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Mean-TVaR Models for Diversified Multi-period Portfolio Optimization with Realistic Factors Based on Uncertainty Theory

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Abstract The focus of any portfolio optimization problem is to imitate the stock markets and propose the optimal solutions to dealing with diverse investor expectations. In this paper, we propose new multi-period portfolio optimization problems when security returns are uncertain variables, given by experts' estimations, and take the Tail value at risk (TVaR) as a coherent risk measure of investment in the framework of uncertainty theory. Real- constraints, in which transaction costs, liquidity of securities, and portfolio diversification, are taken into account. Equivalent deterministic forms of mean–TVaR models are proposed under the assumption that returns and liquidity of the securities obey some types of uncertainty distributions. We adapted the Delphi method in order to evaluate the expected, the standard deviation and the turnover rates values of returns of the given securities. Finally, numerical examples are given to illustrate the effectiveness of the proposed models.

Keywords Uncertain Variable, transaction cost, Liquidity, Multi-period Portfolio Optimization, Tail value at risk, Entropy measure.

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1. Introduction

The portfolio optimization problem has always been an interesting topic that is concerned with the optimal allocation of capital to several securities. The Mean-Variance (MV) models proposed by Markowitz (1952, 1959) [1, 2] have been the most popular way to determine the optimal portfolio. It amalgamates optimization techniques with probability theory in which investment return and portfolio risk are quantified as the mean and the variance of security returns respectively. The Mean-Variance model is formally presented in two manners: maximizing expected return for a given level of variance or, minimizing portfolio variance for a given expected return level that the investor feels satisfactory.

These works and other, assume that security returns are estimated from analysis of past data and modeled by random variables. However, in many cases, unexpected events occur in the financial market such as interest rate drop by the central bank or unexpected events of companies push investors not to believe that the past data of security returns can well reflect their future returns. These complex factors make the probabilistic approaches difficult to apply.
For this reason, Liu (2007) [3] proposed a self-dual measure called uncertain measure, which can be used to

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measure subjective estimation. It is the core of uncertainty theory, refined by Liu [4] in 2010, which is a branch of mathematics for modeling subjective uncertain phenomena. The introduction of uncertain theory has contributed, without doubt, to overcoming several limitations and proposing other solutions to portfolio selection, which has been the main reason for the increasing number of studies on portfolio optimization in recent decennia. Huang (2012) [5] proposed two uncertain mean-variance and mean-semi-variance models subject to experts' estimations. Liu and Qin (2012) [6] presented an uncertain mean-semi-absolute deviation model. Mean-VaR Model for Portfolio Selection with Uncertain Returns was discussed by Ning et al. (2012) [7]. Qin et al. (2014) [8] proposed an uncertain portfolio adjusting model using semi-absolute deviation. In 2016, Qian et al. [9] presented portfolio selection models based on the Cross-entropy of uncertain variables. Huang and Zhao (2014) [10] discuss the Mean-chance model for portfolio selection based on the uncertain measure. Recently, the Mean-variance model for portfolio optimization with background risk based on uncertainty theory was presented by Zhai and Bai (2017) [11]. In their uncertain models, Xue et al. [12] suggested uncertain Portfolio Selection with Mental Accounts and Realistic Constraints. Huang et al. (2014) [13] discussed the capital budgeting problem of projects using annual cash inflows, initial investment outlays, and cash outflows based on experts' evaluations.

All these papers mentioned above proposed a single-period portfolio optimization proplems. They allocate their capital at the beginning of the investment and hold it until the end of the investment period. In a real market, many investors prefer investing long-term to gain more return by adjusting their investment strategies from time to time and taking into consideration novel market conditions. For this reason, the multi-period portfolio selection problems have attracted many researchers such as [14, 15, 16, 17] .

Risk measures are an important pilar in portfolio theory, great attention is given by scholars to studying this field. Peng [18] introduced, for the first time, the concepts of Value at Risk and Tail Value at Risk to the framework of uncertainty theory. Yan [19] introduced Mean-VaR uncertain portfolio selection problem. However, VaR is not a coherent risk measure. It does not give any information about the severity of losses beyond the VaR value. To deal with this insuffisance, Tail value at risk (TVaR) is considered to provide a better measure of risk since it is coherent when independence is satisfied. It quantifies risk beyond value-at-risk. Ning et al. [20] provided Mean-TVaR Model for Portfolio Selection with uncertain Returns.

To ensure diversification allocations and avoid the solutions of the models are concentrated on only some stocks which can bring a great loss, some researchers [21, 22] employed cardinality constraint enforcing solutions diversification, while Chen et al. [23] employed Shannon entropy as a diversification risk measure. Liquidity is also one of the main concerns for investors. For this reason, researchers take it into acount [24, 25, 26] . It measures the degree of chance to convert an investment into cash without any significant loss.

In many cases, after a certain period, existing security may not be gainful. Therefore, investors wish to change their situations in the financial market by buying or selling a risky asset. The cost resulting from these operations is called transaction cost. Some researchers like [27, 28, 29] , etc. extended the works on portfolio optimization problems with transaction costs.

To the best of our knowledge, there are a no studies in the literature that focus on multi-period portfolio model using uncertain Tail value at Risk as a risk measure with some constraints reflecting the market reality. Therefore, based on uncertain variables to describe stocks returns and liquidity constraints and in the presence of transaction costs and diversification constraints, three multi-period portfolio optimization problems are proposed under the assumption that the initial capital is allocated among the assets at the beginning of the first period while the total wealth is generated at the end of the investment. Theses models are transformed into a crisp mathematical problem under the assumption that the security returns and liquidity obey some uncertainty distribution forms. In addition, an examples to illustrate the models and analyze some effects of diversification constraint and liquidity on portfolio selection within uncertainty theory are given. The Delphi method was adapted in order to evaluate the expected, the standard deviation and the turnover rates values of returns of the given securities.

This paper is organized as follows. For a better understanding of the paper, some fundamentals of uncertainty theory will be presented in the second section. In section 3, portfolio return, diversification constraint, transaction costs, and liquidity constraint are formally expressed and new multi-period uncertain portfolio selection models are formulated. Section 4 has been dedicated to introducing the equivalent models ether liquidity and return securities are modelled by uncertain normal variables. In section 5, Delphi method is presented to evaluate the expected, the standard deviation and turnover rates values of returns of the given security. Numerical examples to illustrate our proposed models are presented in section 6. Finally, some conclusions and further works are given in Section 7.

2. Fundamentals

Uncertainty theory was founded by Liu (2007) and further developed by Liu (2010) as a branch of mathematics for modelling subjective uncertain phenomena. For more informations about uncertainty theory, consult [4]. This section recalls the basic contents of uncertainty theory.

Let Γ be a nonempty set, L be σ-algebra of a collection of subsets of Γ. Let be a nonnegative function from L to $[0, 1]$ satisfying normality axiom, self-duality axiom, countable subadditivity axiom and product measure axiom if product space is needed, is called uncertain measure defined on L.

The triplet $(\Gamma, L,)$ is called an uncertainty space.

Definitions 1

An uncertain variable ξ is defined as a measurable function from an uncertainty space (Γ, L, Λ) to the set of real numbers, i.e. for any Borel set B of real numbers, $\{\xi \epsilon B\} = \{\gamma \in \Gamma / \xi(\gamma) \in B\}$ is an event.

Definitions 2

The uncertainty distribution Φ of an uncertain variable is defined by $\Phi(x) = M(\xi \leq x)$ for any real number x.

Peng and Iwamura (2010) [30] proved that a function $\Phi(\mathrm{x}) : \mathrm{IR} \mapsto [0, 1]$ is an uncertainty distribution if and only if it is a monotone increasing function.

Definitions 3

Let ξ be an uncertain variable with continuous and strictly increasing uncertainty distribution $\Phi(x)$. Then the inverse function $\Phi^{-1}(\alpha)$ is called the inverse uncertainty distribution of ξ.

For example, the normal uncertainty distribution of the uncertain variable $\xi \sim \mathrm{N}(\mathrm{e}, \sigma)$ is

$$\Phi(\mathrm{x}) = \left(1 + \exp\left(\frac{\pi(\mathrm{e} - \mathrm{x})}{\sqrt{3}\sigma}\right)\right)^{-1}.$$

The inverse uncertainty distribution of normal uncertain variable ξ is $\Phi^{-1}(\alpha) = \mathrm{e} + \frac{\sqrt{3}\sigma}{\pi} \cdot \ln\left(\frac{\alpha}{1-\alpha}\right)$ where e and σ are real numbers with $\sigma > 0$.

Definitions 4

Let ξ be an uncertain variable. Then the expected value of ξ is defined by $E(\xi) = \int_0^{+\infty}(1 - \Phi(\mathrm{r}))\mathrm{dr} - \int_{-\infty}^0 \Phi(\mathrm{r})\mathrm{dr}$ provided that at least one of the two integrals is finite. $\mathrm{E}(\xi)$ can also be expressed by $\mathrm{E}(\xi) = \int_0^{+\infty} \mathrm{M}(\xi \geq \mathrm{r})\mathrm{dr} - \int_{-\infty}^0 \mathrm{M}(\xi \leq \mathrm{r})\mathrm{dr}$.

Theorem 1
Let ξ be an uncertain variable with regular uncertainty distribution. If the expected value exists, then

$$E(\xi) = \int_0^1 \Phi^{-1}(\alpha) d\alpha$$

Theorem 2
Let a and b be two real numbers, and ξ and η two uncertain variables. Then we have $\mathrm{E}(\mathrm{a}\xi + \mathrm{b}) = aE(\xi) + b$. Further, if and ξ and η are independent, then $E(a\xi + b\eta) = aE(\xi) + bE(\eta)$.

Definitions 5
Let ξ be an uncertain variable that has a finite expected value e. Then the variance of ξ is defined by $\mathrm{V}(\xi) = \mathrm{E}\left[(\xi - \mathrm{e})^2\right]$.

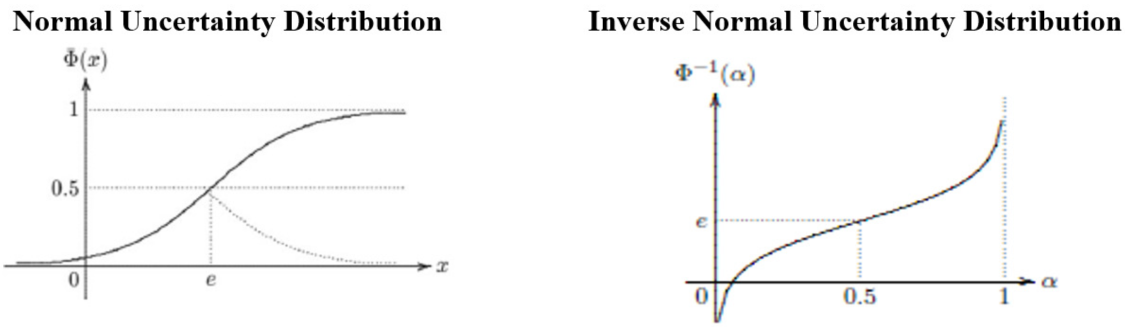

Figure 1. Normal and Inverse Normal Uncertainty Distribution

Value at Risk (VaR) is a measure of the risk of loss for investments in normal market conditions and a set period. For a given portfolio, time horizon t, and confidence level α, the $\mathrm{VaR}_{\alpha,\mathrm{t}}$ can be defined as the maximum possible loss during that time after excluding all α worse outcomes. In other words, VaR is defined as sufficient capital to cover the potential losses of a portfolio over a given period. Formally, in the framework of uncertainty theory, Peng [18] introduced, in 2009 , the following definition:

Let ξ be an uncertain variable and $\alpha \in (0,1]$ be the risk confidence level. Then the value at risk "VaR" of ξ at the confidence level α and instant t, $\mathrm{VaR}_{\alpha,t}$, is the function

$$\mathrm{VaR}_{\alpha,\mathrm{t}} : (0,1] \to \mathrm{R} \text{ such that}$$
$$\mathrm{VaR}_{\alpha,\mathrm{t}} = \inf\{\mathrm{x} \mid \mathrm{M}\{\xi \leq \mathrm{x}\} \geq \alpha\}$$

When the uncertain variable ξ is of the continuous type with distribution function Φ, then $\mathrm{VaR}_{\alpha,t}$ is expressed as

$$\mathrm{VaR}_{\alpha,\mathrm{t}} = -\Phi(1-\alpha)$$

Because of the non-coherence of the VaR risk measure, Peng [18] has been introduced a coherent risk measure called Tail Value at risk "TVaR" as follows:

Let ξ be an uncertain variable and $\alpha \in (0,1)$ be the risk confidence level. Then the tail value at risk "TVaR" of ξ at the confidence level α and instant t, TVaR $_{\alpha,t}$, is the function

$$\mathrm{TVaR}_{\alpha,\mathrm{t}} : (0,1) \to \mathrm{IR} \text{ such that}$$
$$\mathrm{TVaR}_{\alpha,\mathrm{t}} = \frac{1}{1-\alpha} \int_\alpha^1 \mathrm{VaR}_{\beta,\mathrm{t}} \cdot \mathrm{d}\beta$$

3. Portfolio optimization model

In our multi-period portfolio selection models, we assume that investors want to build a portfolio from n risky assets in the financial market. The initial capital W_0 is allocated among n assets at the beginning of the first period while the total wealth of all the investment W_T is obtained at the end of period T. Investor can readjust his wealth among the n risky assets at the beginning of each period along with the investment.

For the sake of discussion convenience, some necessary notations are listed as follows:

Symbols	**Explanation**
ξ_{it}	Uncertain return rate at the beginning of period t to security i;
$\eta_{i,t}$	Uncertain liquidity rate at the beginning of period t to security i;
$\mathbf{W_0}$	The initial capital of the investment;
$\mathbf{W_t}$	The wealth at the end of period t;
T	Lifetime of investment;
x_{it}	The portion of capital allocated at the beginning of period t to security i;
$\mathbf{d_{it}}$	The unit transaction cost of risky asset i at period t;
$\mathbf{e}_{t,i}$	Return of security i at period t;
$\boldsymbol{\sigma}_{\mathrm{t,i}}$	Standard deviation of security i at period t;
$\mathbf{R}_{t,p}$	Net portfolio return rate at period t;

Figure 2. Multiperiod investment

3.1. Investment Return

In the portfolio optimization theory, individual security returns are the basic information for the investors, and every decision is made based on this information. The security return is expressed by the rate of return, which is defined as

$$\frac{\text{Security price at the end of the period Beginning price + dividend}}{\text{Beginning price}}$$

By denoting ξ_{it} as the uncertain variable represents the return of the ith security at the end of period t, then, the portfolio return is $\sum_{\mathrm{i=1}}^{\mathrm{n}} \mathrm{x_{it}} \cdot \xi_{\mathrm{it}}$. It is important to take into account transaction costs in portfolio selection. We suppose that it is a V shape function of the difference between the tth period portfolio and the $t-1$ th period portfolio. Then, the transaction cost for security i at period t can be expressed by

$$\mathbf{d_{it}}\left|\mathbf{x_{it}}-\mathbf{x_{it-1}}\right| \quad ,\mathrm{t}=1,2,\ldots,\mathrm{T}$$

Therefore, the transaction cost for the portfolio at period t is

$$\sum_{\mathbf{i=1}}^{\mathrm{n}} \mathbf{d_{it}}\left|\mathbf{x_{it}}-\mathbf{x}_{\mathbf{it}-1}\right| \quad ,\mathrm{t}=1,2,\ldots,\mathrm{T}$$

Then, the net portfolio return rate at period t, $R_{t,p}$ is expressed as

$$\mathbf{R}_{t,p}=\sum_{i=1}^{n}\left(\xi_{it}x_{it}-\mathbf{d}_{it}\left|\mathbf{x}_{it}-\mathbf{x}_{it-1}\right|\right) \quad ,t=1,2,\ldots,T$$

Therefore, the wealth at the end of period t can be represented as

$$\begin{aligned}\mathbf{W_{t+1}} &= \mathbf{W_t}\left(\mathbf{1}+\mathbf{R}_{\mathbf{t},\mathrm{p}}\right)\\ &=\mathbf{W_t}\left(\mathbf{1}+\sum_{\mathrm{i}=\mathbf{1}}^{\mathrm{n}}\left(\xi_{\mathrm{it}}\mathrm{x}_{\mathrm{it}}-\mathbf{d}_{\mathrm{it}}\left|\mathbf{x}_{\mathrm{it}}-\mathbf{x}_{\mathrm{it}-\mathbf{1}}\right|\right)\right) \quad ,\mathrm{t}=1,2,\ldots,\mathrm{T}\end{aligned}$$

Recursively, the terminal wealth obtained at the end of period T is

$$\mathbf{W}_{\mathrm{T}}=\mathbf{W}_0\prod_{t=1}^{\mathrm{T}}\left[\mathbf{1}+\sum_{i=1}^{\mathrm{n}}\left(\xi_{it}\mathrm{x}_{\mathrm{it}}-\mathbf{d_{it}}\left|\mathbf{x}_{\mathrm{it}}-\mathbf{x_{it}}_{-1}\right|\right)\right]$$

Since portfolio securities returns fluctuate continuously, we propose a deterministic number to characterize the terminal wealth. For this reason, we employ the uncertain expected value of the terminal wealth at the end of investment $\mathbf{E}\left(\mathbf{W}_{\mathrm{T}}\right)$ as follows

$$\mathbf{E}\left(\mathbf{W}_{\mathrm{T}}\right)=\mathbf{W}_0\prod_{t=1}^{\mathrm{T}}\left[1+\sum_{i=1}^{\mathrm{n}}\left(E\left(\xi_{it}\right)\mathbf{x}_{it}-\mathbf{d}_{it}\left|\mathbf{x}_{it}-\mathbf{x}_{it-1}\right|\right)\right]$$

3.2. Diversification constraint

Unlike proverbs: “Don’t risk everything on one endeavor”, and “Don’t put all your eggs in one basket”, which means that we should diversify our portfolio. Traditional models for portfolio optimization always lead to concentrated allocations on a limited number of assets, and the investor suffers risk greatly [31].

Orris Herfindahl’s index of industrial concentration defined as, the sum of the squared shares of each product’s contribution to the firm’s total output, is used also as diversification measure [32].

Recently [23] apply it in the portfolio selection theory as a constraint to ensure diversification portfolio by assuming that x_i is the weight of the security i in the portfolio.

If $\mathbf{A}=\{\mathbf{A_1},..,\mathbf{A_n}\}$ is a partition of the set Ω and $\mathbf{x_i}$ is the probability the event $\mathrm{A_i}$ with $\mathrm{i}=1,\ldots,$ n. The Shannon entropy [33] of $\mathbf{A}$ is formally defined as

$$\mathrm{E}=\sum_{\mathrm{i}=1}^{\mathrm{n}}\mathrm{x_i}\ln\left(\frac{1}{\mathrm{x_i}}\right),\quad \mathrm{t}=1,2,\ldots,\mathrm{T}$$

It’s important to note that the maximum value of E is attained $(\ln(\mathrm{n}))$ when all $\mathrm{x_i}$ are equal and reaches its minimum value (0) if $x_i=1$ where $i=1,..,n$. Then, a great entropy value means that the portfolio is more diversified and a small one means the portfolio is concentrated.

3.3. Liquidity risk

In the practice, liquidity risk is a determinant factor affecting the optimal portfolio selection. It measures the degree of chance to convert an investment into cash without any significant loss. Rational investors prefer to select securities with high liquidity. Turnover rate, which is defined as the percentage of traded shares, is used to measure it. It is modeled by uncertain variable, η, due to the uncertainty of the market. Then, portfolio liquidity at instant t, is represented formally as follows,

$$l_t=E\left(\sum_{i=1}^{n}x_{i,t}\cdot\eta_{i,t}\right)=\sum_{i=1}^{n}x_{i,t}.E\left(\eta_{i,t}\right),\quad t=1,2,\ldots,T$$

3.4. Formulation of portfolio optimization models

Supposing that an investor wants minimizing the cumulative investment risk of the portfolio $\sum_{t=1}^{T} \mathrm{TVaR}_{\alpha,t}$ knowing that the terminal wealth over the T period, $\prod_{t=1}^{T}\left[1+\sum_{i=1}^{n}\left(\mathrm{E}\left(\xi_{it}\right) \mathrm{x}_{it}- \quad \mathrm{d}_{it}\left|\mathrm{x}_{it}-\mathrm{x}_{it-1}\right|\right)\right]$, must be greater than the given minimum return level λ. In addition, to avoid the concentrative allocation and control the maximum portion to invest in each security, we employ entropy constraints. Let β_t and l_t be the minimum levels of diversification and liquidity that the investor preset in each period t respectively. As a result, the multi-period uncertain portfolio selection model is formally expressed as follows,

$$\mathrm{Min} \sum_{t=1}^{T} \mathrm{TVaR}_{\alpha,t}$$
$$\text{Subject to} \begin{cases} \mathrm{W}_0 \prod_{t=1}^{T}\left[1+\sum_{i=1}^{n}\left(\mathrm{E}\left(\xi_{it}\right) x_{it}-\mathrm{d}_{it}\left|\mathrm{x}_{it}-\mathrm{x}_{it-1}\right|\right)\right] \geq \lambda \\ \sum_{i=1}^{n} \mathrm{x}_{it}=1, \mathrm{t}=1, \ldots, \mathrm{T} \\ -\sum_{i=1}^{n} \mathrm{x}_{it} \ln \left(\mathrm{x}_{it}\right) \geqslant \beta_t, \mathrm{t}=1,2, \ldots, \mathrm{T} \\ \sum_{i=1}^{n}\left(\eta_{i,t}\right) \geq \mathrm{l}_t, \mathrm{t}=1,2, \ldots, \mathrm{T} \\ \mathrm{x}_{it} \geqslant 0, \mathrm{t}=1, \ldots, \mathrm{T}; \quad \mathrm{i}=1 \ldots, \mathrm{n} \end{cases} \tag{1}$$

If an investor wants to maximize the final wealth at the maximal given risk level θ, then the adequat programming model is expressed as follows,

$$\mathrm{Max}\ \mathrm{W}_0 \prod_{t=1}^{T}\left[1+\sum_{i=1}^{n}\left(\mathrm{E}\left(\xi_{it}\right) \cdot \mathrm{x}_{it}-\mathrm{d}_{it}\left|\mathrm{x}_{it}-\mathrm{x}_{it-1}\right|\right)\right]$$
$$\text{Subject to} \begin{cases} \sum_{t=1}^{T} \mathrm{TVaR}_{\alpha,t} \leq \theta \\ \sum_{i=1}^{n} \mathrm{x}_{it}=1, \quad \mathrm{t}=1,2, \ldots, \mathrm{T} \\ -\sum_{i=1}^{n} \mathrm{x}_{it} \ln \left(\mathrm{x}_{it}\right) \geqslant \beta_t, \quad \mathrm{t}=1,2, \ldots, \mathrm{T} \\ \sum_{i=1}^{E} \mathrm{E}\left(\eta_{i,t}\right) \geqslant \mathrm{l}_t, \mathrm{t}=1,2, \ldots, \mathrm{T} \\ \mathrm{x}_{it} \geqslant 0, \mathrm{t}=1, \ldots, \mathrm{T}; \quad \mathrm{i}=1 \ldots, \mathrm{n} \end{cases} \tag{2}$$

A rational investor always wants to maximize the terminal wealth and minimize the investment risk, which are completely inconsistent. In order to select an optimal portfolio under a given level of risk aversion, we introduce the following optimization problem,

$$\mathrm{Max}\ \mathrm{W}_0 \prod_{i=1}^{T}\left[1+\sum_{i=1}^{n}\left(\mathrm{E}\left(\xi_{it}\right) \cdot \mathrm{x}_{it}-\mathrm{d}_{it} \mid \mathrm{x}_{it}-\mathrm{x}_{it-1}\right)\right]-\phi \cdot\left(\sum_{t=1}^{T} \mathrm{TVaR}_{\alpha,t}\right)$$
$$\text{Subject to} \begin{cases} \sum_{i=1}^{n} \mathrm{x}_{it}=1, \quad \mathrm{t}=1,2, \ldots, \mathrm{T} \\ -\sum_{i=1}^{n} \mathrm{x}_{it} \ln \left(\mathrm{x}_{it}\right) \geqslant \beta_t, \quad \mathrm{t}=1,2, \ldots, \mathrm{T} \\ \sum_{i=1}^{n} \mathrm{E}\left(\eta_{i,t}\right) \geqslant \mathrm{l}_t, \quad \mathrm{t}=1,2, \ldots, \mathrm{T} \\ \mathrm{x}_{it} \geqslant 0, \mathrm{t}=1, \ldots, \mathrm{T}, \quad \mathrm{i}=1 \ldots, \mathrm{n} \end{cases} \tag{3}$$

Where ϕ is a non-negative number. It means the level of risk aversion. The greater the value of ϕ, the more conservative is the investor.

4. Deterministic Forms of the Uncertain Models

Theorem Suppose the return rates of securities $\xi_{t,i}$ and turnover rates $\eta_{i,t}$ are all normal uncertain variables $\xi_{t,i} \sim N\left(\mathrm{e}_{it}, \sigma_{it}\right)$ and $\eta_{i,t} \sim N\left(\left(\mu_{it}, \tau_{it}\right), i=1,2, \ldots, n; \mathrm{t}=1,2, \ldots, \mathrm{T}\right.$. Then model (2) and model (3) can be converted into the following forms,

$$\text{Max } \mathrm{W_0} \prod_{\mathrm{t=1}}^{\mathrm{T}} \left[1 + \sum_{\mathrm{i=1}}^{\mathrm{n}} \left(\mathrm{e_{it}} \cdot \mathrm{x_{it}} - \mathrm{d_{it}} \left| \mathrm{x_{it}} - \mathrm{x_{it-1}} \right| \right) \right]$$

$$\text{Subject to} \begin{cases} \sum_{\mathrm{t=1}}^{\mathrm{T}} \sum_{\mathrm{i=1}}^{\mathrm{n}} \mathrm{x_{it}} \left[-\mathrm{e_{it}} + \frac{\sqrt{3}}{\pi} \sigma_{\mathrm{it}} \left(-\ln(1-\alpha) - \frac{\alpha}{1-\alpha} \ln(\alpha) \right) \right] \leq \theta \\ \sum_{\mathrm{i=1}}^{\mathrm{n}} \mathrm{x_{it}} = 1, \mathrm{t} = 1, \ldots, \mathrm{T} \\ -\sum_{\mathrm{i=1}}^{\mathrm{n}} \mathrm{x_{it}} \ln(\mathrm{x_{it}}) \geqslant \beta_{\mathrm{t}}, \mathrm{t} = 1, 2, \ldots, \mathrm{T} \\ \sum_{\mathrm{i=1}}^{\mathrm{n}} \mu_{\mathrm{it}} \geqslant \mathrm{l_t}, \mathrm{t} = 1, 2, \ldots, \mathrm{T} \\ \mathrm{x_{it}} \geqslant 0, \mathrm{t} = 1, \ldots, \mathrm{T}; \quad \mathrm{i} = 1 \ldots, \mathrm{n} \end{cases} \tag{4}$$

and

$$\text{Max } \mathrm{W_0} \prod_{\mathrm{t=1}}^{\mathrm{T}} \left[1 + \sum_{\mathrm{i=1}}^{\mathrm{n}} \left(\mathrm{e_{it}} \cdot \mathrm{x_{it}} - \mathrm{d_{it}} \left| \mathrm{x_{it}} - \mathrm{x_{it-1}} \right| \right) \right]$$

$$-\phi \cdot \left(\sum_{\mathrm{t=1}}^{\mathrm{T}} \sum_{\mathrm{i=1}}^{\mathrm{n}} \mathrm{x_{it}} \left[-\mathrm{e_{it}} \cdot \frac{\sqrt{3}}{\pi} \sigma_{\mathrm{it}} \left(-\ln(1-\alpha) - \frac{\alpha}{1-\alpha} \ln(\alpha) \right) \right] \right) \tag{5}$$

$$\text{Subject to} \begin{cases} \sum_{\mathrm{i=1}}^{\mathrm{n}} \mathrm{x_{it}} = 1, \quad \mathrm{t} = 1, 2, \ldots, \mathrm{T} \\ -\sum_{\mathrm{i=1}}^{\mathrm{n}} \mathrm{x_{it}} \ln(\mathrm{x_{it}}) \geqslant \beta_{\mathrm{t}}, \quad \mathrm{t} = 1, 2, \ldots, \mathrm{T} \\ \sum_{\mathrm{i=1}}^{\mathrm{n}} \mu_{\mathrm{it}} \geqslant \mathrm{l_t}, \mathrm{t} = 1, 2, \ldots, \mathrm{T} \\ \mathrm{x_{it}} \geqslant 0, \mathrm{t} = 1, \ldots, \mathrm{T}; \quad \mathrm{i} = 1 \ldots, \mathrm{n} \end{cases}$$

Proof. The returns of securities $\xi_{\mathrm{t,i}}$ are chosen as normal uncertain variables for all $\mathrm{t} = 1, 2, \ldots, \mathrm{T}$ and $\mathrm{i} = 1, 2, \ldots, \mathrm{n}$, with mean denoted by $\mathrm{e_{t,i}}$ and standard deviation $\sigma_{\mathrm{t,i}}$.
According to the Liu (2007), if ξ_1 and ξ_2 be independent normal uncertain variables $\mathrm{N}(\mathrm{m_1}; \sigma_1)$ and $\mathrm{N}(\mathrm{m_2}; \sigma_2)$, respectively. Then the sum $\xi_1 + \xi_2$ is also a normal uncertain variable $\mathrm{N}(\mathrm{m_1}; +\mathrm{m_2};; \sigma_1 + \sigma_2)$. Further, the multiplication of a normal uncertain variable $\mathrm{N}(\mathrm{m}; \sigma)$ and a scalar number $\mathrm{k} > 0$ is also a normal uncertain variable $\mathrm{N}(\mathrm{km}; \mathrm{k}\sigma)$.
Let $\mathrm{R_{t,p}} = \sum_{\mathrm{i=1}}^{\mathrm{n}} \mathrm{x_{it}} \xi_{\mathrm{it}}$ be the portfolio return at the end of period t. $\mathrm{R_{t,p}}$ is a normal uncertain variable with expected value $e_t = \sum_{i=1}^{n} x_{it} e_{it}$ and variance $\sigma_t^2 = \sum_{i=1}^{n} x_{it}^2 \sigma_{it}^2, t = 1, \ldots, n$. The loss, being the negative of this, is given therefore by $-\sum_{i=1}^{n} x_{it} \xi_{it}$. Then

$$\mathrm{VaR}_{\alpha,\mathrm{t}} = \inf \left\{ \mathrm{k} \mid \mathrm{M} \left\{ -\sum_{\mathrm{i=1}}^{\mathrm{n}} \mathrm{x_{it}} \xi_{\mathrm{it}} \leq \mathrm{k} \right\} \geq \alpha \right\}$$
$$= \inf \left\{ \mathrm{k} \mid \mathrm{M} \left\{ \sum_{\mathrm{i=1}}^{\mathrm{n}} \mathrm{x_{it}} \xi_{\mathrm{it}} \leq -\mathrm{k} \right\} \leq 1 - \alpha \right\} = -\sum_{\mathrm{i=1}}^{\mathrm{n}} \mathrm{x_{it}} \Phi_{\mathrm{i}}^{-1}(1-\alpha)$$

When Φ_{i}^{-1} is the inverse function distribution of ξ_{it}. Therefore, Value at risk of security i at the end of period t is expressed as following

$$\mathrm{VaR}_{\alpha,\mathrm{t}} = -\sum_{\mathrm{i=1}}^{\mathrm{n}} \mathrm{x_{it}} \mathrm{e_{it}} + \sum_{\mathrm{i=1}}^{\mathrm{n}} \mathrm{x_{it}} \frac{\sqrt{3} \sigma_{\mathrm{it}}}{\pi} \cdot \ln \left(\frac{\alpha}{1-\alpha} \right), \quad \mathrm{t} = 1, \ldots, \mathrm{T}$$

Based on Tail Value at risk definition, TVaR of security i at the end of period t is

$$\mathrm{TVaR}_{\alpha,\mathrm{t}} = \frac{1}{1-\alpha} \int_{\alpha}^{1} \left[-\sum_{\mathrm{i=1}}^{\mathrm{n}} \mathrm{x_{it}} \mathrm{e_{it}} + \sum_{\mathrm{i=1}}^{\mathrm{n}} \mathrm{x_{it}} \frac{\sqrt{3} \sigma_{\mathrm{it}}}{\pi} \cdot \ln \left(\frac{\beta}{1-\beta} \right) \right] \cdot \mathrm{d}\beta$$
$$= \sum_{\mathrm{i=1}}^{\mathrm{n}} \mathrm{x_{it}} \left[-\mathrm{e_{it}} + \frac{\sqrt{3}}{\pi} \left(-\ln(1-\alpha) - \frac{\alpha}{1-\alpha} \ln(\alpha) \right) \sigma_{\mathrm{it}} \right], \quad \mathrm{t} = 1, \ldots, \mathrm{T}$$

Wherefore, the total loss of the end of investment is $\sum_{t=1}^{T} \mathrm{TVaR}_{\alpha,t}$ and expressed as

$$\sum_{t=1}^{T}\sum_{i=1}^{n} x_{it}\left[-e_{it}+\frac{\sqrt{3}}{\pi}\sigma_{it}\left(-\ln(1-\alpha)-\frac{\alpha}{1-\alpha}\ln(\alpha)\right)\right]$$

The theorem is completed. Theorem Suppose the return rates of securities $\xi_{t,i}$ are all linear uncertain variables $\xi_{t,i} \sim \mathrm{L}(a_{it}, b_{it}), i=1,2,\ldots,n; t=1,2,\ldots,T$ and turnover rates $\eta_{i,t}$ are all zigzag uncertain variables $\eta_{i,t} \sim \mathrm{Z}(a_{it}, b_{it}, c_{it}), i=1,2,\ldots,n; t=1,2,\ldots,T$. Then model (2) and model (3) can be converted into the following forms:

$$\begin{aligned}
&\text{Max} \quad w_0 \prod_{t=1}^{T}\left[1+\sum_{i=1}^{n}\left(\frac{a_{it}+b_{it}}{2}x_{it}-d_{it}\left|x_{it}-x_{it-1}\right|\right)\right]\\
&\text{Subject to}\begin{cases}
-\sum_{t=1}^{T}\sum_{i=1}^{n} x_{it}\left(\frac{1+\alpha}{2}a_{it}+\frac{1-\alpha}{2}b_{it}\right) \leq \theta\\
\sum_{i=1}^{n} x_{it}=1, t=1,\ldots,T\\
-\sum_{i=1}^{n} x_{it}\ln(x_{it}) \geqslant \beta_t, t=1,2,\ldots,T\\
\sum_{i=1}^{n} x_{i,t}\cdot(a_{it}+2b_{it}+c_{it}) \geqslant 4l_t, t=1,2,\ldots,T\\
x_{it} \geqslant 0, t=1,\ldots,T; \quad i=1\ldots,n
\end{cases}
\end{aligned} \tag{6}$$

and

$$\begin{aligned}
&\text{Max} \quad \begin{array}{l} w_0 \prod_{t=1}^{T}\left[1+\sum_{i=1}^{n}\frac{a_{it}+b_{it}}{2}, x_{it}-d_{it}\left|x_{it}-x_{it-1}\right|\right)\right] \\ \quad +\phi\cdot\sum_{t=1}^{T}\sum_{i=1}^{n} x_{it}\left(\frac{1+\alpha}{2}a_{it}+\frac{1-\alpha}{2}b_{it}\right)\end{array}\\
&\text{Subject to}\begin{cases}
\sum_{i=1}^{n} x_{it}=1, \quad t=1,2,\ldots,T\\
-\sum_{i=1}^{n} x_{it}\ln(x_{it}) \geqslant \beta_t, \quad t=1,2,\ldots,T\\
\sum_{i=1}^{n} x_{i,t}\cdot(a_{it}+2b_{it}+c_{it}) \geqslant 4l_t, \quad t=1,2,\ldots,T\\
x_{it} \geqslant 0, t=1,\ldots,T; \quad i=1\ldots,n
\end{cases}
\end{aligned} \tag{7}$$

Proof. The returns of securities $\xi_{t,i}$ are linear uncertain variables $\eta_{t,i} \sim \mathrm{L}(a_{it}, b_{it})$. We have and

$$\mathrm{E}(\xi_{it})=\frac{a_{it}+b_{it}}{2} \text{ and } \mathrm{E}(\eta_{it})=\frac{a_{it}+2b_{it}+c_{it}}{4}$$

$$\mathrm{VaR}_{\alpha,t}=-\sum_{i=1}^{n} x_{it}\Phi_i^{-1}(1-\alpha)=-\sum_{i=1}^{n} x_{it}\cdot(\alpha a_{it}+(1-\alpha)b_{it})$$

When Φ_i^{-1} is the inverse function distribution of ξ_{it}.
Then, TVaR of security i at the end of period t is

$$\begin{aligned}
\mathrm{TVaR}_{\alpha,t} &= \frac{1}{1-\alpha}\int_{\alpha}^{1}\left[-\sum_{i=1}^{n} x_{it}\cdot(\beta a_{it}+(1-\beta)b_{it})\right]\cdot d\beta\\
&= -\sum_{i=1}^{n} x_{it}\left(\frac{1+\alpha}{2}a_{it}+\frac{1-\alpha}{2}b_{it}\right), \quad t=1,\ldots,T
\end{aligned}$$

Wherefore, the total loss of the end of investment is $\sum_{t=1}^{T} \mathbf{TVaR}_{\alpha,\mathbf{t}}$ and expressed as

$$-\sum_{t=1}^{T}\sum_{i=1}^{n} x_{it}\left(\frac{1+\alpha}{2}a_{it}+\frac{1-\alpha}{2}b_{it}\right)$$

The theorem is completed.

5. Expected and standard deviation values estimation

In order to evaluate the expected, the standard deviation and turnover rates values of returns of the given security used in the last model, we will adapt Delphi method introduced by WangGao-Guo [35] as follows:
As a first step, m numbers of domain experts are invited to evaluate the expected values, standard deviation values, and liquidity rate values of the given security respectively, and give their reasons for their evaluations. An anonymous summary of the evaluations and the reasons for them is given to the m experts in order to revise their earlier evaluations. During this process, the opinions of domain experts will converge to appropriate values. The main process is listed as follows:

Step 1. The m experts provide their expert's experimental data. Let $(\mathbf{e}_{i,j}, \sigma_{i,j}, \eta_{i,t}), i = 1, \ldots, n$ and $j = 1, 2, \ldots, m$ be the evaluation data, where $\mathbf{e}_{i,j}$ represent the jth expert's evaluation of the expected value of the ith security return, and $\sigma_{i,j}$ the jth expert's evaluation of the standard deviation value of the ith security returns and $\boldsymbol{\eta}_{\mathbf{i},\mathbf{t}}$ the jth expert's evaluation of the turnover rates of the ith security.

Step 2. By amusing that the experts are considered equally knowledgeable, we calculate the expected returns, the standard deviations, and the turnover rates as follows:

$$\overline{\mathbf{e_1}} = \frac{1}{\mathrm{m}} \sum_{\mathrm{j}=1}^{\mathrm{m}} \mathbf{e}_{\mathbf{i},\mathrm{j}} \quad , \quad \overline{\boldsymbol{\sigma}}_{\mathbf{1}} = \frac{1}{\mathrm{m}} \sum_{\mathrm{j}=1}^{\mathrm{m}} \boldsymbol{\sigma}_{\mathrm{i,j}} \quad \text{and} \quad \overline{\eta_{\mathbf{i}}} = \frac{1}{\mathrm{m}} \sum_{\mathrm{j}=1}^{\mathrm{m}} \boldsymbol{\eta}_{\mathbf{i},\mathrm{t}}, \quad \mathrm{i} = 1, \ldots, \mathrm{n}$$

to Step 4. Otherwise, give the m experts the anonymous summary of the earlier evaluations and the reasons for the evaluations, and asked them to provide a new round of evaluation data. Go to Step 2 .

Step 4. Let $\mathbf{e_i} = \overline{\mathbf{e}}_{\mathbf{1}}$ and $\boldsymbol{\sigma}_{\mathbf{i}} = \overline{\boldsymbol{\sigma}}_{\mathbf{1}}$ and $\boldsymbol{\eta}_{\mathbf{i}} = \overline{\boldsymbol{\eta}}_{\mathbf{1}}, \mathrm{i} = 1, \ldots, n$. The evaluations of the means and standard deviations and turnover rates of all securities are determined.

6. Case study

In order to illustrate the effectiveness and the behavior of our proposed models on portfolio selection, two numerical examples are presented in this section. Suppose an investor plans to adjust his capital four times during the investment among the six stocks chosen from Shanghai Stock Exchange: Stock 1 (code 600019); Stock 2 (code 600115); Stock 3 (code 600150); Stock 4 (code 600229); Stock 5 (code 600295) and Stock 6 (code 600398). Return rates, standard deviations and turnover rates are estimated by experts based on available information. Return rates and turnover rates are considered as normal uncertain variables ($\xi_{\mathrm{it}} \sim \mathrm{N}(\mathrm{e}_{\mathrm{it}}; \sigma_{\mathrm{it}})$ and $\xi_{\mathrm{it}} \sim \mathrm{N}(\eta_{\mathrm{i,t}}; \delta_{\mathrm{it}})$) and are given in **table 1 and table 2** respectively. The following results are obtained by the software LINGO.

Table 1. Uncertain normal return rates

	Stock 1	Stock 2	Stock 3
Period 1	N(0.0303, 0.0088)	N(0.0338, 0.0178)	N(0.0424, 0.0468)
Period 2	N(0.0465, 0.0252)	N(0.0565, 0.0367)	N(0.0530, 0.0516)
Period 3	N(0.0239, 0.0274)	N(0.0347, 0.0158)	N(0.0330, 0.0445)
Period 4	N(0.0648, 0.0297)	N(0.0894, 0.0687)	N(0.0675, 0.0294)
	Stock 4	**Stock 5**	**Stock 6**
Period 1	N(0.0402, 0.0123)	N(0.0524, 0.0994)	N(0.0386, 0.0498)
Period 2	N(0.0654, 0.0315)	N(0.0336, 0.0417)	N(0.0539, 0.0475)
Period 3	N(0.0545, 0.1185)	N(0.0024, 0.0954)	N(0.0386, 0.0074)
Period 4	N(0.0834, 0.0678)	N(0.0536, 0.0157)	N(0.1062, 0.1167)

Table 2. Uncertain normal turnover rates

	Stock 1	Stock 2	Stock 3
Period 1	$N(0.2300, 0.0022)$	$N(0.1201, 0.0065)$	$N(0.3276, 0.0033)$
Period 2	$N(0.2500, 0.0032)$	$N(0.1348, 0.0071)$	$N(0.2917, 0.0045)$
Period 3	$N(0.2750, 0.0045)$	$N(0.1436, 0.0057)$	$N(0.3056, 0.0037)$
Period 4	$N(0.2587, 0.0067)$	$N(0.1574, 0.0063)$	$N(0.2737, 0.0043)$
	Stock 4	Stock 5	Stock 6
Period 1	$N(0.3101, 0.0068)$	$N(0.1010, 0.0070)$	$N(0.1029, 0.0045)$
Period 2	$N(0.2879, 0.0093)$	$N(0.1376, 0.0085)$	$N(0.1250, 0.0065)$
Period 3	$N(0.2896, 0.0085)$	$N(0.1260, 0.0073)$	$N(0.1376, 0.0048)$
Period 4	$N(0.2659, 0.0082)$	$N(0.1525, 0.0072)$	$N(0.1530, 0.0053)$

Example 1. Assuming that the transaction cost rates of the six risky assets are $\mathbf{d}_{it} = 0.003; i = 1, 2, \ldots, 6; t = 1, 2, 3, 4$. Besides that, in the beginning of investment, investor has no security on hand. Therefore, we presumed that $x_{i0} = 0, i = 1, \ldots, 6$. We also set $\alpha = 0.95, \theta = 0.2, l_t = 0.1; t = 1, 2, 3, 4$ and $\beta_t = 1; t = 1, 2, 3, 4$. By investing 10000RMB yuan, the maximum expected total wealth, using model 4, is 12581.37 RMB yuan. We remark that the optimal portfolio, shown in Table 3 , is diversified.

Table 3. Optimal allocation of model (4) $(\%)$

	Stock 1	Stock 2	Stock 3	Stock 4	Stock 5	Stock 6
Period 1	0.57	5.37	71.05	12.30	4.73	0
Period 2	2.29	11.60	5.98	70.73	0.53	9.23
Period 3	0.03	11.60	1.48	30.63	0	56.26
Period 4	1.05	16.25	1.58	15.83	0.27	64.75

In order to emphasize the impact of the realistic constraints on model (4), we will consider two sub-portfolios. One without diversification constraints and the other without liquidity constraints. Numerical results showed that if we do not consider the diversification constraints, the optimal solution of the Mean-TVaR model is concentrated and expected wealth increses at 12694.83 RMB yuan. This means that the existence of diversification constraints in our optimization model minimizes the risk but also the total return.

To investigate the behavior of the terminal wealth as a function of diversification level, we will change the preset entropy value β_t on the interval $(0, \ln 6)$. The result is presented in figure 3 . We remark that terminal wealth decreases as a function of the diversification level.

Table 4. Optimal allocation of model 4 without diversification constraints $(\%)$

	Stock 1	Stock 2	Stock 3	Stock 4	Stock 5	Stock 6
Period 1	0	0	0	100	0	0
Period 2	0	0	0	100	0	0
Period 3	0	0	0	36.46	0	63.54
Period 4	0	0	0	0	0	100

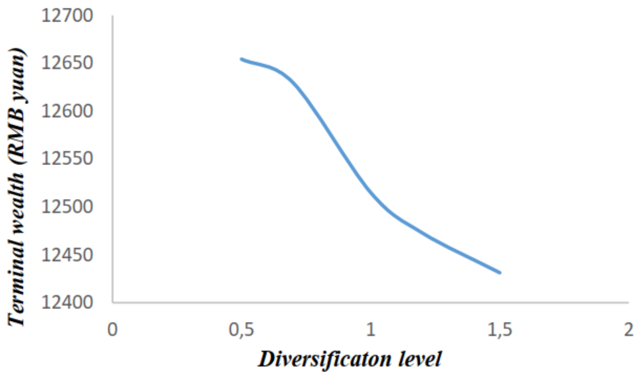

Figure 3. The behavior of the terminal wealth as a function of diversification level of model 4

It can be seen from Figure 3 that, using model 4 without liquidity constraints, when the turnover ratre increases, the investor gains less terminal expected wealth. As a particulate cas, in the absence of liquidity constraints, terminal wealth is 12515.11 RMB yuan, Table 5.

Table 5. Optimal allocation of model 4 without liquidity constraints (%)

	Stock 1	Stock 2	Stock 3	Stock 4	Stock 5	Stock 6
Period 1	16	7.02	16.70	38.01	18.82	3.45
Period 2	1.05	7.65	20.54	65.58	0	5.18
Period 3	0	8.55	7.57	18.85	0	65.03
Period 4	0.84	23.85	1.11	11.42	0.47	62.30

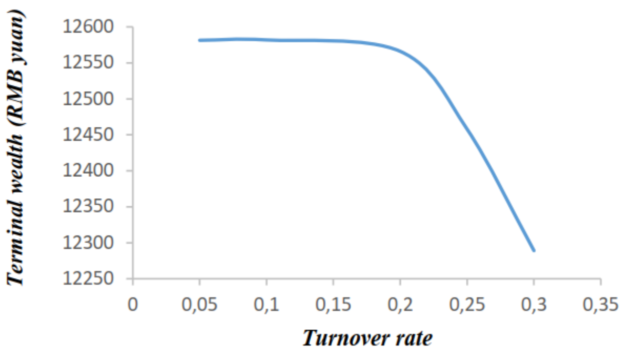

Figure 4. The behavior of the terminal wealth as a function of turnover rate of model 4

In order to further highlight the change of the optimal terminal expected wealth with the change of TVaR, we change the risk tolerant levels in model 4, with and without diversification and liquidity constraints, and obtain different optimal terminal expected wealth. We can see from Figure 5 that the bigger of risk tolerant level, the bigger terminal wealth of the optimal portfolio in the two cases, with diversification and liquidity constraints and without diversification and liquidity constraints.

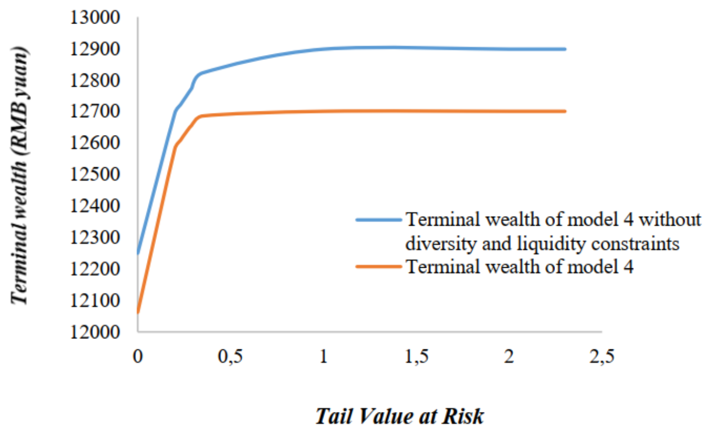

Figure 5. The efficient frontiers of model 4 with and without diversification and liquidity constraints

Example 2. Using the same data of the first example, supposing that an investor searches the optimal solution by apply model (5). We can see that when $\phi = 1.5$, terminal expected wealth is 12699.96 RMI (yuan). The optimal investment strategy for portfolio selection is displayed in table 6. To study the influence of ϕ on the optimal allocation, we will change risk aversion values into $(0, 20)$. Investment return values are shown in Figure 6. It is clear that terminal wealth decreases when risk aversion increases.

Table 6. Optimal allocation of model (5) when $\phi = 1.5$

	Stock 1	Stock 2	Stock 3	Stock 4	Stock 5	Stock 6
Period 1	0.80	2.08	6.45	12.24	70.51	7.92
Period 2	1.56	10.73	6.45	69.15	0.15	11.96
Period 3	1.57	10.73	6.37	69.15	0.15	12.02
Period 4	1.57	13.57	3.00	13.57	0.45	67.84

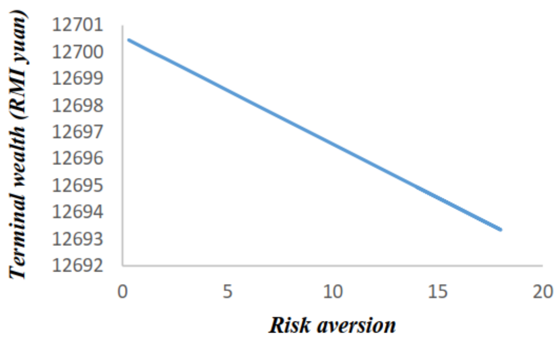

Figure 6. The behavior of the terminal wealth as a function of the preference coefficient of model (5)

7. Conclusion and perspectives

This article proposed three uncertain models for multi-period portfolio optimization. Compared with the existing works, we model the return and the risk of the investment using the uncertain mean value and the Tail Value at Risk, respectively. The security returns and turnover rates are assumed uncertain variables, and taking into account liquidity constraints, diversification constraints and transaction costs. In addition, theses models are transformed into a crisp mathematical problems under the assumption that the security returns and liquidity obey some uncertainty distribution forms. We adapted the Delphi method in order to evaluate the expected, the standard deviation and the turnover rates values of returns of the given securities. The effectiveness of the proposed models is illustrated by examples.

As a perspective of our research, we are interested in proposing multi-period optimization problems considering more general conditions, where security returns are other kinds of uncertain variables and considering other reality factors. We aim in developing algorithms for solving the problem in general cases.

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