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The Marshall-Olkin-Topp-Leone-Gompertz-G Family of Distributions with Applications

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Abstract A new family of distributions called the Marshall-Olkin-Topp-Leone-Gompertz-G (MO-TL-Gom-G) distribution is developed and studied in detail. Some mathematical and statistical properties of the new family of distributions are explored. Statistical properties of the new family of distributions considered are the quantile function, moments and generating function, probability weighted moments, distribution of the order statistics and Rényi entropy. The maximum likelihood technique is used for estimating the model parameters and Monte Carlo simulation is conducted to examine the performance of the model. Finally, we give examples of real-life data applications to show the usefulness of the above mentioned Topp-Leone-Gompertz generalization.

Keywords Marshall-Olkin-G, Topp-Leone-Gompertz-G, Maximum Likelihood Estimation

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1. Introduction

There are many generalizations of the Gompertz [16] distribution in the literature. These new generalizations are used for modeling data in several areas including reliability, economics, finance, and hydrology among others. The importance of generalizing those distributions is that they provide a better fit than the existing ones in modeling real-life data. Some examples of those generalized distributions include, the Marshall-Olkin-Gompertz-G family of distributions by Chipepa and Oluyede [12], the Marshall-Olkin extended generalized Gompertz distribution by Lazhar [23], Topp-Leone Gompertz distribution by Nzei et al. [29] and the Gompertz-G family of distributions by Alizadeh et al. [2].

Some Topp-Leone generalizations in the literature including the Marshall-Olkin Topp-Leone half-logistic-G family of distributions by Sengweni et al. [33], Topp-Leone odd Burr III-G family of distributions by Moakofi et al. [28] and Topp-Leone odd Burr X-G Family of distributions by Oluyede et al. [31].

There are very useful generalizations of distributions in the literature via the Marshall-Olkin generator [27]. Some of those generalizations include the beta Marshall-Olkin-G (BMO-G) distribution by Alizadeh et al. [3], Kumaraswamy Marshall-Olkin-G (KwMO-G) distribution by Alizadeh et al. [4] and Marshall-Olkin Half Logistic-G (MOHL-G) distribution by Makubate et al. [26]. A new class of distributions called the Marshall-Olkin Log-logistic Extended Weibull (MOLLEW) family of distributions was proposed by Lepetu et al. [25]. Chakraborty and Handique [9] presented the generalized Marshall-Olkin Kumaraswamy-G distribution, and the ratio and inverse

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moments of Marshall-Olkin extended Burr Type III distribution based on lower generalized order statistics was proposed by Kumar [22].

The main motivations for developing the MO-TL-Gom-G family of distributions are as follows:

- To model data with heavy tails,
- To obtain flexible distributions for modeling various types of data in practice, and
- To develop a family of distributions that has both monotonic and non-monotonic hazard rate functions including bathtub and upside-down bathtub shapes.

This paper is outlined as follows: In Section 2, we define our new model, Marshall-Olkin-Topp-Leone-Gompertz-G (MO-TL-Gom-G) family of distributions and its sub-models. Section 3 contains some statistical properties of the MO-TL-Gom-G family of distributions including expansion of the probability density function, hazard rate and quantile functions, moments, generating functions, stochastic orders, probability weighted moments, order statistics and Rényi entropy. Maximum likelihood estimates of the model parameters are given in Section 4. Examples of the special cases of the new family of distributions are given in Section 5. A Monte Carlo simulation study to examine the bias and mean square error of the maximum likelihood estimates is presented in Section 6. Section 7 contains applications of the new model to two real data sets. And lastly, concluding remarks are given in Section 8.

2. The Model

In this section, we present the proposed model and its sub-models. This distribution offers flexibility in capturing the skewness present in real-world data, allowing for accurate modeling of a wide range of data. The new distribution can model data that are right-skewed, left-skewed and symmetric. The Topp-Leone Gompertz-G (TL-Gom-G) distribution was introduced by Oluyede et al. [30] with the cumulative distribution function (cdf) and probability density function (pdf) given by

$$F_{TL-Gom-G}(x;b,\theta,\xi) = \left[1-\left(\exp\left[\frac{1}{\theta}(1-(1-G(x;\xi))^{-\theta})\right]\right)^2\right]^b \tag{1}$$

and

$$\begin{aligned} f_{TL-Gom-G}(x;b,\theta,\xi) &= 2bg(x;\xi)(1-G(x;\xi))^{-\theta-1}\left(\exp\left[\frac{1}{\theta}(1-(1-G(x;\xi))^{-\theta})\right]\right)^2 \\ &\times \left[1-\left(\exp\left[\frac{1}{\theta}(1-(1-G(x;\xi))^{-\theta})\right]\right)^2\right]^{b-1}, \end{aligned} \tag{2}$$

respectively, for $x > 0$, $b, \theta > 0$ and parameter vector ξ. Marshall and Olkin [27] developed the distribution with the cdf and pdf given by

$$F_{MO-G}(x;\delta,\xi) = 1-\frac{\delta\bar{G}(x;\xi)}{1-\bar{\delta}\bar{G}(x;\xi)} \tag{3}$$

and

$$f_{MO-G}(x;\delta,\xi) = \frac{\delta g(x;\xi)}{[1-\bar{\delta}\bar{G}(x;\xi)]^2}, \tag{4}$$

respectively, where δ is the tilt parameter, $\bar{\delta} = 1 - \delta$ and $G(x;\xi)$ is the baseline cdf which depends on the parameter vector ξ. Considering the generalization by Marshall and Olkin and inserting equations (1) and (2) into equations (3) and (4) respectively, we obtain the MO-TL-Gom-G family of distributions with the cdf and pdf given by

$$F_{MO-TL-Gom-G}(x;\delta,b,\theta,\xi) = 1 - \frac{\delta\left(1-\left[1-\left(\exp\left[\frac{1}{\theta}(1-(1-G(x;\xi))^{-\theta})\right]\right)^2\right]^b\right)}{1-\bar{\delta}\left(1-\left[1-\left(\exp\left[\frac{1}{\theta}(1-(1-G(x;\xi))^{-\theta})\right]\right)^2\right]^b\right)} \tag{5}$$

and

$$\begin{aligned} f_{MO-TL-Gom-G}(x;\delta,b,\theta,\xi) &= 2b\delta g(x;\xi)(1-G(x;\xi))^{-\theta-1}\left(\exp\left[\frac{1}{\theta}(1-(1-G(x;\xi))^{-\theta})\right]\right)^2 \\ &\times \left[1-\left(\exp\left[\frac{1}{\theta}(1-(1-G(x;\xi))^{-\theta})\right]\right)^2\right]^{b-1} \\ &\times \left(1-\bar{\delta}\left(1-\left[1-\left(\exp\left[\frac{1}{\theta}(1-(1-G(x;\xi))^{-\theta})\right]\right)^2\right]^b\right)\right)^{-2}, \end{aligned} \tag{6}$$

for $\delta, b, \theta > 0$ and parameter vector ξ.

2.1. Sub-Models

In this subsection, we present some of the sub-models of the MO-TL-Gom-G family of distributions.

- When $\delta = 1$, we obtain the TL-Gom-G family of distributions with the cdf given by

$$F(x;b,\theta,\xi) = \left[1-\left(\exp\left[\frac{1}{\theta}(1-(1-G(x;\xi))^{-\theta})\right]\right)^2\right]^b,$$

  for $b, \theta > 0$, and parameter vector ξ. See Oluyede et al. [30] for additional details.
- When $b = 1$, we obtain the distribution with the cdf given by

$$F(x;\delta,\theta,\xi) = 1 - \frac{\delta\left(1-\left(\exp\left[\frac{1}{\theta}(1-(1-G(x;\xi))^{-\theta})\right]\right)^2\right)}{1-\bar{\delta}\left(1-\left(\exp\left[\frac{1}{\theta}(1-(1-G(x;\xi))^{-\theta})\right]\right)^2\right)},$$

  for $\delta, \theta > 0$, and parameter vector ξ.
- When $\delta = b = 1$, we get the distribution with the cdf given by

$$F(x;\theta,\xi) = 1 - \left(\exp\left[\frac{1}{\theta}(1-(1-G(x;\xi))^{-\theta})\right]\right)^2$$

  for $\theta > 0$ and parameter vector ξ.

In addition, several new distributions can be obtained when the baseline cdf $G(x;\xi)$ is specified.

3. Some Statistical Properties

In this section, we present the expansion of the density function, hazard rate and quantile functions, moments, generating function, stochastic orders, probability weighted moments, distribution of order statistics and Rényi entropy of the MO-TL-Gom-G family of distributions.

3.1. Expansion of the Density Function

We apply the general results by Barreto-Souza et al. [7] to obtain the linear representations of the MO-TL-Gom-G family of distributions in this subsection. Equation (6) can be written as

$$f_{MO-TL-Gom-G}(x;\delta,b,\theta,\xi) \quad = \quad \frac{f_{TL-Gom-G}(x;b,\theta,\xi)}{\delta\left[1-\frac{\delta-1}{\delta}F_{TL-Gom-G}(x;b,\theta,\xi)\right]^2}, \tag{7}$$

where $f_{TL-Gom-G}(x;b,\theta,\xi)$ and $F_{TL-Gom-G}(x;b,\theta,\xi)$ are given in equations (2) and (1), respectively.

If $\delta\in(0,1)$, we obtain the pdf of the MO-TL-Gom-G family of distributions as

$$\begin{aligned} f_{MO-TL-Gom-G}(x;\delta,b,\theta,\xi) &= 2b\sum_{j,i,k,l,m=0}^{\infty}\sum_{q=0}^{j}\frac{(-1)^{i+l+m}(2i+2)^k}{\theta^k k!(m+1)}\gamma_{j,q}\binom{b(j-q+1)-1}{i} \\ &\times \binom{-\theta(l+1)-1}{m}\binom{k}{l}(m+1)g(x;\xi)G^m(x;\xi) \\ &= \sum_{m=0}^{\infty}V_{m+1}h_{(m+1)}(x;\xi), \end{aligned} \tag{8}$$

where

$$V_{m+1} \quad = \quad 2b\sum_{j,i,k,l=0}^{\infty}\sum_{q=0}^{j}\frac{(-1)^{i+l+m}(2i+2)^k}{\theta^k k!(m+1)}\gamma_{j,q}\binom{b(j-q+1)-1}{i}\binom{-\theta(l+1)-1}{m}\binom{k}{l}, \tag{9}$$

and $h_{(m+1)}(x;\xi)=(m+1)g(x;\xi)G^m(x;\xi)$. It follows that for $\delta\in(0,1)$, the MO-TL-Gom-G family of distributions can be expressed as a linear combination of the exponentiated-G (Exp-G) densities with power parameter (m+1).

For $\delta>1$, the series representation of the pdf of the MO-TL-Gom-G family of distributions is given by

$$\begin{aligned} f_{MO-TL-Gom-G}(x;\delta,b,\theta,\xi) &= 2b\sum_{j,i,k,l,q=0}^{\infty}\frac{(-1)^{i+l+q}(2i+2)^k}{\theta^k k!(q+1)}e_j\binom{b(j+1)-1}{i} \\ &\times \binom{-\theta(l+1)-1}{q}\binom{k}{l}(q+1)g(x;\xi)G^q(x;\xi) \\ &= \sum_{q=0}^{\infty}W_{q+1}h_{(q+1)}(x;\xi), \end{aligned} \tag{10}$$

where

$$W_{q+1} \quad = \quad 2b\sum_{j,i,k,l=0}^{\infty}\frac{(-1)^{i+l+q}(2i+2)^k}{\theta^k k!(q+1)}e_j\binom{b(j+1)-1}{i}\binom{-\theta(l+1)-1}{q}\binom{k}{l}, \tag{11}$$

and $h_{(q+1)}(x;\xi)=(q+1)g(x;\xi)G^q(x;\xi)$. Again, for $\delta>1$, the MO-TL-Gom-G family of distributions can be expressed as an infinite linear combination of the Exp-G densities with power parameter (q+1). *See Appendix section for more details of the derivations.*

3.2. Hazard Rate and Quantile Functions

In this subsection, we present the hazard rate and quantile functions of the MO-TL-Gom-G family of distributions. The hazard rate function (hrf) is given by

$$\begin{aligned} h_{MO-TL-Gom-G}(x;\delta,b,\theta,\xi) &= 2b\delta g(x;\xi)(1-G(x;\xi))^{-\theta-1}\left(\exp\left[\frac{1}{\theta}(1-(1-G(x;\xi))^{-\theta})\right]\right) \\ &\times \left[1-\left(\exp\left[\frac{1}{\theta}(1-(1-G(x;\xi))^{-\theta})\right]\right)^2\right]^{b-1} \\ &\times \left(1-\bar{\delta}\left(1-\left[1-\left(\exp\left[\frac{1}{\theta}(1-(1-G(x;\xi))^{-\theta})\right]\right)^2\right]^b\right)\right)^{-2} \\ &\times \left(\frac{\delta\left(1-\left[1-\left(\exp\left[\frac{1}{\theta}(1-(1-G(x;\xi))^{-\theta})\right]\right)^2\right]^b\right)}{1-\bar{\delta}\left(1-\left[1-\left(\exp\left[\frac{1}{\theta}(1-(1-G(x;\xi))^{-\theta})\right]\right)^2\right]^b\right)}\right)^{-1}. \end{aligned}$$

We obtain the quantile function of the MO-TL-Gom-G family of distributions by inverting the non-linear equation

$$1-\frac{\delta\left(1-\left[1-\left(\exp\left[\frac{1}{\theta}(1-(1-G(x;\xi))^{-\theta})\right]\right)^2\right]^b\right)}{1-\bar{\delta}\left(1-\left[1-\left(\exp\left[\frac{1}{\theta}(1-(1-G(x;\xi))^{-\theta})\right]\right)^2\right]^b\right)}=u,$$

for $0\leqslant u\leqslant 1$, so that

$$\ln\left[1-\left(1-\left(\frac{(1-u)}{\delta+(1-u)\bar{\delta}}\right)^{\frac{1}{b}}\right)^{\frac{1}{2}}\right]=\frac{1}{\theta}(1-(1-G(x;\xi))^{-\theta}).$$

Consequently, the quantile function of the MO-TL-Gom-G family of distributions is given by

$$Q_G(u)=G^{-1}\left[1-\left(1-\left(\theta\ln\left[1-\left(1-\frac{(1-u)}{\delta+(1-u)\bar{\delta}}\right)^{\frac{1}{b}}\right]^{\frac{1}{2}}\right)\right)^{-\frac{1}{\theta}}\right]. \quad (12)$$

3.3. Moments and Generating Function

In this subsection, we present the moments and the generating function of the MO-TL-Gom-G family of distributions.

- For $\delta\in(0,1)$, we obtain the n^{th} moment of the MO-TL-Gom-G family of distributions as

$$E(X^n)=\int_{-\infty}^{\infty}x^n f_{MO-TL-Gom-G}(x;\delta,b,\theta,\xi)dx=\sum_{m=0}^{\infty}V_{m+1}E(Y_{m+1}^n), \quad (13)$$

where $E(Y_{m+1}^n)$ is the n^{th} moment of Y_{m+1} which follows an Exp-G distribution with power parameter (m+1) and V_{m+1} is given in equation (9).

- For $\delta > 1$, the n^{th} moment of the MO-TL-Gom-G family of distributions is given by

$$E(X^{^n}) = \int_{-\infty}^{\infty} x^n f_{_{MO-TL-Gom-G}}(x;\delta,b,\theta,\xi)dx = \sum_{q=0}^{\infty} W_{q+1} E(Y_{q+1}^{^n}), \tag{14}$$

  where $E(Y_{q+1}^{^n})$ is the n^{th} moment of Y_{q+1} which follows an Exp-G distribution with power parameter (q+1) and W_{q+1} is given in equation (11). The generating function of the MO-TL-Gom-G family of distributions can be obtained as follows:
- For $\delta \in (0,1)$, the moment generating function (MGF) of the MO-TL-Gom-G family of distributions is

$$M_X(h) = E\left(e^{hX}\right) = \sum_{m=0}^{\infty} V_{m+1} E\left(e^{hY_{m+1}}\right),$$

  where $E\left(e^{hY_{m+1}}\right)$ is the mgf of the Exp-G family of distributions with power parameter (m+1).
- For $\delta > 1$, the MGF of the MO-TL-Gom-G family of distributions is

$$M_X(h) = E\left(e^{hX}\right) = \sum_{q=0}^{\infty} W_{q+1} E\left(e^{hY_{q+1}}\right),$$

  where $E\left(e^{hY_{q+1}}\right)$ is the mgf of the Exp-G family of distributions with power parameter (q+1).

3.4. Stochastic Ordering

Stochastic orders for the MO-TL-Gom-G family of distributions are presented in this subsection. Suppose we have two random variables Z and T with distribution functions $F_Z(r)$ and $F_T(r)$, respectively, and $\overline{F}_Z(r) = 1 - F_Z(r)$ the survival function. Note that Z is stochastically smaller than T if $\overline{F}_Z(r) \leq \overline{F}_T(r)$ for all r or $F_Z(r) \geq F_T(r)$ for all r. This is denoted by $Z <_s T$. Hazard rate order and likelihood ratio order are stronger and are given by $Z <_{hr} T$ if $h_Z(r) \geq h_T(r)$ for all r, and $Z <_{lr} T$ if $\frac{f_Z(r)}{f_T(r)}$ is decreasing in r, (Shaked and Shanthikumar [34]). We know that $Z <_{lr} T \Rightarrow Z <_{hr} T \Rightarrow Z <_s T$.

Suppose we let X_1 and X_2 to be two independent random variables following MO-TL-Gom-G$(\delta_1, b, \theta, \xi)$ and MO-TL-Gom-G $(\delta_2, b, \theta, \xi)$ distributions, then

$$\begin{aligned} f_1(x;\delta_1,b,\theta,\xi) &= 2b\delta_1 g(x;\xi)(1-G(x;\xi))^{-\theta-1}\left(\exp\left[\frac{1}{\theta}(1-(1-G(x;\xi))^{-\theta})\right]\right)^2 \\ &\times \left[1-\left(\exp\left[\frac{1}{\theta}(1-(1-G(x;\xi))^{-\theta})\right]\right)^2\right]^{b-1} \\ &\times \left(1-\bar{\delta_1}\left(1-\left[1-\left(\exp\left[\frac{1}{\theta}(1-(1-G(x;\xi))^{-\theta})\right]\right)^2\right]^b\right)\right)^{-2}, \end{aligned}$$

and

$$\begin{aligned} f_2(x;\delta_2,b,\theta,\xi) &= 2b\delta_2 g(x;\xi)(1-G(x;\xi))^{-\theta-1}\left(\exp\left[\frac{1}{\theta}(1-(1-G(x;\xi))^{-\theta})\right]\right)^2 \\ &\times \left[1-\left(\exp\left[\frac{1}{\theta}(1-(1-G(x;\xi))^{-\theta})\right]\right)^2\right]^{b-1} \\ &\times \left(1-\bar{\delta_2}\left(1-\left[1-\left(\exp\left[\frac{1}{\theta}(1-(1-G(x;\xi))^{-\theta})\right]\right)^2\right]^b\right)\right)^{-2}. \end{aligned}$$

Note that

$$\frac{f_1(x;\delta_1,b,\theta,\xi)}{f_2(x;\delta_2,b,\theta,\xi)} = \frac{\delta_1}{\delta_2}\left(\frac{1-\bar{\delta_1}\left(1-\left[1-\left(\exp\left[\frac{1}{\theta}(1-(1-G(x;\xi))^{-\theta})\right]\right)^2\right]^b\right)}{1-\bar{\delta_2}\left(1-\left[1-\left(\exp\left[\frac{1}{\theta}(1-(1-G(x;\xi))^{-\theta})\right]\right)^2\right]^b\right)}\right)^{-2}, \tag{15}$$

and upon differentiating equation (15) with respect to x, we get

$$\begin{aligned} \frac{d}{dx}\left(\frac{f_1(x;\delta_1,b,\theta,\xi)}{f_2(x;\delta_2,b,\theta,\xi)}\right) &= \frac{1-\bar{\delta_2}\left(1-\left[1-\left(\exp\left[\frac{1}{\theta}(1-(1-G(x;\xi))^{-\theta})\right]\right)^2\right]^b\right)}{\left[1-\bar{\delta_1}\left(1-\left[1-\left(\exp\left[\frac{1}{\theta}(1-(1-G(x;\xi))^{-\theta})\right]\right)^2\right]^b\right)\right]^3} \\ &\times \frac{\delta_1}{\delta_2}(\delta_1-\delta_2)\left[1-\left(\exp\left[\frac{1}{\theta}(1-(1-G(x;\xi))^{-\theta})\right]\right)^2\right]^{b-1} \\ &\times 2bg(x;\xi)(1-G(x;\xi))^{-\theta-1}\exp\left[\frac{1}{\theta}(1-(1-G(x;\xi))^{-\theta})\right] \end{aligned} \tag{16}$$

which is ≤ 0 if $\delta_1 \leq \delta_2$. Therefore, $X_1 <_{lr} X_2, X_1 <_{hr} X_2$ and $X_1 <_s X_2$. This proofs that the random variables X_1 and X_2 are stochastically ordered.

3.5. *Probability Weighted Moments*

The probability weighted moments (PWMs) for the MO-TL-Gom-G family of distributions are presented in this subsection. The PWMs were introduced by Greenwood et al. [17]. The PWMs for a random variable X following the MO-TL-Gom-G distributions is given by

$$\eta_{a,r} = E(X^a[F(X)]^r) = \int_{-\infty}^{\infty} x^a f_{MO-TL-Gom-G}(x)[F_{MO-TL-Gom-G}(x)]^r dx. \tag{17}$$

If $\delta \in (0,1)$, we can write

$$f(x)[F(x)]^r = \sum_{m=0}^{\infty} V^*_{m+1} g^*_{(m+1)}(x;\xi), \tag{18}$$

where

$$V^*_{m+1} = 2b \sum_{j,r,i,k,l=0}^{\infty} \sum_{q=0}^{j} \frac{(-1)^{i+l+m}(2i+2)^k}{\theta^k k!(m+1)} B_{j,r,q} \binom{b(j+r-q+1)-1}{i} \binom{-\theta(l+1)-1}{m} \binom{k}{l},$$

and $g^*_{(m+1)}(x;\xi) = (m+1)g(x;\xi)G^m(x;\xi)$ is the Exp-G density with the power parameter (m+1) and parameter vector ξ. Therefore, the PWMs of the MO-TL-Gom-G family of distributions is given by

$$\eta_{a,r} = \sum_{m=0}^{\infty} V^*_{m+1} \int_{-\infty}^{\infty} x^a g^*_{(m+1)}(x;\xi)dx. \tag{19}$$

Note that for $\delta > 1$, we have

$$f(x)[F(x)]^r = \sum_{m=0}^{\infty} a^*_{m+1} g^*_{(m+1)}(x;\xi),$$

where

$$a^*_{m+1} = 2b \sum_{j,r,i,k,l=0}^{\infty} \frac{(-1)^{i+l+m}(2i+2)^k}{\theta^k k!(m+1)} A_{j,r} \binom{b(j+r+1)-1}{i} \binom{-\theta(l+1)-1}{m} \binom{k}{l},$$

and $g^*_{(m+1)}(x;\xi) = (m+1)g(x;\xi)G^m(x;\xi)$ is the Exp-G density with the power parameter (m+1) and parameter vector ξ. Therefore, the PWMs of the MO-TL-Gom-G family of distribution is given by

$$\eta_{a,r} = \sum_{m=0}^{\infty} a^*_{m+1} \int_{-\infty}^{\infty} x^a g^*_{(m+1)}(x;\xi)dx. \tag{20}$$

Derivations follow in the Appendix section.

3.6. Distribution of Order Statistics and Rényi Entropy

In this subsection, we present the pdf of the i^{th} order statistic and Rényi entropy for the MO-TL-Gom-G family of distributions.

3.6.1. Distribution of Order Statistics Let $X_1, ...X_n$ be independent identically distributed (iid) sample from MO-TL-Gom-G family of distributions, then the pdf of the i^{th} order statistic is given by

$$f_{i:n}(x) = \delta n! f_{TL-Gom-G}(x;b,\theta,\xi) \sum_{r=0}^{n-i} \frac{(-1)^r}{(i-1)!(n-i)!} \frac{F^{r+i-1}_{TL-Gom-G}(x;b,\theta,\xi)}{\left[1-\bar{\delta}F_{TL-Gom-G}(x;b,\theta,\xi)\right]^{r+i-1}}.$$

For $\delta \in (0,1)$, we obtain the pdf of the i^{th} order statistic as

$$f_{i:n}(x) = \sum_{m=0}^{\infty} \Omega_{m+1} g^*_{(m+1)}(x;\xi), \tag{21}$$

where

$$\Omega_{m+1} = 2b \sum_{j,q,p=0}^{\infty} \sum_{r=0}^{n-i} \sum_{k=0}^{j} \frac{(-1)^{q+l+m}(2q+2)^p}{\theta^p p!(m+1)} \omega_{j,r,k} \binom{b(j+r-k+i)-1}{q} \binom{-\theta(l+1)-1}{m} \binom{p}{l},$$

and $g^*_{(m+1)}(x;\xi) = (m+1)g(x;\xi)\left[G(x;\xi)\right]^m$ is the Exp-G pdf with power parameter (m+1) and parameter vector ξ.

If $\delta > 1$, we have

$$f_{i:n}(x) = \sum_{m=0}^{\infty} D_{m+1} g^*_{(m+1)}(x;\xi), \tag{22}$$

where $g^*_{(m+1)}(x;\xi) = (m+1)g(x;\xi)G^m(x;\xi)$ and

$$D_{m+1} = 2b \sum_{j,q,p=0}^{\infty} \sum_{r=0}^{n-i} \frac{(-1)^{q+l+m}(2q+2)^p}{\theta^p p!(m+1)} D_{j,r} \binom{b(j+r+i)-1}{q} \binom{-\theta(l+1)-1}{m} \binom{p}{l}.$$

Also for $\delta > 1$, the pdf of the i^{th} order statistic of the MO-TL-Gom-G family of distributions can also be expressed as an infinite linear combination of the Exp-G densities with power parameter (m+1)and parameter vector ξ. *See Appendix section for derivations.*

3.6.2. Rényi Entropy Rényi entropy [32] of a random variable X following the MO-TL-Gom-G family of distributions is given by

$$I_R(\nu) = (1-\nu)^{-1} \log \left[\int_0^{\infty} f^{\nu}_{MO-TL-Gom-G}(x;\delta,b,\theta,\xi)dx \right] \text{ for } \nu > 0 \ and \ \nu \neq 1.$$

For $\delta \in (0,1)$, we obtain Rényi entropy of the MO-TL-Gom-G family of distributions as

$$I_R(\nu) = (1-\nu)^{-1} \log \left[\sum_{p=0}^{\infty} C^*_m \exp(1-\nu) I_{REG} \right], \tag{23}$$

where $I_{REG} = \int_0^{\infty} \left(\left(\frac{m}{\nu}+1\right) g(x;\xi) G^{\frac{m}{\nu}}(x;\xi) \right)^{\nu} dx$ is Rényi entropy of Exp-G densities with power parameter $(\frac{m}{\nu}+1)$ and

$$C^*_m = \sum_{j,s,q,p,l=0}^{\infty} C_j \int_0^{\infty} (2b)^{\nu} (-1)^{s+q+l+m} \binom{j}{s} \binom{b(s+\nu)-\nu}{q} \frac{(2q+2\nu)^p}{\theta^p p!(\frac{m}{\nu}+1)^{\nu}} \binom{p}{l} \binom{-\theta(l+\nu)-\nu}{m}.$$

If $\delta > 1$, the Rényi entropy is given by

$$I_R(\nu) = (1-\nu)^{-1} \log \left[\sum_{p=0}^{\infty} e^*_m \exp(1-\nu) I_{REG} \right], \tag{24}$$

where $I_{REG} = \int_0^{\infty} \left(\left(\frac{m}{\nu}+1\right) g(x;\xi) G^{\frac{m}{\nu}}(x;\xi) \right)^{\nu} dx$ is Rényi entropy of Exp-G densities with power parameter $(\frac{m}{\nu}+1)$ and

$$e_m^* = \sum_{j,q,p,l,m=0}^{\infty} e_j \int_0^{\infty} (2b)^{\nu}(-1)^{q+l+m}\binom{b(j+\nu)-\nu}{q}\frac{(2q+2\nu)^p}{\theta^p p!(\frac{m}{\nu}+1)^{\nu}}\binom{p}{l}\binom{-\theta(l+\nu)-\nu}{m}.$$

The full derivations are given in the Appendix section.

4. Maximum Likelihood Estimation

Let $X \sim MO-TL-Gom-G(\delta, b, \theta, \xi)$ and $\Theta = (\delta, b, \theta, \xi)^T$ be the parameter vector, then the log-likelihood function (ℓ_n) of a random sample of size n from the $MO-TL-Gom-G(x; \delta, b, \theta, \xi)$ family of distributions is given by

$$\begin{aligned}
\ell_n(\Theta) &= n\log(2b\delta) + \sum_{i=1}^{n}\log[g(x_i;\xi)] - (\theta+1)\sum_{i=1}^{n}\log[(1-G(x_i;\xi))] \\
&+ \frac{2}{\theta}\sum_{i=1}^{n}\left[(1-(1-G(x_i;\xi))^{-\theta})\right] \\
&+ (b-1)\sum_{i=1}^{n}\log\left[1-\left(\exp\left[\frac{1}{\theta}(1-(1-G(x_i;\xi))^{-\theta})\right]\right)^2\right] \\
&- 2\sum_{i=1}^{n}\log\left[1-\bar{\delta}\left(1-\left[1-\left(\exp\left[\frac{1}{\theta}(1-(1-G(x_i;\xi))^{-\theta})\right]\right)^2\right]^b\right)\right].
\end{aligned}$$

Elements of the score vector $U(\mathbf{\Theta})=\left(\frac{\partial\ell_n}{\partial\delta}, \frac{\partial\ell_n}{\partial b}, \frac{\partial\ell_n}{\partial\theta}, \frac{\partial\ell_n}{\partial\xi_k}\right)$ can be readily obtained. The maximum likelihood estimates (mles) of the parameters can be obtained by solving a system of non-linear equations $\left(\frac{\partial\ell_n}{\partial\delta}, \frac{\partial\ell_n}{\partial b}, \frac{\partial\ell_n}{\partial b}, \frac{\partial\ell_n}{\partial\xi_k}\right)^T = 0$ by numerical methods. Elements of the score vector are presented in the appendix.

5. Some Special cases of the MO-TL-Gom-G Family of Distributions

In this section, we present some of the special models of the MO-TL-Gom-G family of distributions. We consider when the baseline cdf $G(x;\xi)$ are log-logistic, Weibull and Burr III distributions.

5.1. Marshall-Olkin-Topp-Leone-Gompertz-Log-Logistic (MO-TL-Gom-LLo) Distribution

Suppose the baseline distribution is the log-logistic distribution with the cdf and pdf given by $G(x;c) = 1-(1+x^c)^{-1}$ and $g(x;c) = cx^{c-1}(1+x^c)^{-2}$, respectively, for $c, x > 0$. Then the cdf and pdf of the MO-TL-Gom-LLo distribution are given by

$$F(x;\delta,b,\theta,c) = 1-\frac{\delta\left(1-\left[1-\left(\exp\left[\frac{1}{\theta}(1-((1+x^c)^{-1})^{-\theta})\right]\right)^2\right]^b\right)}{1-\bar{\delta}\left(1-\left[1-\left(\exp\left[\frac{1}{\theta}(1-((1+x^c)^{-1})^{-\theta})\right]\right)^2\right]^b\right)},$$

and

$$
\begin{aligned}
f(x;\delta,b,\theta,c) &= 2b\delta c x^{c-1}(1+x^c)^{-2}((1+x^c)^{-1})^{-\theta-1}\left(\exp\left[\frac{1}{\theta}(1-((1+x^c)^{-1})^{-\theta})\right]\right)^2 \\
&\times \left[1-\left(\exp\left[\frac{1}{\theta}(1-((1+x^c)^{-1})^{-\theta})\right]\right)^2\right]^{b-1} \\
&\times \left(1-\bar{\delta}\left(1-\left[1-\left(\exp\left[\frac{1}{\theta}(1-((1+x^c)^{-1})^{-\theta})\right]\right)^2\right]^b\right)\right)^{-2},
\end{aligned}
$$

respectively, for $\delta, b, \theta, c > 0$.

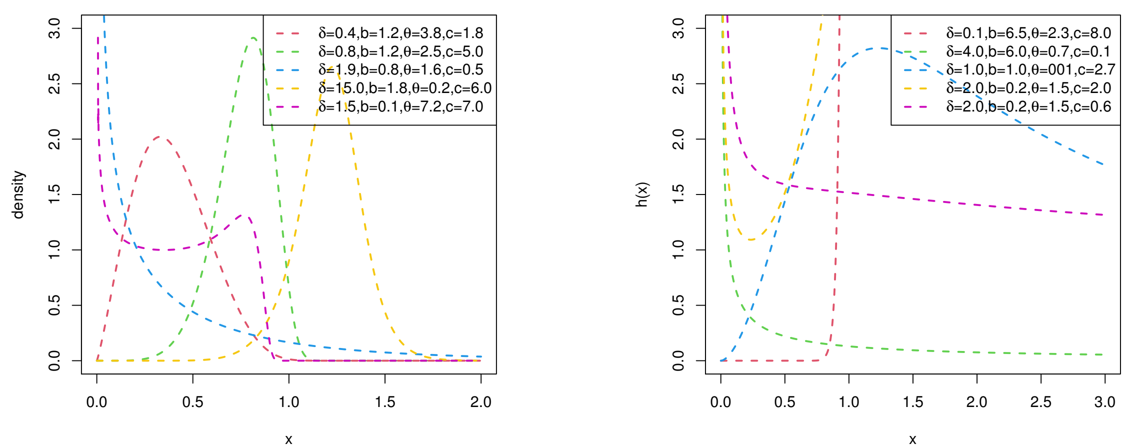


Figure 1. Pdf and hrf plots for the MO-TL-Gom-LLo distribution

Figure 1 gives the plots of the pdf and hrf of the MO-TL-Gom-LLo distribution. The pdf can take several shapes including right-skewed, left-skewed and almost symmetric shapes whereas the hrf displays decreasing, J-shape, upside-down bathtub and bathtub shapes. In Tables 1 and 2, some quantile values and moments for MO-TL-Gom-LLo distribution are presented for different parameter values. The 3D plots for MO-TL-Gom-LLo distribution are given in Figures 2 and 3. The plots indicate that MO-TL-Gom-LLo distribution can model data with different levels of kurtosis and that the skewness can either be positive and negative. Some moments for selected parameters values are given in Table 2

Table 1. **Quantiles for MO-TL-Gom-LLo distribution**

			(δ, b, θ, c)		
u	$(1, 2, 1.3, 0.2)$	$(0.7, 1, 3, 1.5)$	$(2.4, 1, 1, 2)$	$(2.1, 1, 1.9, 1)$	$(1.5, 1, 1.2, 3)$
0.1	0.0009	0.1092	0.3439	0.1004	0.4244
0.2	0.0017	0.1778	0.4849	0.1943	0.5394
0.3	0.0078	0.2391	0.5949	0.2848	0.6236
0.4	0.0231	0.2987	0.6912	0.3751	0.6953
0.5	0.0605	0.3597	0.7822	0.4685	0.7609
0.6	0.1491	0.4237	0.8735	0.5685	0.8251
0.7	0.3700	0.4952	0.9714	0.6822	0.8917
0.8	0.9973	0.5809	1.0865	0.8226	0.9672
0.9	3.5238	0.6996	1.2486	1.0304	1.0686

Table 2. **Moments for MO-TL-Gom-LLo distribution**

			(δ, b, θ, c)		
μ'_s	(1, 0.5, 0.3, 1.2)	(0.5, 1.5, 0.5, 1.5)	(1.3, 0.1, 1.8, 0.6)	(1.1, 1, 0.1, 2)	(0.2, 0.8, 0.6, 0.9)
μ'_1	0.4249	0.6168	0.0493	0.7819	0.1679
μ'_2	0.5715	0.5877	0.0360	0.9116	0.1812
μ'_3	1.5646	0.7969	0.0499	1.5415	0.4835
μ'_4	7.6037	1.4462	0.1006	3.9038	2.2387
μ'_5	60.8931	3.3474	0.2652	15.9183	15.5135
μ'_6	760.8010	9.5244	0.8615	114.2684	148.7158
SD	0.6253	0.4553	0.1832	0.5479	0.3912
CV	1.4716	0.7381	3.7164	0.7008	2.3297
CS	4.0479	1.8943	7.2863	2.1836	6.7115
CK	35.7593	9.0185	80.9886	14.4801	82.9597

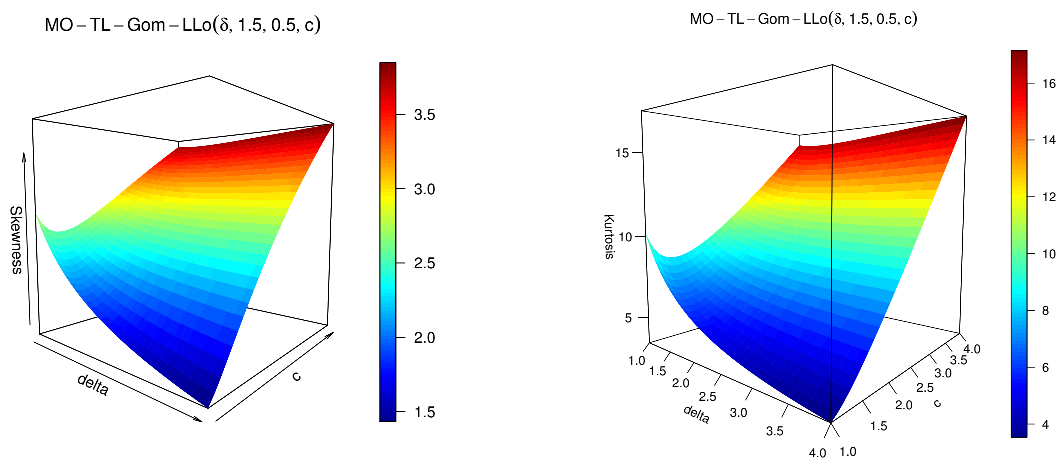


Figure 2. Plots of Skewness and Kurtosis for MO-TL-Gom-LLo distribution.

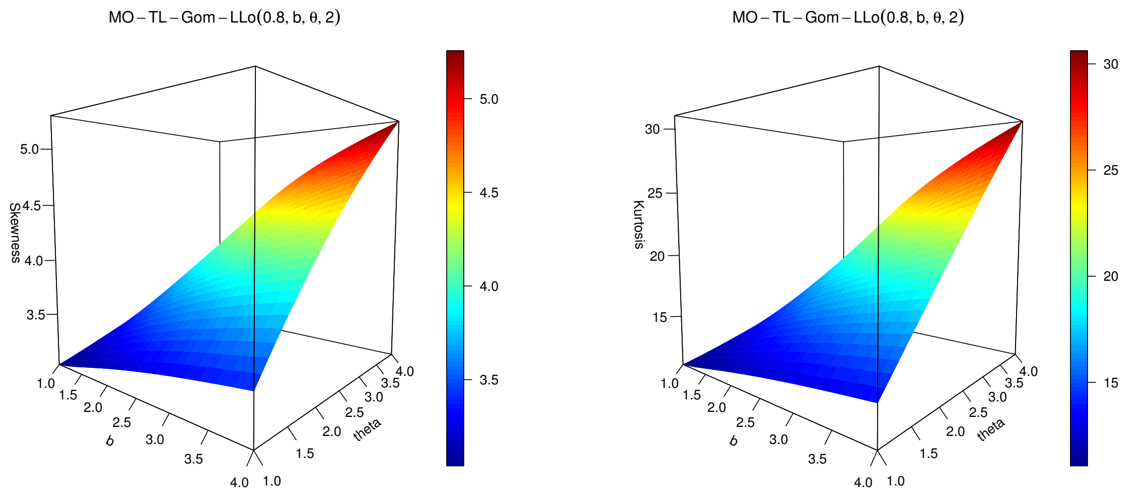


Figure 3. Plots of Skewness and Kurtosis for MO-TL-Gom-LLo distribution.

5.2. Marshall-Olkin-Topp-Leone-Gompertz-Weibull (MO-TL-Gom-W) Distribution

If we take the baseline distribution to be the Weibull distribution with the cdf and pdf given by $G(x;\lambda) = 1 - e^{-x^{\lambda}}$ and $g(x;\lambda) = \lambda x^{\lambda-1} e^{-x^{\lambda}}$, for $\lambda, x > 0$, then we obtain the cdf and pdf of the MO-TL-Gom-W distribution as

$$F(x;\delta,b,\theta,\lambda) = 1 - \frac{\delta\left(1-\left[1-\left(\exp\left[\frac{1}{\theta}(1-e^{\theta x^{\lambda}})\right]\right)^{2}\right]^{b}\right)}{1-\bar{\delta}\left(1-\left[1-\left(\exp\left[\frac{1}{\theta}(1-e^{\theta x^{\lambda}})\right]\right)^{2}\right]^{b}\right)},$$

and

$$\begin{aligned} f(x;\delta,b,\theta,\lambda) &= 2b\delta\lambda x^{\lambda-1}e^{-x^{\lambda}}(e^{-x^{\lambda}})^{-\theta-1}\left(\exp\left[\frac{1}{\theta}(1-e^{\theta x^{\lambda}})\right]\right)^{2} \\ &\times \left[1-\left(\exp\left[\frac{1}{\theta}(1-e^{\theta x^{\lambda}})\right]\right)^{2}\right]^{b-1} \\ &\times \left(1-\bar{\delta}\left(1-\left[1-\left(\exp\left[\frac{1}{\theta}(1-e^{\theta x^{\lambda}})\right]\right)^{2}\right]^{b}\right)\right)^{-2}, \end{aligned}$$

respectively, for $\delta, b, \theta, \lambda > 0$.

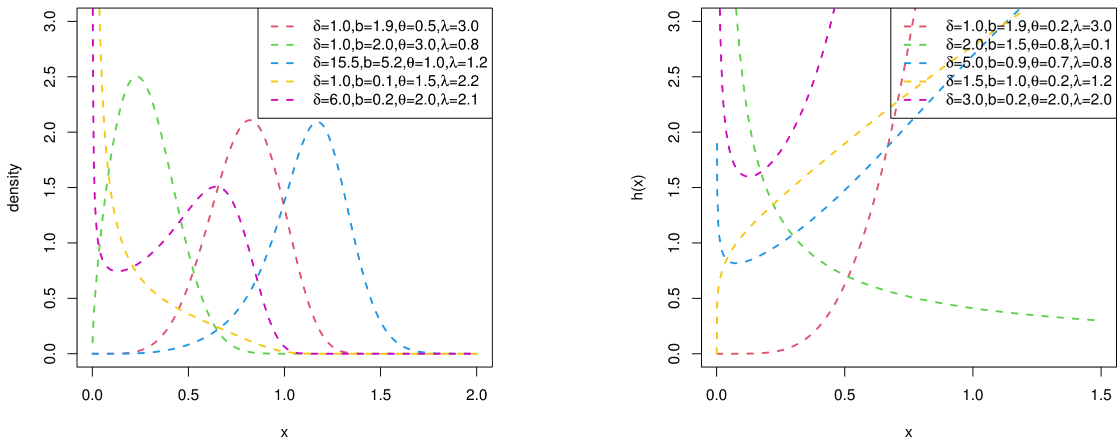


Figure 4. Plots of pdf and hrf for the MO-TL-Gom-W distribution

Table 3. **Quantiles for MO-TL-Gom-W distribution**

	$(\delta, b, \theta, \lambda)$				
u	$(1, 2, 1.3, 0.2)$	$(0.7, 1, 3, 1.5)$	$(2.4, 1, 1, 2)$	$(2.1, 1, 1.9, 1)$	$(1.5, 1, 1.2, 3)$
0.1	0.0005	0.1079	0.3342	0.0957	0.4193
0.2	0.0009	0.1734	0.4595	0.1774	0.5263
0.3	0.0035	0.2307	0.5503	0.2505	0.6011
0.4	0.0083	0.2839	0.6247	0.3185	0.6618
0.5	0.0187	0.3365	0.6909	0.3841	0.7146
0.6	0.0382	0.3901	0.7530	0.4501	0.7638
0.7	0.0769	0.4473	0.8151	0.5200	0.8123
0.8	0.1597	0.5121	0.8831	0.6003	0.8639
0.9	0.3870	0.5964	0.9692	0.7082	0.9274

Table 4. **Moments for MO-TL-Gom-W distribution**

	$(\delta, b, \theta, \lambda)$				
μ'_s	(0.1, 1.5, 0.5, 1.5)	(1.1 ,0.1, 0.2, 1.5)	(3, 1.2, 0.1, 3)	(0.5, 0.2, 0.7, 0.9)	(0.5, 1.5, 3, 0.6)
μ'_1	0.3233	0.1360	1.0740	0.0837	0.1733
μ'_2	0.1929	0.1274	1.3238	0.0906	0.0699
μ'_3	0.2000	0.2490	1.8508	0.2110	0.0439
μ'_4	0.3192	0.8665	2.9370	0.7864	0.0366
μ'_5	0.6986	5.1000	5.3433	4.1553	0.0375
μ'_6	1.9404	49.2592	11.3559	29.0809	0.0449
SD	0.2973	0.3300	0.4127	0.2891	0.1997
CV	0.9195	2.4265	0.3843	3.4543	1.1521
CS	3.0636	5.6221	0.8988	7.8373	2.2573
CK	19.0461	62.7446	5.3868	102.9504	10.1035

The pdf of the MO-TL-Gom-W distribution can be right-skewed, left-skewed and almost symmetric, whereas the hrf displays bathtub, decreasing and decreasing shapes. In Tables 3 and 4, some quantile values and moments for MO-TL-Gom-W distribution are presented for different parameter values. Figures 5 and 6 shows that MO-TL-Gom-W distribution can model data sets with different levels of skewness and kurtosis.

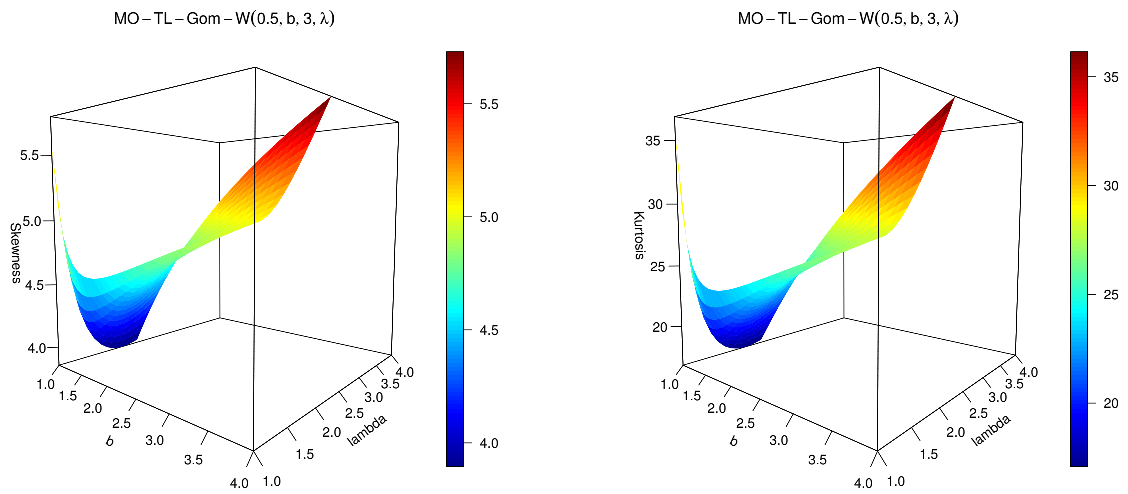


Figure 5. Plots of Skewness and Kurtosis for MO-TL-Gom-W distribution

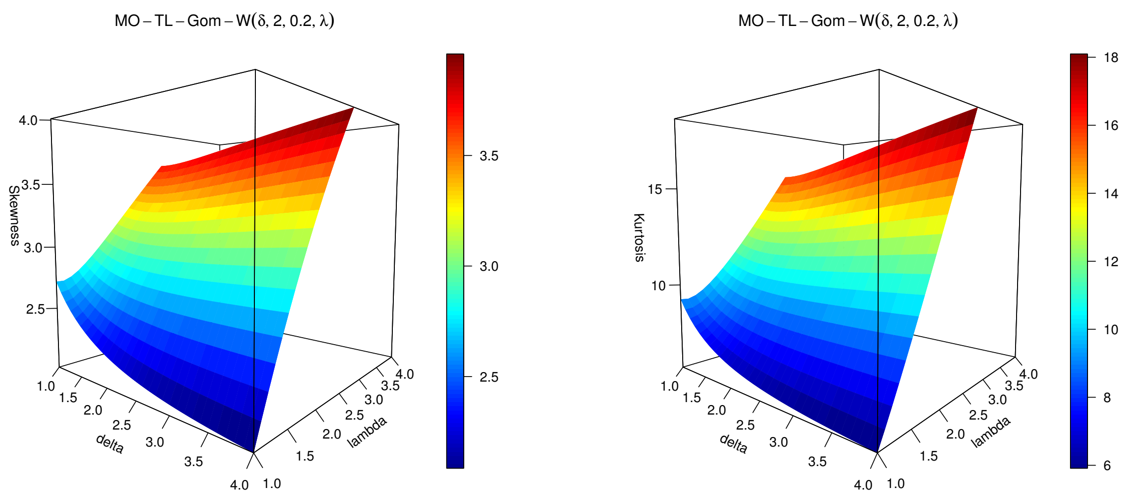


Figure 6. Plots of Skewness and Kurtosis for MO-TL-Gom-W distribution

5.3. Marshall-Olkin-Topp-Leone-Gompertz-Burr III (MO-TL-Gom-BIII) Distribution

Suppose the baseline distribution is Burr III distribution with the cdf and pdf given by $G(x;\lambda,\beta) = (1+x^{-\beta})^{-\lambda}$ and $g(x;\lambda,\beta) = \beta x^{-\beta-1}(1+x^{-\beta})^{-\lambda-1}$, for $\lambda, \beta, x > 0$, then the cdf and pdf of the MO-TL-Gom-BIII distribution are given by

$$F(x;\delta,b,\theta,\lambda,\beta) = 1-\frac{\delta\left(1-\left[1-\left(\exp\left[\frac{1}{\theta}(1-(1-(1+x^{-\beta})^{-\lambda})^{-\theta})\right]\right)^2\right]^b\right)}{1-\bar{\delta}\left(1-\left[1-\left(\exp\left[\frac{1}{\theta}(1-(1-(1+x^{-\beta})^{-\lambda})^{-\theta})\right]\right)^2\right]^b\right)},$$

and

$$\begin{aligned} f(x;\delta,b,\theta,\lambda,\beta) &= 2b\delta\lambda\beta x^{-\beta-1}(1+x^{-\beta})^{-\lambda-1}(1-(1+x^{-\beta})^{-\lambda})^{-\theta-1} \\ &\times \left(\exp\left[\frac{1}{\theta}(1-(1-(1+x^{-\beta})^{-\lambda})^{-\theta})\right]\right)^2 \\ &\times \left[1-\left(\exp\left[\frac{1}{\theta}(1-(1-(1+x^{-\beta})^{-\lambda})^{-\theta})\right]\right)^2\right]^{b-1} \\ &\times \left(1-\bar{\delta}\left(1-\left[1-\left(\exp\left[\frac{1}{\theta}(1-(1-(1+x^{-\beta})^{-\lambda})^{-\theta})\right]\right)^2\right]^b\right)\right)^{-2}, \end{aligned}$$

respectively, for $\delta,b,\theta,\lambda,\beta>0$.

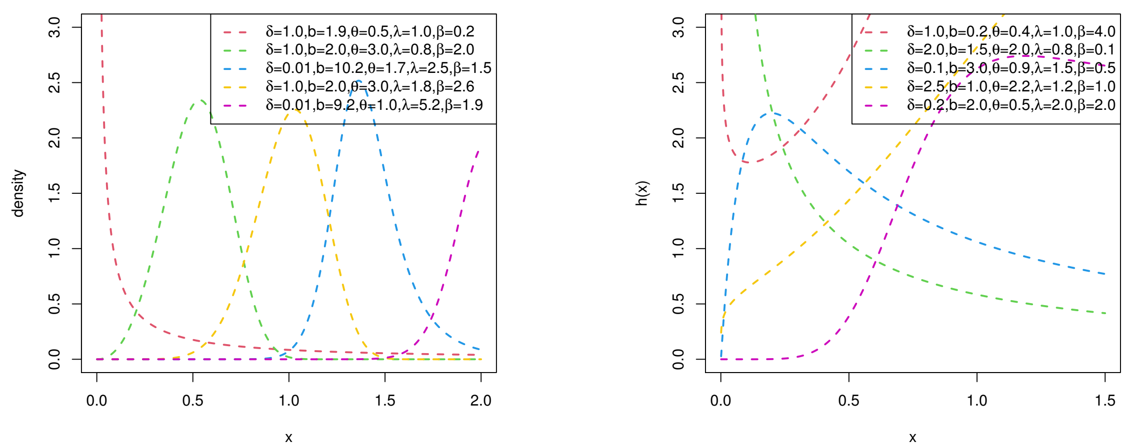


Figure 7. Plots of pdf and hrf for the MO-TL-Gom-BIII distribution

The MO-TL-Gom-BIII distribution has the pdf that displays right-skewed, left-skewed, and reverse-J shapes, whereas the hrf exhibit bathtub, upside-down bathtub, increasing and decreasing shapes. In Tables 5 and 6, some quantile values and moments for MO-TL-Gom-BIII distribution are presented for different parameter values.

Table 5. **Quantiles for MO-TL-Gom-BIII distribution**

	$(\delta, b, \theta, \lambda, \beta)$				
u	$(1,2,1.3,0.2,2)$	$(0.7,1,3,1,1.5)$	$(2.4,1,1,2.5,2)$	$(2.1,1,2,1.9,1)$	$(1.5,1,2,1.2,3)$
0.1	0.0097	0.1092	0.8285	0.3945	0.4946
0.2	0.0232	0.1778	1.0303	0.6205	0.6055
0.3	0.0392	0.2391	1.1862	0.8191	0.6839
0.4	0.0579	0.2987	1.3234	1.0073	0.7486
0.5	0.0798	0.3597	1.4537	1.1954	0.8059
0.6	0.1058	0.4237	1.5853	1.3925	0.8607
0.7	0.1375	0.4952	1.7273	1.6118	0.9164
0.8	0.1796	0.5809	1.8959	1.8790	0.9772
0.9	0.2444	0.6996	2.1356	2.2669	1.0561

Table 6. **Moments for MO-TL-Gom-BIII distribution**

	$(\delta, b, \theta, \lambda, \beta)$				
μ'_s	(0.8, 1, 2, 0.5, 2)	(2, 0.5, 1, 1.5, 3, 1)	(1, 2, 0.8, 0.4, 0.8)	(0.5, 0.5, 0.3, 0.2, 0.9)	(1.5, 2, 1.3, 3, 2.5)
μ'_1	0.2391	0.8348	0.1383	0.0100	1.5255
μ'_2	0.0860	0.7966	0.0633	0.0051	2.4252
μ'_3	0.0375	0.8318	0.0553	0.0083	4.0023
μ'_4	0.0188	0.9297	0.0758	0.0313	6.8343
μ'_5	0.0103	1.0977	0.1470	0.2345	12.0417
μ'_6	0.0061	1.3575	0.3784	3.2109	40.7003
SD	0.1698	0.3158	0.2102	0.0707	0.3131
CV	0.7102	0.3783	1.5197	7.0711	0.2053
CS	0.6435	0.0103	3.6975	23.0489	0.1142
CK	3.1631	2.6195	26.3291	1238.8412	2.9343

6. Simulation Results

The performance of the MO-TL-Gom-LLo distribution is examined by conducting various simulations for different sizes (n=25, 50, 100, 200, 400, 800, 1000) via the R package. We simulate $N = 1000$ samples for the true parameters values given in Table 7. Additional simulation results are available upon request or in the appendix. The tables list the mean MLEs of the model parameters along with the respective average bias (ABIAS) and root mean squared errors (RMSEs). The average bias and RMSE for the estimated parameter, say, $\hat{\theta}$, say, are given by:

$$\text{ABIAS}(\hat{\theta}) = \frac{\sum_{i=1}^{N} \hat{\theta}_i}{N} - \theta, \text{ and RMSE } (\hat{\theta}) = \sqrt{\frac{\sum_{i=1}^{N} (\hat{\theta}_i - \theta)^2}{N}},$$

respectively. As we can see from the results, RMSEs and ABIAS decrease as the sample size n increases, and the mean estimates of the parameter are closer to the true parameter values.

Table 7. Monte Carlo simulation results for MO-TL-Gom-LLo distribution: mean, average bias and RMSE

		$\delta = 1.5, b = 0.2, \theta = 1.0, c = 0.1$			$\delta = 0.9, b = 0.2, \theta = 0.2, c = 2.0$		
Parameter	n	Mean	ABIAS	RMSE	Mean	ABIAS	RMSE
δ	25	1.7891	0.2891	2.1656	1.1146	0.2146	1.4260
	50	1.7868	0.2868	2.0244	0.9730	0.0730	0.8375
	100	1.5696	0.0696	0.6909	0.8871	-0.0129	0.2656
	200	1.5205	0.0205	0.3177	0.8926	-0.0074	0.1610
	400	1.4939	-0.0061	0.1312	0.8932	-0.0068	0.0677
	800	1.4949	-0.0051	0.0830	0.8998	-0.0002	0.0390
	1000	1.4994	-0.0006	0.0698	0.9000	0.0000	0.0304
b	25	0.4356	0.2356	0.3218	0.7657	0.5657	3.1847
	50	0.4068	0.2068	0.3041	0.4294	0.2294	0.6390
	100	0.3384	0.1384	0.2306	0.2965	0.0965	0.2003
	200	0.2582	0.0582	0.1344	0.2231	0.0231	0.0821
	400	0.2282	0.0282	0.0844	0.2050	0.0050	0.0396
	800	0.2089	0.0089	0.0404	0.2011	0.0011	0.0217
	1000	0.2055	0.0055	0.0346	0.2010	0.0010	0.0205
θ	25	1.2173	0.2173	0.7363	1.2219	1.0219	1.7382
	50	1.1851	0.1851	0.6504	0.8872	0.6872	1.1212
	100	1.1452	0.1452	0.4890	0.5015	0.3015	0.5559
	200	1.0595	0.0595	0.2990	0.2716	0.0716	0.2441
	400	1.0205	0.0205	0.1638	0.2158	0.0158	0.1035
	800	1.0040	0.0040	0.0977	0.2028	0.0028	0.0419
	1000	0.9992	-0.0008	0.0873	0.2014	0.0014	0.0312
c	25	0.5795	0.4795	0.6727	1.3333	-0.6667	0.9528
	50	0.5393	0.4393	0.6431	1.4927	-0.5073	0.8137
	100	0.3763	0.2763	0.5010	1.7227	-0.2773	0.5343
	200	0.1973	0.0973	0.2588	1.9327	-0.0673	0.2608
	400	0.1337	0.0337	0.1150	1.9969	-0.0031	0.1157
	800	0.1075	0.0075	0.0350	2.0022	0.0022	0.0517
	1000	0.1033	0.0033	0.0166	1.9999	-0.0001	0.0121

7. Applications

This section provides illustrations of the flexible nature and usefulness of the MO-TL-Gom-LLo distribution in data modeling. We fit the MO-TL-Gom-LLo distribution to the data set in subsections 7.1 and 7.2. These fits are contrasted with several competing non-nested distributions with the same number of parameters. The

MO-TL-Gom-LLo distribution is compared with Topp-Leone generated Weibull (TLGW) (Aryal et al. [5]), Topp-Leone-Marshall-Olkin Weibull (TLMOW) (Chipepa et al. [13]), beta-Gompertz (BGom) (Jafari et al. [19]), Kumaraswamy Gompertz (KGom) (Silva et al. [35]), Marshall-Olkin extended generalized Gompertz (MOEGGom) (Benkhelifa [8]), generalized Weibull Gompertz (GWGom) (El-Damcese et al. [15]), extended generalized Gompertz (EGGom) (Karamikabir et al. [20]) and odd generalized exponential Gompertz (OGEGom) (El-Damcese et al. [14]) distributions. The pdfs of the TLGW, TLMOW, BGom, KGom, MOEGGom, GWGom, EGGom and OGEGom distributions are given in appendix.

Our model parameters were estimated using NLmixed in SAS and our goodness-of-fit test was conducted using the package AdequacyModel in R software. The estimated values of the parameters (standard error in parenthesis), -2log-likelihood statistic $(-2\ln(L))$, Akaike Information Criterion $(AIC = 2p - 2\ln(L))$, Bayesian Information Criterion $(BIC = p\ln(n) - 2\ln(L))$ and Consistent Akaike Information Criterion $\left(AICC = AIC + 2\frac{p(p+1)}{n-p-1}\right)$, where $L = L(\hat{\mathbf{\Theta}})$ is the value of the likelihood function evaluated at the parameter estimates, n is the number of observations, and p is the number of estimated parameters are presented.

We also obtain the following goodness-of-fit statistics: Crameŕ-von Mises (W^*) and Anderson-Darling (A^*) statistics described by Chen and Balakrishnan [11], as well as Kolmogorov-Smirnov (K-S) statistic and its P-value. Note that for the value of the log-likelihood function at its maximum (ℓ_n), larger value is good and preferred, and for AIC, AICC, BIC, and the goodness-of-fit statistics W^*, A^* and K-S, smaller values are preferred. The results are shown in Tables 8 and 9.

Plots of the fitted densities, the histogram of the data and probability plots (Chambers et al [10]) are given for each example to show how well the new model fits the observed data sets. For the probability plot, we plotted $F_{MO-TL-Gom-G}(x_{(j)};\hat{\delta},\hat{b},\hat{\theta},\hat{\xi})$ against $\dfrac{j-0.375}{n+0.25}$, $j = 1, 2, \cdots, n$, where $x_{(j)}$ are the ordered values of the observed data. The measures of closeness are given by the sum of squares $SS = \sum_{j=1}^{n}\left[F_{MO-TL-Gom-G}(x_{(j)};\hat{\delta},\hat{b},\hat{\theta},\hat{\xi}) - \left(\dfrac{j-0.375}{n+0.25}\right)\right]^2$. These plots are shown in Figures 8 and 10.

7.1. *Fatigue Fracture Data*

The first data set represents the life of fatigue fracture of Kevlar 373/ epoxy subjected to constant pressure at 90% stress level until all had failed. The data set has previously been used by Barlow et al. [6]. The observations are as follows:
0.0251, 0.0886, 0.0891, 0.2501, 0.3113, 0.3451, 0.4763, 0.5650, 0.5671, 0.6566, 0.6748, 0.6751, 0.6753, 0.7696, 0.8375, 0.8391, 0.8425, 0.8645, 0.8851, 0.9113, 0.9120, 0.9836, 1.0483, 1.0596, 1.0773, 1.1733, 1.2570, 1.2766, 1.2985, 1.3211, 1.3503, 1.3551, 1.4595, 1.4880, 1.5728, 1.5733, 1.7083, 1.7263, 1.7460, 1.7630, 1.7746, 1.8275, 1.8375, 1.8503, 1.8808, 1.8878, 1.8881, 1.9316, 1.9558, 2.0048, 2.0408, 2.0903, 2.1093, 2.1330, 2.2100, 2.2460, 2.2878, 2.3203, 2.3470, 2.3513, 2.4951, 2.5260, 2.9911, 3.0256, 3.2678, 3.4045, 3.4846, 3.7433, 3.7455, 3.9143, 4.8073, 5.4005, 5.4435, 5.5295, 6.5541, 9.0960.

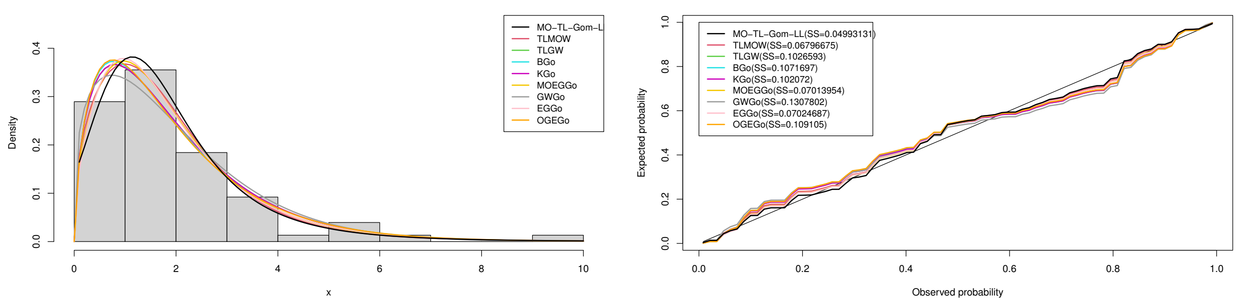
The estimated variance-covariance matrix for MO-TL-Gom-LLo model on fatigue fracture data set is given by

$$\begin{pmatrix} 0.0726 & 6.3647 & 0.2039 & -0.0192 \\ 6.3647 & 558.1588 & 17.8829 & -1.6819 \\ 0.2039 & 17.8829 & 0.5781 & -0.0547 \\ -0.0192 & -1.6819 & -0.0547 & 0.0053 \end{pmatrix},$$

and the 95% two-sided asymptotic confidence intervals for δ, b, θ and c are given by 119.39 ± 0.5280, 10.5298 ± 46.3058, 3.6039 ± 1.4902 and 0.1531 ± 0.1431, respectively.

Table 8. Parameter estimates and goodness-of-fit statistics for various models fitted for fatigue fracture data

Model	Estimates				Statistics							
	$\hat{\delta}$	$\hat{b}$	$\hat{\theta}$	$\hat{c}$	$-2\log L$	AIC	$AICC$	BIC	W^*	A^*	$K-S$	p-value
MO-TL-Gom-LLo	119.39	10.5298	3.6039	0.1531	240.2	248.2	248.8	257.7	0.0575	0.3359	0.0754	0.7513
	(0.2694)	(23.6254)	(0.7603)	(0.0730)								
	$\hat{\alpha}$	$\hat{\theta}$	$\hat{\beta}$	$\hat{\eta}$								
TLGW	0.8382	1.9633	0.9636	0.4227	244.2	252.2	252.7	261.5	0.1144	0.6778	0.0976	0.4358
	(1.1722)	(3.3561)	(0.3835)	(0.3173)								
	$\hat{b}$	$\hat{\delta}$	$\hat{\lambda}$	$\hat{\gamma}$								
TLMOW	0.6326	0.0006	4.48×10^{-5}	2.2203	242.1	250.1	250.7	259.4	0.0785	0.4713	0.0813	0.6661
	(7.46×10^{-10})	(7.05×10^{-7})	(8.68×10^{-6})	(3.09×10^{-10})								
	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\gamma}$	$\hat{\theta}$								
BGom	1.6797	1.5086	7.06×10^{-8}	0.4849	244.5	252.5	253.0	261.8	0.1169	0.6938	0.0957	0.4608
	(0.3153)	(0.0020)	(0.0409)	(1.2715)								
	$\hat{a}$	$\hat{b}$	$\hat{\gamma}$	$\hat{\theta}$								
KGom	1.5869	2.0031	1.48×10^{-8}	0.3896	244.2	252.2	252.8	261.5	0.1146	0.6791	0.0975	0.4374
	(0.3708)	(3.9509)	(0.0217)	(0.6509)								
	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\lambda}$	$\hat{\theta}$								
MOEGGom	1.4534	0.0084	0.4669	0.0033	242.3	250.3	250.8	259.6	0.0900	0.5318	0.0875	0.5756
	(0.2919)	(0.0242)	(0.2195)	(0.0114)								
	$\hat{a}$	$\hat{b}$	$\hat{c}$	$\hat{d}$								
GWGom	0.4482	1.3256	0.5974	1.25×10^{-7}	245.0	253.0	253.6	262.4	0.1306	0.7672	0.1099	0.2953
	(0.0380)	(0.7974)	(0.0379)	(0.5942)								
	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\lambda}$	$\hat{\sigma}$								
EGGom	1.2974	0.0173	0.3058	0.0441	242.2	250.2	250.8	259.5	0.0891	0.5251	0.0870	0.5819
	(0.2314)	(0.0706)	(0.2185)	(0.1192)								
	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\lambda}$	$\hat{c}$								
OGEGom	418.97	1.7006	0.0017	1.55×10^{-7}	244.5	252.5	253.1	261.8	0.1169	0.6943	0.0946	0.4754
	(2.26×10^{-6})	(0.3043)	(0.0003)	(0.0427)								



(a) (b)

Figure 8. Histogram, fitted density and probability plots for fatigue fracture data

Table 8 indicates that MO-TL-Gom-LLo distribution has the highest p-value for the K-S statistic and the lowest values for all goodness-of-fit statistics. Thus, we conclude that the MO-TL-Gom-LLo model performs better on fatigue fracture data than the non-nested TLGW, TLMOW, BGom, KGom, MOEGGom, GWGom, EGGom and OGEGom models. Further, Figure 8 shows that our model outperforms the competing non-nested models.

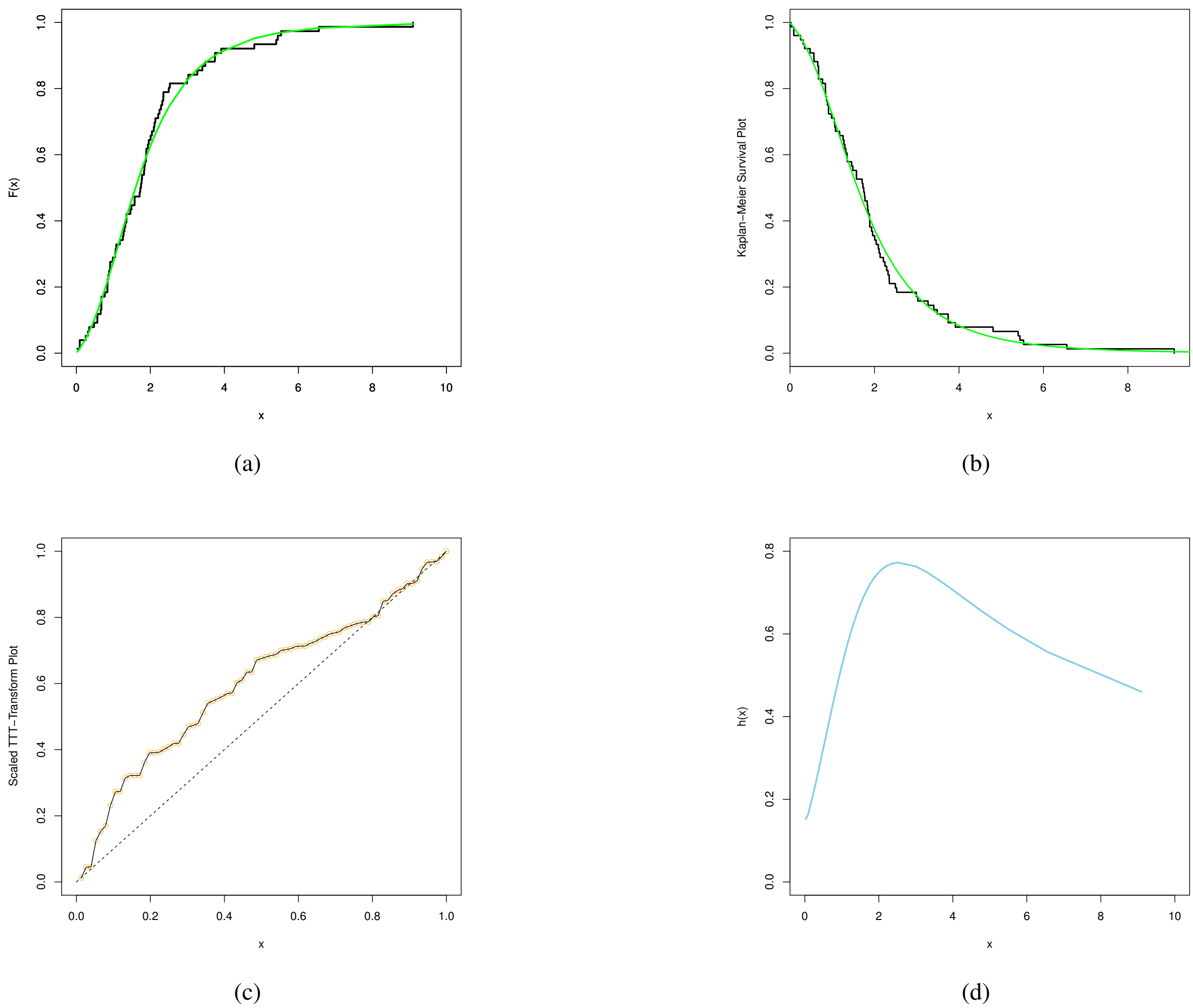


Figure 9. Estimated cdf, K-M survival, TTT-Transform and estimated hrf plots of the MO-TL-Gom-LLo distribution for fatigue fracture data

Figure 9 gives the estimated cdf, Kaplan-Meier survival, the scaled TTT-Transform plots and the estimated hrf plot for fatigue fracture data. We see that the estimated cdf for the MO-TL-Gom-LLo distribution indicated in green (a) is close to the empirical cdf while the survival function (b) in green is also close to the Kaplan-Meier(K-M) curve which shows that indeed our model is better in explaining the fatigue fracture data. The scaled TTT-Transform plot for fatigue fracture with the upside-down bathtub shaped hazard rate function are also given in (c) and (d), respectively.

7.2. Bladder Cancer Data

For the second example, we consider remission times (in months) of a random sample of 128 bladder cancer patients reported in Lee and Wang [24]. The observations are as follows:
0.08, 4.98, 25.74, 3.70, 10.06, 2.69, 7.62, 1.26, 7.87, 4.4, 2.02, 21.73, 2.09, 6.97, 0.50, 5.17, 14.77, 4.18, 10.75, 2.83, 11.64, 5.85, 3.31, 2.07, 3.48, 9.02, 2.46, 7.28, 32.15, 5.34, 16.62, 4.33, 17.36, 8.26, 4.51, 3.36, 4.87, 13.29, 3.64, 9.74, 2.64, 7.59, 43.01, 5.49, 1.40, 11.98, 6.54, 6.93, 6.94, 0.40, 5.09, 14.76, 3.88, 10.66, 1.19, 7.66, 3.02,

19.13, 8.53, 8.65, 8.66, 2.26, 7.26, 26.31, 5.32, 15.96, 2.75, 11.25, 4.34, 1.76, 12.03, 12.63, 13.11, 3.57, 9.47, 0.81, 7.39, 36.66, 4.26, 17.14, 5.71, 3.25, 20.28, 22.69, 23.63, 5.06, 14.24, 2.62, 10.34, 1.05, 5.41, 79.05, 7.93, 4.50, 2.02, 0.20, 7.09, 25.82, 3.82, 14.83, 2.69, 7.63, 1.35, 11.79, 6.25, 3.36, 2.23, 9.22, 0.51, 5.32, 34.26, 4.23, 17.12, 2.87, 18.1, 8.37, 6.76, 3.52, 13.8, 2.54, 7.32, 0.90, 5.41, 46.12, 5.62, 1.46, 12.02, 12.07.

The estimated variance-covariance matrix for MO-TL-Gom-LLo model on remission times data set is given by

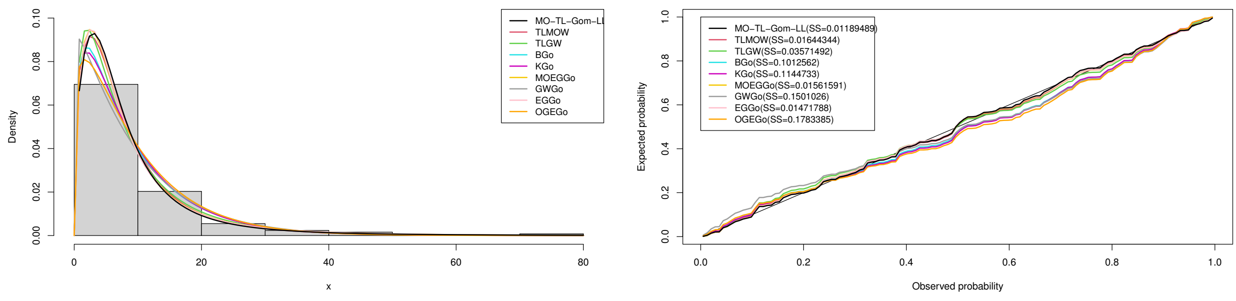
$$\begin{pmatrix} 6.52\times10^{-10} & 1.17\times10^{-11} & -9.54\times10^{-7} & 7.47\times10^{-8} \\ 1.17\times10^{-11} & 2.08\times10^{-13} & -1.71\times10^{-8} & 1.34\times10^{-9} \\ -9.54\times10^{-7} & -1.71\times10^{-8} & 1.40\times10^{-3} & -1.09\times10^{-4} \\ 7.47\times10^{-8} & 1.34\times10^{-9} & -1.09\times10^{-4} & 1.12\times10^{-5} \end{pmatrix},$$

and the 95% two-sided asymptotic confidence intervals for δ, b, θ and c are given by $219.84 \pm 5.00\times10^{-5}$, $13074 \pm 8.94\times10^{-7}$, 4.9427 ± 0.0733 and 0.0444 ± 0.0067, respectively.

Table 9. Parameter estimates and goodness-of-fit statistics for various models fitted for bladder cancer data

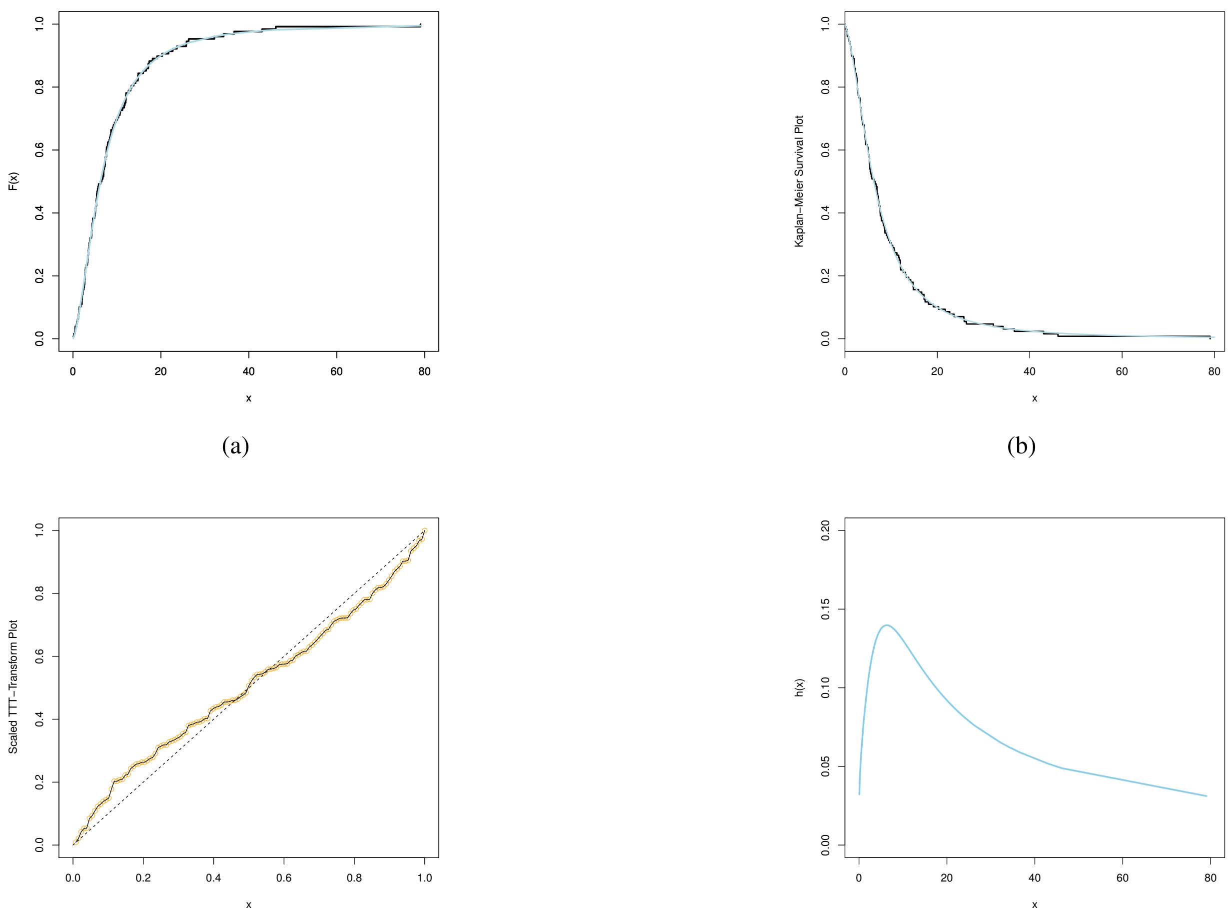
	Estimates				Statistics							
Model	$\hat{\delta}$	$\hat{b}$	$\hat{\theta}$	$\hat{c}$	$-2\log L$	AIC	$AICC$	BIC	W^*	A^*	$K-S$	p-value
MO-TL-Gom-LLo	219.84 (2.55 $\times10^{-5}$)	13074.0 (4.56 $\times10^{-7}$)	4.9427 (0.0374)	0.0444 (0.0034)	818.9	826.9	827.2	838.3	0.0140	0.0991	0.0293	0.9999
	$\hat{\alpha}$	$\hat{\theta}$	$\hat{\beta}$	$\hat{\eta}$								
TLGW	1.1652 (1.9707)	2.9000 (6.2302)	0.5427 (0.2222)	0.2039 (0.3177)	821.2	829.2	829.5	840.6	0.0414	0.2735	0.0444	0.9627
	$\hat{b}$	$\hat{\delta}$	$\hat{\lambda}$	$\hat{\gamma}$								
TLMOW	1.7215 (0.5055)	6516.74 (5.8 $\times10^{-5}$)	6.5826 (0.4824)	0.1371 (0.0206)	819.3	827.3	827.6	838.7	0.0190	0.1221	0.0338	0.9986
	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\gamma}$	$\hat{\theta}$								
BGom	1.4485 (0.3102)	0.1788 (0.1412)	1.27 $\times10^{-9}$ (0.0062)	0.6454 (0.4809)	824.7	832.7	833.0	844.1	0.5276	3.3945	0.0674	0.6053
	$\hat{a}$	$\hat{b}$	$\hat{\gamma}$	$\hat{\theta}$								
KGom	1.4165 (0.3490)	0.3708 (0.3813)	6.58 $\times10^{-7}$ (0.0064)	0.3190 (0.3124)	825.1	833.1	833.4	844.5	0.0925	0.5632	0.0705	0.5487
	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\lambda}$	$\hat{\theta}$								
MOEGGom	1.4920 (0.1772)	0.0024 (0.0067)	0.0387 (0.0215)	0.0022 (0.0084)	819.3	827.3	827.7	838.8	0.0218	0.1534	0.0322	0.9994
	$\hat{a}$	$\hat{b}$	$\hat{c}$	$\hat{d}$								
GWGom	0.6474 (0.0049)	1.0478 (0.4798)	0.1354 (0.0248)	5.57 $\times10^{-8}$ (0.4443)	828.2	836.2	836.5	847.6	0.1314	0.7865	0.0701	0.5552
	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\lambda}$	$\hat{\sigma}$								
EGGom	1.2844 (0.2531)	0.2173 (1.0982)	2.97 $\times10^{-8}$ (0.0502)	0.0600 (0.1756)	819.2	827.2	827.5	838.6	0.0189	0.1335	0.0313	0.9996
	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\lambda}$	$\hat{c}$								
OGEGom	45.7510 (0.0001)	1.2054 (0.1702)	0.0025 (0.0004)	3.00$\times10^{-10}$ (0.0062)	827.7	835.7	836.1	847.1	0.1283	0.7696	0.0829	0.3420

Based on Table 9, MO-TL-Gom-LLo distribution has the highest p-value for the K-S statistic and the lowest goodness-of-fit statistics compared to other non-nested models. Thus, we conclude that the MO-TL-Gom-LLo model performs better with bladder cancer data than the non-nested TLGW, TLMOW, BGom, KGom, MOEGGom, GWGom, EGGom and OGEGom models. Moreover, Figure 10 shows that our model outperforms the competing non-nested models on bladder cancer data.



(a) (b)

Figure 10. Histogram, fitted density and probability plots for bladder cancer data



(a) (b) (c) (d)

Figure 11. Estimated cdf, K-M survival, TTT-Transform and estimated hrf plots of the MO-TL-Gom-LLo distribution for bladder cancer data

In Figure 11 again, we see that the estimated cdf for the MO-TL-Gom-LLo distribution indicated in blue in (a) is closer to the empirical cdf while the survival function in blue (b) is also close to the K-M curve which indicate that our model is the best in explaining the bladder cancer data. The scaled TTT-Transform plot for bladder cancer is also presented in (c) and a uni-modal hazard rate function in (d).

8. Concluding Remarks

A new generalized distribution called the Marshall-Olkin Topp-Leone Gompertz-G (MO-TL-Gom-G) family of distributions is developed and presented. The MO-TL-Gom-G distribution has several new and known distributions as special cases or sub-models. The behaviour of the hazard rate function is flexible. We also obtained closed form expressions for the moments and generating function, distribution of order statistics and entropy. Maximum likelihood estimation technique is used to estimate the model parameters. The performance of a special case of the MO-TL-Gom-G family of distributions was examined by conducting various simulations for different sizes. Finally, we illustrated that the MO-TL-Gom-G family of distributions is useful for lifetime applications by fitting its special case, MO-TL-Gom-LLo distribution to two real data sets.

Appendix

The link below contains the appendix
https://drive.google.com/file/d/1BLQjGat89t5VbGJyaQibDK66WmC8xIol/view?usp=sharing

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