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Grey Median Problem and Vertex Optimality

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Abstract The median problem is a basic model in location theory and transportation sciences. This problem deals with locating a facility on a network, to minimize the sum of weighted distances between the facility and the vertices of the network. In this paper, the cases that weights of vertices, edge lengths or both of them are grey numbers, are considered. For all these cases, we show that the set of vertices of network contains a solution of the median problem. This property is called vertex optimality. Median problem with grey parameters and its properties are first considered in this paper.

Keywords facility location, grey number, median problem

AMS 2010 subject classifications 90B80, 90C70

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1. Introduction

Location problems play important roles in operations research and transportation science. In the most of location models, the location of one or more facilities for servicing the clients should be found such that some criteria such as transportation cost, service times and etc are optimal. These models have many applications in the real world and attracted the attention of many authors. The main study in location problems on a network has begun by the paper of Hakimi [6] in 1964. He introduced two basic location models, namely median and center problems. In the median problem, the goal is locating a facility in the network to minimize the sum of weighted distances from this facility to the vertices of the network. The aim of the center problem is determining the location of a facility in the network such that the maximum weighted distance between clients on the vertices of the network and the facility is minimized. An important property for the median problem, namely vertex optimality, has been showed by Hakimi [6]. This property indicates that there is a solution of the median problem on the vertices of network. Since then many kind of facility location problems have been considered by researchers. For more details in location models and their applications see e.g. the books of Handler and Mirchandani [7], Mirchandani and Francis [16] and Daskin [2].

In the real world, there is many ambiguity and uncertainty and most of decisions are not made on the deterministic conditions. Based on this observation, in the last decades, many researchers have been considered the optimization models in uncertain environments such as fuzzy and grey systems. Among them fuzzy location problems have been considered by many authors. Canos et al. [1] proposed one of the first fuzzy algorithms for the fuzzy p-median problem. Mayoya and Verdegay [15] considered the p-median problem with fuzzy vertex weights and edge lengths. Perez et al. [20] and Taghi-Nezhad [21] showed that vertex optimality holds for the median problem with fuzzy parameters. There are many other researches on fuzzy location problems, see e.g. [19, 11, 25, 23, 22, 21, 24].

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Although location problems with fuzzy parameters have been investigated by many authors, these problems in grey systems have not been considered by any researcher. Optimization in grey environment is an applicable approach to accomplish the uncertainty and ambiguity in decision making. In a mathematical programming, uncertainty can be represented by grey numbers. The first work on applying grey numbers in mathematical programming is due to Huang et al. [8]. Then linear and integer programming models with grey parameters have been investigated by Huang and Moore [10] and Huang et al. [9], respectively. Since then several approaches have been extended to solve some cases of grey linear programming models (see e.g. [13, 18, 17, 3, 4]). Recently, a comprehensive survey on grey linear programming has been presented by Darvishi et al. [5].

This paper, deals with the grey median problem. We show that in the cases that the vertex weights, edge lengths or both of them are grey numbers, the vertex optimality holds.

In what follows of this paper, basic definition and some necessary concepts of grey systems are presented in Section 2. Section 3 contains median problem with gray parameters and showing vertex optimality for this problem. Finally the conclusion and suggestions for the further works are given in Section 4.

2. Grey numbers

This section contains some basic concepts of grey systems.

Definition 1
Suppose U is the universal set. A grey set X is defined by the maps $\underline{\mu}_X : U \to [0,1]$ and $\bar{\mu}_X : U \to [0,1]$, where for each $u \in U$, $\bar{\mu}_X(u) \geq \underline{\mu}_X(u)$. The maps $\underline{\mu}_X(u)$and $\bar{\mu}_X(u)$ are the lower and upper membership functions in U, respectively.

Note that in the case that $\bar{\mu}_X(u) = \underline{\mu}_X(u)$, then the grey set X converted to a fuzzy set.

Definition 2
A grey number which is denoted by $\otimes A \in [\underline{a}, \bar{a}] = \{t | \underline{a} \leq t \leq \bar{a}\}$ is a number with unknown position in an interval with clear bounds.

Definition 3
A grey number $\otimes A \in [\underline{a}, \bar{a}]$ is nonnegative, if $\underline{a} \geq 0$.

Definition 4
If $\underline{a} = \bar{a}$, then the grey number $\otimes A \in [\underline{a}, \bar{a}]$ is called white.

Definition 5
For a given grey number $\otimes A$, the notation $\tilde{\otimes} A$ is used for the white number of $\otimes A$ with the greatest probability. The process of evaluating appropriate value for $\tilde{\otimes} A$, is called whitenization of $\otimes A$.

The following basic whitenization method has been introduced by Liu and Lin [14].

Remark 1
The grey number $\otimes A$ can be evaluate by the whitening function $\tilde{\otimes} A = \delta \underline{a} + (1 - \delta)\bar{a}$, where $\delta \in [0,1]$ is whitening coefficient.

Definition 6
Let $\otimes A \in [\underline{a}, \bar{a}]$ and $\otimes B \in [\underline{b}, \bar{b}]$ be two grey numbers. Then

1. The main arithmetic operations for grey numbers, are defined as follow;

$$\otimes A + \otimes B \in [\underline{a} + \underline{b}, \bar{a} + \bar{b}],$$

$$\otimes A - \otimes B \in [\underline{a} - \bar{b}, \bar{a} - \underline{b}].$$

2. $\otimes A \leq \otimes B$ if $\bar{a} \leq \bar{b}$ and $\underline{a} \leq \underline{b}$

3. The median problem

Let $N = (V, E)$ be a given network. Where V and E are the sets of vertices and edges of N, respectively. Let $|V| = n$. Vertices are supposed to be the location of clients and to each vertex $v_j \in V$ a weight $w_j \geq 0$ is assigned, which is the demand of client on v_j. In the crisp case of the median problem, we should locate a facility x on the network N, to minimize the sum of weighted distances between x and all vertices in N. Therefore, the following objective function should be minimized:

$$f_1(x) = \sum_{j=1}^{n} w_j d(x, v_j),$$

where for any two points x and z in N, $d(x, z)$ is the length of minimum distance between points x and z.

An important property of the median problem which is shown by Hakimi [6] is vertex optimality:

Theorem 1
For a given network N, the median problem has a solution on the vertices of N.

Using this property, the median problem on a network could be easily solved by enumeration method with time complexity $O(n^2)$.

In the following of this section, we show that vertex optimality property holds for the cases that any demands of clients, edge lengths or both are grey numbers.

3.1. The problem with grey vertices weights

Assume that for $j = 1, ..., n$ the weight of vertex v_j is a nonnegative grey number $\otimes w_j \in [\underline{w_j}, \bar{w}_j]$. Then the median problem converted to the following:

$$min \otimes f_1(x) = \sum_{j=1}^{n} \otimes w_j d(x, v_j).$$

With definition of the whitening function, we say $m \in V$ is the median of network N, if for each vertex $v \in V$,

$$\tilde{\otimes} f_1(m) \leq \tilde{\otimes} f_1(v).$$

Theorem 2
The vertices of network N contains a solution of the median problem with nonnegative grey vertices weights.

Proof Let $y \notin V$ be a nonvertex of N. Assume $e_{ab} = (a, b)$ is the edge contains y, where $a, b \in V$ (see Figure 1). Let

$$V_1 = \{v \in V | d(y, v) = d(y, a) + d(a, v)\},$$
$$V_2 = V \setminus V_1.$$

Note that for each $u \in V_2$, $d(y, u) = d(y, b) + d(b, u)$, moreover $a \in V_1$ and $b \in V_2$.

We show either $\tilde{\otimes} f_1(a) \leq \tilde{\otimes} f_1(y)$ or $\tilde{\otimes} f_1(b) \leq \tilde{\otimes} f_1(y)$.

By definition of the median problem;

$$\begin{aligned} \otimes f_1(y) = \sum_{v_j \in V} \otimes w_j d(y, v_j) &= \sum_{v_j \in V_1} \otimes w_j d(y, v_j) + \sum_{v_j \in V_2} \otimes w_j d(y, v_j) \\ &= \sum_{v_j \in V_1} \otimes w_j (d(y, a) + d(a, v_j)) + \sum_{v_j \in V_2} \otimes w_j (d(y, b) + d(b, v_j)) \end{aligned}$$

$$= d(y, a) \sum_{v_j \in V_1} \otimes w_j + \sum_{v_j \in V_1} \otimes w_j d(a, v_j) + d(y, b) \sum_{v_j \in V_2} \otimes w_j + \sum_{v_j \in V_2} \otimes w_j d(b, v_j). \quad (1)$$

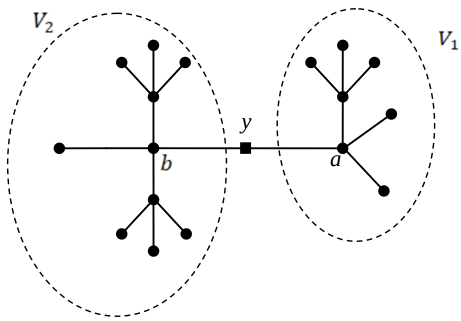

Figure 1. The sets V_1 and V_2.

Now, by whitening function let $\tilde{\otimes} w_j = \delta \underline{w_j} + (1-\delta)\bar{w}_j$, for $j = 1, ..., n$, where $\delta \in [0,1]$. Then for $i = 1, 2$

$$\sum_{v_j \in V_i} \tilde{\otimes} w_j = \sum_{v_j \in V_i} (\delta \underline{w_j} + (1-\delta)\bar{w}_j) = \delta \sum_{v_j \in V_i} \underline{w_j} + (1-\delta) \sum_{v_j \in V_i} \bar{w}_j.$$

Let for $i = 1, 2$, $\bar{W}_i = \sum_{v_j \in V_i} \bar{w}_j$ and $\underline{W_i} = \sum_{v_j \in V_i} \underline{w_j}$. Thus

$$\sum_{v_j \in V_i} \tilde{\otimes} w_j = \delta \underline{W_i} + (1-\delta)\bar{W}_i.$$

Two cases may happen:

1. $\delta \underline{W_1} + (1-\delta)\bar{W}_1 \leq \delta \underline{W_2} + (1-\delta)\bar{W}_2$.
2. $\delta \underline{W_2} + (1-\delta)\bar{W}_2 < \delta \underline{W_1} + (1-\delta)\bar{W}_1$.

We follow the proof in the first case. The proof in the second case, is the same. Using (1) we obtain

$$\begin{aligned}
\tilde{\otimes} f_1(y) &= (d(b,a) - d(b,y)) \sum_{v_j \in V_1} \tilde{\otimes} w_j + \sum_{v_j \in V_1} \tilde{\otimes} w_j (d(b,v_j) - d(b,a)) \\
&\quad + d(y,b) \sum_{v_j \in V_2} \tilde{\otimes} w_j + \sum_{v_j \in V_2} \tilde{\otimes} w_j d(b,v_j) \\
&= (-d(b,y))(\delta \underline{W_1} + (1-\delta)\bar{W}_1) + \sum_{v_j \in V_1} \tilde{\otimes} w_j d(b,v_j) \\
&\quad + d(b,y)(\delta \underline{W_2} + (1-\delta)\bar{W}_2) + \sum_{v_j \in V_2} \tilde{\otimes} w_j d(b,v_j) \\
&\geq \sum_{v_j \in V} \tilde{\otimes} w_j d(b,v_j) = \tilde{\otimes} f_1(b).
\end{aligned}$$

□

3.2. The problem with grey edge lengths

Now for any edge $e_{uv} \in E$, let the length of e_{uv} be a nonnegative grey number $\otimes l_{uv} \in [\underline{l_{uv}}, \bar{l_{uv}}]$. Then the length of any path between two vertices of the network is also a grey number and defined as follows;

Definition 7
Suppose P is a path between two vertices a and b in V. Then we define the length of P as

$$\otimes L_P = \sum_{e_i \in P} \otimes l_i.$$

For the inner points of an edge the distance to the vertices can be defined as follow.

Definition 8
Let $e_{uv} \in E$ and x is an inner point on this edge. Then the distance between x and vertex u is defined as follow;

$$\otimes l_{ux} = \lambda \otimes l_{uv}, \ \lambda \in [0, 1].$$

Note that in the crisp case, the length of shortest path between any two points x and y in N, is denoted by $d(x, y)$. However, in the grey systems, since in some cases the grey numbers may not be comparable with second part of Definition 6. For example $\otimes[0, 3] \leq \otimes[1, 4]$, but we can not directly compare two grey numbers $\otimes[0, 4]$ and $\otimes[1, 2]$ by using Definition 6. Therefore, in a network may exist two paths between two vertices that can not compare their lengths. Thus the shortest path is meaningless. So, we consider the whitening function and assume $\tilde{\otimes} d(x, y)$ is the shortest whited distances between x and y. If the underlying network is a tree, then there is just one path between any two vertices. Therefore, in this case the notation $\otimes d(x, y)$ can be used.

The median problem with grey edge lengths is minimizing the following function;

$$\tilde{\otimes} f_2(x) = \sum_{j=1}^{n} w_j \tilde{\otimes} d(x, v_j).$$

Theorem 3
There is a vertex of network N, which is a solution of the median problem with grey edge lengths.

Proof The same as proof of Theorem 2, let $y \notin V$ be a nonvertex of N and $e_{ab} = (a, b)$ be the edge which contains x, where $a, b \in V$. Let

$$V_1 = \{u \in V | \tilde{\otimes} d(y, u) = \tilde{\otimes} d(y, a) + \tilde{\otimes} d(a, u)\},$$

$$V_2 = V \setminus V_1.$$

Then $a \in V_1$ and $b \in V_2$. We observe

$$\begin{aligned} \tilde{\otimes} f_2(y) &= \sum_{u_j \in V} w_j \tilde{\otimes} d(y, u_j) = \sum_{u_j \in V_1} w_j \tilde{\otimes} d(y, u_j) + \sum_{u_j \in V_2} w_j \tilde{\otimes} d(y, u_j) \\ &= \sum_{u_j \in V_1} w_j (\tilde{\otimes} d(y, a) + \tilde{\otimes} d(a, u_j)) + \sum_{u_j \in V_2} w_j (\tilde{\otimes} d(y, b) + \tilde{\otimes} d(b, u_j)) \end{aligned}$$

$$= \tilde{\otimes} d(y, a) \sum_{u_j \in V_1} w_j + \sum_{u_j \in V_1} w_j \tilde{\otimes} d(a, u_j) + \tilde{\otimes} d(y, b) \sum_{u_j \in V_2} w_j + \sum_{u_j \in V_2} w_j \tilde{\otimes} d(b, u_j). \tag{2}$$

Without loss of generality, let $\sum_{u_j \in V_1} w_j \leq \sum_{u_j \in V_2} w_j$. Then by definitions 7 and 8,

$$\tilde{\otimes} d(b, a) = \tilde{\otimes} d(b, y) + \tilde{\otimes} d(y, a),$$

and for any vertex $u_j \in V_1$

$$\tilde{\otimes} d(b, u_j) = \tilde{\otimes} d(b, a) + \tilde{\otimes} d(a, u_j).$$

Using (2)

$$\begin{aligned}\tilde{\otimes} f(y) &= (\tilde{\otimes} d(b,a) - \tilde{\otimes} d(b,y)) \sum_{u_j \in V_1} w_j + \sum_{u_j \in V_1} w_j (\tilde{\otimes} d(b,u_j) - \tilde{\otimes} d(b,a)) \\ &\quad + \tilde{\otimes} d(y,b) \sum_{u_j \in V_2} w_j + \sum_{u_j \in V_2} w_j \tilde{\otimes} d(b,u_j) \\ &= \tilde{\otimes} d(b,y) (\sum_{u_j \in V_2} w_j - \sum_{u_j \in V_1} w_j) + \sum_{u_j \in V_1} w_j \tilde{\otimes} d(b,u_j) + \sum_{u_j \in V_2} w_j \tilde{\otimes} d(b,u_j) \\ &\geq \sum_{u_j \in V} w_j \tilde{\otimes} d(b,u_j) = \tilde{\otimes} f_2(b).\end{aligned}$$

□

3.3. *The problem with grey parameters*

Let the weights of vertices and edge lengths of the given network be nonnegative gery numbers. Suppose the notations of grey weights, edge lengths, length of a path and distances are defined the same as Sections 3.1 and 3.2. Then consider the following 1-median problem on a network with grey weights of vertices and edge lengths,

$$min \otimes f_3(x) = \sum_{j=1}^{n} \otimes w_j \tilde{\otimes} d(x, v_j).$$

With the combination technique of theorems 2 and 3, the vertex optimality property can be shown for this case.

Theorem 4
There is a solution on the vertices of the network N for the median problem with grey parameters.

4. Conclusion

Median location is a basic problem in facility location theory. Since in real life problems there exist many ambiguous and uncertainty, then the grey systems can be helped for modeling location problems in the real situations. In this paper, we considered a network with grey vertex weights and edge lengths. We showed that vertex optimality holds for the median problem with grey parameters. This property indicates that there is a solution of grey median problem on the vertices of network.

Other kinds of location problems such as continuous location, covering and center problems with grey parameters can be considered in the future works.

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