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A New Odd Log- Logistic Inverse Lindley Distribution with Properties and Applications

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Abstract In this article, we introduce a new three-parameter odd log-logistic power inverse Lindley distribution and discuss some of its properties. These include the shapes of the density and hazard rate functions, mixture representation, the moments, the quantile function, and order statistics. Maximum likelihood estimation of the parameters and their estimated asymptotic standard errors are derived. Three algorithms are proposed for generating random data from the proposed distribution. A simulation study is carried out to examine the bias and root mean square error of the maximum likelihood estimators of the parameters. An application of the model to three real data sets is finally presented and compared with the fit attained by some other well-known two and three-parameter distributions for illustrative purposes. It is observed that the proposed model has some advantages in analyzing lifetime data as compared to other popular models in the sense that it exhibits varying shapes and shows more flexibility than many currently available distributions.

Keywords Lambert function, maximum likelihood estimation, order statistics, power inverse Lindley distribution, Stochastic ordering

AMS 2010 subject classifications 62E10

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1. Introduction

Reliability and survival analysis have several applications as an important branch of statistics, in different applied fields, such as actuarial science, engineering, demography, biomedical studies, and industrial reliability. Several lifetime distributions have been proposed in the statistical literature to model data in many applied sciences.

Lindley [20] proposed the Lindley distribution in the context of the Bayes theorem as a counter-example of fiducial statistics with the probability density function (pdf):

$$f\left(y;\beta\right)=\frac{\beta^{2}}{1+\beta}\left(1+y\right)e^{-\beta y};y,\ \beta>0. \tag{1}$$

Ghitany et al. [12] discussed the Lindley distribution and its applications extensively and showed that the Lindley distribution is a better fit than the exponential distribution based on the waiting time at the bank for service. The Lindley distribution has been extended by different researchers, including Zakerzadeh and Dolati [27], Nadarajah et al. [22], Shanker and Mishra [24], Ghitany et al. [13], Ashour and Eltehiwy [4], Eltehiwy [10], and Alizadeh et al. [2]. The inverse Lindley distribution was proposed by Sharma et al. [25] using the transformation $X=\frac{1}{Y}$ with the pdf:

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$$f(x;\beta) = \frac{\beta^2}{1+\beta}\left(\frac{1+x}{x^3}\right)e^{-\frac{\beta}{x}};\beta, x > 0, \tag{2}$$

where Y is a random variable having pdf (1).
Another two-parameter inverse Lindley distribution introduced by Sharma et al. [26], called “the generalized inverse Lindley distribution,” is a new statistical inverse model for upside-down bathtub survival data that uses the transformation $X = Y^{-\frac{1}{\alpha}}$ with the pdf:

$$f(x;\beta,\alpha) = \frac{\alpha\beta^2}{1+\beta}\left(\frac{1+x^{\alpha}}{x^{2\alpha+1}}\right)e^{-\frac{\beta}{x^{\alpha}}};\beta,\alpha, x > 0, \tag{3}$$

with Y being a random variable and having pdf (1). Note that Barco et al. [5] also obtained the generalized inverse Lindley distribution by taking the transformation $X = Y^{-\frac{1}{\alpha}}$ where Y follows inverse Lindley distribution, known as the power inverse Lindley distribution (PIL), with the same pdf.
The pdf (3) can be shown as a mixture of two distributions as follows:

$$f(x;\beta,\alpha) = pf_1(x) + (1-p)f_2(x),$$

where,

$$p = \tfrac{\beta}{\beta+1},\ f_1 = \tfrac{\alpha\beta}{x^{\alpha+1}}e^{-\frac{\beta}{x^{\alpha}}},\ x > 0 \text{ and } f_2 = \tfrac{\alpha\beta^2}{x^{2\alpha+1}}e^{-\frac{\beta}{x^{\alpha}}},\ x > 0.$$

with mixing proportion $p = \beta/(\beta+1)$, we see that, PIL is a two-component mixture of inverse Weibull distribution (shape α and scale β) and generalized inverse gamma distribution (with shape parameters 2, α and scale β).

Several methods to generate new distributions by adding one or more parameters have been proposed in the statistical literature. Some well-known generators are the Marshall-Olkin generator (MO-G) by Marshall and Olkin [21], beta-G by Eugene et al. [11], Kumaraswamy-G (Kw-G) by Cordeiro and de Castro [7], Weibull-G by Bourguignon et al. [6], exponentiated half-logistic-G by Cordeiro et al. [8] and among others.

Gleaton and Lynch [14, 15, 16] introduced a new family of distributions called the Generalized log-logistic family of distributions. The cumulative distribution function (CDF) of this family is given by

$$F(x;\theta,\xi) = \frac{G(x;\xi)^{\theta}}{G(x;\xi)^{\theta} + \overline{G}(x;\xi)^{\theta}}, \tag{4}$$

where $\theta > 0$ is the shape parameter, $G(x;\xi)$ is the CDF of the baseline distribution, $\overline{G}(x;\xi) = 1 - G(x;\xi)$ is the survival function and ξ is the set of the parameters of the baseline distribution $G(.)$. In addition, the pdf of the family is

$$f(x;\theta,\xi) = \frac{\theta \mathrm{g}(x;\xi)\, G(x;\xi)^{\theta-1}\overline{G}(x;\xi)^{\theta-1}}{\left[G(x;\xi)^{\theta} + \overline{G}(x;\xi)^{\theta}\right]^2}$$

This family was later called the odd log-logistic family of distributions. If the baseline distribution has a closed form CDF; the newly generated distribution will also have a closed form CDF. One can easily show that

$$\frac{\log\left[\frac{F(x;\theta,\xi)}{\overline{F}(x;\theta,\xi)}\right]}{\log\left[\frac{G(x;\xi)}{\overline{G}(x;\xi)}\right]} = \theta.$$

Therefore, θ is the quotient of the log-odds ratio for the generated and baseline distributions.

Now, by letting $G(x;\xi)$ in (4) as the CDF of the power inverse Lindley distribution, where $\xi = (\beta,\alpha)$ is the set of parameters, we can obtain a new extension of the power inverse Lindley distribution, called the odd log-logistic power inverse Lindley (henceforth, OLL-PIL) distribution. The CDF, pdf, and hazard rate function of this distribution are given by

$$F\left(x;\alpha,\beta,\theta\right)=\frac{\left(1+\frac{\beta}{(1+\beta)x^{\alpha}}\right)^{\theta}e^{-\frac{\theta\beta}{x^{\alpha}}}}{\left(1+\frac{\beta}{(1+\beta)x^{\alpha}}\right)^{\theta}e^{-\frac{\theta\beta}{x^{\alpha}}}+\left[1-\left(1+\frac{\beta}{(1+\beta)x^{\alpha}}\right)e^{-\frac{\beta}{x^{\alpha}}}\right]^{\theta}}, \tag{5}$$

for $x>0,\ \theta,\ \beta,\alpha>0$ and the corresponding pdf and hazard rate function are provided by

$$f\left(x;\alpha,\beta,\theta\right)=\frac{\alpha\theta\beta^{2}\left(\frac{1+x^{\alpha}}{x^{2\alpha+1}}\right)e^{-\frac{\beta\theta}{x^{\alpha}}}\left[\left(1+\frac{\beta}{(1+\beta)x^{\alpha}}\right)\right]^{\theta-1}\left[1-\left(1+\frac{\beta}{(1+\beta)x^{\alpha}}\right)e^{-\frac{\beta}{x^{\alpha}}}\right]^{\theta-1}}{(1+\beta)\left\{\left(1+\frac{\beta}{(1+\beta)x^{\alpha}}\right)^{\theta}e^{-\frac{\theta\beta}{x^{\alpha}}}+\left[1-\left(1+\frac{\beta}{(1+\beta)x^{\alpha}}\right)e^{-\frac{\beta}{x^{\alpha}}}\right]^{\theta}\right\}^{2}}, \tag{6}$$

$$h\left(x;\alpha,\beta,\theta\right)=\frac{\alpha\theta\beta^{2}\left(\frac{1+x^{\alpha}}{x^{2\alpha+1}}\right)e^{-\frac{\beta\theta}{x^{\alpha}}}\left[\left(1+\frac{\beta}{(1+\beta)x^{\alpha}}\right)\right]^{\theta-1}}{(1+\beta)\left[1-\left(1+\frac{\beta}{(1+\beta)x^{\alpha}}\right)e^{-\frac{\beta}{x^{\alpha}}}\right]\left\{\left(1+\frac{\beta}{(1+\beta)x^{\alpha}}\right)^{\theta}e^{-\frac{\theta\beta}{x^{\alpha}}}+\left[1-\left(1+\frac{\beta}{(1+\beta)x^{\alpha}}\right)e^{-\frac{\beta}{x^{\alpha}}}\right]^{\theta}\right\}}. \tag{7}$$

We write $X\sim OLL-PIL(\alpha,\beta,\theta)$if the pdf of X can be written as (6). The new distribution is very flexible in the sense that it can be skewed or symmetric depending on the specific choices of the parameters. Furthermore, the associated CDF is in closed form. Consequently, this distribution can be applied to modelling censored data too. This is a major motivation to carry out this work. Furthermore, in reliability engineering and lifetime analysis, we often assume that the failure times of the components within each system follow the exponential lifetimes; see, for example, Adamidis and Loukas [1], among others, and the references therein. This assumption may seem unreasonable because, for the exponential distribution, the hazard rate is a constant, whereas many real-life systems do not have constant hazard rates, and the components of a system are often more rigid than the system itself, such as the bones in a human body, the balls of a steel pipe, etc. Accordingly, it becomes reasonable to consider the components of a system to follow a distribution with a non-constant hazard function that has flexible hazard function shapes.

A motivation of this family can be explained as follows: Let X be a lifetime random variable having power inverse Lindley distribution. The odds ratio that an individual (or component) following the lifetime X will die (fail) at time x is $y=G(x;\alpha,\beta)/\overline{G}(x;\alpha,\beta)$. Here, one can consider this odd of death as a random variable, say Y. Now, if we model the randomness of the "odds of death" using the log-logistic distribution with scale parameter 1 and shape parameter θ, $\left(F_{Y}\left(y\right)=y^{\theta}/\left[1+y^{\theta}\right]\right)$ for $y>0$. Then we can write

$$Pr\left(Y\leq y\right)=F_{Y}\left(G\left(x;\alpha,\beta\right)/\overline{G}(x;\alpha,\beta)\right),$$

which is given by (5), see Cooray [9] for more details regarding this interpretation.

Plots of the pdf are shown in Fig. 1. The pdfs appear always unimodal. The mode moves more to the right and the pdf becomes less peaked with increasing values of β. The mode moves more to the right and the pdf becomes less peaked with increasing values of θ. The pdf becomes more peaked with increasing values of α . The behavior of $h(x)$ in (7) of the OLL-PIL for different values of the parameters $\alpha,\ \beta$ and $\theta$are showed graphically in Fig. 2.

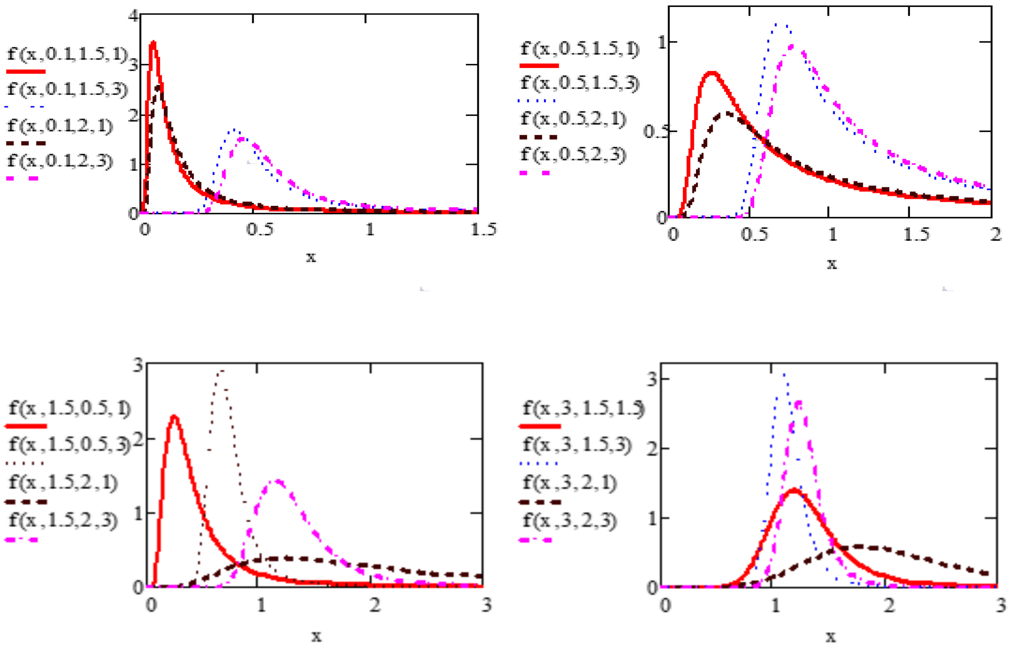

Figure 1. Pdfs of the OLL-PIL model for selected θ, β and α.

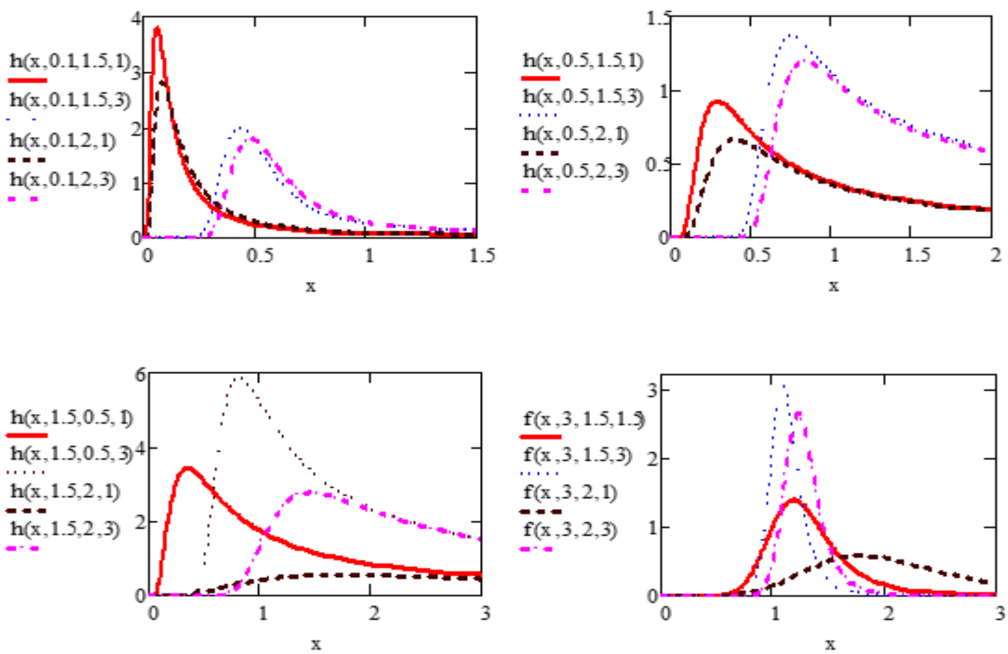

Figure 2. Hazard rate functions of the OLL-PIL model for selected θ, β and α.

Because the hazard rate function of extended inverse Lindley distribution is always a unimodal function in x, the new distribution is also unimodal. Figure 2 illustrates the behavior of the hazard rate function of the OLL-PIL distribution at different values of the parameters involved. Concerning the hazard rate function of the odd log logistic power inverse Lindley distribution, which is shown in Fig. 2; it notably has the shape of an upside-down bathtub, therefore being unimodal in x.This attractive flexibility makes the OLL-PIL hazard rate function useful and suitable for non-monotone empirical hazard behaviors which are more likely to be encountered or observed in real life situations.

We hope that this new distribution can be applied to describing lifetime data more properly than the existing distributions. The major motivation for introducing the OLL-PIL distribution can be summarized as follows: (i) The OLL-PIL distribution contains several lifetime distributions as special cases, such as the power inverse Lindley (PIL) distribution due to Barco et al. [5] for $\theta = 1$: (ii) It is shown in Section 2 that the OLL-PIL distribution can be viewed as a mixture of exponentiated power inverse Lindley (EPIL) distributions introduced by Jan et al. [19]. (iii) The OLL-PIL distribution is a flexible model, that can be widely used for modelling lifetime data. (iv) The OLL-PIL distribution exhibits non-monotone hazard rates but does not exhibit a constant hazard rate, which makes this distribution superior to other lifetime distributions. (v) The OLL-PIL distribution outperforms several of the well-known lifetime distributions with respect to some real data examples.

Special cases:

For $\theta = 1$, we obtain the power inverse Lindley distribution.

For $\alpha = 1$, we obtain the odd log-logistic inverse Lindley distribution.

For $\theta = \alpha = 1$, we obtain the inverse Lindley distribution.

The rest of the article is organized as follows: in Section 2, we discuss some structural properties of the OLL-PIL distribution. Section 3 deals with the classical method of estimation (using maximum likelihood) of the model parameters of the OLL-PIL distribution. In Section 4, three real data sets are considered as an example to illustrate the applicability of OLL-PIL distribution. In Section 5, a simulation study is conducted to verify the efficacy of the said estimation procedure. In Section 6, we provide some concluding remarks.

2. Structural properties

In this section, we discuss some structural properties of the OLL-PIL distribution.

2.1. Mixture representations for the pdf and CDF

The EPIL distribution, introduced by Jan et al. [19] has the pdf

$$f_{EPIL}\left(x;\alpha,\beta,\theta\right)=\frac{\alpha\theta\beta^{2}}{\beta+1}\left(\frac{1+x^{\alpha}}{x^{2\alpha+1}}\right)e^{-\frac{\beta\theta}{x^{\alpha}}}\left[\left(1+\frac{\beta}{\left(1+\beta\right)x^{\alpha}}\right)\right]^{\theta-1},x>0,\ \theta,\ \beta,\alpha>0 \tag{8}$$

We write $EPIL\left(\alpha,\beta,\theta\right)$ if the pdf of X can be expressed as (8). In addition, the CDF of the EPIL model is

$$F_{EPIL}\left(x;\alpha,\beta,\theta\right)=\left[\left(1+\frac{\beta}{\left(1+\beta\right)x^{\alpha}}\right)e^{-\frac{\beta}{x^{\alpha}}}\right]^{\theta},x>0,\ \theta,\ \beta,\alpha>0 \tag{9}$$

Now, we show that the OLL-PIL distribution can be viewed as a mixture of EPIL distributions. Using the generalized binomial expansion, the numerator of (5) can be

$$\left[\left(1+\frac{\beta}{\left(1+\beta\right)x^{\alpha}}\right)\right]^{\theta}e^{-\frac{\beta\theta}{x^{\alpha}}}=\sum_{k=0}^{\infty}a_{k}\left[\left(1+\frac{\beta}{\left(1+\beta\right)x^{\alpha}}\right)e^{-\frac{\beta}{x^{\alpha}}}\right]^{k}$$

Where $a_k = \sum_{j=k}^{\infty} -1^k \binom{\theta}{k}\binom{j}{k}$ and the denominator of (5) can be written as

$$\left(1+\frac{\beta}{(1+\beta)\,x^{\alpha}}\right)^{\theta} e^{-\frac{\theta\beta}{x^{\alpha}}} + \left[1-\left(1+\frac{\beta}{(1+\beta)\,x^{\alpha}}\right)e^{-\frac{\beta}{x^{\alpha}}}\right]^{\theta} = \sum_{k=0}^{\infty} b_k \left[\left(1+\frac{\beta}{(1+\beta)\,x^{\alpha}}\right)e^{-\frac{\beta}{x^{\alpha}}}\right]^{k}$$

where $b_k = a_k + -1^k \binom{\theta}{k}$. Therefore, the CDF of the OLL-PIL distribution can be expressed as

$$F(x) = \frac{\sum_{k=0}^{\infty} a_k \left[\left(1+\frac{\beta}{(1+\beta)x^{\alpha}}\right)e^{-\frac{\beta}{x^{\alpha}}}\right]^{k}}{\sum_{k=0}^{\infty} b_k \left[\left(1+\frac{\beta}{(1+\beta)x^{\alpha}}\right)e^{-\frac{\beta}{x^{\alpha}}}\right]^{k}} = \sum_{k=0}^{\infty} c_k \left[\left(1+\frac{\beta}{(1+\beta)\,x^{\alpha}}\right)e^{-\frac{\beta}{x^{\alpha}}}\right]^{k},$$

where $c_0 = \frac{a_0}{b_0} = 0$ and for $k \geq 1$ we have

$$c_k = b_0^{-1}\left[a_k - b_0^{-1}\sum_{r=1}^{k} b_r c_{k-r}\right].$$

Or equivalently, we can write the CDF of OLL-PIL as

$$F(x) = \sum_{k=1}^{\infty} c_k F_{EPIL}(x;k,\alpha,\beta) = \sum_{k=0}^{\infty} c_{k+1} F_{EPIL}(x;k+1,\alpha,\beta), \tag{10}$$

where $F_{EPIL}(x;k+1,\alpha,\beta)$ denotes the CDF of the EPIL distribution with parameters $k+1$, α and β. We note that $\sum_{k=0}^{\infty} c_{k+1} = 1$. By differentiating equation (10), the pdf of the OLL-PIL distribution can be expanded as

$$f(x) = \sum_{k=0}^{\infty} c_{k+1} f_{EPIL}(x;k+1,\alpha,\beta), \tag{11}$$

where $f_{EPIL}(x;k+1,\alpha,\beta)$ denotes the pdf of the EPIL distribution with parameters $k+1$, α and β.

2.2. Moments

The r^{th} ordinary moment of X is given by $\mu_r^{'} = E(X^r) = \int_{-\infty}^{\infty} x^r f(x)\,dx$. Then, using Eq.(11), we obtain

$$\mu_r^{'} = (\beta)^{\frac{r}{\alpha}} \sum_{k=0}^{\infty}\sum_{i=0}^{\infty}\binom{k}{i} c_{k+1} (k+1)^{\frac{r}{\alpha}} \frac{\left[i+1-\frac{r}{\alpha}+(k+1)\beta\right]\Gamma\left(i+1-\frac{r}{\alpha}\right)}{\left[(k+1)(\beta+1)\right]^{i+1}}$$

For r^{th} moment to exist, the constraint $\alpha > r$ must be satisfied.
The moment generating function $M_X(t) = E(e^{tx})$ of X can be derived from Eq. (11) as follows:

$$M_X(t) = \sum_{k=0}^{\infty}\sum_{i=0}^{\infty}\sum_{n=0}^{\infty} \frac{t^n}{n!}\left[\beta(k+1)\right]^{\frac{n}{\alpha}} \binom{k}{i} c_{k+1} \frac{\left[i+1-\frac{n}{\alpha}+(k+1)\beta\right]\Gamma\left(i+1-\frac{n}{\alpha}\right)}{\left[(k+1)(\beta+1)\right]^{i+1}}$$

2.3. Incomplete moments

The main applications of the first incomplete moment refer to the mean deviations and the Bonferroni and Lorenz curves. These curves are very useful in economics, reliability, demography, insurance, and medicine. The s^{th} incomplete moment, say $\eta_s(t)$, of the OLL-PIL distribution is given by

$$\eta_s(t) = \int_0^t x^s f(x)\,dx,$$

$$\eta_s\left(t\right) = \left(\beta\right)^{\frac{s}{\alpha}} \sum_{k=0}^{\infty}\sum_{i=0}^{\infty}\binom{k}{i} c_{k+1}\left(k+1\right)^{\frac{s}{\alpha}} \frac{\left[i+1-\frac{s}{\alpha}+\left(k+1\right)\beta\right]\gamma\left(i+1-\frac{s}{\alpha},\frac{(k+1)(\beta+1)}{t^{\alpha}}\right)}{\left[\left(k+1\right)\left(\beta+1\right)\right]^{i+1}}, \qquad (12)$$

where $\gamma\left(.,.\right)$ is the lower incomplete moments. The first incomplete moment of the OLL-PIL distribution can be obtained by setting $s=1$ in (12). The first incomplete moment is related to the Bonferroni and Lorenz curves, the mean residual, and the mean waiting times. The Bonferroni and Lorenz curves are important in economics, reliability, demography, insurance, and medicine. The Lorenz curve, say $LO(x)$, and Bonferroni curve, say $BO(x)$, are defined by

$$LO\left(x\right) = \frac{\eta_1\left(t\right)}{E(X)}$$

and

$$BO\left(x\right) = \frac{LO(x)}{F_{OLL-PIL}\left(x;\theta,\beta,\alpha\right)}$$

2.4. Stochastic Orders

Stochastic ordering of positive continuous random variables is an important tool for judging comparative behavior. Suppose X_i is distributed according to (Eqs. 5 and 6) with common parameter β and parameters θ_i and α_i for $i=1,2$. Let F_i denote the cumulative distribution of X_i and let f_i denote the probability density function of X_i. A random variable X_1 is said to be smaller than a random variable X_2 in the

I. Stochastic order $(X_1 \leq_{st} X_2)$ if $F_1\left(x\right) \geq F_2\left(x\right)$ for all x.
II. Hazard rate order $(X_1 \leq_{hr} X_2)$if $h_1\left(x\right) \geq h_2\left(x\right)$ for all x.
III. Likelihood ratio order $(X_1 \leq_{Lr} X_2)$ if $\frac{f_1(x)}{f_2(x)}$ decreases in x.

The following results due to Shaked and Shanthikumar [23] are well known for establishing stochastic ordering of distributions:

$$X_1 \leq_{Lr} X_2 \Rightarrow X_1 \leq_{hr} X_2 \Rightarrow X_1 \leq_{st} X_2$$

The OLL-PILD is ordered with respect to the strongest "likelihood ratio" ordering as shown in the following theorem:

Theorem 2.1. Let $X_1 \sim OLLPILD\left(\theta_1,\beta_1,\alpha_1\right)$ and $X_2 \sim OLL-PILD\left(\theta_2,\beta_2,\alpha_2\right)$. If $\beta_1=\beta_2$, and$\theta_2 \geq \theta_1$ (or if $\beta_2 \geq \beta_1$and $\theta_1=\theta_2$), then $X_1 \leq_{Lr} X_2$ and hence $X_1 \leq_{hr} X_2$ and $X_1 \leq_{st} X_2$.
Proof. Straight forward and hence omitted.
Setting $\alpha_1=\alpha_2$
Case 1: $\beta_1=\beta_2$ and $\theta_2 \geq \theta_1$ we obtained $\frac{d}{dx}\left(\frac{f_2(x)}{f_1(x)}\right)$ as an increasing function of x.
Case 2: $\beta_1 \geq \beta_2$ and $\theta_2=\theta_1$ we obtained $\frac{d}{dx}\left(\frac{f_2(x)}{f_1(x)}\right)$ as an increasing function of x.
This implies $X_1 \leq_{Lr} X_2$ and hence $X_1 \leq_{hr} X_2$ and $X_1 \leq_{st} X_2$.

2.5. Quantile Function

Let X denotes a random variable with the probability density function (Eq. 6). The quantile function, say $Q\left(p\right)$, defined by $F\left(Q\left(p\right)\right)=p$ is the root of the equation

$$\left(1+\frac{\beta}{\left(1+\beta\right)Q(p)^{\alpha}}\right)e^{-\frac{\beta}{Q(p)^{\alpha}}} = \frac{-\left(1+\beta\right)p^{1/\theta}}{p^{1/\theta}+\left(1-p\right)^{1/\theta}} \qquad (13)$$

for $0 < p < 1$. Multiplying (13) both sides by $e^{-1-\beta}$ we get,

$$-\left(1+\beta+\frac{\beta}{Q\left(p\right)^{\alpha}}\right)e^{-\left(1+\beta+\frac{\beta}{Q(p)^{\alpha}}\right)}=\frac{-\left(1+\beta\right)p^{1/\theta}e^{-(1+\beta)}}{p^{1/\theta}+\left(1-p\right)^{1/\theta}}$$

Using the Lambert W function which is the solution of the equation $W\left(z\right)e^{W(z)}$, where z is a complex number, we have

$$W\left(\frac{-\left(1+\beta\right)p^{1/\theta}e^{-(1+\beta)}}{p^{1/\theta}+\left(1-p\right)^{1/\theta}}\right)=-\left(1+\beta+\frac{\beta}{Q\left(p\right)^{\alpha}}\right)$$

The negative Lambert W function of the real argument $\frac{-(1+\beta)p^{1/\theta}e^{-(1+\beta)}}{p^{1/\theta}+(1-p)^{1/\theta}}$ is

$$W_{-1}\left(\frac{-\left(1+\beta\right)p^{1/\theta}e^{-(1+\beta)}}{p^{1/\theta}+\left(1-p\right)^{1/\theta}}\right)=-\left(1+\beta+\frac{\beta}{Q\left(p\right)^{\alpha}}\right),$$

which upon solving for $Q\left(p\right)$ results in

$$Q\left(p\right)=\left[-1-\frac{1}{\beta}-\frac{1}{\beta}W_{-1}\left(\frac{-\left(1+\beta\right)p^{1/\theta}e^{-(1+\beta)}}{p^{1/\theta}+\left(1-p\right)^{1/\theta}}\right)\right]^{-\frac{1}{\alpha}}.$$

Using above equation, the quartiles of the OLL-PIL distribution can be determined.

2.6. *Asymptotic properties*

Let X ∼OLL-PIL then the asymptotic of equations (5), (6) and (7) as $x \to 0$ are given by

$$F(x) \sim \left(\tfrac{\beta}{x^{\alpha}}\right)^{\theta} \text{ as } x \to 0$$
$$f(x) \sim \tfrac{\alpha\theta\beta^{\theta}}{x^{\alpha\theta+1}} \text{ as } x \to 0$$
$$h(x) \sim \tfrac{\alpha\theta\beta^{\theta}}{x^{\alpha\theta+1}} \text{ as } x \to 0$$

The asymptotic of equations (5), (6) and (7) as $x \to \infty$ are given by

$$1-F\left(x\right) \sim \left(\tfrac{\beta}{1+\beta}\right)^{\theta}\tfrac{e^{-\frac{\theta\beta}{x^{\alpha}}}}{x^{\theta\alpha}} \text{ as } x \to \infty$$
$$f(x) \sim \theta\beta\alpha\left(\tfrac{\beta}{1+\beta}\right)^{\theta}\tfrac{e^{-\frac{\theta\beta}{x^{\alpha}}}}{x^{\alpha(\theta+1)+1}} \text{ as } x \to \infty$$
$$h(x) \sim \tfrac{\theta\beta\alpha}{x^{\alpha+1}} \text{ as } x \to \infty$$

This attractive flexibility makes the OLL-PIL hazard rate function useful and suitable for non-monotone empirical hazard behaviors that are more likely to be encountered or observed in real life situations.

2.7. *Distribution of order statistics*

Order statistics make their appearance in many areas of statistical theory and practice. Suppose that $X_1, \ldots, X_n$ are a random sample from an OLL-PIL distribution. Let $X_{i:n}$ denote the i-th order statistic. The pdf of $X_{i:n}$ can be expressed as (see Arnold et al. [3]).

$$f_{i:n}\left(x\right)=Kf\left(x\right)F^{i-1}\left(x\right)\left\{1-F\left(x\right)\right\}^{n-i}=K\sum_{j=0}^{n-i}(-1)^{j}\left(\begin{array}{c}n-j\\ j\end{array}\right)f(x)F(x)^{j+i-1},$$

where $K = \frac{n!}{(i-1)!(n-i)!}$
We use the result 0.314 of Gradshteyn and Ryzhik [17] for a power series raised to a positive integer n $(n \geq 1)$

$$\left(\sum_{i=0}^{\infty} a_i u^i\right)^n = \sum_{i=0}^{\infty} d_{n,i} u^i,$$

where the coefficients $d_{n,i}$ (for $i = 1, 2, \ldots$) are determined from the recurrence equation (with $d_{n,0} = a_0^n$)

$$d_{n,i} = (ia_0)^{-1} \sum_{m=1}^{i} \left[m\left(n+1\right) - i\right] a_m d_{n,i-m}.$$

We can demonstrate that the density function of the *i*-th order statistics of an OLL-PIL distribution can be expressed as

$$f_{i:n} = \sum_{r,k=0}^{\infty} \sum_{j=0}^{\infty} m^{*}_{r,k,j} f_{EPIL}\left(x, r+k+i+j, \alpha, \beta\right), \tag{14}$$

where $f_{EPIL}\left(x; \alpha, \beta, \theta\right)$ denotes the density of EPIL distribution with parameters α, β and θ and the coefficient $m^{*}_{r,k,j} \equiv m^{*}_{r,k,j}(i,n)'s$ are given by

$$m^{*}_{r,k,j} = \frac{n!\left(r+1\right) c_{r+1} (-1)^{j} a^{*}_{j+i-1.k}}{(i-1)!\left(n-i-j\right)! j! \left(r+k+i+j\right)},$$

in which the coefficients $c_r's$ are defined in subsection 2.1 and quantities $a^{*}_{j+i-1,k}$ can be determined such that $a^{*}_{j+i-1,0} = c_1^{j+i-1}$ and for $k \geq 1$

$$a^{*}_{j+i-1.k} = (kc_1)^{-1} \sum_{q=1}^{k} \left[q\left(j+i\right) - k\right] c_{q+1} a^{*}_{j+i-1,k-q}.$$

Equation (14) is the main result of this section. It reveals that the pdf of the OLL-PIL order statistic is a linear combination of EPIL distributions. Therefore, several mathematical quantities of these order statistics, like ordinary and incomplete moments, factorial moments, the moment generating function, mean deviations, and others, can be derived using this result.

3. Maximum Likelihood Estimation of Parameters

Let $X_1, \ldots, X_n$ be a random sample of size n from OLL-PIL. Then, the log-likelihood function is given by

$$\mathcal{L}(\alpha, \beta, \gamma, \theta) = \sum_{i=1}^{n} \ln f\left(x_i\right),$$

$$= n\left[\ln\left(\alpha\right) + 2\ln\left(\beta\right) + \ln\left(\theta\right) - \ln\left(1+\beta\right)\right] + \sum_{i=1}^{n} \ln\left(1+x_i^{\alpha}\right) - \left(2\alpha+1\right) \sum_{i=1}^{n} \ln\left(x_i\right)$$

$$-\beta \sum_{i=1}^{n} x_i^{-\alpha} + \left(\theta-1\right) \sum_{i=1}^{n} \ln\left[t_i\left(1-t_i\right)\right] - 2\sum_{i=1}^{n} \ln\left[t_i^{\theta} + \left(1-t_i\right)^{\theta}\right], \tag{15}$$

where $t_i = \left(1 + \frac{\beta}{(1+\beta)x^{\alpha}}\right) e^{-\frac{\beta}{x^{\alpha}}}$.

The MLEs $\widehat{\alpha}, \widehat{\beta}, \widehat{\theta}$ of α, β, θ are then the solutions of the following non-linear equations:

$$\frac{\partial}{\partial \alpha} \mathcal{L}\left(\alpha, \beta, \gamma, \theta\right) = \frac{n}{\alpha} + \sum_{i=1}^{n} \frac{x_i^{\alpha} \ln\left(x_i\right)}{x_i^{\alpha}+1} - 2\sum_{i=1}^{n} \ln\left(x_i\right) + \beta \sum_{i=1}^{n} x_i^{-\alpha}.\ln\left(x_i\right)$$

$$+(\theta-1)\sum_{i=1}^{n}\frac{t_i^{(\alpha)}}{t_i}+(1-\theta)\sum_{i=1}^{n}\frac{t_i^{(\alpha)}}{1-t_i}-2\theta\sum_{i=1}^{n}t_i^{(\alpha)}\frac{t_i^{\theta-1}-(1-t_i)^{\theta-1}}{t_i^{\theta}+(1-t_i)^{\theta}}=0, \quad (16)$$

$$\frac{\partial}{\partial\beta}\mathcal{L}(\alpha,\beta,\gamma,\theta)=\frac{n(\beta+2)}{\beta(\beta+1)}-\sum_{i=1}^{n}x_i^{-\alpha}+(\theta-1)\sum_{i=1}^{n}\frac{t_i^{(\beta)}}{t_i}+(1-\theta)\sum_{i=1}^{n}\frac{t_i^{(\beta)}}{1-t_i}$$

$$-2\theta\sum_{i=1}^{n}t_i^{(\beta)}\frac{t_i^{\theta-1}-(1-t_i)^{\theta-1}}{t_i^{\theta}+(1-t_i)^{\theta}}=0 \quad (17)$$

$$\frac{\partial}{\partial\theta}\mathcal{L}(\alpha,\beta,\gamma,\theta)=\frac{n}{\theta}+\sum_{i=1}^{n}\ln[t_i(1-t_i)]-2\sum_{i=1}^{n}\frac{t_i^{\theta}\ln(t_i)+(1-t_i)^{\theta}\ln(1-t_i)}{t_i^{\theta}+(1-t_i)^{\theta}}=0, \quad (18)$$

where

$$t_i^{(\alpha)}=\frac{\beta^2}{1+\beta}\left(\frac{1+x_i^{\alpha}}{x_i^{2\alpha+1}}\right)e^{-\frac{\beta}{x_i^{\alpha}}}\ln(x_i),$$

$$t_i^{(\beta)}=\frac{e^{-\frac{\beta}{x_i^{\alpha}}}}{x_i^{\alpha}(1+\beta)^2}-\frac{e^{-\frac{\beta}{x_i^{\alpha}}}}{x_i^{\alpha}}\left(\frac{\beta}{x_i^{\alpha}(1+\beta)}+1\right)$$

The above non-linear system of equations is solved by the numerical iteration technique, and maximum likelihood estimates are obtained.For the three parameters of the OLL-PIL distribution, all the second order derivatives exist. Thus, we have the inverse dispersion matrix, as follows:

$$\begin{pmatrix}\widehat{\theta}\\ \widehat{\beta}\\ \widehat{\alpha}\end{pmatrix}\sim N\left[\begin{pmatrix}\theta\\ \beta\\ \alpha\end{pmatrix},\begin{pmatrix}\hat V_{11} & \hat V_{12} & \hat V_{13}\\ \hat V_{21} & \hat V_{22} & \hat V_{23}\\ \hat V_{31} & \hat V_{32} & \hat V_{33}\end{pmatrix}\right]$$

$$V^{-1}=-E\left[\begin{pmatrix}V_{11} & \dots & V_{13}\\ \dots & \dots & \dots\\ V_{31} & \dots & V_{33}\end{pmatrix}\right]=-E\begin{pmatrix}\frac{\partial^2\mathcal{L}}{\partial\theta^2} & \dots & \frac{\partial^2\mathcal{L}}{\partial\theta\partial\gamma}\\ \dots & \dots & \dots\\ \frac{\partial^2\mathcal{L}}{\partial\theta\partial\gamma} & \dots & \frac{\partial^2\mathcal{L}}{\partial\gamma^2}\end{pmatrix} \quad (19)$$

Equation (19) is the variance covariance matrix of the $\mathrm{OLL}-\mathrm{PIL}\ (\theta,\beta,\alpha)$.

$$V_{11}=\frac{\partial^2\mathcal{L}}{\partial\theta^2}\quad V_{12}=\frac{\partial^2\mathcal{L}}{\partial\theta\partial\beta}\quad V_{13}=\frac{\partial^2\mathcal{L}}{\partial\theta\partial\alpha}\quad V_{22}=\frac{\partial^2\mathcal{L}}{\partial\beta^2}$$

$$V_{23}=\frac{\partial^2\mathcal{L}}{\partial\beta\partial\alpha}\qquad V_{33}=\frac{\partial^2\mathcal{L}}{\partial\alpha^2}$$

The second derivatives of $\mathcal{L}$ can be derived as follows:

$$\frac{\partial^2\mathcal{L}}{\partial\theta^2}=\frac{-n}{\theta^2}-2\sum_{i=1}^{n}\frac{t_i^{\theta}(1-t_i)^{\theta}\ln(t_i)\ln\left(\frac{t_i}{1-t_i}\right)+t_i^{\theta}(1-t_i)^{\theta}\ln(1-t_i)\ln\left(\frac{1-t_i}{t_i}\right)}{\left[t_i^{\theta}+(1-t_i)^{\theta}\right]^2},$$

$$\frac{\partial^2\mathcal{L}}{\partial\alpha^2}=-\frac{n}{\alpha^2}+\sum_{i=1}^{n}\frac{x_i^{\alpha}\ln(x_i)^2}{(1+x_i^{\alpha})^2}-\beta\sum_{i=1}^{n}\frac{(\ln x_i)^2}{x_i^{\alpha}}+(\theta-1)\sum_{i=1}^{n}\frac{t_i^{(\alpha\alpha)}t_{i-}\left[t_i^{(\alpha)}\right]^2}{t_i^2}$$

$$+(1-\theta)\sum_{i=1}^{n}\frac{t_i^{(\alpha\alpha)}(1-t_i)+\left[t_i^{(\alpha)}\right]^2}{(1-t_i)^2}-2\theta\sum_{i=1}^{n}t_i^{(\alpha\alpha)}\frac{t_i^{\theta-1}-(1-t_i)^{\theta-1}}{t_i^{\theta}+(1-t_i)^{\theta}}$$

$$-2\theta(1-\theta)\sum_{i=1}^{n}\left[t_i^{(\alpha)}\right]^2\frac{t_i^{\theta-2}+(1-t_i)^{\theta-2}}{t_i^{\theta}+(1-t_i)^{\theta}}+2\theta^2\sum_{i=1}^{n}\left[t_i^{(\alpha)}\frac{t_i^{\theta-1}-(1-t_i)^{\theta-1}}{t_i^{\theta}+(1-t_i)^{\theta}}\right]^2$$

$$\frac{\partial^2\mathcal{L}}{\partial\beta^2}=\left[\frac{-2n}{\beta^2}+\frac{n}{(\beta+1)^2}\right]+(\theta-1)\sum_{i=1}^{n}\frac{t_i^{(\beta\beta)}t_i-\left[t_i^{(\beta)}\right]^2}{t_i^2}+(1-\theta)\sum_{i=1}^{n}\frac{t_i^{(\beta\beta)}(1-t_i)+\left[t_i^{(\beta)}\right]^2}{(1-t_i)^2}$$

$$-2\theta\sum_{i=1}^{n}t_i^{(\beta\beta)}\frac{t_i^{\theta-1}-(1-t_i)^{\theta-1}}{t_i^{\theta}+(1-t_i)^{\theta}}-2\theta(\theta-1)\sum_{i=1}^{n}\left[t_i^{(\beta)}\right]^2\frac{t_i^{\theta-2}+(1-t_i)^{\theta-2}}{t_i^{\theta}+(1-t_i)^{\theta}}$$

$$+2\theta^2\sum_{i=1}^{n}\left[t_i^{(\beta)}\frac{t_i^{\theta-1}-(1-t_i)^{\theta-1}}{t_i^{\theta}+(1-t_i)^{\theta}}\right]^2.$$

$$\frac{\partial^2\mathcal{L}}{\partial\theta\partial\alpha}=\sum_{i=1}^{n}\frac{t_i^{(\alpha)}}{t_i}-\sum_{i=1}^{n}\frac{t_i^{(\alpha)}}{1-t_i}-2\sum_{i=1}^{n}t_i^{(\alpha)}\frac{t_i^{\theta-1}-(1-t_i)^{\theta-1}}{t_i^{\theta}+(1-t_i)^{\theta}}$$

$$-2\theta\sum_{i=1}^{n}\frac{t_i^{(\alpha)}t_i^{\theta-1}(1-t_i)^{\theta-1}\ln\left(\frac{t_i}{1-t_i}\right)}{\left[t_i^{\theta}+(1-t_i)^{\theta}\right]^2},$$

$$\frac{\partial^2\mathcal{L}}{\partial\theta\partial\beta}=\sum_{i=1}^{n}\frac{t_i^{(\beta)}}{t_i}-\sum_{i=1}^{n}\frac{t_i^{(\beta)}}{1-t_i}-2\sum_{i=1}^{n}t_i^{(\beta)}\frac{t_i^{\theta-1}-(1-t_i)^{\theta-1}}{t_i^{\theta}+(1-t_i)^{\theta}}$$

$$-2\theta\sum_{i=1}^{n}t_i^{(\beta)}\frac{t_i^{\theta-1}\ln(t_i)-(1-t_i)^{\theta-1}\ln(1-t_i)}{t_i^{\theta}+(1-t_i)^{\theta}}+2\theta\sum_{i=1}^{n}t_i^{(\beta)}\frac{\left[t_i^{\theta}\ln(t_i)+(1-t_i)^{\theta}\ln(1-t_i)\right]\left[t_i^{\theta-1}-(1-t_i)^{\theta-1}\right]}{\left[t_i^{\theta}+(1-t_i)^{\theta}\right]^2},$$

$$\frac{\partial^2\mathcal{L}}{\partial\alpha\partial\beta}=\sum_{i=1}^{n}x_i^{-\alpha}\ln(x_i)+(\theta-1)\sum_{i=1}^{n}\frac{t_i^{(\alpha\beta)}t_i-t_i^{(\alpha)}t_i^{(\beta)}}{t_i^2}$$

$$+(1-\theta)\sum_{i=1}^{n}\frac{t_i^{(\alpha\beta)}(1-t_i)-t_i^{(\alpha)}t_i^{(\beta)}}{(1-t_i)^2}-2\theta\sum_{i=1}^{n}t_i^{(\alpha\beta)}\frac{t_i^{\theta-1}-(1-t_i)^{\theta-1}}{t_i^{\theta}+(1-t_i)^{\theta}}$$

$$-2\theta(\theta-1)\sum_{i=1}^{n}t_i^{(\alpha)}t_i^{(\beta)}\frac{t_i^{\theta-2}+(1-t_i)^{\theta-2}}{t_i^{\theta}+(1-t_i)^{\theta}}+2\theta^2\sum_{i=1}^{n}t_i^{(\alpha)}t_i^{(\beta)}\left[\frac{t_i^{\theta-1}+(1-t_i)^{\theta-1}}{t_i^{\theta}+(1-t_i)^{\theta}}\right]^2,$$

in which;

$$t_i^{(\alpha\alpha)}=\frac{\beta^2}{1+\beta}\left(\frac{1}{x_i^{2\alpha+1}}\right)e^{-\frac{\beta}{x_i^{\alpha}}}\left[\ln(x_i)\right]^2\left[-(x^{\alpha}+2)+\beta\left(1+x^{-\alpha}\right)\right],$$

$$t_i^{(\beta\beta)}=\frac{e^{-\frac{\beta}{x_i^{\alpha}}}}{x_i^{2\alpha}}\left(\frac{\beta}{x_i^{\alpha}(1+\beta)}+1\right)-\frac{2e^{-\frac{\beta}{x_i^{\alpha}}}}{x_i^{\alpha}(1+\beta)^3}-\frac{2e^{-\frac{\beta}{x_i^{\alpha}}}}{x_i^{2\alpha}(1+\beta)^2},$$

$$t_i^{(\alpha\beta)}=-\frac{\beta e^{-\frac{\beta}{x_i^{\alpha}}}\ln(x_i)(1+x_i^{\alpha})}{x_i^{2\alpha+1}(1+\beta)^2}\left[\frac{\beta(1+\beta)}{x^{\alpha}}-\beta-2\right].$$

These solutions will yield the asymptotic variance and co-variances of these ML estimators for $\widehat{\theta}$, $\widehat{\beta}$ *and* $\widehat{\alpha}$ by solving this inverse dispersion matrix. By using Eq.19 approximately the $100(1-\alpha)\%$ confidence intervals for θ,β,α and γ can be determined as

$$\widehat{\theta}\pm Z_{\frac{\alpha}{2}}\sqrt{\hat{V}_{11}}\qquad\widehat{\beta}\pm Z_{\frac{\alpha}{2}}\sqrt{\hat{V}_{22}}\qquad\widehat{\alpha}\pm Z_{\frac{\alpha}{2}}\sqrt{\hat{V}_{33}}$$

where $Z_{\frac{\alpha}{2}}$ is the upper α-th percentile of the standand normal distribution.

4. Data Analysis

In this section, we illustrate the power of OLL-PIL distribution using three real data sets. Tables 1 and 2 contain average wind speed data from Demak, Indonesia, as reported by Hibatullah et al. [18]. The third dataset in table 3 is from Nadarajah et al. [22]. The data shows the relief times of 20 patients receiving an analgesic. The estimation of the parameters for the OLL-PIL distribution was conducted using the maximum likelihood method. The comparison of OLL-PIL with the other models is conducted in terms of the log-likelihood values (log L), the Akaike information criterion (AIC) defined by $-2\log\, L + 2q$, and the Bayesian information criterion (BIC) defined by $-2\log\, L + q\, \log(n)$, where q is the number of estimated parameters and n is the sample size. The best model would be given by the highest value of $\log\, L$ and the lowest values of the AIC and BIC. Thus, the OLL-PIL distribution is compared with the Lindley (L) distribution, the power Lindley (PL) distribution, the inverse Lindley (IL) distribution, the Power inverse Lindley (PIL) distribution, the Weibull (W) distribution, and the Gamma (G) distribution.

Table 1. Data set 1.

1.04525	2.28740	2.44529	2.68460	1.50003	3.33749
2.78426	4.79976	13.1893	5.45061	2.01266	1.27453
2.54918	1.32359	2.16495	1.32353	1.74341	2.29751
6.90446	1.71967	3.78884	1.48582	3.11761	3.26983
2.46577	3.52471	2.20266	5.10102	0.80668	2.65993
2.83905	0.38095	0.71543	3.00342	2.65187	4.53323
2.09819	10.9028	16.4941	1.77735	4.64156	5.73434
0.47927	1.38314	3.14792	4.88295	1.65586	2.09596
1.41378	1.89628	7.72747	0.80280	6.95507	1.52554
4.77888	1.03046	2.84926	5.02584	5.83996	2.71060

TABLE 2. Data set 2.

61.80	43.54	6.53	6.53	3.72	2.74	3.00	3.10	5.26	17.56
32.40	14.00	24.90	6.40	8.90	5.90	7.60	10.60	87.50	40.30
57.09	49.65	34.58	34.58	28.35	20.56	25.54	7.62	10.60	87.51
62.44	48.78	18.92	18.92	11.54	14.84	30.89	8.85	9.90	20.98
25.60	37.55	12.52	12.52	7.00	13.48	23.25	24.41	28.83	377.09
29.92	28.9	40.25	26.44	31.43	18.90	19.20	28.85	34.66	17.34

TABLE 3. Data set 3.

1.1	1.4	1.3	1.7	1.9	1.8	1.6	2.2	1.7	2.7
4.1	1.8	1.5	1.2	1.4	3.0	1.7	2.3	1.6	2.0

TABLE 4. Summary of fitted distributions for data set 1

Models	α	β	θ	log$L$	AIC	BIC
Lindley (L)	—-	0.49297	—-	-129.586	261.172	263.266
Inverse Lindley (IL)	1	2.50067	1	-132.532	267.063	269.157
Power Lindley (PL)	1.09454	0.43377	1	-129.022	262.044	266.233
Power Inverse Lindley (PIL)	1.26995	2.68507	1	-129.671	263.343	267.531
Gamma (G)	1.95473	1.73284	—	-129.639	257.278	261.467
Weibull (W)	1.33872	3.72481	—-	-128.960	259.920	266.109
OLL-PIL	0.22158	1.31936	7.162	**-123.611**	**253.222**	**259.506**

TABLE 5. Summary of fitted distributions for data set 2

Models	α	β	θ	log$L$	AIC	BIC
Lindley (L)	—-	0.06303	—-	-269.795	541.590	543.684
Inverse Lindley (IL)	1	13.6935	1	-262.256	526.512	528.606
Power Lindley (PL)	0.77956	0.14195	1	-263.387	530.773	534.962
Power Inverse Lindley (PIL)	1.11928	17.3911	1	-261.593	527.186	531.375
Gamma (G)	1.21189	25.4074	—	-264.977	533.954	538.143
Weibull (W)	0.99265	30.6667	—-	-265.630	533.259	539.448
OLL-PIL	0.53839	4.04225	2.44	**-257.303**	**520.606**	**526.889**

TABLE 6. Summary of fitted distributions for data set 3

Models	α	β	θ	log$L$	AIC	BIC
Lindley (L)	—-	0.8161	—-	-30.2496	62.4991	63.4948
Inverse Lindley (IL)	1	2.2547	1	-31.7572	65.5144	66.5101
Power Lindley (PL)	2.2529	0.3445	1	-20.4320	44.8640	46.8554
Power Inverse Lindley (PIL)	3.9812	6.7190	1	**-15.4132**	**34.8263**	**36.8178**
Gamma (G)	9.6695	0.1965	—	-17.8186	39.6372	43.8259
Weibull (W)	2.7870	2.1300	—-	-20.5864	43.1728	49.3615
OLL-PIL	3.55142	5.54344	1.15	-15.3889	36.7779	39.7652

Table 4 presents the parameter estimation, log L, AIC, and BIC from dataset 1 for the fitted OLL-PIL distribution and its special cases (IL, PIL), L, PL, the Gamma distribution, and the Weibull distribution. Table 5 presents log L, AIC, and BIC from dataset 2 for the fitted OLL-PIL distribution and its special cases (IL, PIL), L, PL, the Gamma distribution, and the Weibull distribution. From the value of log L, we see that OLL-PIL has higher values than the other models. From the AIC and BIC, OLL-PIL has the lowest values of all the models. Therefore, OLL-PIL provides the best fit to this data. Fig. 3 gives a graphical representation of the histogram for dataset 1 and the graphs of OLL-PIL, Gamma, Weibull, and PL. Fig. 4 gives a graphical representation of the histogram for dataset 2 and the graphs of OLL-PIL, Gamma, Weibull and IL. Table 6 presents log L, AIC, and BIC from dataset 3 for the fitted OLL-PIL and its special cases (IL, PIL), L, PL, the Gamma distribution, and the Weibull distribution. From the values of log L, AIC, and BIC, the PIL model was a better fit than the other models. Fig. 5 gives a graphical representation of the histogram for dataset 3 and the graphs of OLL-PIL, Gamma, Weibull, and PIL.

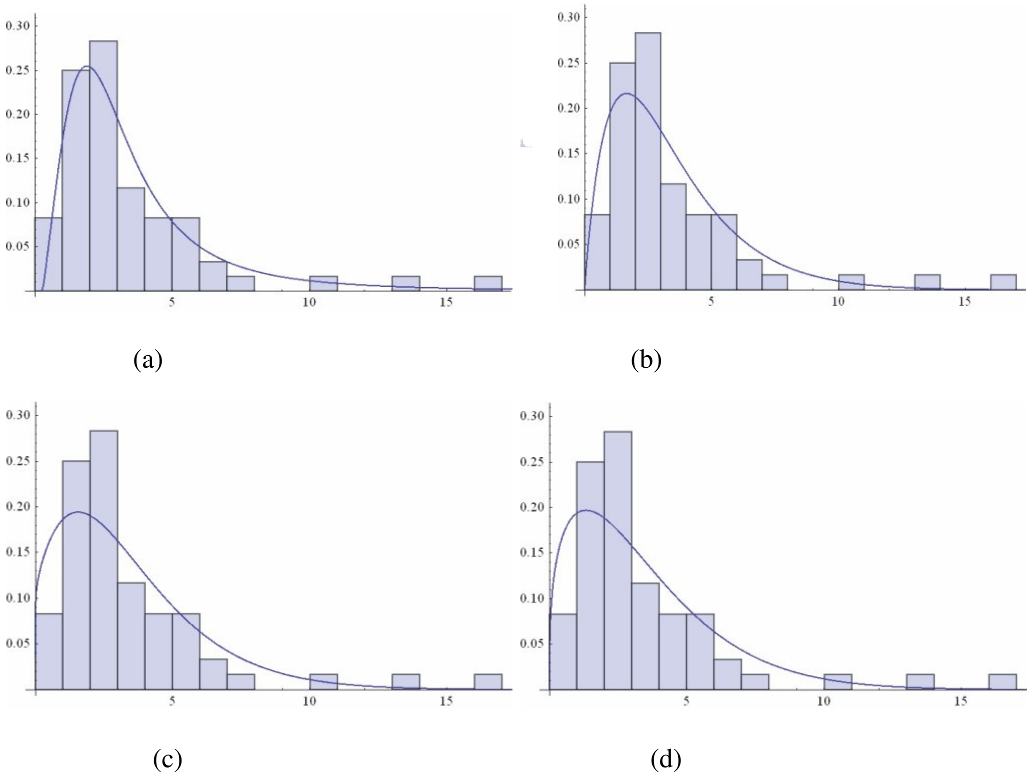

FIGURE 3. plots of the histogram from data set 1 and (a) OLL-PIL, (b) Gamma, (c) Weibull, and (d) PL

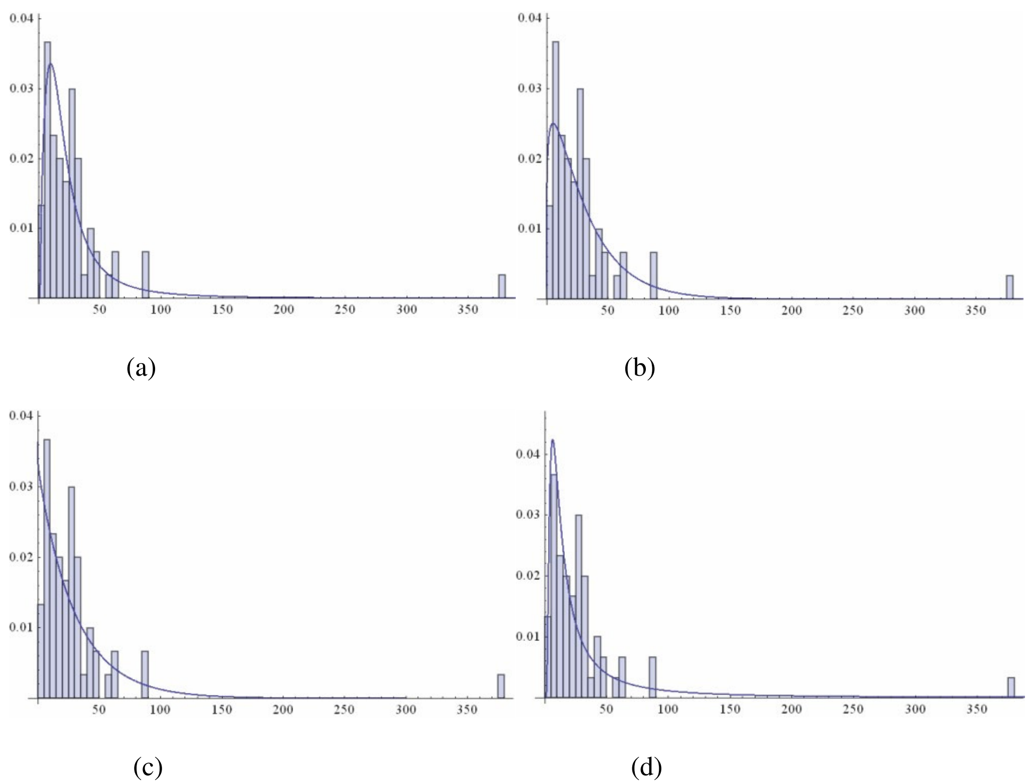

FIGURE 4. Plots of the histogram from data set 2 and (a) OLL-PIL, (b) Gamma, (c) Weibull, and (d) IL

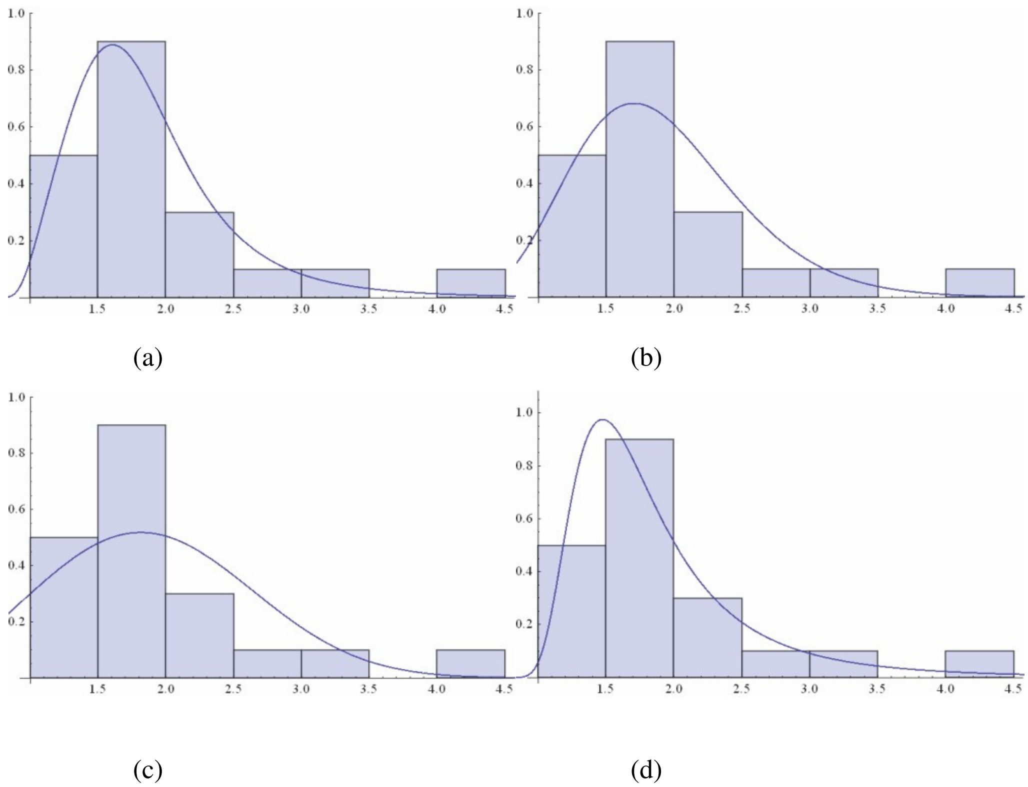

FIGURE 5. Plots of the histogram from data set 3 and (a) OLL-PIL, (b) Gamma, (c) Weibull, and (d) PIL.

5. Generation Algorithms and Monte Carlo Simulation Study

In this section, the algorithms for generating random data from OLL-PIL distribution are given. A simulation study was also conducted to check the performance and accuracy of maximum likelihood estimates of the OLL-PIL model parameters. The Simulation study was performed using the statistical software Mathcad 14.

5.1. Generation algorithms

In this subsection, different algorithms that can be used to generate random data from OLL-PIL distribution are presented.

Algorithm I. (mixture form of the inverse Lindley distribution)

1. Generate $U_i \sim uniform\ (0,1)\ ,\ i = 1,\dots,n;$
2. Generate $V_i \sim\ inverse\ Exponential\ (\beta)\ ,\ i = 1,\dots,n;$
3. Generate $G_i \sim inverse\ Gamma\ (2,\beta)\ , i = 1,\dots,n.$
4. if $\frac{U_i^{1/\theta}}{U_i^{1/\theta}+(1-U_i)^{1/\theta}} \leq \frac{\beta}{1+\beta}$, then set $X_i = V_i^{1/\alpha}$, otherwise, set $X_i = G_i^{1/\alpha}, i = 1,\dots,n.$

Algorithm II. (mixture form of the Extended inverse Lindley distribution)

1. Generate $U_i \sim uniform\ (0,1)\ ,\ i = 1,\dots,n;$
2. Generate $Y_i \sim\ inverse\ Weibull\ (\alpha,\beta)\ ,\ i = 1,\dots,n;$
3. Generate $S_i \sim Generalized\ inverse\ Gamma\ (2,\alpha,\beta)\ , i = 1,\dots,n.$
4. if $\frac{U_i^{1/\theta}}{U_i^{1/\theta}+(1-U_i)^{1/\theta}} \leq \frac{\beta}{1+\beta}$, then set $X_i = Y_i$, otherwise, set $X_i = S_i, i = 1,\dots,n.$

Algorithm III: (inverse CDF)

1. Generate $U_i \sim uniform\,(0,1)\,,\ i = 1,\ldots,n;$
2. Set
$X_i = \left[-1 - \frac{1}{\beta} - \frac{1}{\beta} W_{-1}\left(\frac{-(1+\beta)p^{1/\theta}e^{-(1+\beta)}}{p^{1/\theta}+(1-p)^{1/\theta}}\right)\right]^{-\frac{1}{\alpha}}$

5.2. Monte Carlo simulation study

In this subsection, we study the performance and accuracy of maximum likelihood estimates of the OLL-PIL model parameters by conducting various simulations for different combinations of 5 sample sizes with two sets of parameter values. Algorithm II was used to generate random data from the OLL-PIL distribution. The simulation study was repeated N = 10,000 times each with samples of size n = 25, 50, 100, 200, 400 combined with parameter values *(I)*: $\theta = 0.7,\ \beta$ =4, $\alpha = 0.8$, and *(II)*: $\theta = 1.5,\ \beta = 0.6,\ \alpha = 2$. Four quantities were computed in this simulation study: (i) Average bias of the MLE $\widehat{\vartheta}$ of the parameter $\vartheta = \alpha, \beta, \theta$: $\frac{1}{N}\sum_{i=1}^{N}\left(\widehat{\vartheta} - \vartheta\right)$; (ii) Root mean squared error (RMSE) of the MLE $\widehat{\vartheta}$ of the parameter $\vartheta = \alpha, \beta, \theta$: $\left[\frac{1}{N}\sum_{i=1}^{N}\left(\widehat{\vartheta} - \vartheta\right)^2\right]^{0.5}$;(iii) Coverage probability (CP) of 95% confidence intervals of the parameter $\vartheta = \alpha, \beta, \theta$; (iv) Average width (AW) of 95% confidence intervals of the parameter $\vartheta = \alpha, \beta, \theta$. Table 7 presents the average Bias, RMSE, CP and AW values of the parameters $\alpha, \beta$and θ for different sample sizes. According to the results, it can be concluded that, as the sample size n increases, the RMSEs decrease toward zero. We also observe that, for all the parameters, the biases decrease as the sample size n increases. The results show that the coverage probabilities of the confidence intervals are quite close to the nominal level of 95% and that the average confidence widths decrease as the sample size increases. Consequently, the MLEs and their asymptotic results can be used for estimating and constructing confidence intervals even for reasonably small sample sizes.

Table 7: Monte Carlo simulation results: Average Bias, RMSE, CP and AW

		I				II			
Parameter	n	Average bias	RMSE	CP	AW	Average bias	RMSE	CP	AW
θ	25	0.6485	0.3878	0.9613	0.7842	0.6562	0.8655	0.9441	3.8412
	50	0.6277	0.3766	0.9631	0.6851	0.6549	0.8466	0.9423	2.4051
	100	0.5921	0.3611	0.9635	0.4723	0.6516	0.8411	0.9453	1.8763
	200	0.5851	0.3541	0.9653	0.4514	0.6415	0.8381	0.9465	1.5791
	400	0.5745	0.3348	0.9741	0.3649	0.5713	0.7972	0.9626	0.3678
β	25	2.4498	2.1978	0.9632	1.2381	0.5866	0.6925	0.9632	5.0502
	50	2.3834	2.1198	0.9561	0.4227	0.5751	0.6807	0.9636	1.5696
	100	1.9259	2.1756	0.9573	0.3289	0.5561	0.6212	0.9675	0.8471
	200	1.9111	2.1767	0.9616	0.2464	0.5452	0.6112	0.9691	0.5452
	400	1.8482	1.9856	0.9645	0.2032	0.4418	0.4947	0.9701	0.2535
α	25	0.6631	0.7442	0.9426	2.1786	0.6256	0.9531	0.9216	1.8816
	50	0.5111	0.6561	0.9402	1.5187	0.5224	0.6932	0.9390	1.4735
	100	0.4482	0.4999	0.9312	1.1469	0.4347	0.5945	0.9258	0.8399
	200	0.4412	0.4658	0.9356	0.8867	0.4305	0.3788	0.9276	0.7119
	400	0.4271	0.4411	0.9468	0.3386	0.3497	0.3431	0.9496	0.4187

6. Concluding Remarks

In this paper, we have proposed a new family of distributions called the odd log-logistic power inverse Lindley distribution. As special cases from OLL-PIL, we obtain the probability density functions for odd log-logistic inverse Lindley and power inverse Lindley distributions. Some mathematical properties and estimation issues are addressed. The hazard rate function behavior of the odd-logistic power inverse Lindley distribution shows that the subject distribution can be used to model reliability data. The estimation of parameters is approached by the method of maximum likelihood. We present a simulation study to exhibit the performance and accuracy of maximum likelihood estimates of the OLL-PIL model parameters. A data application was also presented to illustrate the usefulness and applicability of the OLL-PIL distribution.

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