STATISTICS, OPTIMIZATION AND INFORMATION COMPUTING
Stat., Optim. Inf. Comput., Vol. **11**, September 2023, pp 829–852.
Published online in International Academic Press (www.IAPress.org)

Statistical inferences for the Weibull distribution under adaptive progressive type-II censoring plan and their application in wind speed data analysis

Jaber Kazempoor, Arezou Habibirad*, Adel Ahmadi Nadi, and Gholam Reza Mohtashami Borzadaran

Department of Statistics, Faculty of Mathematical Sciences, Ferdowsi University of Mashhad, Mashhad, Iran

Abstract This paper provides four well-known statistical inferences for the principal parameters regarding the two-parameter Weibull distribution including its hazard, quantile, and survival function based on an adaptive progressive type-II censoring plan. The statistical inferences involve the likelihood and approximate likelihood methods, the Bayesian approach, the bootstrap procedure, and a new conditional technique. To construct Bayesian point estimators and credible intervals, Markov chain Monte Carlo, Metropolis-Hastings, and Gibbs sampling algorithms were used. The Bayesian estimators are developed under conjugate and non-conjugate priors and in the presence of symmetric and asymmetric loss functions. In addition, a conditional estimation technique with interesting distributional characteristics has been introduced. The aforementioned methods are compared extensively through a series of simulations. The results of comparative study showed the superiority of the conditional approach over the other ones. Finally, the developed methods are applied to analyze well-known wind speed data.

Keywords Adaptive progressively type-II censoring, Conditional inferences, Weibull distribution, MCMC, Smallest extreme value distribution.

AMS 2010 subject classifications 62E15, 62F15, 62F25.

DOI: 10.19139/soic-2310-5070-1501

1. Introduction

Although the Weibull distribution is a classical distribution introduced in the early 1950s by [42], it has attracted more and more attention in recent decades from modern engineers, statisticians, and researchers. This is because its probability density function (*p.d.f.*) is well-formed and flexible enough to describe many types of data. This model is broadly utilized in reliability engineering [7,20], quality engineering [3,15,19,28], survival analysis [10,22,31,39], and statistical inference [6,30,43] for modeling the failure time or failure rate of a product, component, or an alive creature. While the two-parameter Weibull model has two original parameters, i.e., a scale parameter and a shape parameter; there are some interesting functions related to the distribution like the cumulative distribution function (*c.d.f.*), survival function (*s.f.*), hazard rate function (*h.r.f.*), and quantile function (*q.f.*) that their estimations are of importance for the researchers. In brief, the main challenges around this model are as follows: (i) How to estimate its parameters or its related interesting functions, and (ii) How to provide lifetime data to make an analysis. To tackle concern (i), several inferential methods like maximum likelihood (ML) and approximate maximum likelihood (AML), least square, best linear unbiased, bootstrap, and Bayesian approaches are implemented in literature. In addition, censoring plans, especially progressively censored samples, are commonly used to relax concern (ii) when trying to collect lifetime data for reliability tests, quality inspections, or clinical trials. The reader can find excellent topics in these areas in [5,8,22,34] and references therein.

*Correspondence to: Arezou Habibirad (Email: ahabibi.um.ac.ir). Department of Statistics, Faculty of Mathematical Sciences, Ferdowsi University of Mashhad, Mashhad, Iran.

ISSN 2310-5070 (online) ISSN 2311-004X (print)

In the literature, the estimation of the Weibull original parameters as well as its related functions has been widely investigated. Lin et al. [27] provided point estimators of the original parameters through ML and AML methods according to a progressive type-II hybrid censoring plan. Aboeleneen [2] investigated the Bayesian and ML estimation of the original parameters along with the *s.f.* and *h.r.f.* under generalized order statistics. Given a progressively type-II hybrid censoring sample, ML, AML and Bayesian estimators of the original parameters were studied by Bayatmokhtari et al. [29]. Elmahdy and Aboutahoun [13] considered finite mixtures of three parameters Weibull distribution and derived ML estimation of the original parameters when data are completely available. The entropy of the Weibull distribution was estimated by utilizing the ML and Bayesian estimation under the generalized progressively hybrid censoring scheme by Cho et al. [12]. Jia et al. [21] presented the point and interval estimations in accordance with the ML, AML, and Bayesian approaches for original parameters and *s.f.* under multiply type-II censored data. Recently, Tiefeng [45] developed the original parameters estimations using the Newton-Raphson and EM algorithms, least square, weighted least square, and maximum product of spacing methods under a generalized progressively hybrid censored sample.

On the other hand, in the conditional estimation perspective, there exists great attention from the researchers in this filed. Among the earliest work, two comprehensive papers have dealt with the conditional interval estimation of the Weibull distribution. Firstly, Lawless [24] provided conditional interval estimations of the Weibull original parameters as well as *q.f.* in the presence of a type-II censoring scheme. They also enumerated some reasons for considering this strategy instead of classical methods. Later, Viveros and Balakrishnan [38] extended the conditional estimations of the original parameters and *q.f.* for the location-scale family of distributions. From the application point of view, Haghighi et al. [19] presented conditional control charts for monitoring the *q.f.* of the Weibull distribution under the type-II censoring plan. A similar work was also done by Haghighi [18] based on the Bayesian-conditional method. Recently, Wang et al. [41] compared conditional and some existing monitoring procedures of the Weibull *q.f.* under complete and type-II censored data.

Although researchers have addressed separately some estimation methods of the Weibull *s.f.*, *h.r.f.* and *q.f.*, to the best of our knowledge, there is no unified study in the literature concerning the point and interval estimators of these interesting functions based on the ML, AML, Bayesian, and bootstrap approaches. In addition, this study develops conditional estimators of *s.f.* and *h.r.f.*. The motivation for constructing conditional confidence intervals for the *s.f.* and *h.r.f.* has arisen from the fact that the conditional confidence intervals for the original parameters as well as *q.f.* are reported in literature as more efficient than the traditional methods even in the small sample sizes. Thus, this study also tries to answer the question: what are the performances of the conditional confidence intervals for *s.f.* and *h.r.f.* of the Weibull distribution? Another favourite challenge is generalizing sampling plan to a general scheme that includes ordinary and progressive type-I and type-II censored schemes. This generalization helps the practitioner to investigate and compare all the previous stuff and provides the matter to make inference about the Weibull original parameters and its relating functions under progressive and ordinary type-I and/or type-II censoring plans. This flexible censoring plan is called adaptive progressive type-II censoring (*APC-II*) scheme, which is completely discussed in the next section.

The rest of the paper is organized as follow. The *p.d.f.*, *s.f*, *h.r.f.*, and *q.f.* of the Weibull and smallest extreme value (SEV) models as well as the definition of *APC-II* plan are presented in Section 2. The point and interval estimators of *p.d.f.*, *s.f*, *h.r.f.*, and *q.f.* based on ML and AML methods, percentile bootstrap procedure, Bayesian approach, and conditional technique are developed in Sections 3-7, respectively. Section 8 is devoted to the conduct simulation of studies to assess the behavior of the presented estimation methods. In section 9, to analyze a well-known wind speed dataset, the conditional method has been applied for assessing the treatment of the parameters of *s.f*, *h.r.f.*, and *q.f.* according to the Weibull distribution. Finally, in the last section, some conclusions are provided.

2. Models and Notation

Some of the introductory symbols and definitions applied throughout the paper are provided in this section. Especially, the *APC-II* plan has been described in detail and some of its interesting features are discussed. In addition, the related functions of both the Weibull and SEV distributions and their interactions are presented.

Finally, four loss functions including two symmetric and two asymmetric functions along with their corresponding Bayesian estimators are presented for the Bayesian approach.

2.1. Adaptive progressive Type-II censoring scheme

Inference based on complete datasets usually takes a lot of time and may cost lots of money as well [23,37]. To tackle this obstacle, researchers introduce several sampling plans to reduce the sampling time and cost. For example, the ordinary and progressive Type-I and Type-II censoring schemes have been introduced to deal with time and cost limitations of censorship, respectively [17,32]. In a Type-I scheme, the time for the test is pre-fixed, whereas in a Type-II scheme, the number of observed failures is pre-fixed and the test termination time is random. Both of these sampling plans do not provide the flexibility and freedom of action to the experimenter. To tackle these restrictions, many sampling plans are suggested as extensions or combinations of these two basic censoring schemes. As some excellent examples, in [32,33,37], the *APC-II* censoring scheme was introduced that combines the progressive Type-I and Type-II plans. In this plan, the experimenter has a prefixed time like the progressive Type-I plan and a censoring scheme like the progressive Type-II plan. Until the prefix time, the experimenter acts like a progressive Type-II censoring scheme, and after that, he/she only wants to terminate the experiment as soon as possible. In the next step, the failure times are observed, and no unit is removed. In continue, the process of this experiment is presented and some of its related notations are provided.

Consider threshold time T, sample size n, number of failed units $m\,(0 \leq m \leq n)$ and censoring scheme $R = (R_1, R_2, \ldots, R_m)$ such that $n = m + \sum_{i=1}^{m} R_i$ as prefixed values available for the experimenter before the test. Assume that n units are putted on a life testing experiment and let $X_1, X_2, \ldots, X_n$ be their corresponding random lifetimes. The experiment is set to begin until the first failure, which is denoted by $X_{1:m:n}$. After that, immediately, R_1 units from $n-1$ unobserved ones are randomly removed from the process. The experiment continues until the second failure, denoted by $X_{2:m:n}$, is occurred, and similarly, the experiment is followed until progressively type-II censored order statistics (*PCOS-II*), $X_{1:m:n}, X_{2:m:n}, \ldots, X_{m:m:n}$ are completely observed. Just like the process of *PCOS-II*, in *APC-II*, $X_{i:m:n}$'s are observed until the first of ones exceeds the threshold T or equivalently $j = \inf\{j_1 : x_{j_1-1:m:n} < T < x_{j_1:m:n}\}$ is supposed. Here, the experimenter must reset the censoring scheme such that the experiment is done as soon as possible, or consequently the new censoring scheme is $R^* = (R_1, R_2, \ldots, R_{j-1}, 0, 0, \ldots, 0, 0, n-m-\sum_{i=1}^{j-1} R_i)$. It is easy to check that for $T = -\infty$ and $T = \infty$, *APC-II* reduces to progressive Type I and progressive Type-II censoring schemes, respectively.

Given the integer sizes m and n, censoring scheme $R = (R_1, R_2, \ldots, R_m)$, and time threshold T and according to [32,33], the joint *p.d.f.* of the random sample $X_{1:m:n}, X_{2:m:n}, \ldots, X_{m:m:n}$ arising from a *APC-II* plan is given by

$$f_{X^R_{1:m:n},\ldots,X^R_{m:m:n}}(x_1, \ldots, x_m) = \prod_{i=1}^{m} \gamma_i f(x_i) \times S^{C_j}(x_m) \prod_{i=1}^{J} S^{R_i}(x_i), \quad (1)$$

where $\gamma_i = \prod_{k=1}^{m}(n-k+1-\sum_{l=1}^{min(i-1,j)} R_l)$ and $C_j = n-m-\sum_{i=1}^{j} R_i$.

The relation (1), has a complicated form and it is better to altered. In accordance with a *APC-II* plan, one can replace $R = (R_1, R_2, \ldots, R_m)$ and γ_i with $R^* = (R^*_1, R^*_2, \ldots, R^*_j, 0, 0, \ldots, 0)$ and $\gamma^*_i = \sum_{j=i}^{m} R^*_j$ where $J = j$. Consequently, the joint *p.d.f.* in (1) can be represented as [8]:

$$f_{X^R_{1:m:n},\ldots,X^R_{m:m:n}}(x_1, \ldots, x_m) = \prod_{i=1}^{m} \gamma^*_i f(x_i) \times S^{R^*_i}(x_i). \quad (2)$$

In what follows, relation (2) would be used instead of (1) because of its simplicity in manipulation and simulation process.

2.2. Models

Suppose that the lifetime X follows the two parameters Weibull distribution with shape parameter α and scale parameter β. It is shown here by $X \sim W(\alpha, \beta)$. The *p.d.f.*, *s.f.*, *h.r.f.*, and *q.f.* of X are, respectively, given by:

$$f_X(x) = \frac{\alpha}{\beta}\left(\frac{x}{\beta}\right)^{\alpha-1} e^{-\left(\frac{x}{\beta}\right)^{\alpha}}, \tag{3}$$

$$S_X(x) = e^{-\left(\frac{x}{\beta}\right)^{\alpha}}, \tag{4}$$

$$H_X(x) = \frac{\alpha}{\beta}\left(\frac{x}{\beta}\right)^{\alpha-1}, \quad x>0, \quad \alpha, \beta > 0, \tag{5}$$

and

$$Q_X(p) = \beta[-\log(1-p)]^{\alpha}, \quad 0<p<1. \tag{6}$$

As a suitable transformation to the location-scale family of distributions, set $Y = \log(X)$. It is known that Y follows the SEV distribution with location parameter $\mu = \log(\beta)$ and scale parameter $\sigma = \frac{1}{\alpha}$ that is shown by $Y \sim SEV(\mu, \sigma)$. Then, the *p.d.f.*, *s.f.*, *h.r.f.*, and *q.f.* of Y are, respectively, given by:

$$f_Y(y) = \frac{1}{\sigma} e^{\frac{y-\mu}{\sigma} - e^{\frac{y-\mu}{\sigma}}}, \tag{7}$$

$$S_Y(y) = e^{-e^{\frac{y-\mu}{\sigma}}}, \tag{8}$$

$$H_Y(y) = \frac{1}{\sigma} e^{\frac{y-\mu}{\sigma}}, \quad y, \mu \in \mathbb{R}, \quad \sigma > 0, \tag{9}$$

and

$$Q_Y(p) = \mu + \sigma \log(-\log(1-p)), \quad 0<p<1. \tag{10}$$

This transformation allows conditional estimation for the reliability parameters of the Weibull distribution. It's due to the direct derivations between the aforementioned parameters in the SEV and Weibull distributions discussed in the following Lemmas.

Lemma 1. Let $(X^R_{1:m:n}, \ldots, X^R_{m:m:n})$ be an *APC-II* sample arising from the $W(\alpha, \beta)$ distribution with censoring scheme $R = (R_1, R_2, \ldots, R_m)$ and the threshold T and let the transformations $Y_i = \log(X^R_{i:m:n})$ for $i = 1, 2, \ldots, m$. Then $Y_i \overset{d}{=} Y^R_{i:m:n}$ where the notation $\overset{d}{=}$ represents that X and Y follow the same distribution and where $(Y^R_{1:m:n}, \ldots, Y^R_{m:m:n})$ is an *APC-II* sample arising from the $SEV(\mu, \sigma)$ distribution with the same censoring scheme and threshold $\log(T)$.

Proof
Using relation (2) and monotone property of transformation $Y = \log(X)$, the proof would be resulted. □

Lemma 2. If $X \sim W(\alpha, \beta)$ and $Y \sim SEV(\mu, \sigma)$, then

$$Q_X(x) = e^{Q_Y(x)}, \quad S_X(x) = S_Y(\log(x)),$$

and

$$H_X(x) \simeq \lim_{h\to 0} \log \frac{S_X(x)}{S_X(x+h)} = \lim_{h\to 0} \log \frac{S_Y(\log(x))}{S_Y(\log(x+h))}.$$

Proof
Applying the transformation $Y = \log(X)$ and considering relations (3)-(6) and (7)-(10), the proof would be resulted after some mathematical manipulations. □

Table 1. Four kinds of loss functions and their corresponding Bayesian estimators

Loss Function	Formula $L_i(\delta,\theta)$	Bayesian Estimator δ^*_{iB}
Squared Error Loss (SEL)	$(\delta-\theta)^2$	$E(\theta \mid \underline{x})$
Linear Exponential Loss (LINEXL)	$e^{(\delta-\theta)}-(\delta-\theta)-1$	$-\log(E(e^{-\theta} \mid \underline{x}))$
Entropy Loss (EL)	$\frac{\delta}{\theta}-\log(\frac{\delta}{\theta})-1$	$\frac{1}{E(\frac{1}{\theta}\mid\underline{x})}$
Stein's Loss (SL)	$\frac{\delta}{\theta}+\frac{\theta}{\delta}-2$	$\sqrt{\frac{E(\theta\mid\underline{x})}{E(\frac{1}{\theta}\mid\underline{x})}}$

2.3. *Loss Functions*

Two kinds of loss functions are considered in Bayesian estimation, symmetric and asymmetric ones. Let θ be an unknown parameter and δ be an arbitrary estimator for θ. These loss functions and the Bayesian estimator regarding each of them are presented in Table (1). It is easy to see that Squared Error Loss (SEL) and Stein's Loss (SL) are symmetric, i.e., $L(\delta,\theta)=L(\theta,\delta)$ and Linear Exponential Loss (LINEX) and Entropy Loss (EL) are asymmetric, i.e., $L(\delta,\theta)\neq L(\theta,\delta)$.

3. MLE

The most widely used estimator that researchers are interested in, is MLE. The strong theoretical background, easy way of understanding, having asymptotic normality property, closed-form in many cases, and the availability in most of the mathematical software have made this type of estimator so popular. In this section, the method of deriving MLEs and related asymptotic confidence intervals according to these estimators for the original parameters and *s.f.*, *h.r.f.*, and *q.f.*of the Weibull model under an *APC-II* scheme is investigated.

Let $(X^R_{1:m:n},\ldots,X^R_{m:m:n})$ be an adaptive progressive type-II censored sample of size m from a sample of size n that are taken from the model (3) with the associated progressive censoring scheme $R=(R_1,\ldots,R_m)$. Given $J=j$, the corresponding likelihood function based on these data using relation (1), is given as

$$L(\mu,\sigma)=\frac{\alpha^m\prod_{i=1}^m\gamma^*_i}{\beta^{\alpha m}}(\prod_{i=1}^m x_i)^{\alpha-1}e^{-\sum_{i=1}^m(R^*_i+1)(\frac{x_i}{\beta})^\alpha}, \tag{11}$$

where $R^*=(R^*_1,\ldots,R^*_m)=(R_1,\ldots,R_j,0^{(m-j-1)},n-m-\sum_{i=1}^j R_i)$ and consequently $\gamma^*_i=\sum_{l=i}^m R^*_l$. The MLEs of α and β denoted by $\hat\alpha$ and $\hat\beta$ can be calculated by maximizing relation (11). To calculate them, the log-likelihood function of α and β is

$$l(\alpha,\beta)=m\log(\alpha)-m\alpha\log(\beta)+\sum_{i=1}^m\log(\gamma^*_i)+(\alpha-1)\sum_{i=1}^m\log(x_i)-\sum_{i=1}^m(R^*_i+1)(\frac{x_i}{\beta})^\alpha. \tag{12}$$

Taking the first partial derivatives of $l(\alpha,\beta)$ in (12) with respect to α and β, the following log-likelihood equations are derived

$$\frac{\partial l(\alpha,\beta)}{\partial\alpha}=\frac{m}{\alpha}+\sum_{i=1}^m\log(x_i)-m\log(\beta)-\sum_{i=1}^m(R^*_i+1)\log(\frac{x_i}{\beta})(\frac{x_i}{\beta})^\alpha, \tag{13}$$

$$\frac{\partial l(\alpha,\beta)}{\partial\beta}=\frac{\alpha}{\beta}[-m+\sum_{i=1}^m(R^*_i+1)(\frac{x_i}{\beta})^\alpha]. \tag{14}$$

The estimators $\hat\alpha$ and $\hat\beta$ can be calculated by equating (13) and (14) to zeros, respectively. However, because of the complicated form of these equations, $\hat\alpha$ and $\hat\beta$ can be obtained in closed forms and should be calculated numerically. Accordingly, the MLEs of $S_X(x), H_X(x)$, and $Q_X(x)$ in (4), (5), and (6), can be obtained by the invariance property of MLEs as

$$\hat{S}_X(x) = e^{-\left(\frac{x}{\hat{\beta}}\right)^{\hat{\alpha}}},$$

$$\hat{H}_X(x) = \frac{\hat{\alpha}}{\hat{\beta}}\left(\frac{x}{\hat{\beta}}\right)^{\hat{\alpha}-1},$$

and

$$\hat{Q}_X(x) = \hat{\beta}[-\log(1-x)]^{\hat{\alpha}},$$

respectively. Once the point estimations are determined, we aim to build confidence intervals. Based on the asymptotic normality of the MLE, the asymptotic distribution of $(\hat{\alpha}, \hat{\beta})$ is $N(\Pi, \Sigma)$ such that

$$\Pi = \begin{bmatrix}\alpha \\ \beta\end{bmatrix} \tag{15}$$

and

$$\Sigma = \begin{bmatrix}\Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22}\end{bmatrix} = \begin{bmatrix}-\frac{\partial^2 l(\alpha,\beta|\underline{x})}{\partial\alpha^2} & -\frac{\partial^2 l(\alpha,\beta|\underline{x})}{\partial\alpha\partial\beta} \\ -\frac{\partial^2 l(\alpha,\beta|\underline{x})}{\partial\alpha\partial\beta} & -\frac{\partial^2 l(\alpha,\beta|\underline{x})}{\partial\beta^2}\end{bmatrix}^{-1}_{\substack{\alpha=\hat{\alpha}\\ \beta=\hat{\beta}}}. \tag{16}$$

In order to construct the confidence intervals, we applied the delta method (see [26] and [14]) and used the facts that

$$var(\hat{\alpha}) = \Sigma_{11}, \quad var(\hat{\beta}) = \Sigma_{22},$$

$$var(\hat{Q}_X(x)) = \Sigma_{11} + w_p^2\Sigma_{22} + 2w_p\Sigma_{12},$$

$$var(\hat{S}_X(x)) = f_X^2(x)[(\frac{x}{\alpha}\log(\frac{x}{\beta}))^2\Sigma_{11} - 2(\frac{x^2}{\alpha\beta}\log(\frac{x}{\beta}))\Sigma_{12} + (\frac{x}{\beta})^2\Sigma_{22}],$$

$$var(\hat{H}_X(x)) = H_X^2(x)[(\frac{1}{\alpha} + \log(\frac{x}{\beta}))^2\Sigma_{11} - 2\frac{\alpha}{\beta}(\frac{1}{\alpha} + \log(\frac{x}{\beta}))\Sigma_{12} + (\frac{\alpha}{\beta})^2\Sigma_{22}].$$

We have,

$$\begin{aligned}
1-(\epsilon_1+\epsilon_2) &= P\left(\hat{\alpha} - z_{\epsilon_1}\sqrt{var(\hat{\alpha})} \le \alpha \le \hat{\alpha} + z_{\epsilon_2}\sqrt{var(\hat{\alpha})}\right) \\
&= P\left(\hat{\beta} - z_{\epsilon_1}\sqrt{var(\hat{\beta})} \le \beta \le \hat{\beta} + z_{\epsilon_2}\sqrt{var(\hat{\beta})}\right) \\
&= P\left(\hat{Q}_X(x) - z_{\epsilon_1}\sqrt{var(\hat{Q}_X(x))} \le Q_X(x) \le \hat{Q}_X(x) + z_{\epsilon_2}\sqrt{var(\hat{Q}_X(x))}\right) \\
&= P\left(\hat{S}_X(x) - z_{\epsilon_1}\sqrt{var(\hat{S}_X(x))} \le S_X(x) \le \hat{S}_X(x) + z_{\epsilon_2}\sqrt{var(\hat{S}_X(x))}\right) \\
&= P\left(\hat{H}_X(x) - z_{\epsilon_1}\sqrt{var(\hat{H}_X(x))} \le H_X(x) \le \hat{H}_X(x) + z_{\epsilon_2}\sqrt{var(\hat{H}_X(x))}\right),
\end{aligned}$$

where $0 < \epsilon_1, \epsilon_2 < 1$ and also $0 < \epsilon_1 + \epsilon_2 < 1$.

4. AMLE

Despite that MLEs have some many features, which are mentioned in the previous section, they have has also some defects in practice. Including these shortcomings, the bad performances for small sample size and their needs for numerical resolution in some sampling schemes and some distributions can be highlighted. In addition, the optimization problem in most mathematical software needs to start for solving the problem at some start points. In statistical software R, [36], the function *optim* can be useful for calculating MLE. Moreover, there are some famous packages such as *bbmle*, *maxlik*, *fitdistrplus*, and so on, that each of them is so applicable with the aim of solving likelihood equations. For those packages and their functions related to the MLE, there is one effective weakness that may be confusing in some cases. The problem of choosing the start point is that weakness. This problem becomes more serious in situations that there is no moderate initial guess like a moment estimator for the start point. One way to tackle this practical problem is utilizing AMLE that has most of the MLE properties and also can be derived in a closed-form without need of any numerical experiment. It is necessary to mention that, in the present study, the *bbmle* package in statistical software R have been used to calculate MLEs; see [36].

In the previous section, it was seen that $\hat{\alpha}$ and $\hat{\beta}$ can not be obtained in closed forms form the derivatives of the log-likelihood function. Thus, an approximated log-likelihood function is desired. To this end, similar to [9], the Weibull data transformed to the SEV data and then by utilizing the invariant property of MLE, AMLE, are derived. Assume that the failure times $X^R_{1:m:n}, X^R_{2:m:n}, \ldots, X^R_{m:m:n}$ come from the Weibull model (3) under an *APC-II* plan with censoring scheme $R = (R_1, R_2, \ldots, R_m)$ and switching time T. After transforming of these data, i.e., $Y^{RT}_{i:m:n} = \log(X^{RT}_{i:m:n})$, it is clear to check that $(Y^{RT}_{1:m:n}, Y^{RT}_{2:m:n}, \ldots, Y^R_{m:m:n})$ come from the SEV model (7) under the same censoring scheme R and switching time $\log(T)$. The log-likelihood of the transformed sample is

$$\log l(\mu, \sigma) = \sum_{j=1}^{m} \gamma^*_j - m\log(\sigma) + \sum_{j=1}^{m} \frac{y_j - \mu}{\sigma} - \sum_{j=1}^{m} (R^*_j + 1) e^{\frac{y_j - \mu}{\sigma}}, \tag{17}$$

where $\mu = \log(\beta)$ and $\sigma = \frac{1}{\alpha}$. Taking the first partial derivatives of $l(\mu, \sigma)$ in (17)) with respect to μ and σ and then setting $z_j = \frac{y_j - \mu}{\sigma}$, $\frac{\partial}{\partial\sigma} z_j = -\frac{z_j}{\sigma}$, $\frac{\partial}{\partial\mu} z_j = -\frac{1}{\sigma}$, $\frac{\partial}{\partial\mu} \log(l(\mu, \sigma)) = 0$, and $\frac{\partial}{\partial\mu} \log(l(\mu, \mu)) = 0$, the following equations can be derived

$$\sum_{j=1}^{m} (R^*_j + 1) e^{z_j} - m = 0, \quad \sum_{j=1}^{m} (R^*_j + 1) z_j e^{z_j} - \sum_{j=1}^{m} z_j - m = 0.$$

From here onwards, with some changes in the notation, we do the same task as before done in [9]. Finally, the AMLEs of μ and σ can be calculated in closed forms as

$$\tilde{\sigma} = \frac{A + \sqrt{A^2 + 4mB}}{2m}, \quad \tilde{\mu} = K + L\tilde{\sigma},$$

where

$$K = \frac{\sum_{j=1}^{m} (R^*_j + 1) c_j y_j}{\sum_{j=1}^{m} (R^*_j + 1) c_j}, \quad L = \frac{\sum_{j=1}^{m} (R^*_j + 1) d_j - m}{\sum_{j=1}^{m} (R^*_j + 1) c_j},$$

$$A = \sum_{j=1}^{m} [(R^*_j + 1) d_j - 1](y_j - K), \quad B = \sum_{j=1}^{m} (R^*_j + 1) c_j (y_j - K)^2,$$

$$c_j = e^{E(Z_{j:m:n})}, \quad d_j = e^{E(Z_{j:m:n})}(1 - E(Z_{j:m:n})).$$

Consequently, the AMLEs of α and β denoted by $\tilde{\alpha}$ and $\tilde{\beta}$ can be obtained as $\tilde{\alpha} = \frac{1}{\tilde{\sigma}}$ and $\tilde{\beta} = e^{\tilde{\mu}}$, respectively.

5. Bootstrap Methods

Percentile bootstrap is another alternative method for constructing a confidence interval for any given parameter. The simplicity procedure, good performances in the presence of small sample sizes, and its convergence to the true value of parameter are some advantages of this estimation method. In this section, the percentile bootstrap confidence interval of Weibull original parameters and its *s.f.*, *h.r.f.*, and *q.f.* are provided. Suppose that, given the *APC-II* sample $X^{RT}_{1:m:n}, X^{RT}_{2:m:n}, \ldots, X^{R}_{m:m:n}$, with censoring scheme $R = (R_1, R_2, \ldots, R_m)$ and switching time T, MLEs $\hat{\alpha}, \hat{\beta}, \hat{S}_X(x), \hat{H}_X(x)$, and $\hat{Q}_X(x)$ are calculated. The next step is generating B (for example $B = 1000$) *APC-II* random samples of size m with censoring scheme R and switching time T from Weibull distribution with estimated parameters $\hat{\alpha}$ and $\hat{\beta}$. These samples are called Bootstrap samples. Then, the MLEs of $\alpha, \beta, S_X(x), H_X(x)$, and $Q_X(x)$ based on B Bootstrap samples, denoted here by $(\alpha^*_i, \beta^*_i, S^*_{Xi}(x), H^*_{Xi}(x), Q^*_{Xi}(x))$ for $i = 1, 2, ..., B$, should be calculated. Finally, after sorting calculated bootstrap MLEs, for example $\alpha^*_{(1)} < \alpha^*_{(2)} <, ..., < \alpha^*_{(B)}$, $100\,(1 - (\epsilon_1 + \epsilon_2))\,\%$ percentile bootstrap confidence intervals are calculated as follows:

$$\begin{aligned}1 - (\epsilon_1 + \epsilon_2) &= P(\alpha^*_{(B\epsilon_1)} \leq \alpha \leq \alpha^*_{(B(1-\epsilon_2))}) \\ &= P(\beta^*_{(B\epsilon_1)} \leq \beta \leq \beta^*_{(B(1-\epsilon_2))}) \\ &= P(S^*_{X(B\epsilon_1)}(x) \leq S_X(x) \leq S^*_{X(B(1-\epsilon_2))}) \\ &= P(H^*_{X(B\epsilon_1)} \leq H_X(x) \leq H^*_{X(B(1-\epsilon_2))}) \\ &= P(Q^*_{X(B\epsilon_1)} \leq Q_X(x) \leq Q^*_{X(B(1-\epsilon_2))}).\end{aligned}$$

6. Bayesian Estimators

In contrast to the classical estimation approaches, the Bayesian method has been arisen. In this section, the Bayesian strategy of estimating Weibull original parameters and its related functions have been investigated. The use of this method allows incorporating given knowledge of parameters through the priors. Therefore, as a deep grasp of the behavior of choosing prior densities, two kinds of densities have been considered. For the first prior, it is assumes that α and β are independence and follow different gamma distributions. It can be mentioned as a backing up of choosing this prior [23,32,33]. The second prior is chosen in a way to corporate the sample information, and moreover, the corresponding posterior density is also conjugate. In addition to these assumptions, the desired Bayesian estimators are calculated with respect to four loss functions including two symmetric and two asymmetric ones, which are listed in Table 1. Based on the above assumptions about the prior densities of (α, β), the first and the second prior densities are

$$\pi_1(\alpha, \beta) = \alpha^{a-1} e^{-b\alpha} \beta^{c-1} e^{-d\beta}, \quad (a, b, c, d, \alpha, \beta) \in \mathbb{R}^+, \tag{18}$$

$$\pi_2(\alpha, \beta) = \frac{[\sum_{i=1}^m (R^*_i + 1) x_i^\alpha]^{m - \frac{1}{\alpha}}}{\Gamma(m - \frac{1}{\alpha})} \frac{(-\log(b))^a}{\Gamma(a)} \alpha^a b^\alpha \beta^{-m\alpha} e^{-\sum_{i=1}^m (R^*_i + 1)(\frac{x_i}{\beta})^\alpha}. \tag{19}$$

It can be clearly checked that $\pi_2(\alpha, \beta)$ is a true joint *p.d.f.* The corresponding posterior densities are proportional to:

$$\pi_1(\alpha, \beta \mid \underline{x}) \propto \alpha^{a+m-1} e^{-\alpha[b - \sum_{i=1}^m \log(\frac{x_i}{\beta})]} \beta^{c-1} e^{-d\beta} e^{-\sum_{i=1}^m (R^*_i + 1)(\frac{x_i}{\beta})^\alpha}$$

and

$$\pi_2(\alpha, \beta \mid \underline{x}) \propto \frac{[\sum_{i=1}^m (R^*_i + 1) x_i^\alpha]^{m - \frac{1}{\alpha}}}{\Gamma(m - \frac{1}{\alpha})} \alpha^{m+a} (b \prod_{i=1}^m x_i)^\alpha \beta^{-2m\alpha} e^{-2\sum_{i=1}^m (R^*_i + 1)(\frac{x_i}{\beta})^\alpha},$$

respectively. For more simplification in simulation study, the second posterior density can be rewritten as

$$\begin{aligned}\pi_2(\alpha, \beta \mid \underline{x}) \propto{}& \frac{[2\sum_{i=1}^m (R^*_i + 1) x_i^\alpha]^{2m - \frac{1}{\alpha}}}{\Gamma(2m - \frac{1}{\alpha})} \alpha^{m+a} \left(b \prod_{i=1}^m x_i\right)^\alpha \beta^{-2m\alpha} \\ &\times e^{-2\sum_{i=1}^m (R^*_i + 1)(\frac{x_i}{\beta})^\alpha} \frac{\Gamma(2m - \frac{1}{\alpha})}{\Gamma(m - \frac{1}{\alpha})} 2^{2m - \frac{1}{\alpha}} (\sum_{i=1}^m (R^*_i + 1) x_i^\alpha)^m.\end{aligned}$$

Table 2. The functions in equation (20).

Abbreviation	Function
$h_{11}(\alpha \mid \beta, \underline{x})$	$\alpha^{a+m-1}[\frac{e^{-b}\prod_{j=1}^{m} x_i}{\beta^m}]^{\alpha} e^{-\sum_{i=1}^{m}(R_i^*+1)(\frac{x_i}{\beta})^{\alpha}}$
$h_{12}(\beta \mid \alpha, \underline{x})$	$\beta^{c-m\alpha-1} e^{-[d\beta+\sum_{i=1}^{m}(R_i^*+1)(\frac{x_i}{\beta})^{\alpha}]}$
$h_{13}(\alpha, \beta \mid \underline{x})$	1
$h_{21}(\alpha \mid \beta, \underline{x})$	$\alpha^{m+a-1}(b\prod_{i=1}^{m} x_i)^{\alpha}$
$h_{22}(\beta \mid \alpha, \underline{x})$	$\frac{\alpha[2\sum_{i=1}^{m}(R_i^*+1)x_i^{\alpha}]^{2m-\frac{1}{\alpha}}}{\Gamma(2m-\frac{1}{\alpha})} \beta^{-2m\alpha} e^{-2\sum_{i=1}^{m}(R_i^*+1)(\frac{x_i}{\beta})^{\alpha}}$
$h_{23}(\alpha, \beta \mid \underline{x})$	$\frac{\Gamma(2m-\frac{1}{\alpha})}{\Gamma(m-\frac{1}{\alpha})} 2^{2m-\frac{1}{\alpha}} (\sum_{i=1}^{m}(R_i^*+1)x_i^{\alpha})^m$

The posterior density functions $\pi_1(\alpha, \beta \mid \underline{x})$ and $\pi_2(\alpha, \beta \mid \underline{x})$ can be rewritten as

$$\pi_i(\alpha, \beta \mid \underline{x}) \propto h_{i1}(\alpha \mid \beta, \underline{x}) h_{i2}(\beta \mid \alpha, \underline{x}) h_{i3}(\alpha, \beta \mid \underline{x}), \quad i = 1, 2,$$

or equivalently

$$\pi_i(\alpha, \beta \mid \underline{x}) = \frac{h_{i1}(\alpha \mid \beta, \underline{x}) h_{i2}(\beta \mid \alpha, \underline{x}) h_{i3}(\alpha, \beta \mid \underline{x})}{\int_0^{\infty}\int_0^{\infty} h_{i1}(\alpha \mid \beta, \underline{x}) h_{i2}(\beta \mid \alpha, \underline{x}) h_{i3}(\alpha, \beta \mid \underline{x}) d\alpha d\beta}, \quad i = 1, 2. \tag{20}$$

The functions $h_{i1}(\alpha \mid \beta, \underline{x})$ and $h_{i2}(\beta \mid \alpha, \underline{x})$, presented in Table (2), show the full conditional density function corresponding to the prior density functions $\pi_i(\alpha, \beta \mid \underline{x})$.

Thus, the Bayesian estimator of any function of α and β, say, $u(\alpha, \beta)$ considering the prior density function $\pi_i(\alpha, \beta),\ i = 1, 2$ under the SEL, LINEXL, EL, and SL functions are given by

$$u_{1Bi}^*(\alpha, \beta) = \frac{\int_0^{\infty}\int_0^{\infty} u(\alpha, \beta) h_{i1}(\alpha \mid \beta, \underline{x}) h_{i2}(\beta \mid \alpha, \underline{x}) h_{i3}(\alpha, \beta \mid \underline{x}) d\alpha d\beta}{\int_0^{\infty}\int_0^{\infty} h_{i1}(\alpha \mid \beta, \underline{x}) h_{i2}(\beta \mid \alpha, \underline{x}) h_{i3}(\alpha, \beta \mid \underline{x}) d\alpha d\beta}, \tag{21}$$

$$u_{2Bi}^*(\alpha, \beta) = -\log\Big[\frac{\int_0^{\infty}\int_0^{\infty} e^{-u(\alpha,\beta)} h_{i1}(\alpha \mid \beta, \underline{x}) h_{i2}(\beta \mid \alpha, \underline{x}) h_{i3}(\alpha, \beta \mid \underline{x}) d\alpha d\beta}{\int_0^{\infty}\int_0^{\infty} h_{i1}(\alpha \mid \beta, \underline{x}) h_{i2}(\beta \mid \alpha, \underline{x}) h_{i3}(\alpha, \beta \mid \underline{x}) d\alpha d\beta}\Big], \tag{22}$$

$$u_{3Bi}^*(\alpha, \beta) = \frac{\int_0^{\infty}\int_0^{\infty} h_{i1}(\alpha \mid \beta, \underline{x}) h_{i2}(\beta \mid \alpha, \underline{x}) h_{i3}(\alpha, \beta \mid \underline{x}) d\alpha d\beta}{\int_0^{\infty}\int_0^{\infty} \frac{1}{u(\alpha,\beta)} h_{i1}(\alpha \mid \beta, \underline{x}) h_{i2}(\beta \mid \alpha, \underline{x}) h_{i3}(\alpha, \beta \mid \underline{x}) d\alpha d\beta}, \tag{23}$$

$$u_{4Bi}^*(\alpha, \beta) = \sqrt{\frac{\int_0^{\infty}\int_0^{\infty} u(\alpha, \beta) h_{i1}(\alpha \mid \beta, \underline{x}) h_{i2}(\beta \mid \alpha, \underline{x}) h_{i3}(\alpha, \beta \mid \underline{x}) d\alpha d\beta}{\int_0^{\infty}\int_0^{\infty} \frac{1}{u(\alpha,\beta)} h_{i1}(\alpha \mid \beta, \underline{x}) h_{i2}(\beta \mid \alpha, \underline{x}) h_{i3}(\alpha, \beta \mid \underline{x}) d\alpha d\beta}}, \tag{24}$$

respectively. It is not possible here to calculate the mentioned relation analytically. Consequently, these integrals should be approximated by some numerical methods. For such a similar calculation in Bayesian inferences, the Monte Carlo methods are usually used to perform these ends. On the other hand, given the dependence of the parameters in both posterior densities, it has been needed to use the Gibbs sampling. Because in the case of Gibbs sampling, the generated variables are based on the variables that were generated in the previous steps, the Monte Carlo Markov Chain (MCMC) method is utilized. After that by utilizing the MCMC and Gibbs sampling method, some pairs of random variables have been generated from each of the posterior densities. For generating random numbers from full conditional posteriors related to first posterior densities, one should use Metropolis–Hastings algorithm. The step-by-step procedures to generate random numbers from the posterior densities and to calculate Bayesian estimators are presented in Algorithms (1) and (2).

Algorithm 1: Posterior Density 1

1: Start with an initial values of each parameters $(\alpha_1^{(0)}, \beta_1^{(0)})$.
2: Set $j = 1$.
3: Generate $\alpha_1^{(j)}$ from $h_{11}(\alpha \mid \beta_1^{(j-1)}, \underline{x})$ with the $N(\alpha_1^{(j-1)}, Var(\hat{\alpha}))$ as a proposal distribution.
4: Generate $\beta_1^{(j)}$ from $h_{22}(\beta \mid \alpha_1^{(j)}, \underline{x})$ with the $N(\beta_1^{(j-1)}, Var(\hat{\alpha}))$ as a proposal distribution.
5: Compute $\alpha_i^{(j)}$ and $\beta_i^{(j)}$.
6: Set $j = j + 1$.
7: Repeat steps $3 - 6$ N times.
8: Obtain the Bayesian estimators of $S_X(x)$, $Q_X(x)$, and $H_X(x)$ based on SEL, LINEX, EL, and SL functions, using the following numerical integral approximation

$$\int_0^\infty \int_0^\infty h(\alpha,\beta) h_{11}(\alpha \mid \beta, \underline{x}) h_{12}(\beta \mid \alpha, \underline{x}) \times h_{13}(\alpha, \beta \mid \underline{x}) d\alpha d\beta = \frac{\sum_{j=M+1}^{N} h(\alpha_1^{(j)}, \beta_1^{(j)})}{N - M} \quad (25)$$

for any two variables function $h(\cdot,\cdot)$ and M as the burn-in number for eliminating the effects of choosing initial values.
9: For any given prefix value of x, to compute the credible intervals of $S_X(x)$, $Q_X(x)$, and $H_X(x)$ increasingly sort the computed Bayesian estimators of these parameters as $S_X^{(i1)}(x) \leq S_X^{(i2)}(x) \leq \cdots \leq S_X^{(i(N-M))}(x)$, $Q_X^{(i1)}(x) \leq Q_X^{(i2)}(x) \leq \cdots \leq Q_X^{(i(N-M))}(x)$, and $H_X^{(i1)}(x) \leq H_X^{(i2)}(x) \leq \cdots \leq H_X^{(i(N-M))}(x)$.
10: Now, the $100(1 - \epsilon_1 - \epsilon_2)$ percent credible intervals of $S_X(x)$, $Q_X(x)$, and $H_X(x)$, respectively, are given by $(S_X^{(i([\epsilon_1 \frac{N-M}{2}]))}(x), S_X^{(i([\epsilon_2 \frac{N-M}{2}]))}(x))$, $(Q_X^{(i([\epsilon_1 \frac{N-M}{2}]))}(x), Q_X^{(i([\epsilon_2 \frac{N-M}{2}]))}(x))$, and $(H_X^{(i([\epsilon_1 \frac{N-M}{2}]))}(x), H_X^{(i([\epsilon_2 \frac{N-M}{2}]))}(x))$, where $[x]$ denotes the greatest integer number less than or equal to the value of x.
11: The End.

Algorithm 2: Posterior Density 2

1: Set $j = 1$.
2: Generate $\alpha_2^{(j)}$ from a gamma distribution with shape parameter $a + m$ and scale parameter $\frac{1}{\log(b \prod_{i=1}^m x_{i:m:n})}$.
3: Generate β^* from a gamma distribution with shape parameter $2m - \frac{1}{\alpha_2^{(j)}}$ and scale parameter $\frac{1}{2\sum_{i=1}^m (R_i^*+1) x_{i:m:n}^{\alpha_2^{(j)}}}$ and set $\beta_2^{(j)} = (\beta^*)^{-\frac{1}{\alpha_2^{(j)}}}$.
4: Set

$$h_{23}(\alpha_2^{(j)}, \beta_2^{(j)} \mid \underline{x}) = 2^{2m - \frac{1}{\alpha_2^{(j)}}} \times \frac{\Gamma(2m - \frac{1}{\alpha_2^{(j)}})}{\Gamma(m - \frac{1}{\alpha_2^{(j)}})} (\sum_{i=1}^m (R_i^* + 1) x_i^{\alpha_2^{(j)}})^m \quad (26)$$

5: Compute $\alpha_2^{(j)}$, $\beta_2^{(j)}$ and $h_{23}(\alpha_2^{(j)}, \beta_2^{(j)} \mid \underline{x})$.
6: Set $j = j + 1$.
7: Repeat steps $3 - 6$ N times.
8: Obtain the Bayesian estimators of $S_X(x)$, $Q_X(x)$, and $H_X(x)$ based on SEL, LINEX, EL, and SL functions, using the following numerical integral approximation

$$\int_0^\infty \int_0^\infty h(\alpha,\beta) h_{21}(\alpha \mid \beta, \underline{x}) h_{22}(\beta \mid \alpha, \underline{x}) \times h_{23}(\alpha, \beta \mid \underline{x}) d\alpha d\beta = \frac{\sum_{j=1}^{N^*} h(\alpha_1^{(j)}, \beta_1^{(j)}) \times h_{23}(\alpha_2^{(j)}, \beta_2^{(j)} \mid \underline{x})}{N^*} \quad (27)$$

for any two variables function $h(\cdot,\cdot)$ and $N^* = N - M$ (Regarding the comparable results of two prior density, the number of simulated random variables are equally considered.).
9: Do the same as the steps 9 and 10 in Algorithm (2) with the new constructed desired parameters.
10: The End.

7. Conditional Estimators

As it mentioned earlier, conditional confidence intervals for the original parameters as well as for *q.f.* of the Weibull distribution in the literature were developed under complete or Type-II censored samples. In this section, the exact *c.d.f.* of the conditional estimators of *s.f.*, *h.r.f.*, and *q.f.* of the SEV distribution given a *APC-II* censored sample are provided. To this end, let $\boldsymbol{a}$ be a $1 \times m$ vector of pivotal quantities with elements $a_i = \frac{Y_{i:m:n}^R - \hat{\mu}}{\hat{\sigma}}$, $i = 1, 2, \ldots, m$. In addition, let us define new parameters $Z_1 = \frac{\hat{\mu} - \mu}{\hat{\sigma}}$ and $Z_2 = \frac{\hat{\sigma}}{\sigma}$. Given the vector $\boldsymbol{a}$, the likelihood function regarding these parameters for

$$f_{Z_1,Z_2|\boldsymbol{a}}(z_1, z_2) \propto z_2^{m-2} e^{\sum_{i=1}^m (z_1 + a_i) z_2} e^{-\sum_{i=1}^m (R_i^*+1) e^{(z_1+a_i)z_2}}. \quad (28)$$

The conditional *c.d.f.* of *s.f.*, *h.r.f.*, and *q.f.* in (8), (9), and (10) are calculated in the following subsections.

7.1. *Conditional estimation of s.f.*

The role of *s.f.* in medical and engineering studies is undeniable. In medicine, it is very important to know how long a patient is alive from a certain time and also it is so functional to know the probability of failure for any component or system after its work to a specific time. It is so obvious to understand the importance of estimation in the mentioned situations. The following theorem provides the statistical matter to construct conditional confidence interval of *s.f.* in the SEV model under an *APC-II* plan.

Theorem 1. Let $c = \frac{y-\hat{\mu}}{\hat{\sigma}}$. The conditional *c.d.f.* of *s.f.* in the SEV model under an *APC-II* plan is

$$P(S_Y(y) \leq z \mid \boldsymbol{a}) = \frac{\int_0^{\infty} \frac{w^{m-2} e^{\sum_{i=1}^m (a_i - c)w}}{[\sum_{i=1}^m (R_i^*+1) e^{(a_i-c)w}]^m} \left[\int_{-\log(z) \sum_{i=1}^m (R_i^*+1) e^{(a_i-c)w}}^{\infty} \frac{s^{m-1} e^{-s}}{(m-1)!} ds \right] dw}{\int_0^{\infty} \frac{w^{m-2} e^{\sum_{i=1}^m (a_i - c)w}}{[\sum_{i=1}^m (R_i^*+1) e^{(a_i-c)w}]^m} dw}.$$

Proof
Let we define the new auxiliary random variable $U = Z_2(Z_1 + \frac{y-\hat{\mu}}{\hat{\sigma}})$. The joint density function of U and Z_2 can be obtained after some slight mathematical calculations as

$$f_{U,Z_2|\boldsymbol{a}}(u,w) \propto w^{m-2} e^{mu + \sum_{i=1}^m (a_i-c)w} e^{-e^u \sum_{i=1}^m (R_i^*+1) e^{(a_i-c)w}}.$$

Now, we have

$$\begin{aligned} P\left(S_Y(y) \leq z \mid \boldsymbol{a}\right) &= P\left(e^{-e^{\frac{y-\mu}{\sigma}}} \leq z \mid \boldsymbol{a} \right) \\ &= P\left(U \geq \log(-\log(z)) \mid \boldsymbol{a}\right) \\ &= \int_{\log(-\log(z))}^{\infty} f_{U|\boldsymbol{a}}(u) du \\ &= \int_{\log(-\log(z))}^{\infty} \int_0^{\infty} f_{U,Z_2|\boldsymbol{a}}(u,w) dw du \\ &= \int_0^{\infty} \int_{\log(-\log(z))}^{\infty} f_{U,Z_2|\boldsymbol{a}}(u,w) du dw. \end{aligned}$$

Finally, substituting the transformation $s = e^u$ in the last integrand completes the proof. □

It is clear that $(S_{\epsilon_1}(y), S_{1-\epsilon_2}(y))$ is a $100(1-(\epsilon_1+\epsilon_2))\%$ confidence interval of the SEV's *s.f.* at any point $y \in \mathbb{R}$ such that $P(S_Y(y) \leq S_\epsilon(y) \mid a) = \epsilon$ for all $\epsilon \in [0,1]$.

7.2. *Conditional estimation of h.r.f.*

Throughout the studying of life span of a component, item, system, or even an alive creature, it is very useful to know the dynamic behavior of event rate at a given time t conditioning on the survival until or later t, i.e., $X \geq t$. This information is provided through the *h.r.f.* which is defined as $H_X(t) = \lim_{dt \longrightarrow 0} \frac{Pr(X<t+dt|X>t)}{dt} = \frac{f_X(x)}{S_X(t)}$ for any continuous random variable X. The following theorem presents the exact conditional *c.d.f.* of *h.r.f.* in SEV model under an *APC-II* plan.

Theorem 2. Given the assumptions of Theorem (1), the conditional *c.d.f.* of *h.r.f.* in SEV model under an *APC-II* plan is

$$P(H_Y(y) \leq z \mid \boldsymbol{a}) = \frac{\int_0^{\infty} \frac{w^{m-2} e^{\sum_{i=1}^m a_i w}}{[\sum_{i=1}^m (R_i^*+1) e^{a_i w}]^m} \left[\int_0^{\hat{\sigma} z \frac{e^{-cw} \sum_{i=1}^m (R_i^*+1) e^{a_i w}}{w}} \frac{s^{m-1} e^{-s}}{(m-1)!} ds \right] dw}{\int_0^{\infty} \frac{w^{m-2} e^{\sum_{i=1}^m a_i w}}{[\sum_{i=1}^m (R_i^*+1) e^{a_i w}]^m} dw}.$$

Proof
Let define the auxiliary random variable $U = Z_2 e^{(c+Z_1)Z_2}$. Applying a two-dimensional transformation and using

the Jacobian rule, the joint *p.d.f.* of variables U and Z_2 is derived as

$$f_{U,Z_2|\boldsymbol{a}}(u,w) \propto w^{-2}u^{m-1}e^{-cmw}e^{\sum_{i=1}^{m} a_i w}e^{-u\frac{e^{-cw}\sum_{i=1}^{m}(R_i^*+1)e^{a_i w}}{w}}.$$

Now, the conditional marginal *c.d.f.* of $H_Y(y)$ is obtained as

$$\begin{aligned}
P(H_Y(y) \le z \mid \boldsymbol{a}) &= P\left(\frac{1}{\sigma}e^{\dfrac{y-\mu}{\sigma}} \le z \mid \boldsymbol{a}\right) \\
&= P(U \le \hat{\sigma}z \mid \boldsymbol{a}) \\
&= \int_0^{\hat{\sigma}z} f_{U|\boldsymbol{a}}(u)du \\
&= \int_0^{\hat{\sigma}z}\int_0^{\infty} f_{U,Z_2|\boldsymbol{a}}(u,w)dwdu \\
&= \int_0^{\infty}\int_0^{\hat{\sigma}z} f_{U,Z_2|\boldsymbol{a}}(u,w)dudw.
\end{aligned}$$

Substituting $s = u\frac{e^{-cw}\sum_{i=1}^{m}(R_i^*+1)e^{a_i w}}{w}$ in the last integrand and some naive manipulations result in the desired result. □

It is clear that $(H_{\epsilon_1}(y), H_{1-\epsilon_2}(y))$ is a $100(1-(\epsilon_1+\epsilon_2))\%$ confidence interval for the SEV's *h.r.f.* at given point $y \in \mathbb{R}$ such that $P(H_Y(y) \le H_\epsilon(y) \mid a) = \epsilon$ for all $\epsilon \in [0,1]$.

7.3. Conditional estimation of q.f.

While *s.f.* gives us a probability for a given point within the domain of random variable, *q.f.* gives us the point of the domain that the given probability occurred. Moreover, the extreme quantiles have the ability to express the behavior of the tails of the random variable and also are helpful in finding its natural tolerance range.

Theorem 3. Given the assumptions of Theorem (1), the conditional *c.d.f.* of *q.f.* in SEV model under an *APC-II* plan is

$$P(Q_Y(y) \le z \mid \boldsymbol{a}) = \frac{\int_0^\infty \frac{w^{m-2}e^{\sum_{i=1}^{m} a_i w}}{[\sum_{i=1}^{m}(R_i^*+1)e^{a_i w}]^m}\left[\int_{e^{-w\frac{z-\hat{\mu}}{\hat{\sigma}}+w(y)}\sum_{i=1}^{m}(R_i^*+1)e^{a_i w}}^{\infty} \frac{s^{m-1}e^{-s}}{(m-1)!}ds\right]dw}{\int_0^\infty \frac{w^{m-2}e^{\sum_{i=1}^{m} a_i w}}{[\sum_{i=1}^{m}(R_i^*+1)e^{a_i w}]^m}dw}.$$

Proof
Let us define the auxiliary random variable $U = \frac{w(y)}{Z_2} - Z_1$ and auxiliary value $w(y) = \log(-\log(1-y))$. Therefore, the joint *p.d.f.* of U and Z_2 is given by

$$f_{U,Z_2|\boldsymbol{a}}(u,w) \propto w^{m-2}e^{-muw+mw(y)+\sum_{i=1}^{m} a_i w}e^{-e^{w(y)-uw}\sum_{i=1}^{m}(R_i^*+1)e^{a_i w}}.$$

Now, the conditional marginal *c.d.f.* of $Q_Y(y)$ is obtained as

$$
\begin{aligned}
P(Q_Y(y) \leq z \mid \boldsymbol{a}) &= P\left(\mu + \sigma \log(-\log(1-y)) \leq z \mid \boldsymbol{a}\right) \\
&= P\left(w(y) - \frac{\hat{\mu} + \hat{\sigma} w(y) - \mu - \sigma w(y)}{\hat{\sigma}} \leq \frac{z - \hat{\mu}}{\hat{\sigma}} \mid \boldsymbol{a}\right) \\
&= P\left(\frac{w(y)}{Z_2} - Z_1 \leq \frac{z - \hat{\mu}}{\hat{\sigma}} \mid \boldsymbol{a}\right) \\
&= \int_{-\infty}^{\frac{z-\hat{\mu}}{\hat{\sigma}}} f_{U|\boldsymbol{a}}(u) du \\
&= \int_{-\infty}^{\frac{z-\hat{\mu}}{\hat{\sigma}}} \int_0^{\infty} f_{U,Z_2|\boldsymbol{a}}(u, w) dw du \\
&= \int_0^{\infty} \int_{-\infty}^{\frac{z-\hat{\mu}}{\hat{\sigma}}} f_{U,Z_2|\boldsymbol{a}}(u, w) du dw.
\end{aligned}
$$

Substituting $s = e^{-wu+w(y)}$ in the last integral completes the proof. □

It is straightforward to check that $(Q_{\epsilon_1}(y), Q_{1-\epsilon_2}(y))$ is a $100(1-(\epsilon_1+\epsilon_2))\%$ confidence interval of the SEV's *q.f.* given the point $0 < y < 1$ such that $P(Q_Y(y) \leq Q_\epsilon(y) \mid a) = \epsilon$ for all $\epsilon \in [0, 1]$.

8. Simulation Study

This section provides a comparison study for investigating the behaviour of the presented ML, AML, Percentile-Bootstrap, Bayesian, and conditional estimators using the Monte Carlo simulations. These ends are assessed through four popular criteria including mean and MSE of point estimators and the average length and empirical coverage probability of confidence intervals. During the simulations, we consider sample of sizes $n = 54,\ m = 18, 27$ and $n = 108,\ m = 36, 54$ for two different Weibull distributions with common scale parameter $\beta = 1$ and shape parameters $\alpha = 0.5$ and $\alpha = 1.5$. The elements of censoring scheme are considered equally and to investigate the performances of the time threshold, the values $0, \infty$ mode and mean of the corresponding distribution are provided in each case. The number Bootstrap and Monte Carlo's repetitions in the Bayesian estimation are considered as $B = 1000$ and 10000, respectively. For the Bayesian part, the conjugate prior with parameters $a = 1$ and $b = 1$ and the SEL has been considered.

The choice of conjugate prior and SEL have been made in accordance with the fact that conjugate priors are more efficient in generating method and precise estimation. Moreover, under these priors, there is no notable difference in estimation based on different loss functions. In addition, the comparisons are tabulated under censoring schemes $R = (1^{(27)}), (1^{(54)}), (2^{(18)}), (2^{(36)})$ in Tables $3 - 6$.

Table 3. Comparison results: small sample size and decreasing failure rate

$n = 54$	$\alpha = 0.5$	$\beta = 1$	Significant level $= 0.95$		
Threshold	Parameter	MLE	P-Bootstrap	Bayes	Conditional
$m = 18$	$R = (2^{(18)})$		$m = 27$	$R = (1^{(27)})$	
T=0	$h_X(0.5)$	(4.1800)[1.6990]	(0.8826)[0.2424]	(-0.4473)[-0.2781]	(0.3769)[0.2116]
		(20.8512)[4.4595]	(3.5053)[0.2163]	(0.2017)[0.0807]	(0.1998)[0.1801]
		(12.4579)[3.3636]	(4.5084)[1.3522]	(0.4012)[0.4627]	(0.3126)[0.3458]
		(0.9899)[0.5059]	(0.8560)[0.8961]	(0.0211)[0.4524]	(0.4859)[0.5563]
	$S_X(2)$	(-0.2287)[-0.2120]	(-0.0654)[-0.0392]	(0.2690)[0.1296]	(0.0961)[0.0625]
		(0.0531)[0.0461]	(0.0143)[0.0078]	(0.0732)[0.0177]	(0.0326)[0.0258]
		(0.0774)[0.1078]	(0.3642)[0.2981]	(0.3562)[0.2760]	(0.4682)[0.3976]
		(0.0414)[0.0361]	(0.8534)[0.8896]	(0.0370)[0.6072]	(0.8423)[0.8637]
	$q_X(0.9)$	(-4.6218)[-4.2401]	(4.391124)[0.87968]	(5.2369)[1.2574]	(0.9621)[0.7563]
		(21.6346)[18.2696]	(29.6149)[21.3926]	(32.6895)[3.2478]	(1.0256)[0.9865]
		(1.5381)[1.6443]	(48.3359)[17.3078]	(35.6289)[4.1298]	(1.1236)[1.0358]
		(0.0220)[0.0149]	(0.8533)[0.8890]	(0.7896)[0.8159]	(0.6523)[0.5986]
T=0.480453	$h_X(0.5)$	(0.2684)[0.2484]	(0.4443)[0.1690]	(-0.3562)[-0.1691]	(0.2983)[0.1629]
		(0.4329)[0.1674]	(0.7014)[0.1073]	(0.1292)[0.0327]	(0.1126)[0.1046]
		(1.4829)[0.9835]	(2.2859)[1.0173]	(0.5295)[0.5293]	(0.2679)[0.2987]
		(0.9785)[0.9577]	(0.8631)[0.8935]	(0.3701)[0.8736]	(0.8956)[0.9324]
	$S_X(2)$	(-0.0497)[-0.0648]	(-0.0532)[-0.0293]	(0.1953)[0.0736]	(0.0456)[0.0568]
		(0.0126)[0.0101]	(0.0110)[0.0060]	(0.0388)[0.0063]	(0.0375)[0.0158]
		(0.3616)[0.2634]	(0.3337)[0.2748]	(0.3773)[0.2626]	(0.3258)[0.2378]
		(0.8248)[0.7628]	(0.8580)[0.8976]	(0.5130)[0.9016]	(0.8964)[0.9158]
	$q_X(0.9)$	(-0.3442)[-1.4919]	(1.0929)[0.2317]	(2.2356)[1.0123]	(0.8962)[0.6821]
		(19.0946)[6.9034]	(35.2691)[9.3217]	(4.2589)[1.9687]	(0.8951)[0.7924]
		(15.0707)[7.3678]	(21.4803)[11.9523]	(2.5879)[2.0125]	(0.8714)[0.6312]
		(0.7157)[0.6211]	(0.8504)[0.8837]	(0.7981)[0.8219]	(0.8692)[0.8761]
T=1.992439	$h_X(0.5)$	(0.1740)[0.0933]	(0.4385)[0.1633]	(-0.3527)[-0.1570]	(0.1841)[0.1025]
		(0.2534)[0.0636]	(0.8227)[0.0993]	(0.1267)[0.0288]	(0.1362)[0.1163]
		(1.2850)[0.7782]	(2.2620)[0.9878]	(0.5332)[0.5336]	(0.2479)[0.2641]
		(0.9675)[0.9668]	(0.8673)[0.8967]	(0.3907)[0.8991]	(0.9687)[0.9597]
	$S_X(2)$	(-0.0302)[-0.0220]	(-0.0532)[-0.0309]	(0.1931)[0.0678]	(-0.0365)[0.0249]
		(0.0108)[0.0061]	(0.0109)[0.0061]	(0.0379)[0.0056]	(0.0269)[0.0199]
		(0.3696)[0.2759]	(0.3326)[0.2714]	(0.3775)[0.2609]	(0.2698)[0.2189]
		(0.8668)[0.8923]	(0.8575)[0.8894]	(0.5285)[0.9136]	(0.9357)[0.9586]
	$q_X(0.9)$	(0.3109)[-0.1721]	(1.0278)[0.2149]	(0.5896)[0.6215]	(0.7962)[0.6321]
		(22.4499)[8.0703]	(30.2841)[11.3698]	(0.2489)[0.3289]	(0.7321)[0.6258]
		(17.2937)[10.4117]	(20.9341)[11.5985]	(0.2219)[0.2986]	(0.8152)[0.6028]
		(0.7846)[0.8094]	(0.8572)[0.8794]	(0.9014)[0.9246]	(0.9214)[0.9453]
T=∞	$h_X(0.5)$	(0.1801)[0.0756]	(0.4392)[0.1628]	(-0.3529)[-0.1544]	(0.1936)[0.1125]
		(0.2651)[0.0580]	(0.7000)[.0970]	(0.1268)[0.0280]	(0.1324)[0.1216]
		(1.2974)[0.7555]	(2.2562)[0.9852]	(0.5328)[0.5344]	(0.2314)[0.2519]
		(0.9709)[0.9621]	(0.8614)[0.8958]	(0.3857)[0.9040]	(0.9547)[0.9632]
	$S_X(2)$	(-0.0291)[-0.0157]	(-0.0530)[-0.0298]	(0.1926)[0.0679]	(-0.0258)[-0.0201]
		(0.0109)[0.0057]	(0.0111)[0.0061]	(0.0377)[0.0056]	(0.0361)[0.0258]
		(0.3701)[0.2768]	(0.3320)[0.2712]	(0.3777)[0.2601]	(0.2754)[0.2169]
		(0.8641)[0.9078]	(0.8542)[0.8905]	(0.5371)[0.9071]	(0.9589)[0.9609]
	$q_X(0.9)$	(0.3845)[0.1063]	(0.8868)[0.1933]	(1.0298)[1.0327]	(0.8324)[0.7986]
		(23.9362)[8.3225]	(29.9631)[10.2587]	(1.2147)[1.3294]	(0.7982)[0.7163]
		(17.5475)[11.0895]	(20.3291)[11.4727]	(0.9968)[0.9852]	(0.2896)[0.1986]
		(0.7896)[0.8412]	(0.8418)[0.8745]	(0.7146)[0.7329]	(0.9327)[0.9583]

Table 4. Comparison results: large sample size and decreasing failure rate

$n=108$	$\alpha=0.5$	$\beta=1$	Significant level $=0.95$		
Threshold	Parameter	MLE	P-Bootstrap	Bayes	Conditional
$m=36$	$R=(2^{(36)})$		$m=54$	$R=(1^{(54)})$	
T=0	$h_X(0.5)$	(3.0738)[1.5075]	(4.6035)[1.7766]	(5.0258)[3.2598]	(0.1254)[0.1058]
		(14.3990)[2.7661]	(17.8516)[5.2381]	(20.2419)[8.3627]	(0.1412)[0.1123]
		(6.1195)[2.1165]	(13.2153)[3.1091]	(11.2418)[7.2395]	(0.1189)[0.0986]
		(0.4497)[0.0143]	(0.0057)[0.0011]	(0.1247)[0.1127]	(0.9023)[0.9142]
	$S_X(2)$	(-0.2339)[-0.2168]	(-0.2305)[-0.2154]	(4.1239)[2.1749]	(0.1142)[0.1028]
		(0.0549)[0.0475]	(0.0534)[0.0469]	(13.2698)[3.2174]	(0.1025)[0.1003]
		(0.0448)[0.0752]	(0.0593)[0.0795]	(6.2175)[3.0012]	(0.0985)[0.0902]
		(0.0026)[0.0007]	(0.0037)[0.0013]	(0.1495)[0.1028]	(0.9123)[0.9208]
	$q_X(0.9)$	(-4.6783)[-4.2957]	(1.5636)[0.3451]	(3.2785)[1.0256]	(0.1526)[0.1428]
		(21.9772)[18.5719]	(62.3574)[30.2197]	(7.6328)[1.2058]	(0.1149)[0.1029]
		(0.9451)[1.0738]	(21.5669)[10.4083]	(5.3274)[1.0029]	(0.0952)[0.0902]
		(0.0005)[0.0003]	(0.9030)[0.9195]	(0.1426)[0.1003]	(0.8927)[0.9009]
T=0.480453	$h_X(0.5)$	(0.1783)[0.2056]	(0.2816)[0.2601]	(4.2356)[2.1456]	(0.1057)[0.0859]
		(0.1119)[0.0825]	(0.1835)[0.1155]	(15.3628)[6.3589]	(0.0745)[0.0628]
		(0.9080)[0.6532]	(1.1604)[0.7503]	(13.2578)[8.2584]	(0.0526)[0.0412]
		(0.9716)[0.8707]	(0.8127)[0.6449]	(0.2036)[0.1785]	(0.9327)[0.9527]
	$S_X(2)$	(-0.0457)[-0.0635]	(-0.0599)[-0.0698]	(3.6958)[2.1589]	(0.0845)[0.0714]
		(0.0077)[0.0072]	(0.0087)[0.0079]	(11.2578)[2.9986]	(0.0529)[0.0418]
		(0.2719)[0.1928]	(0.2545)[0.1880]	(5.2896)[2.0247]	(0.0421)[0.0401]
		(0.8405)[0.7032]	(0.8025)[0.6632]	(0.2147)[0.1852]	(0.9341)[0.9396]
	$q_X(0.9)$	(-0.8821)[-1.7089]	(0.4088)[0.1074]	(2.2859)[1.0058]	(0.1125)[0.0964]
		(7.2756)[4.8064]	(48.5698)[21.5826]	(5.6398)[0.9952]	(0.0859)[0.0748]
		(9.0133)[4.7834]	(12.5161)[8.0206]	(3.2569)[0.9852]	(0.0528)[0.0419]
		(0.7228)[0.5364]	(0.8944)[0.9157]	(0.3214)[0.2689]	(0.9251)[0.9379]
T=1.992439	$h_X(0.5)$	(0.0736)[0.0623]	(0.1657)[0.1008]	(0.9865)[0.8512]	(0.0321)[0.0204]
		(0.0548)[0.0256]	(0.0975)[0.0353]	(1.2351)[1.0215]	(0.0052)[0.0041]
		(0.7633)[0.5207]	(0.9790)[0.5887]	(0.8562)[0.8324]	(0.1254)[01103]
		(0.9647)[0.9548]	(0.9014)[0.8886]	(0.4895)[0.3961]	(0.9425)[0.9508]
	$S_X(2)$	(-0.0170)[-0.0172]	(-0.0303)[-0.0247]	(1.9526)[1.2413]	(0.0278)[0.0325]
		(0.0056)[0.0031]	(0.0057)[0.0039]	(2.3695)[2.1478]	(0.0123)[0.0109]
		(0.2761)[0.1997]	(0.2662)[0.1986]	(1.9965)[1.3358]	(0.0102)[0.0092]
		(0.9080)[0.9142]	(0.9010)[0.8924]	(0.7562)[0.7103]	(0.9452)[0.9501]
	$q_X(0.9)$	(0.1271)[-0.3058]	(0.4278)[0.0929]	(0.7856)[0.5263]	(0.0815)[0.0521]
		(9.0214)[3.5948]	(35.9124)[17.6289]	(0.5123)[0.4125]	(0.0421)[0.0369]
		(11.3723)[7.0580]	(12.4145)[7.8041]	(0.3215)[0.2986]	(0.0125)[0.0109]
		(0.8459)[0.8363]	(0.8984)[0.9201]	(0.6283)[0.5981]	(0.9416)[0.9527]
T=∞	$h_X(0.5)$	(0.0739)[0.0352]	(0.1595)[0.707]	(2.2563)[1.9562]	(0.0785)[0.0562]
		(0.0552)[0.0193]	(0.0899)[0.0269]	(5.2365)[3.2471]	(0.0038)[0.0256]
		(0.7630)[0.4966]	(0.9683)[0.5602]	(4.2518)[3.6528]	(0.0096)[0.0081]
		(0.9645)[0.9595]	(0.9005)[0.9197]	(0.4856)[0.5125]	(0.8962)[0.9263]
	$S_X(2)$	(-0.0158)[-0.0079]	(-0.0300)[-0.0152]	(2.1254)[1.8254]	(0.0563)[0.0895]
		(0.0055)[0.0028]	(0.0057)[0.0029]	(6.3254)[4.9652]	(0.0296)[0.0276]
		(0.2764)[0.2001]	(0.2665)[0.2001]	(4.2518)[3.8524]	(0.0302)[0.0284]
		(0.9098)[0.9278]	(0.8971)[0.9204]	(0.6142)[0.6895]	(0.9226)[0.9321]
	$q_X(0.9)$	(0.1693)[0.0750]	(0.4240)[0.0642]	(1.9584)[1.3562]	(0.1124)[0.0962]
		(9.4741)[4.0082]	(29.3674)[21.3217]	(3.9527)[2.0281]	(0.0985)[0.0741]
		(11.4549)[7.6985]	(12.3950)[7.7387]	(2.2518)[1.8652]	(0.0421)[0.0325]
		(0.8452)[0.8838]	(0.8967)[0.9116]	(0.4759)[0.5123]	(0.9235)[0.9367]

Table 5. Comparison results: small sample size and increasing failure rate

$n = 54$	$\alpha = 1.5$	$\beta = 1$	Significant level $= 0.95$		
Threshold	Parameter	MLE	P-Bootstrap	Bayes	Conditional
$m = 18$	$R = (2^{(18)})$		$m = 27$	$R = (1^{(27)})$	
T=0	$h_X(0.5)$	(3.0553)[1.3740]	(0.5222)[0.1555]	(-0.5211)[-0.2555]	(-0.5029)[0.1136]
		(19.9875)[2.6183]	(1.0212)[0.1262]	(0.2747)[0.0701]	(0.2369)[0.0628]
		(6.6918)[2.2252]	(2.7909)[1.1957]	(0.6434)[0.6237]	(0.5693)[0.5129]
		(0.7720)[0.2242]	(0.8794)[0.9166]	(0.1285)[0.7616]	(0.6981)[0.7893]
	$S_X(2)$	(-0.0583)[-0.0579]	(-0.0010)[-0.0029]	(0.3056)[0.1417]	(0.2516)[0.1216]
		(0.0034)[0.0033]	(0.0030)[0.0018]	(0.0951)[0.0214]	(0.0896)[0.0462]
		(0.0064)[0.0082]	(0.1968)[0.1547]	(0.4286)[0.2962]	(0.2639)[0.1796]
		(0.0183)[0.0157]	(0.8575)[0.8843]	(0.0426)[0.4869]	(0.8426)[0.8963]
	$q_X(0.9)$	(-0.9046)[-0.7522]	(-0.1074)[-0.0817]	(9.6284)[1.5278]	(0.5623)[0.3417]
		(0.8511)[0.5901]	(0.3378)[0.1286]	(175.5635)[2.8341]	(0.3976)[0.3128]
		(0.5834)[0.4913]	(2.1390)[1.3604]	(36.5327)[4.9061]	(0.8621)[0.7126]
		(0.0387)[0.0241]	(0.8543)[0.8830]	(0.0408)[0.5369]	(0.8639)[0.9128]
T=0.783219	$h_X(0.5)$	(0.2212)[0.1962]	(0.3152)[0.1137]	(-0.3815)[-0.1396]	(0.1165)[0.0986]
		(0.2796)[0.1383]	(0.3742)[0.0887]	(0.1485)[0.0290]	(0.0986)[0.0325]
		(1.4124)[0.9978]	(1.8723)[1.0440]	(0.7646)[0.6403]	(0.3217)[0.2678]
		(0.9666)[0.9308]	(0.8850)[0.9201]	(0.7062)[0.8835]	(0.9367)[0.9512]
	$S_X(2)$	(-0.0120)[-0.0263]	(-0.0048)[-0.0046]	(0.2133)[0.0755]	(0.0125)[0.0149]
		(0.0026)[0.0017]	(0.0021)[0.0014]	(0.0469)[0.0067]	(0.0036)[0.0025]
		(0.1818)[0.1118]	(0.1650)[0.1364]	(0.4195)[0.2489]	(0.1628)[0.1139]
		(0.6784)[0.6004]	(0.8523)[0.8783]	(0.4430)[0.9236]	(0.9258)[0.9651]
	$q_X(0.9)$	(-0.1492)[-0.2327]	(-0.1314)[-0.0787]	(3.8905)[0.6714]	(0.2986)[0.1423]
		(0.1853)[0.1249]	(0.1810)[0.0929]	(21.4451)[0.5864]	(0.0562)[0.0236]
		(1.4948)[0.9333]	(1.5673)[1.1310]	(14.7554)[2.7320]	(0.1128)[0.0963]
		(0.7967)[0.6962]	(0.8490)[0.8808]	(0.4792)[0.9331]	(0.9258)[0.9562]
T=0.902745	$h_X(0.5)$	(0.1710)[0.1469]	(0.3157)[0.1120]	(-0.3792)[-0.1322]	(0.0984)[0.0714]
		(0.2130)[0.1089]	(0.3766)[0.0905]	(0.1468)[0.0273]	(0.0625)[0.0247]
		(1.3405)[0.9553]	(1.8686)[1.0389]	(0.7616)[0.6461]	(0.2578)[0.2375]
		(0.9676)[0.9389]	(0.8872)[0.9236]	(0.7025)[0.8928]	(0.9489)[0.9589]
	$S_X(2)$	(-0.0074)[-0.0193]	(-0.0056)[-0.0050]	(0.2116)[0.0719]	(0.0098)[0.0101]
		(0.0028)[0.0016]	(0.0020)[0.0013]	(0.0462)[0.0061]	(0.0025)[0.0021]
		(0.1925)[0.1269]	(0.1636)[0.1352]	(0.4189)[0.2459]	(0.1745)[0.1204]
		(0.7107)[0.6832]	(0.8526)[0.8812]	(0.4518)[0.9366]	(0.9347)[0.9508]
	$q_X(0.9)$	(-0.1087)[-0.1621]	(-0.1263)[-0.0779]	(3.8208)[0.6358]	(0.2187)[0.1749]
		(0.1806)[0.1033]	(0.1842)[0.0917]	(22.8739)[0.5321]	(0.0471)[0.0365]
		(1.5417)[1.0028]	(1.5669)[1.1211]	(14.4287)[2.6437]	(0.1357)[0.1257]
		(0.8265)[0.7800]	(0.8541)[0.8812]	(0.4916)[0.9449]	(0.9583)[0.9593]
T=∞	$h_X(0.5)$	(0.1389)[0.0565]	(0.3238)[0.1122]	(-0.3760)[-0.1273]	(0.0914)[0.0874]
		(0.1913)[0.0669]	(0.4144)[0.09164]	(0.1444)[0.0271]	(0.0716)[0.0629]
		(1.2960)[0.8745]	(1.8798)[1.0334]	(0.7680)[0.6451]	(0.2491)[0.2235]
		(0.9635)[0.9499]	(0.8759)[0.9236]	(0.7174)[0.8862]	(0.9458)[0.9561]
	$S_X(2)$	(-0.0023)[-0.0026]	(-0.0052)[-0.0055]	(0.2105)[0.0675]	(0.0091)[0.0084]
		(0.0029)[0.0016]	(0.0020)0.0013	(0.0457)[0.0055]	(0.0024)[0.0019]
		(0.2052)[0.1572]	(0.1639)[0.1333]	(0.4187)[0.2415]	(0.1569)[0.1102]
		(0.7484)[0.8284]	(0.8558)[0.8820]	(0.4611)[0.9484]	(0.9582)[0.9602]
	$q_X(0.9)$	(-0.0660)[-0.0426]	(-0.1205)[-0.0746]	(3.7444)[0.5934]	(0.2058)[0.1657]
		(0.1785)[0.0871]	(0.1844)[0.0882]	(22.1279)[0.4685]	(0.0415)[0.0362]
		(1.6020)[1.1232]	(1.5694)[1.1079]	(14.1100)[2.5437]	(0.1125)[0.1016]
		(0.8516)[0.8870]	(0.8469)[0.8815]	(0.5133)[0.9487]	(0.9598)[0.9624]

Table 6. Comparison results: large sample size and increasing failure rate

$n = 108$	$\alpha = 1.5$	$\beta = 1$	Significant level $= 0.95$		
Threshold	Parameter	MLE	P-Bootstrap	Bayes	Conditional
$m = 36$	$R = (2^{(36)})$		$m = 54$	$R = (1^{(54)})$	
T=0	$h_X(0.5)$	(2.5833)[1.3052]	(0.1994)[0.06894]	(-0.5270)[-0.2555]	(0.1528)[0.0985]
		(8.7511)[1.9945]	(0.1581)[0.04011]	(0.2793)[0.0676]	(0.1862)[0.1149]
		(3.8965)[1.4983]	(1.2974)[0.7193]	(0.4421)[0.4376]	(0.2289)[0.1976]
		(0.0309)[0.0123]	(0.9124)[0.9329]	(0.0000)[0.3928]	(0.9327)[0.9398]
	$S_X(2)$	(-0.0589)[-0.0586]	(0.0002)[-0.0018]	(0.3078)[0.1427]	(0.0014)[0.0093]
		(0.0034)[0.0034]	(0.0021)[0.0011]	(0.0956)[0.0210]	(0.0022)[0.0016]
		(0.0016)[0.0028]	(0.1685)[0.1272]	(0.3057)[0.2159]	(0.0176)[0.0116]
		(0.0004)[0.0002]	(0.8984)[0.9208]	(0.0001)[0.0507]	(0.9257)[0.9367]
	$q_X(0.9)$	(-0.9101)[-0.7535]	(-0.0576)[-0.0407]	(1.1258)[1.0068]	(0.0657)[0.0584]
		(0.8447)[0.5800]	(0.1545)[0.0638]	(1.6274)[1.3628]	(0.1126)[0.0926]
		(0.4065)[0.3473]	(1.5148)[0.9835]	(1.0149)[0.9984]	(0.8692)[0.7129]
		(0.0020)[0.0006]	(0.9019)[0.9210]	(0.0147)[0.0128]	(0.9146)[0.9207]
T=0.783219	$h_X(0.5)$	(0.1584)[0.1770]	(0.1264)[0.0533]	(-0.3785)[-0.1238]	(0.1023)[0.1238]
		(0.1038)[0.0747]	(0.0842)[0.0322]	(0.1447)[0.0192]	(0.0689)[0.0.0981]
		(0.9270)[0.6864]	(0.9858)[0.6501]	(0.5366)[0.4508]	(0.1852)[0.1871]
		(0.9467)[0.8705]	(0.9184)[0.9331]	(0.1461)[0.8723]	(0.9347)[0.9476]
	$S_X(2)$	(-0.0171)[-0.0297]	(-0.0029)[-0.0024]	(0.2100)[0.0728]	(0.0012)[0.0016]
		(0.0016)[0.0014]	(0.0014)[0.0008]	(0.0448)[0.0058]	(0.0075)[0.0105]
		(0.1326)[0.0797]	(0.1393)[0.1106]	(0.3086)[0.1823]	(0.0126)[0.0109]
		(0.7023)[0.5360]	(0.8956)[0.9145]	(0.0144)[0.6977]	(0.9356)[0.9397]
	$q_X(0.9)$	(-0.1514)[-0.2308]	(-0.0701)[-0.0417]	(1.3526)[1.1527]	(0.0592)[0.0501]
		(0.1056)[0.0880]	(0.0895)[0.0469]	(1.7895)[1.0256]	(0.0487)[0.0415]
		(1.0426)[0.6599]	(1.1441)[0.8184]	(2.0147)[1.8624]	(0.0874)[0.0709]
		(0.7965)[0.6300]	(0.8983)[0.9100]	(0.0986)[0.0896]	(0.9257)[0.9289]
T=0.902745	$h_X(0.5)$	(0.1089)[0.1257]	(0.1308)[0.0533]	(-0.3753)[-0.1175]	(0.0785)[0.0841]
		(0.0786)[0.0529]	(0.0848)[0.0320]	(0.1423)[0.0180]	(0.0476)[0.0781]
		(0.8798)[0.6549]	(0.9866)[0.6479]	(0.5368)[0.4520]	(0.1124)[0.1135]
		(0.9553)[0.9163]	(0.9140)[0.9317]	(0.1526)[0.8754]	(0.9412)[0.9475]
	$S_X(2)$	(-0.0101)[-0.0227]	(-0.0026)[-0.0033]	(0.2080)[0.0698]	(0.0124)[0.0149]
		(0.0016)[0.0011]	(0.0014)[0.0008]	(0.0440)[0.0054]	(0.0135)[0.0198]
		(0.1453)[0.0912]	(0.1390)[0.1088]	(0.3084)[0.1803]	(0.0241)[0.0253]
		(0.7626)[0.6583]	(0.8977)[0.9125]	(0.0182)[0.7339]	(0.9348)[0.9487]
	$q_X(0.9)$	(-0.0971)[-0.1694]	(-0.0670)[-0.0397]	(1.0258)[1.0049]	(0.0589)[0.0528]
		(0.0936)[0.0664]	(0.0903)[0.0452]	(1.2185)[1.1795]	(0.0417)[0.0498]
		(1.0912)[0.7022]	(1.1409)[0.8123]	(1.8974)[1.9928]	(0.0719)[0.0711]
		(0.8530)[0.7443]	(0.8935)[0.9141]	(0.8965)[0.9127]	(0.9458)[0.9496]
T=∞	$h_X(0.5)$	(0.0618)[0.0243]	(0.1310)[0.0523]	(-0.3732)[-0.1100]	(0.0519)[0.0327]
		(0.0597)[0.0262]	(0.0867)[0.0314]	(0.1407)[0.0165]	(0.0512)[0.0328]
		(0.8360)[0.5935]	(0.9834)[0.6448]	(0.5385)[0.4504]	(0.4182)[0.2896]
		(0.9502)[0.9488]	(0.9146)[0.9334]	(0.1670)[0.8798]	(0.9416)[0.9527]
	$S_X(2)$	(-0.0019)[-0.0017]	(-0.0027)[-0.0031]	(0.2066)[0.0655]	(0.0149)[0.0105]
		(0.0017)[0.0009]	(0.0014)[0.0008]	(0.0434)[0.0047]	(0.0103)[0.0098]
		(0.1594)[0.1181]	(0.1384)[0.1079]	(0.3083)[0.1773]	(0.1102)[0.0982]
		(0.8260)[0.8759]	(0.9125)[0.9183]	(0.0181)[0.7888]	(0.9412)[0.9508]
	$q_X(0.9)$	(-0.0266)[-0.0197]	(0.0785)[0.0618]	(0.9851)[0.9816]	(-0.0127)[-0.0109]
		(0.0905)[0.0435]	(0.0762)[0.0416]	(0.9125)[0.9781]	(0.0352)[0.0325]
		(1.1586)[0.8066]	(1.0125)[1.002]	(1.1369)[1.0952]	(0.0627)[0.0621]
		(0.9059)[0.9201]	(0.8982)[0.9113]	(0.9016)[0.9113]	(0.9417)[0.9508]

In what follows, we made some conclusions from Tables $3-6$, in accordance with the numerical results of the provided estimation strategies.

- The threshold T clearly affects the values of all criteria. This means that in each table and for any estimating methods, there exists a specific T that has better performances. The advantages of feature and optimization necessaries finding the best threshold are also consequences. The optimization problem is exhaustively assessed for the real data set in the next section.
- Large sample sizes (m and n) clearly improve the performance of all estimation strategies.
- The MLE, for small threshold (T placed at the beginning of variables domain), has the fragile deed, especially for non-increasing hazard rate or whenever the shape parameter is less than 1. Moreover, it can be shown that by increasing the scale parameter, this capability also becomes much weaker. Consequently, utilizing the MLE in the above situations does not make sense at all. Despite the fact that Bayesian and bootstrap ways perform better than MLE, but it worth mentioning that these strategies also have poor actions and they are not reliable in that positions. Here, the advantages of the conditional estimation arise. In the aforementioned cases, the performances of the conditional estimating method are clearly understood through the tables.
- As we expected, the conditional method has stable performance in all cases, even in positions that other statistical inferences have strong or weak actions. This strategy has logical estimates in accordance with any of mentioned criteria and seems to be a reasonable way instead of MLE, AMLE, bootstrap, and Bayesian method.

9. Application in Wind Speed Data

In this section, the mentioned algorithms have been considered on a real data set. In practice, when we deal with a data set, in contrast to simulation procedures, the reputation of MLE is not possible. Moreover, for a small sample size, the performances of MLE may lead to biasedness. Besides, in a generation PCOS-II from a fixed sample, it is obvious that the result can change in every process. If we aim to discuss a prefixed CS on this sample, then the reputation of these newly generated samples can provide stable results. As mentioned previously, the Bootstrap method solves the reputation problem for a fixed data set. Here, a real data set of 30 monthly averaged wind speed values from 2006 and 2008 is available. The data set is presented in Table (7).

The Weibull distribution is used to model a wide range of data types. One of these disciplines is wind speed data that arises in wind energy, linked to power generation. Investigation around the sustainable energy source seems to be substantial due to making a better balance between energy demands and environmental protection rather than the conventional sources. It is clear that the reliable wind energy data and good inferential methods can be so useful with respect to the performing wind potential projects as soon as possible. One of the most important and frequently measured quantities dealing with wind energy is wind speed data. Implementation of statistical inference according to progressively censoring may be beneficial and effective, regarding the fact that accessing these data is often problematic and also costly. At the same time, the test time also plays an important role in the sampling process, cost of test, and so forth. Both of the mentioned justification leads us to utilize APCOS-II in this situation. In continue, for the wind speed data collected in [1], the comparison of confidence bounds for the survival, quantile and hazard rate functions based on the presented inferential methods is provided. In addition, a cost-time function is introduced. Consequently, the best choice of CS and threshold are derived, given a random sample of size 14 coming from these data.

The use of Weibull distribution or various types of its modified distributions is widespread to analyze wind speed data among researchers. Garcia et. al., [16] investigated the Weibull assumption for these kinds of data. According to their research "Results reveal that the use of a Weibull probability distribution has a moderate impact in the energy calculation as the largest estimation errors are, on average, no larger than 10 percent of the total monthly energy produced". Wais [40] studied two and three parameters Weibull model for such data. The Weibull assumption in the same cases were also considered in [4,11,35].

Abd-Elfattah [1] showed that the fitting generalized Rayleigh distribution with shape parameter one on this data at five percent significant level is reasonable. This data is analyzed under the same assumption in [44]. On the

Table 7.
Monthly average wind speed values recorded by [1].

2.0	2.0	2.5	2.5	2.6	2.6	2.7	2.8	2.8	2.9	3.2	3.3	3.3	3.4	3.5
3.7	3.8	3.8	3.9	4.0	4.0	4.1	4.1	4.7	5.3	5.4	5.5	5.7	6.7	6.9

other hand, the generalized Rayleigh distribution with shape parameter one is a special case of Rayleigh and even Weibull distribution. The $q-q$ plot of this data under the Weibull assumption is given in Figure (1). In addition, the corresponding p-value of the Kolmogorov–Smirnov test is equal to 0.372. Consequently, the proposed Weibull assumption also provides a good fit for this data.

The MLE and log-likelihood of this data based on the Weibull assumption are $(3.140950, 4.236439)$ and 49.5134. Their AMLE and log-likelihood are $(3.134850, 4.198256)$ and 49.5249. Consequently, it is clear that the AMLE has a little better performance than MLE in this case.

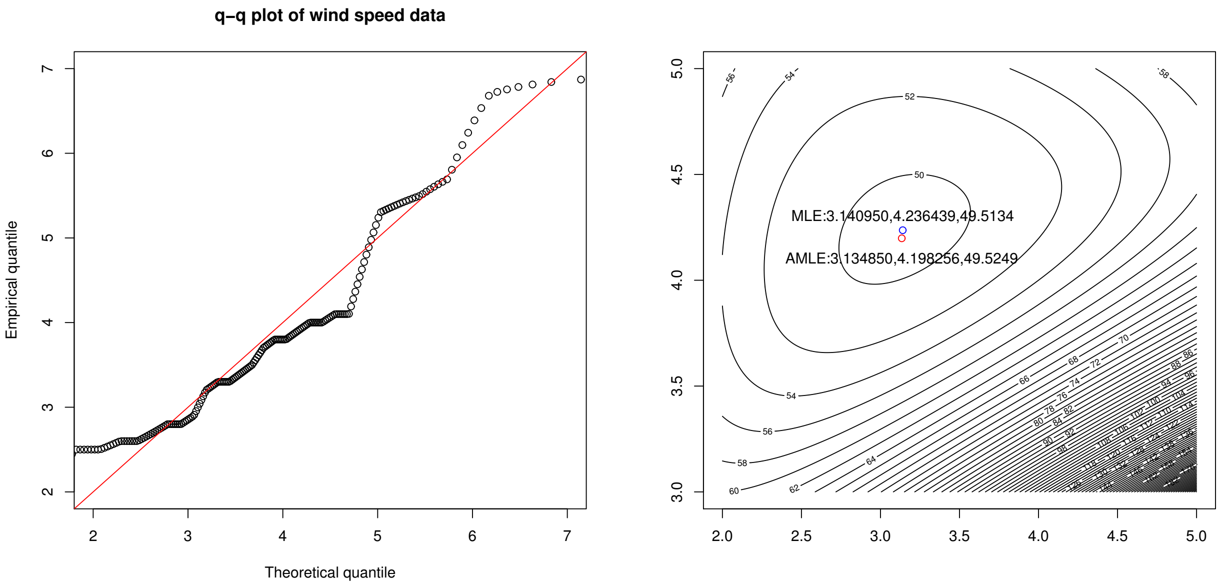

Figure 1. The $q-q$ and contour plot of wind speed data with W$(3.134850, 4.198256)$ assumption in left and right, respectively.

It is problematic and costly to reach real-time wind speed data. Hence, the implementation of the statistical inference methods can be useful in assessing wind potential. On the other hand, collecting a large sample size is expensive and even making trouble. Thereby, it is logical to utilize APCOS-II, which helps in deciding according to both small sample size and the test time. indeed, the problem arises in choosing CS for a fixed sample size. To overcome these problems, including the cost of sampling, time test, and best possible inference based on an available data set, we provide a linear cost function (LCF) of these variables, aiming to minimize this function for each prefixed sample size.

Now, consider a data set $d = (d_1, d_2, \ldots, d_n)$. The assumption is d following a Weibull distribution with unknown shape and scale parameters (γ, λ), respectively. Since for every $1 \leq m \leq n, m \in \mathbb{N}$, there exist $\binom{n-1}{m-1}$ different CSs, we are planned to choose the best of them. It also remains to discuss on the threshold of T. Since T effects on the time test, it is reasonable to consider $E(X_{m:m:n})$, which indicates the expected time on test as an affecting factor of T. Because there exists no information about which threshold can be proper, the element of data set d can itself be considered as T, and consequently the best of these values can be chosen as the proper threshold.

The proper threshold for every m is also considered to be $T_\alpha^{(m)}$. Afterward the LCF is defined as

$$LCF(d,\alpha,m,C_m,C_t \mid R,T) = \alpha_1 \frac{\sum_{j=1}^{B}(var(\hat{\gamma}^j)var(\hat{\lambda}^j) - (cov(\hat{\gamma}^j,\hat{\lambda}^j))^2)}{B} + \alpha_2 C_m + \alpha_3 C_t \frac{\sum_{j=1}^{B} x_{m:m:n}^{(j)}}{B}. \quad (29)$$

In the end, the optimal CS and threshold based on α, called by $R_\alpha^{(m)}$ and $T_\alpha^{(m)}$, are chosen in such a way that the minimum of LCF over all possible CSs and Ts should be $LCF(D,\alpha,m \mid R_\alpha^{(m)},T_\alpha^{(m)})$, which is called Optimal LCF. In addition, C_m and C_t indicate the cost of test for a sampling of size m and the cost of testing per unit of time, respectively. It is also clear that $\frac{\sum_{j=1}^{B}(var(\hat{\gamma}^j)var(\hat{\lambda}^j) - (cov(\hat{\gamma}^j,\hat{\lambda}^j))^2)}{B}$ and $\frac{\sum_{j=1}^{B} x_{m:m:n}^{(j)}}{B}$ are Bootstrap estimations of the determinant of observed Fisher information matrix and $E(X_{m:m:n})$, respectively (Since the MLE may not be calculated especially for large B and small m, the AMLE is utilized to fix this problem.). The vector $\underline{\alpha}$ is chosen based on the researcher's desire according to his need for a balance between good estimation (α_1), cost of sampling (α_2), and expected time on the test (α_3). It is also worth mentioning that Optimal LCF (29) is nonmonotone and that there is no problem related to finding its minimizing values. The reason is according to the nonincreasing property of $\frac{\sum_{j=1}^{B}(var(\hat{\gamma}^j)var(\hat{\lambda}^j) - (cov(\hat{\gamma}^j,\hat{\lambda}^j))^2)}{B}$ and nondecreasing feature of $\frac{\sum_{j=1}^{B} x_{m:m:n}^{(j)}}{B}$ with respect to increasing the m value.

Table 8. Optimal choice of CS and threshold based on a random sample of size 14 comes from wind speed data with $\alpha = (100, 0, 10)$

m	$R_\alpha^{(m)}$	$T_\alpha^{(m)}$	Optimal LCF	Optimal LCF+m
1	13	3.8	545.4813	546.4813
2	$12, 0$	2	115.5552	117.5552
3	$10, 1, 0$	2	71.36596	74.36596
4	$2, 8, 0^{(2)}$	2.8	53.42253	57.42253
5	$0, 9, 0^{(3)}$	2.9	45.2951	50.2951
6	$0, 8, 0^{(4)}$	3.2	43.40121	49.40121
7	$0, 7, 0^{(5)}$	3.7	43.60255	50.60255
8	$0, 6, 0^{(6)}$	4.1	44.7038	52.7038
9	$0, 5, 0^{(7)}$	5.3	45.62643	54.62643
10	$0^{(2)}, 3, 1, 0^{(6)}$	5.4	47.41706	57.41706
11	$0^{(2)}, 3, 0^{(8)}$	6.7	50.64084	61.64084
12	$0^{(2)}, 1^{(2)}, 0^{(8)}$	2	55.0956	67.0956
13	$0^{(7)}, 1, 0^{(5)}$	2.5	61.56019	74.56019
14	$0^{(14)}$	6.9	69.52823	83.52823

In continue, a proposed optimal plan (29) is applied for some different m. Since we are going to choose the best CS in the present case, the cost of sampling C_m has been omitted. Because the values of observed fisher information matrix are small, α_1 has been considered to be 100 . In addition $\alpha_3 = 10$ and $C_t = 1$ are deemed. The number of Bootstrap iteration is also 1000. According to these assumptions, the optimal choice of CS and threshold are available in Table 8.

Finally, it is clear that for $m = n$, there is only one CS $R = (0^{(30)})$ and that no threshold can affect this censoring plan. Moreover, as mentioned previously, APCOS-II, in this case, results in complete sampling. Here, based on the wind speed data, the comparison of asymptotic (based on AMLE), percentile Bootstrap, HPD credible, and conditional intervals for its hazard rate, survival and quantile functions is provided, respectively, in Figure (3).

For constructing HPD credible intervals, first of all, we should choose a prior and also a loss function. As a result, it is clear that the Bayes estimator is close to the MLE. In addition, the conjugate prior performs better, and for this latter, all the loss functions end up with similar estimators. Hence in continue, we utilize HPD credible intervals based on the conjugate prior and SEL function. It is also worth mentioning that the method of constructing HPD credible intervals is taken from Theorem 2 of [25].

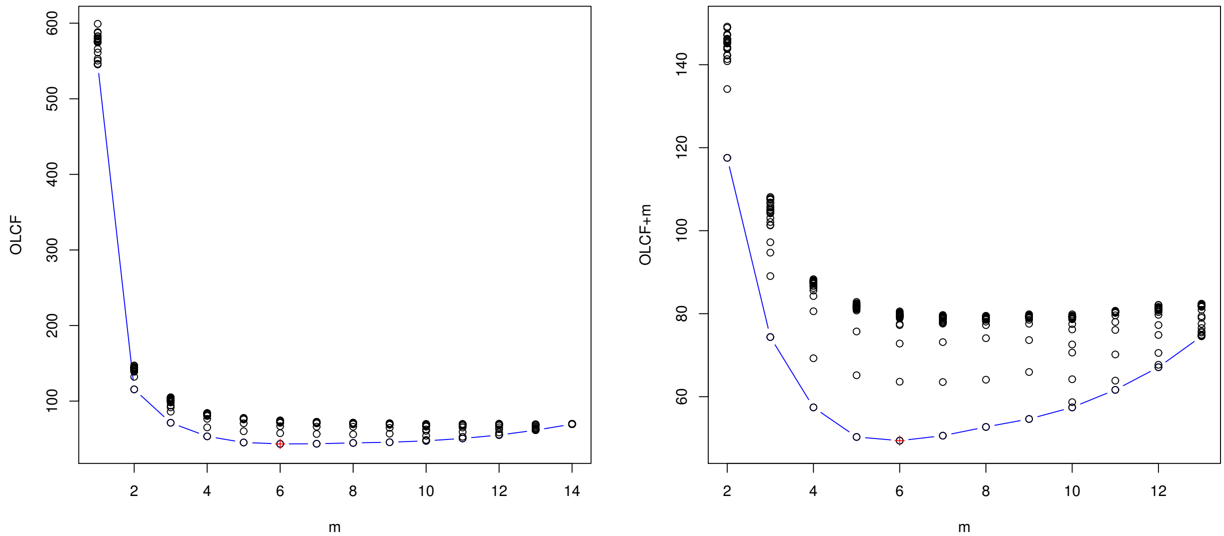

Figure 2. Optimal LCF and Optimal LCF+m of any m and all Threshold, respectively, in right and left

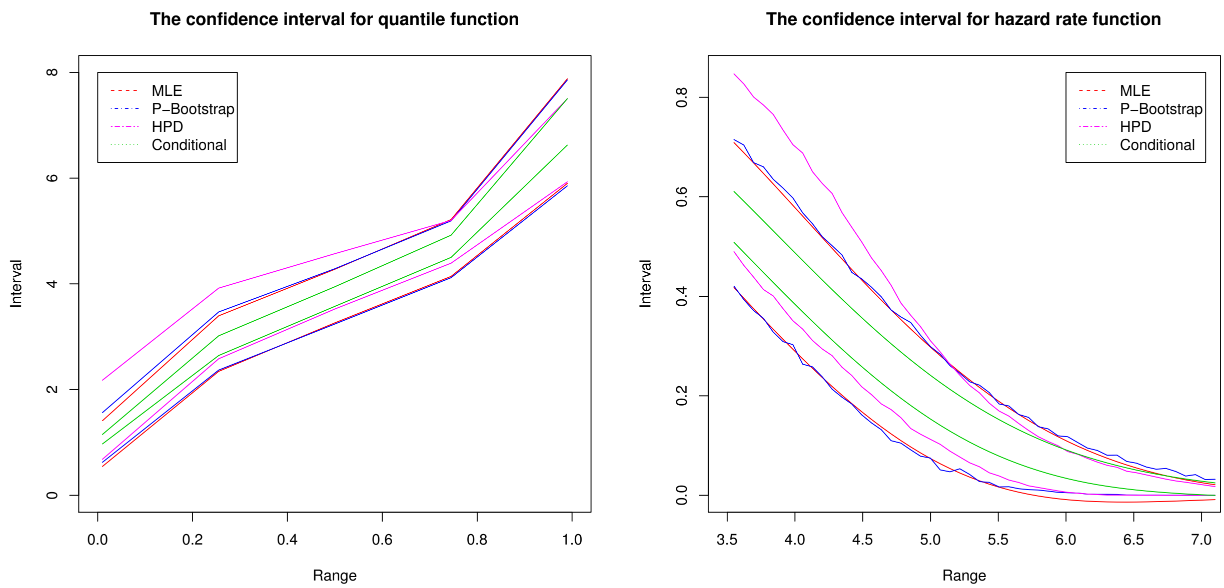

Figure 3. The confidence interval for the quantile and survival functions of wind speed data, respectively, in left and right

10. Conclusion

The paper dealt with the estimation problem of survival, quantile, and hazard rate functions of the Weibull distribution under an *APC-II* scheme. This censoring method covers the ordinary and progressive Type-I and Type II censoring plans as well as complete sample. For the estimation problems, the classical method including MLE, AMLE, and percentile bootstrap were investigated. Besides, the Bayesian method was provided under some kinds of symmetric and asymmetric loss functions. The weaknesses of the mentioned approaches were demonstrated via extensive numerical studies. However, the mentioned methods don't have the same and stable performance. These strategies may work well in estimating survival and quantile functions, but not so well for estimation of hazard rate function. Furthermore, the extreme quantile values don't work well in classical and Bayesian estimation, and consequently, the corresponding estimators have quite large errors. In contrast, we introduced conditional estimating for all of these target parameters (survival, quantile, and hazard rate function). The superiority of the proposed method over the other approaches were clearly demonstrated through a comparison study. The comparision has been conducted based on the following benchmarks: the mean of bias, square error, absolute error, and coverage probability. Finally, these estimation methods were applied to a set of wind speed data. The survival, quantile, and hazard rate functions of this data set were estimated, and moreover, their confidence bounds were also calculated. The excellent behavior of the conditional strategy goes without saying regarding the provided figures.

REFERENCES

[1] ABD-ELFATTAH, A. Goodness of fit test for the generalized rayleigh distribution with unknown parameters. *Journal of Statistical Computation and Simulation 81*, 3 (2011), 357–366.

[2] ABOELENEEN, Z. Inference for weibull distribution under generalized order statistics. *Mathematics and Computers in Simulation 81*, 1 (2010), 26–36.

[3] AHMADI NADI, A., AND SADEGHPOUR GILDEH, B. A group multiple dependent state sampling plan using truncated life test for the weibull distribution. *Quality Engineering 31*, 4 (2019), 553–563.

[4] ALI, S., LEE, S.-M., AND JANG, C.-M. Statistical analysis of wind characteristics using weibull and rayleigh distributions in deokjeok-do island–incheon, south korea. *Renewable energy 123* (2018), 652–663.

[5] ALMALKI, S. J., AND NADARAJAH, S. Modifications of the weibull distribution: A review. *Reliability Engineering & System Safety 124* (2014), 32–55.

[6] ASGHARZADEH, A., FERNÁNDEZ, A. J., AND ABDI, M. Confidence sets for the two-parameter rayleigh distribution under progressive censoring. *Applied Mathematical Modelling 47* (2017), 656–667.

[7] BAI, X., SHI, Y., LIU, Y., AND LIU, B. Reliability inference of stress–strength model for the truncated proportional hazard rate distribution under progressively type-ii censored samples. *Applied Mathematical Modelling 65* (2019), 377–389.

[8] BALAKRISHNAN, N., AND CRAMER, E. The art of progressive censoring: Applications to reliability and quality. *Springer* (2014).

[9] BALAKRISHNAN, N., KANNAN, N., LIN, C., AND WU, S. Inference for the extreme value distribution under progressive type-ii censoring. *Journal of Statistical Computation and Simulation 74*, 1 (2004), 25–45.

[10] BHATT, V., AND TIWARI, N. A spatial scan statistic for survival data based on weibull distribution. *Statistics in medicine 33*, 11 (2014), 1867–1876.

[11] CHAURASIYA, P. K., AHMED, S., AND WARUDKAR, V. Comparative analysis of weibull parameters for wind data measured from met-mast and remote sensing techniques. *Renewable energy 115* (2018), 1153–1165.

[12] CHO, Y., SUN, H., AND LEE, K. Estimating the entropy of a weibull distribution under generalized progressive hybrid censoring. *Entropy 17*, 1 (2015), 102–122.

[13] ELMAHDY, E. E., AND ABOUTAHOUN, A. W. A new approach for parameter estimation of finite weibull mixture distributions for reliability modeling. *Applied Mathematical Modelling 37*, 4 (2013), 1800–1810.

[14] FERGUSON, T. S. *A course in large sample theory.* Routledge, 2017.

[15] FERNÁNDEZ, A. J., PÉREZ-GONZÁLEZ, C. J., ASLAM, M., AND JUN, C.-H. Design of progressively censored group sampling plans for weibull distributions: An optimization problem. *European Journal of Operational Research 211*, 3 (2011), 525–532.

[16] GARCÍA-BUSTAMANTE, E., GONZÁLEZ-ROUCO, J., JIMÉNEZ, P., NAVARRO, J., AND MONTÁVEZ, J. The influence of the weibull assumption in monthly wind energy estimation. *Wind Energy: An International Journal for Progress and Applications in Wind Power Conversion Technology 11*, 5 (2008), 483–502.

[17] GUNASEKERA, S. Inference for the burr xii reliability under progressive censoring with random removals. *Mathematics and Computers in Simulation 144* (2018), 182–195.

[18] HAGHIGHI, F. Bayes-conditional control charts for weibull quantiles under type ii censoring. *Quality and Reliability Engineering International 33*, 5 (2017), 959–968.

[19] HAGHIGHI, F., PASCUAL, F., AND CASTAGLIOLA, P. Conditional control charts for weibull quantiles under type ii censoring. *Quality and Reliability Engineering International 31*, 8 (2015), 1649–1664.

[20] JIA, X., NADARAJAH, S., AND GUO, B. Bayes estimation of $p(y < x)$ for the weibull distribution with arbitrary parameters. *Applied Mathematical Modelling 47* (2017), 249–259.

[21] JIA, X., NADARAJAH, S., AND GUO, B. Exact inference on weibull parameters with multiply type-i censored data. *IEEE Transactions on Reliability 67*, 2 (2018), 432–445.

[22] KAZEMPOOR, J., HABIBIRAD, A., AND OKHLI, K. Bounds for cdfs of order statistics arising from inid random variables. *Journal of The Iranian Statistical Society 19*, 1 (2020), 39–57.

[23] KOTB, M. S., AND RAQAB, M. Z. Statistical inference for modified weibull distribution based on progressively type-ii censored data. *Mathematics and Computers in Simulation 162* (2019), 233–248.

[24] LAWLESS, J. F. Confidence interval estimation for the weibull and extreme value distributions. *Technometrics 20*, 4 (1978), 355–364.

[25] LE, H., PHAM, U., NGUYEN, P., AND PHAM, T. B. Improvement on monte carlo estimation of hpd intervals. *Communications in Statistics-Simulation and Computation 49*, 8 (2020), 2146–2180.

[26] LI, B., AND BABU, G. J. *Graduate Course On Statistical Inference.* Springer, 2019.

[27] LIN, C.-T., NG, H. K. T., AND CHAN, P. S. Statistical inference of type-ii progressively hybrid censored data with weibull lifetimes. *Communications in Statistics—Theory and Methods 38*, 10 (2009), 1710–1729.

[28] MOHARIB ALSARRAY, R. M., KAZEMPOOR, J., AND AHMADI NADI, A. Monitoring the weibull shape parameter under progressive censoring in presence of independent competing risks. *Journal of Applied Statistics* (2021), 1–18.

[29] MOKHTARI, E. B., RAD, A. H., AND YOUSEFZADEH, F. Inference for weibull distribution based on progressively type-ii hybrid censored data. *Journal of Statistical Planning and Inference 141*, 8 (2011), 2824–2838.

[30] MONTOYA, J. A., DÍAZ-FRANCÉS, E., AND FIGUEROA, G. Estimation of the reliability parameter for three-parameter weibull models. *Applied Mathematical Modelling 67* (2019), 621–633.

[31] ORTEGA, E. M., CORDEIRO, G. M., CAMPELO, A. K., KATTAN, M. W., AND CANCHO, V. G. A power series beta weibull regression model for predicting breast carcinoma. *Statistics in medicine 34*, 8 (2015), 1366–1388.

[32] PANAHI, H., AND ASADI, S. On adaptive progressive hybrid censored burr type iii distribution: application to the nano droplet dispersion data. *Quality Technology & Quantitative Management 18*, 2 (2021), 179–201.

[33] PANAHI, H., AND MORADI, N. Estimation of the inverted exponentiated rayleigh distribution based on adaptive type ii progressive hybrid censored sample. *Journal of Computational and Applied Mathematics 364* (2020), 112345.

[34] RINNE, H. *The Weibull distribution: a handbook*. Chapman and Hall/CRC, 2008.

[35] SOULOUKNGA, M., DOKA, S., REVANNA, N., DJONGYANG, N., AND KOFANE, T. Analysis of wind speed data and wind energy potential in faya-largeau, chad, using weibull distribution. *Renewable energy 121* (2018), 1–8.

[36] TEAM, R. C. R: A language and environment for statistical computing.

[37] VALIOLLAHI, R., RAQAB, M. Z., ASGHARZADEH, A., AND ALQALLAF, F. Estimation and prediction for power lindley distribution under progressively type ii right censored samples. *Mathematics and Computers in Simulation 149* (2018), 32–47.

[38] VIVEROS, R., AND BALAKRISHNAN, N. Interval estimation of parameters of life from progressively censored data. *Technometrics 36*, 1 (1994), 84–91.

[39] WAHED, A. S., LUONG, T. M., AND JEONG, J.-H. A new generalization of weibull distribution with application to a breast cancer data set. *Statistics in medicine 28*, 16 (2009), 2077–2094.

[40] WAIS, P. Two and three-parameter weibull distribution in available wind power analysis. *Renewable energy 103* (2017), 15–29.

[41] WANG, F.-K., BIZUNEH, B., AND CHENG, X.-B. New control charts for monitoring the weibull percentiles under complete data and type-ii censoring. *Quality and Reliability Engineering International 34*, 3 (2018), 403–416.

[42] WEIBULL, W., ET AL. A statistical distribution function of wide applicability. *Journal of applied mechanics 18*, 3 (1951), 293–297.

[43] XU, A., FU, J., TANG, Y., AND GUAN, Q. Bayesian analysis of constant-stress accelerated life test for the weibull distribution using noninformative priors. *Applied Mathematical Modelling 39*, 20 (2015), 6183–6195.

[44] ZHANG, Z., AND GUI, W. Statistical inference of reliability of generalized rayleigh distribution under progressively type-ii censoring. *Journal of Computational and Applied Mathematics 361* (2019), 295–312.

[45] ZHU, T. Statistical inference of weibull distribution based on generalized progressively hybrid censored data. *Journal of Computational and Applied Mathematics* (2019), 112705.