azzalini's lead in establishing a general approach for skewing symmetric circular models. Ameijeiras-Alonso and [**?**] introduced the sine-skewed toroidal distributions.

The literature has devoted great attention to the creation of flexible models for circular data. The general constructions proposed include conditional derivation [16]; Moebius transformation [17]; argument transformation [16, 2]; Brownian motion [18]; and parametric extension of the characteristic function of an existing circular model [19].

The von Mises distribution is probably the most well-known unimodal model in circular statistics, owing to its desirable features and the fact that it may be obtained by a variety of constructions; [22] discuss its genesis and

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identify five such constructions. The von Mises distribution has a symmetric and unimodel density as

$$f_{vM}(\theta;\eta,\kappa)=\frac{e^{\kappa\cos(\theta-\eta)}}{2\pi I_0(\kappa)}, \tag{1}$$

for $\theta\in[-\pi,\pi)$, $\eta\in[-\pi,\pi)$, $\kappa\geq 0$ and where $I_r(\kappa)$ is the modified Bessel function of the first kind of order r, defined as

$$I_r(\kappa)=\frac{1}{2\pi}\int_{-\pi}^{\pi}\cos r\theta e^{\kappa\cos\theta}\mathrm{d}\theta,\quad r=0,\pm1,\pm2,....$$

In order to create multimodal models, two well-known approaches have been used to expand the von Mises distribution, which are based on the expansion of the exponential function proposed in Equation (1) and on mixing von Mises distributions. The generalized von Mises distribution can be also constructed using these approaches [20, 11]. To model bimodal data, the exponent term in the generalized von Mises (GvM_2) density can be expanded as follows:

$$f_{GvM_2}(\theta;\eta_1,\eta_2,\kappa_1,\kappa_2)=\frac{\exp\{\kappa_1\cos(\theta-\eta_1)+\kappa_2\cos 2(\theta-\eta_2)\}}{2\pi G_0(\delta,\kappa_1,\kappa_2)},$$

where $\eta_1,\eta_2\in[-\pi,\pi)$, $\delta=(\eta_1-\eta_2)\ (\mathrm{mod}\ 2\pi)$, $\kappa_1,\kappa_2>0$ and the normalizing constant is given by

$$G_0(\delta,\kappa_1,\kappa_2)=\frac{1}{2\pi}\int_{-\pi}^{\pi}\exp\{\kappa_1\cos\theta+\kappa_2\cos 2(\theta+\delta)\}\mathrm{d}\theta.$$

Furthermore, the two-component von Mises mixture distribution (vMM) which can also be used to model bimodal data, has the following density

$$f_{vMM}(\theta;p,\eta_1,\eta_2,\kappa_1,\kappa_2)=pf_{vM}(\theta;\eta_1,\kappa_1)+(1-p)f_{vM}(\theta;\eta_2,\kappa_2),$$

where $(0\leq p\leq 1)$ is the mixing probability.

Weighted sampling occurs when the sampling process converts the proportion of unit samples to a non-negative function termed the weight function. Weighted sampling is an extension of random sampling in which the recorded data is a weighted sample rather than the original sample. As a result, conventional statistical approaches provide invalid results that must be modified. [9] conducted a pioneering study on this biased sampling.

We adapted [24] weighting approach to creating a new flexible family of circular distributions for the symmetric, asymmetric, unimodal, and multimodal cases. The study is divided into six sections. Section 2 demonstrates how to develop a generic structure for constructing the weighted circular density. Section 3 discusses the general features of the sine-cosine weighted circular distribution. In Section 4, we offer results for three specific cases: the sine-cosine weighted von Mises, cardioid, and wrapped Cauchy distributions, all of which are members of the sine-cosine weighted circular family. In Section 5, we consider the maximum likelihood and moments estimation methods for the sine-cosine weighted circular distribution parameters and use a likelihood-ratio test to check the symmetry. A simulation study is presented in Section 6 to assess performance of the maximum-likelihood estimators. Finally, Section 7 employs inferential methods to analyze two data sets.

2. General method

Suppose X is a continuous random variable with the probability density function, pdf, $f(x)$. The pdf of the weighted random variable X^w is given by

$$f^w(x)=\frac{w(x)f(x)}{E[w(X)]},$$

where $w(x)$ is a non-negative weight function and

$$E[w(X)] = \int w(x)f(x)dx < \infty.$$

Theorem 1 adapts this idea to the circular case, resulting in a weighted circular distribution. Remember that a circular pdf is a non-negative periodic function with the period 2π, which integrates to one over intervals of length 2π. For specificity, we will consider the distribution in this discussion as being specified over the interval $[-\pi, \pi)$.

Theorem 1. Suppose that θ is a circular random variable with pdf $f(\theta)$. The pdf of the weighted circular random variable Θ^w is given by

$$f^w(\theta) = \frac{w(\theta)f(\theta)}{E[w(\Theta)]},$$

where $w(\theta)$ is non-negative and periodic, i.e. $w(\theta) = w(\theta + 2k\pi)$ for all integers k.

Proof. $f^w(\theta)$ is a density on $[-\pi, \pi)$ because $f(\theta)$ is a density on $[-\pi, \pi)$ and $w(\theta) > 0$ for $\theta \in [-\pi, \pi)$. However $f^w(\theta)$ is also a circular density because

$$f^w(\theta + 2k\pi) = \frac{w(\theta + 2k\pi)f(\theta + 2k\pi)}{E[w(\Theta + 2k\pi)]} = \frac{w(\theta)f(\theta)}{E[w(\Theta)]} = f^w(\theta)$$

for all integers k.

Let sine-cosine weight function, $w(\theta) = 1 + \lambda_1 \sin j\theta + \lambda_2 \cos j\theta$ for $\lambda_1, \lambda_2 \in [-1, 1]$, with $|\lambda_1| + |\lambda_2| \leq 1$, $\theta \in [-\pi, \pi)$ and an integer j. Accordingly, we define the density of a weighted family of symmetric circular distributions by

$$f^w(\theta) = \frac{(1 + \lambda_1 \sin j\theta + \lambda_2 \cos j\theta)f(\theta)}{1 + \lambda_1\beta_j + \lambda_2\alpha_j}, \tag{2}$$

where $f(\theta)$ is the density function of the base distribution and $\alpha_j = E(\cos j\Theta)$, $\beta_j = E(\sin j\Theta)$ are the cosine and sine moments of the base distribution.

$f(\theta)^w$ belongs to the new flexible family of circular distributions for the symmetric, asymmetric, unimodal and multimodal cases.

This family includes the original distribution ($\lambda_1 = \lambda_2 = 0$), Sine-skewed circular distribution ($\lambda_2 = 0$) and cosine weighted circular distribution ($\lambda_1 = 0$), otherwise it is skewed to the left ($\lambda_1 > 0$) or the right ($\lambda_1 < 0$) and symmetric ($\lambda_1 = 0$). If $j \geq 2$, f^w will be multimodal.

3. General properties of sine-cosine weighted circular distribution

This section considers the properties of any given sine-cosine weighted circular distribution functions with density, trigonometric moment, and other circular measures.

The distribution function $F^w(\theta)$ of any distribution with the density (2) is given by

$$\begin{aligned} F^w(\theta) &= \int_{-\pi}^{\theta} \frac{(1 + \lambda_1 \sin j\phi + \lambda_2 \cos j\phi)f(\phi)}{E[1 + \lambda_1 \sin j\Phi + \lambda_2 \cos j\Phi]} d\phi \\ &= \frac{F(\theta) + \lambda_1 \int_{-\pi}^{\theta} \sin j\phi f(\phi)d\phi + \lambda_2 \int_{-\pi}^{\theta} \cos j\phi f(\phi)d\phi}{1 + \lambda_1\beta_j + \lambda_2\alpha_j}. \end{aligned}$$

Assume $f(\theta)$ $(-\pi \leq \theta < \pi)$ is a unimodal circular density that is symmetric about zero distribution. Then, for $p = 0, \pm 1, \pm 2, ...,$ the trigonometric moments of $f^w(\theta)$ are given by

$$\alpha_p^w = \frac{\frac{\lambda_2}{2}(\alpha_{p-j} + \alpha_{p+j}) + \alpha_p}{1 + \lambda_2\alpha_j}$$

and

$$\beta_p^w = \frac{\frac{\lambda_1}{2}(\alpha_{p-j} - \alpha_{p+j})}{1 + \lambda_2 \alpha_j}.$$

Therefore, the pth mean resultant length and the pth mean direction can be obtained, respectively, by

$$\rho_p^w = \frac{1}{1 + \lambda_2 \alpha_j} \sqrt{(\frac{\lambda_2}{2}(\alpha_{p-j} + \alpha_{p+j}) + \alpha_p)^2 + \frac{\lambda_1^2}{4}(\alpha_{p-j} - \alpha_{p+j})^2}$$

and

$$\mu_p^w = \arg\{\lambda_2(\alpha_{p-j} + \alpha_{p+j}) + 2\alpha_p + i\lambda_1(\alpha_{p-j} - \alpha_{p+j})\}.$$

Thus,

$$\rho^w \equiv \rho_1^w = \frac{1}{1 + \lambda_2 \alpha_j} \sqrt{(\frac{\lambda_2}{2}(\alpha_{1-j} + \alpha_{1+j}) + \alpha_1)^2 + \frac{\lambda_1^2}{4}(\alpha_{1-j} - \alpha_{1+j})^2}$$

and

$$\mu^w \equiv \mu_1^w = \arg\{\lambda_2(\alpha_{1-j} + \alpha_{1+j}) + 2\alpha_1 + i\lambda_1(\alpha_{1-j} - \alpha_{1+j})\}.$$

Hence, the circular variance and circular standard deviation are given by

$$V^w = 1 - \rho^w = 1 - \frac{1}{1 + \lambda_2 \alpha_j} \sqrt{(\frac{\lambda_2}{2}(\alpha_{1-j} + \alpha_{1+j}) + \alpha_1)^2 + \frac{\lambda_1^2}{4}(\alpha_{1-j} - \alpha_{1+j})^2},$$

$$\begin{aligned} \sigma^w &= \{-2\log(1 - V^w)\}^{\frac{1}{2}} \\ &= \big\{ -\log\{\frac{1}{(1 + \lambda_2 \alpha_j)^2}[(\frac{\lambda_2}{2}(\alpha_{1-j} + \alpha_{1+j}) + \alpha_1)^2 + \frac{\lambda_1^2}{4}(\alpha_{1-j} - \alpha_{1+j})^2]\big\}^{\frac{1}{2}}. \end{aligned}$$

The second cosine and sine moments about the mean direction μ, $\bar{\alpha}_2 = E[\cos 2(\Theta - \mu)]$ and $\bar{\beta}_2 = E[\sin 2(\Theta - \mu)]$, can be represented as

$$\bar{\alpha}_2 = \frac{1}{1 + \lambda_2 \alpha_j}\big\{ \cos 2\mu[\alpha_2 + \frac{\lambda_2}{2}(\alpha_{2+j} + \alpha_{2-j})] + \sin 2\mu \frac{\lambda_1}{2}(\alpha_{2+j} - \alpha_{2-j})\big\}$$

and

$$\bar{\beta}_2 = \frac{1}{1 + \lambda_2 \alpha_j}\big\{ \frac{\lambda_1}{2} \cos 2\mu(\alpha_{2-j} - \alpha_{2+j}) + \sin 2\mu[\alpha_2 + \frac{\lambda_2}{2}(\alpha_{2+j} + \alpha_{2-j})]\big\}$$

where,

$$\cos 2\mu = \frac{1}{\rho^{w2}(1 + \lambda_2 \alpha_j)^2}\{[\alpha_1 + \frac{\lambda_2}{2}(\alpha_{1-j} + \alpha_{1+j})]^2 - \frac{\lambda_1^2}{4}(\alpha_{1-j} - \alpha_{1+j})^2\}$$

and

$$\sin 2\mu = \frac{1}{\rho^{w2}(1 + \lambda_2 \alpha_j)^2}\lambda_1(\alpha_{1-j} + \alpha_{1+j})(\alpha_1 + \frac{\lambda_2}{2}(\alpha_{1-j} + \alpha_{1+j})).$$

Therefore, the measure of circular skewness and kurtosis, $\gamma_1^w = \frac{\bar{\beta}_2^w}{V^{w\frac{3}{2}}}$ and $\gamma_2^w = \frac{(\bar{\alpha}_2^w - \rho^{w4})}{V^{w2}}$, are given by

$$\begin{aligned}\gamma_1^w &= \frac{1}{\rho^{w2}(1-\rho^w)^{\frac{3}{2}}}[\frac{\lambda_1}{2}(\alpha_{2-j}-\alpha_{j+2})\{(\alpha_1+\frac{\lambda_2}{2}(\alpha_{1-j}+\alpha_{1+j}))^2 \\ &- \frac{\lambda_1^2}{4}(\alpha_{1-j}-\alpha_{1+j})^2)\} \\ &+ \lambda_1(\alpha_{1-j}-\alpha_{1+j})(\alpha_1+\frac{\lambda_2}{2}(\alpha_{1-j}+\alpha_{1+j}))(\alpha_2+\frac{\lambda_2}{2}(\alpha_{2+j}-\alpha_{2-j}))]\end{aligned}$$

and

$$\begin{aligned}\gamma_2^w &= \frac{1}{\rho^{w2}(1-\rho^w)^2}[(\alpha_2+\frac{\lambda_2}{2}(\alpha_{2-j}+\alpha_{2+j}))\{(\alpha_1+\frac{\lambda_2}{2}(\alpha_{1-j}+\alpha_{1+j}))^2 \\ &- \frac{\lambda_1^2}{4}(\alpha_{1-j}-\alpha_{1+j})^2)\} \\ &+ \frac{\lambda_1^2}{2}(\alpha_{1-j}-\alpha_{1+j})(\alpha_1+\frac{\lambda_2}{2}(\alpha_{1-j}+\alpha_{1+j}))(\alpha_{2-j}-\alpha_{2+j})-\rho^{w6}],\end{aligned}$$

respectively.

4. Special cases of sine-cosine weighted circular distributions

This section discusses the weighted equivalents of some of these well-known distributions.

4.1. Sine-cosine weighted von Mises distribution

Substituting (1), with $\eta = 0$, for f in (2), the density of the sine-cosine weighted von Mises, $SCWvM$, family of distributions is given by

$$f^w(\theta) = \frac{(1+\lambda_1 \sin j\theta + \lambda_2 \cos j\theta)e^{\kappa\cos\theta}}{2\pi(I_0(\kappa)+\lambda_2 I_j(\kappa))}, \qquad -\pi \le \theta < \pi. \tag{3}$$

Figures 1 and 2 depict the family's flexibility.

For $\theta \in [-\pi, \pi)$, the distribution function is determined by

$$F^w(\theta) = \frac{1}{1+\lambda_2 A_j(\kappa)}(F(\theta)+\lambda_2 A_j(\kappa)),$$

where $F(\theta)$ is the distribution function of the base von Mises distribution.

The pth moment of the von Mises distribution is $\alpha_p = A_p(\kappa) = \frac{I_p(\kappa)}{I_0(\kappa)}$ for $p = 0, \pm1, \pm2, ...$ Using the results obtained in Section 2 and the following relationship [3]

$$I_{\nu-1}(z) - I_{\nu+1}(z) = \frac{2\nu}{z}I_\nu(z),$$

the pth cosine and sine moments belonging to sine-cosine weighted von Mises distribution are, respectively, obtained as

$${\alpha_p}^w = \frac{I_p(\kappa)+\frac{\lambda_2}{2}(I_{p+j}(\kappa)+I_{p-j}(\kappa))}{I_0(\kappa)+\lambda_2 I_j(\kappa)}$$

and

$${\beta_p}^w = \frac{\lambda_1(I_{p+j}(\kappa)-I_{p-j}(\kappa))}{2(I_0(\kappa)+\lambda_2 I_j(\kappa))}.$$

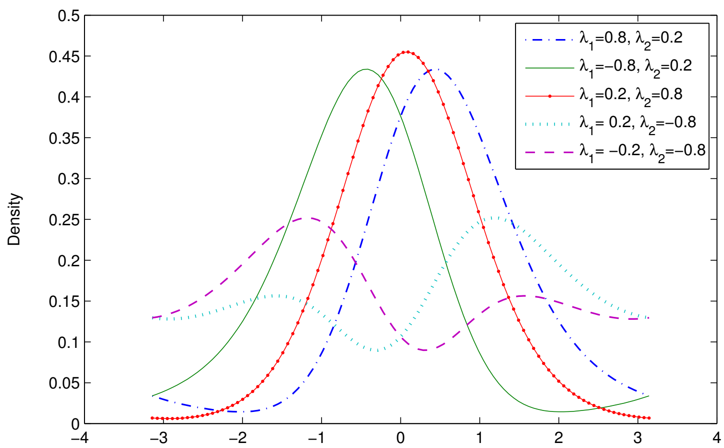

Figure 1. Sine-cosine weighted von Mises densities with $j = 1$ and $\kappa = 1$.

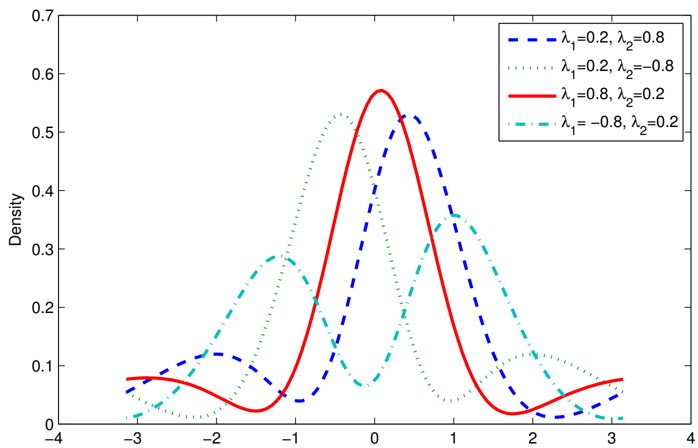

Figure 2. Sine-cosine weighted von Mises densities with $j = 2$ and $\kappa = 1$.

The mean resultant length and mean direction are given by

$$\rho_p{}^w = \frac{1}{I_0(\kappa) + \lambda_2 I_j(\kappa)} \sqrt{\left[I_P(\kappa) + \frac{\lambda_2}{2}(I_{p+j}(\kappa) + I_{p-j}(\kappa))\right]^2 + \frac{{\lambda_1}^2}{4}(I_{p+j}(\kappa) - I_{p-j}(\kappa))^2}$$

and

$$\mu_p{}^w = \arg\{I_p(\kappa) + \frac{\lambda_2}{2}(I_{p+j}(\kappa) + I_{p-j}(\kappa)) + i\frac{\lambda_1}{2}(I_{p-j}(\kappa) - I_{p+j}(\kappa)),$$

respectively. In particular,

$$\rho^w \equiv \rho_1^w = \frac{1}{I_0(\kappa) + \lambda_2 I_j(\kappa)} \sqrt{[I_1(\kappa) + \frac{\lambda_2}{2}(I_{j+1}(\kappa) + I_{j-1}(\kappa))]^2 + \frac{{\lambda_1}^2 j^2}{\kappa} I_j(\kappa)^2},$$

$$\mu^w \equiv \mu_1^w = \arg\{I_1(\kappa) + \frac{\lambda_2}{2}(I_{j+1}(\kappa) + I_{j-1}(\kappa)) + i\frac{\lambda_1 j}{\kappa} I_j(\kappa)\}.$$

Using the general results from Section 3, the second-order central moments, $\bar{\alpha}_2$ and $\bar{\beta}_2$, are obtained as

$$\bar{\alpha}_2 = \frac{1}{1+\lambda_2\alpha_j}\Big\{\cos 2\mu(A_2(\kappa) + \frac{\lambda_2}{2}(A_{2+j}(\kappa) + A_{2-j}(\kappa))) + \sin 2\mu(A_2(\kappa) + \frac{\lambda_1}{2}(A_{2+j}(\kappa) - A_{2-j}(\kappa)))\Big\}$$

and

$$\bar{\beta}_2 = \frac{1}{1+\lambda_2\alpha_j}\Big\{\frac{\lambda_1}{2}\cos 2\mu(A_{2-j}(\kappa) - A_{2+j}(\kappa)) + \sin 2\mu(A_2(\kappa) + \frac{\lambda_2}{2}(A_{2+j}(\kappa) + A_{2-j}(\kappa)))\Big\}$$

where,

$$\cos 2\mu = \frac{1}{\rho^{w2}(1+\lambda_2 A_j(\kappa))^2}\Big\{[A_1(\kappa) + \frac{\lambda_2}{2}(A_{1-j}(\kappa) + A_{1+j}(\kappa))]^2 - \frac{\lambda_1^2}{4}(A_{1-j}(\kappa) - A_{1+j}(\kappa))^2\Big\},$$

and

$$\sin 2\mu = \frac{1}{\rho^{w2}(1+\lambda_2 A_j(\kappa))^2}\lambda_1\big[A_{1-j}(\kappa) + A_{1+j}(\kappa)\big]\big[A_1(\kappa) + \frac{\lambda_2}{2}\big(A_{1-j}(\kappa) + A_{1+j}(\kappa)\big)\big].$$

It follows that

$$\gamma_1^w = \frac{1}{\rho^{w2}(1-\rho^w)^{\frac{3}{2}}}[\frac{\lambda_1}{2}(A_{2-j}(\kappa) - A_{2+j}(\kappa))\{(A_1(\kappa) + \frac{\lambda_2}{2}(A_{1-j}(\kappa) + A_{1+j}(\kappa)))^2 - \frac{\lambda_1^2}{4}(A_{1-j}(\kappa) - A_{1+j}(\kappa))^2)\} + \lambda_1(A_{1-j}(\kappa) - A_{1+j}(\kappa))(A_1(\kappa) + \frac{\lambda_2}{2}(A_{1-j}(\kappa) + A_{1+j}(\kappa))(A_2(\kappa) + \frac{\lambda_2}{2}(A_{2+j}(\kappa) - A_{2-j}(\kappa)))]$$

and

$$\gamma_2^w = \frac{1}{\rho^{w2}(1-\rho^w)^2}\big[(A_2(\kappa) + \frac{\lambda_2}{2}\big(A_{2-j}(\kappa) + A_{2+j}(\kappa)\big))\big\{(A_1(\kappa) + \frac{\lambda_2}{2}(A_{1-j}(\kappa) + A_{1+j}(\kappa)))^2 - \frac{\lambda_1^2}{4}(A_{1-j}(\kappa) - A_{1+j}(\kappa))^2\big\} + \frac{\lambda_1^2}{2}(A_{1-j}(\kappa) - A_{1+j}(\kappa))(A_1(\kappa) + \frac{\lambda_2}{2}(A_{1-j}(\kappa) + A_{1+j}(\kappa)))(A_{2-j}(\kappa) - A_{2+j}(\kappa)) - \rho^{w6}\big].$$

4.2. Sine-cosine weighted wrapped Cauchy distribution

By replacing f in (2) with the density of a wrapped Cauchy distribution, one can obtain the density of a sine cosine weighted wrapped Cauchy, $SCWWC$, distribution as

$$f^w(\theta) = \frac{1 + \lambda_1 \sin j\theta + \lambda_2 \cos j\theta}{2\pi(1+\lambda_2\rho^{|j|})}\frac{1-\rho^2}{1+\rho^2-2\rho\cos\theta}, \tag{4}$$

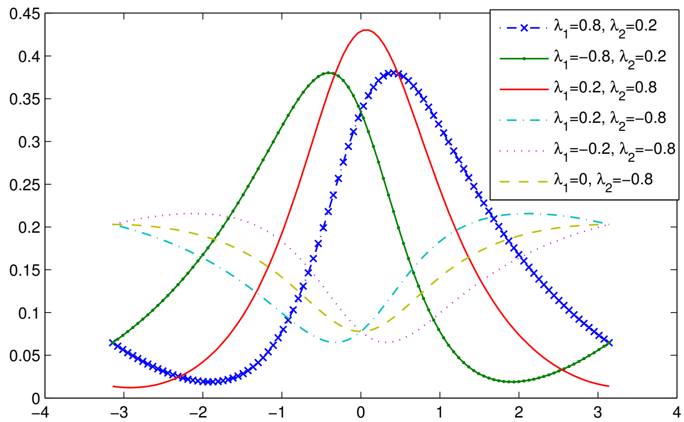

Figure 3. Sine-cosine weighted wrapped Cauchy densities with $j = 1$ and $\rho = 0.5$.

where $\theta \in [-\pi, \pi)$, $\rho \in [0, 1]$.
The flexibility of this family is clarified in Figure 3. For $\theta \in [-\pi, \pi)$ and $j = 1$, the distribution function is derived

$$F^w(\theta) = \frac{1}{1+\lambda_2\rho}\Big\{(1+\frac{\lambda_2(1+\rho^2)}{2\rho})F(\theta) - \frac{1-\rho^2}{4\pi\rho}\big[\lambda_2(\theta+\pi) - \lambda_1\log\{\frac{1+\rho^2-2\rho\cos\theta}{(1+\rho^2)}\}\big]\Big\},$$

where $F(\theta)$ denotes the distribution function of the wrapped Cauchy distribution, as obtained by[21] and
Thus,

$$\rho^w = \frac{1}{1+\lambda_2\rho}\sqrt{(\frac{\lambda_2}{2}+\rho)^2 + \frac{\lambda_1^2}{4}}, \qquad \rho_2^w = \frac{1}{1+\lambda_2\rho}\sqrt{\frac{\rho^2}{4}(\lambda_1^2+\lambda_2^2)}, \qquad \rho_p^w = 0 \ (p \geq 3)$$

and

$$\mu^w = \arg\{\lambda_2 + 2\rho + i\lambda_1\}, \qquad \mu_2^w = \arg\{\lambda_2\rho + i\lambda_1\rho\}, \qquad \mu_p^w = 0 \ (p \geq 3).$$

The circular variance and circular standard deviation are expressed by

$$V^w = 1 - \rho^w = 1 - \frac{1}{1+\lambda_2\rho}\sqrt{(\frac{\lambda_2}{2}+\rho)^2 + \frac{\lambda_1^2}{4}},$$

$$\begin{aligned}\sigma^w &= \{-2\log(1-V^w)\}^{\frac{1}{2}} \\ &= \big\{-2\log\{\frac{1}{1+\lambda_2\rho}\sqrt{(\frac{\lambda_2}{2}+\rho)^2+\frac{\lambda_1^2}{4}}\}\big\}^{\frac{1}{2}}.\end{aligned}$$

The second cosine and sine moments about the mean direction μ, $\bar{\alpha}_2 = E[\cos 2(\Theta - \mu)]$ and $\bar{\beta}_2 = E[\sin 2(\Theta - \mu)]$, can be expressed as

$$\bar{\alpha}_2 = \frac{\rho}{2(1+\lambda_2\rho)}(\lambda_2\cos 2\mu + \lambda_1\sin 2\mu), \qquad \bar{\beta}_2 = \frac{\rho}{2(1+\lambda_2\rho)}(\lambda_1\cos 2\mu + \lambda_2\sin 2\mu)$$

where,

$$\cos 2\mu = \frac{1}{\rho^{w2}(1+\lambda_2\rho)^2}\{(\rho+\frac{\lambda_2}{2})^2 - \frac{\lambda_1^2}{4}\}, \qquad \sin 2\mu = \frac{1}{\rho^{w2}(1+\lambda_2\rho)^2}\lambda_1(\rho+\frac{\lambda_2}{2}).$$

Hence the measure of circular skewness and kurtosis are given by

$$\gamma_1^w = \frac{1}{\rho^{w2}(1-\rho^w)^{\frac{3}{2}}}[\frac{\lambda_1}{2}\rho\{(\rho+\frac{\lambda_2}{2})^2 - \frac{\lambda_1^2}{4} - \lambda_2(\rho+\frac{\lambda_2}{2}))\}]$$

and

$$\gamma_2^w = \frac{1}{\rho^{w2}(1-\rho^w)^2}[\frac{\lambda_2}{2}\rho\{(\rho+\frac{\lambda_2}{2})^2 - \frac{\lambda_1^2}{4})\} + \frac{\lambda_1^2}{2}\rho(\rho+\frac{\lambda_2}{2}) - \rho^{w6}].$$

5. Parameter estimation

This section discusses parameter estimation for a random sample of size n, $\theta_1, \theta_2, ..., \theta_n$, draown from a sine-cosine weighted circular distribution with density (2). General results are presented for the maximum likelihood point estimates. Finally, we use a likelihood-ratio test to check the symmetry.

5.1. Maximum likelihood estimation

Suppose that the density $f(\theta-\eta;\boldsymbol{\beta})$ depends on the vector-valued parameter $\boldsymbol{\beta} = (\beta_1, \beta_2, ..., \beta_l)$ as well as the location parameter. Accordingly, the log-likelihood function can be expressed as

$$\begin{aligned}
\ell(\lambda_1, \lambda_2, \eta, \boldsymbol{\beta}) &= \sum_{i=1}^{n} \log\{1 + \lambda_1 \sin j(\theta_i - \eta) + \lambda_2 \cos j(\theta_i - \eta)\} + \ell(0, 0, \eta, \boldsymbol{\beta}) \\
-n &\quad \log(1 + \lambda_1 \bar{\alpha}_j + \lambda_2 \bar{\beta}_j) \\
&= \sum_{i=1}^{n} \log\{1 + \lambda_1 \sin j(\theta_i - \eta) + \lambda_2 \cos j(\theta_i - \eta)\} \\
&+ \sum_{i=1}^{n} \log f(\theta_i - \eta, \boldsymbol{\beta}) - n \log(1 + \lambda_1 \bar{\alpha}_j + \lambda_2 \bar{\beta}_j).
\end{aligned}$$

Maximizing $\ell(\lambda_1, \lambda_2, \eta, \boldsymbol{\beta})$ with respect to the parameters yields the maximum likelihood estimates (MLEs). MLEs lack closed-form expressions, and one must thus employ numerical methods capable of solving nonlinear optimization problems in order to meet the constraint, $|\lambda_1| + |\lambda_2| \leq 1$. Our result is based on the solver proposed by Ye (1987), implemented in the $Rsolnp$ package [12] of the statistical software R.

5.2. Testing for symmetry

One of the questions that naturally arise in our study is whether or not the sine-cosine weighted models improve significantly on their symmetric antecedents for a given sample $\boldsymbol{\theta}$. The answer to this question involves testing whether a sine-cosine weighted circular distribution is symmetric or not, which can be formulated as the hypothesis testing problem $H_0 : \lambda_1 = 0$ versus $H_1 : \lambda_1 \neq 0$. Here, we consider a likelihood-ratio test based on the test statistics

$$D = -2\big\{\max \ell(0, \lambda_2, \eta, \boldsymbol{\beta}) - \max \ell(\lambda_1, \lambda_2, \eta, \boldsymbol{\beta})\big\}.$$

Hence, the likelihood ratio test rejects the null hypothesis at asymptotic level α whenever the test statistic D exceeds $\chi_1^2(\alpha)$ where $\chi_m^2(\alpha)$ denotes the α-upper quantile of a chi-square distribution with m degrees of freedom.

6. Simulation study

Here, we assess the performance of the maximum-likelihood estimates for the sine-cosine weighted von Mises distribution with respect to sample size n. The assessment is based on a simulation study:
(1) Generate 10,000 samples of size n from Equation (3). the acceptance-rejection method of simulation is used to generate samples.
(2) Compute the maximum-likelihood estimates for the 10,000 samples, say $(\hat{\mu}_i, \hat{\kappa}_i, \hat{\lambda_1}_i, \hat{\lambda_2}_i)$ for $i = 1, 2, ..., 10000$.
(3) Compute the biases and mean squared errors given by

$$bias_\beta(n) = \frac{1}{10000} \sum_{i=1}^{10000} (\hat{\beta}_i - \beta), \quad MSE_\beta(n) = \frac{1}{10000} \sum_{i=1}^{10000} (\hat{\beta}_i - \beta)^2,$$

for $\beta = \mu, \kappa, \lambda_1, \lambda_2$.
We repeat these steps for $n = 10, 20, ..., 1000$ with $\mu = 2$, $\kappa = 1$, $\lambda_1 = 0.04$, $\lambda_2 = 0.5$ and $j = 2$, so computing bias$_\beta(n)$ and MSE$_\beta(n)$ for $\beta = \mu, \kappa, \lambda_1, \lambda_2$ and $n = 10, 20, ..., 1000$.

Figures 5 and 6 show how the four biases and the four mean squared errors vary with respect to n. The broken line in Figure 5 corresponds to the biases being zero. The following observations can be made: the bias for κ is positive; the bias for λ_2 is negative; the biases for each parameter either decrease or increase to zero as $n \to \infty$; the mean squared errors for each parameter decrease to zero as $n \to \infty$; the mean squared errors appear smallest for the parameters κ and λ_2.

7. Applications

In this section two circular data sets are analyzed based on the inferential methods described in the preceding sections in R software.

Ant data

In our first example, we analyze the ant data in Appendix B.7 of [10]. The sample of 100 observations was drawn randomly from the original data collected during the animal orientation experiment described in [15]. The data consist of the directions chosen by the ants in response to an unevenly illuminated black target located at a position of 180° from the zero direction.

Table 1 presents the MLEs and maximum log-likelihood (MLL), Akaike information criterion (AIC), and Bayesian information criterion (BIC) values from fitting sine-cosine weighted von Mises, cardioid and Cauchy distributions with $j = 1$, and the sine-skew von Mises and von Mises distributions.

The fitted sine-cosine weighted, the sine-skewed von Mises and von Mises densities are superimposed on a histogram of the data in Figures 5 and 6.

When the MLL values for the sine-cosine weighted von Mises, sine-skewed von Mises, and von Mises distributions are considered, the test statistics for the typical likelihood-ratio test are 4.48 and 7.96. When these values are compared to the quantiles of the χ^2_1 and χ^2_2 distributions, the test's p-values become 0.034 and 0.0186. Thus, the fit for the sine-cosine weighted von Mises distribution improves more significantly than for the sine-skewed von Mises and von Mises distributions.

The sine-cosine weighted wrapped Cauchy distribution has the greatest MLL value, as seen in Table 1. AIC and BIC criteria are used to confirm it. The p-values for Rayleigh goodness-of-fit test indicate that these are the only two viable models, sine-cosine weighted wrapped Cauchy and sine-cosine weighted von Mises, among the five considered.

Table 1. MLEs, MLL, AIC and BIC values for three sine-cosine weighted (SCW) models and sine-skewed (SS) von Mises and von Mises fitted to the ant data. The final column contains the p-values for the Rayleigh goodness-of-fit test.

Model	$\hat{\eta}$	$\hat{\kappa}$ or $\hat{\rho}$	$\hat{\lambda}_1$	$\hat{\lambda}_2$	MLL	AIC	BIC	p-value
SCW wrapped Cauchy	3.30	0.50	-0.25	0.26	-134.81	277.62	288.04	0.121
SCW cardioid	2.35	0.45	0.70	-0.30	-142.65	293.31	303.72	0.0014
SCW von Mises	3.70	1.72	-0.59	-0.40	-138.38	284.76	295.18	0.064
SS von Mises	3.614	1.36	-0.66		-140.62	287.24	295.05	0.037
von Mises	3.20	1.56			-142.36	288.72	293.45	0.022

Wind direction

We consider a data set taken from [4] as our second example. A place named ”Col de la Roa” in the Italian Alps, a meteorological station records via data-logger several parameters. Measurements are taken every 15 minutes. We report the wind direction recorded daily from January 29, 2001, to March 31, 2001, for the hours of 3.00 am to 4.00 am. This implies that observations are made for 310 measurements taken daily. This data frame contains one variable (wind direction) in radians.

The maximum likelihood fits of three sine-cosine weighted distributions with $j = 1$ are presented in Table 2. Each fit also includes the MLE, MLL, AIC and BIC. In light of these findings, the SCW wrapped Cauchy model is determined to be the best fitting model. The three fitted densities appear superimposed on the histogram in Figure 7.

Table 3 presents all diagnostics as the ones displayed in Table 2 for the fits of three extensions of the von Mises distribution, namely, the four-parameter GvM_2 distribution, the five-parameter vMM distribution, and the four-parameter $SCWvM$ distribution. with $j = 2$. Although the AIC and BIC rank the five-parameter vMM fit as the best, the four-parameter $SCWvM$ and GvM_2 fits have AIC and BIC values that are only slightly greater. The p-values for the Watson goodness-of-fit test indicate that vMM model provides adequate fit to the data.

Table 2. MLEs, MLL, AIC and BIC values for the fits to the wind direction data of three sine-cosine weighted (SCW) models. The final column contains the p-values for the Watson goodness-of-fit test.

Model	$\hat{\eta}$	$\hat{\kappa}$ or $\hat{\rho}$	$\hat{\lambda}_1$	$\hat{\lambda}_2$	MLL	AIC	BIC	p-value
SCW wrapped Cauchy	6.23	0.5	0.77	0.22	-386.04	780.08	795.02	¡0.01
SCW cardioid	5.74	0.363	1.00	0.00	-416.82	841.64	856.58	¡0.01
SCW von Mises	5.98	1.39	0.91	-0.08	-400.68	809.36	824.30	¡0.01

Table 3. MLEs, MLL, AIC and BIC values for the fits to the wind direction data of the sine-cosine weighted von Mises ($SCWvM$), generalized von Mises (GvM), and von Mises mixture (vMM) models. The final column contains the p-values for the Watson goodness-of-fit test.

Model	$\hat{\eta_1}$	$\hat{\eta_2}$	$\hat{\kappa_1}$	$\hat{\kappa_2}$	$\hat{p}$	$\hat{\lambda}_1$	$\hat{\lambda}_2$	MLL	AIC	BIC	p-value
$SCWvM$	0.64		1.82			-0.74	-0.25	-377.42	762.84	777.78	¡0.01
GvM_2	0.60	0.00	1.39	0.94				-380.33	768.66	783.60	¡0.01
vMM	0.08	0.73	18.20	0.94	0.47			-370.51	751.02	769.70	¿0.1

8. Conclusion

We proposed a new general method to obtain more flexible circular distributions. The obtained distributions may have an asymmetric property with multimodal. We have also focused our attention primarily on the sine-cosin weighted von Mises, cardioid and wrapped cauchy models. The examples presented illustrate the potential of sine-cosin weighted distributions as models for real circular data. It would, nevertheless, be of interest to investigate other combinations of the components within the general construction with the intention of obtaining families of distributions which were both mathematically tractable as well as highly flexible.
Finally, whilst (2) provides one means of modeling multimodality. It is conceivable that there will be situations in which the latter approach will provide circular data analysts with greater insight into the mechanisms generating their data.

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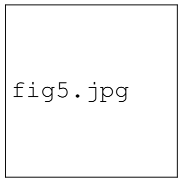

Figure 4. $\text{bias}_{\mu}(n)$ (top left), $\text{bias}_{\kappa}(n)$ (top right), $\text{bias}_{\lambda 1}(n)$ (bottom left) and $\text{bias}_{\lambda 2}(n)$ (bottom right) versus n = 10, 20, . . . , 1000.

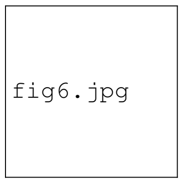

Figure 5. $\text{MSE}_{\mu}(n)$ (top left), $\text{MSE}_{\kappa}(n)$ (top right), $\text{MSE}_{\lambda 1}(n)$ (bottom left) and $\text{MSE}_{\lambda 2}(n)$ (bottom right) versus n = 10, 20, . . . , 1000.

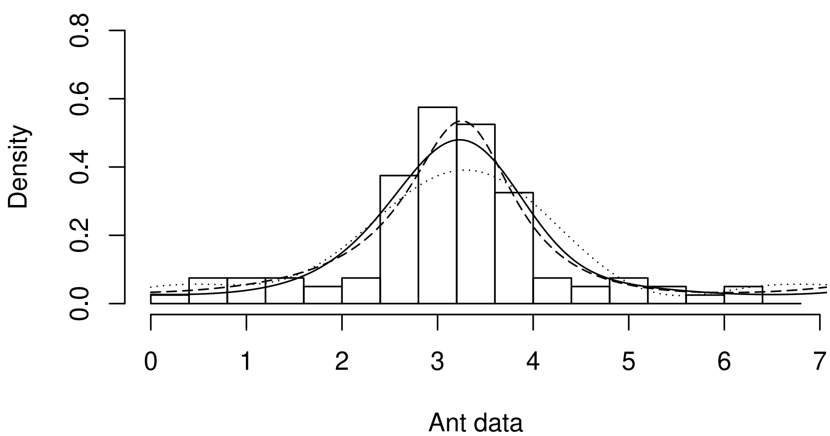

Figure 6. Histogram of the Ant data (in radians), together with the fitted densities for the sine-cosine weighted von Mises (solid), sine-cosine weighted wrapped Cauchy (long dashed), and sine-cosine weighted cardioid (dotted) distributions. The data are plotted over $[0, 2\pi)$.

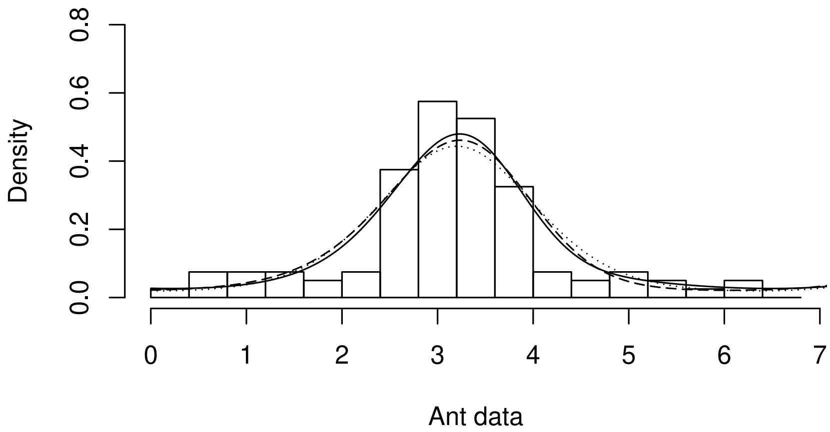

Figure 7. Histogram of the ant data (in radians), together with the fitted densities for the sine-cosine weighted von Mises (solid), sine-skew von Mises (long dashed), and von Mises (dotted) distributions. The data are plotted over $[0, 2\pi)$.

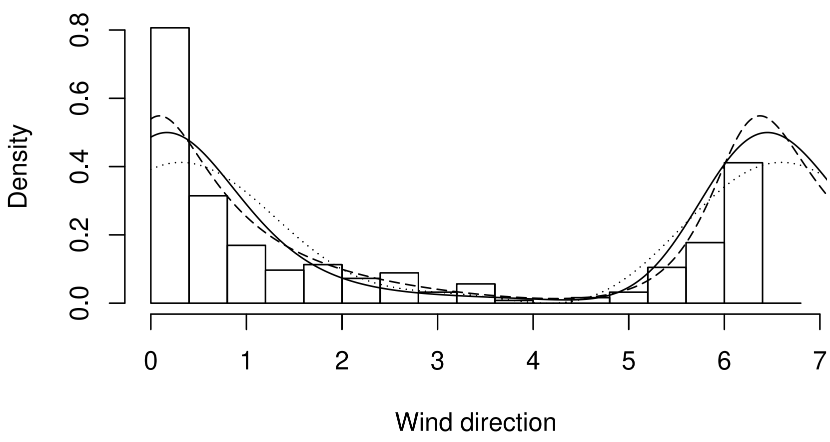

Figure 8. Histogram of the wind direction data (in radians), together with the fitted densities for the sine-cosine weighted von Mises (solid), sine-cosine weighted wrapped Cauchy (long dashed), and sine-cosine weighted cardioid (dotted) distributions. The data are plotted over $[0, 2\pi)$.

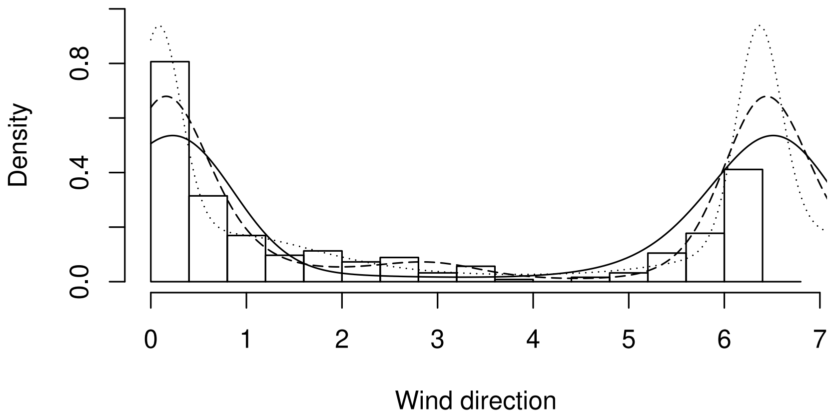

Figure 9. Histogram of the wind direction data (in radians), together with the fitted densities for the sine-cosine weighted von Mises (solid), generalized von Mises (long dashed), and von Mises mixture (dotted) distributions. The data are plotted over $[0, 2\pi)$.