STATISTICS, OPTIMIZATION AND INFORMATION COMPUTING
Stat., Optim. Inf. Comput., Vol. **12**, March 2024, pp 400–404.
Published online in International Academic Press (www.IAPress.org)

On the Jajte Law of Large Numbers for Exchangeable Random Variables

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Abstract In this paper, we prove an extension of the Jajte strong law of large numbers for exchangeable random variables, we make a simulation study for the asymptotic behavior in the sense of convergence almost surly for weighted sums of exchangeable weighted random variables.

Keywords strong law of large numbers, exchangeable random variables

AMS 2010 subject classifications 60F05, 60F15

DOI: 10.19139/soic-2310-5070-1492

1. Introduction

Exchangeable random variables are of great interest in probability theory since they are a natural generalization of independent random variables with common distribution, and share a number of properties with such random variables. For examples, sampling with out replacement (exchangeable) has several properties in common with sampling with replacement (independent). A sequence of random variables $\{X_n, n \geq 1\}$ on the probability space $(\Omega, \mathcal{G}, \mathbb{P})$ is said to be exchangeable if for each $n \geq 1$,

$$\mathbb{P}(X_1 \leq x_1, \ldots, X_n \leq x_n) = \mathbb{P}(X_{\pi(1)} \leq x_1, \ldots, X_{\pi(n)} \leq x_n)$$

for any premutation π of $\{1, 2, \ldots, n\}$ any $x_i \in \mathcal{R}$, $i = 1, 2, \ldots, n$. Let $\mathcal{F}$ be the class of one dimensional distribution functions and $\mathcal{U}$ be the σ - field generated by the topology of weak convergence of the distribution functions. Thus, de Finetti's theorem assert that for an infinite sequence of exchangeable random variables $\{X_n, n \geq 1\}$ there exists a probability measure μ on $(\mathcal{F}, \mathcal{U})$ such that

$$\mathbb{P}(f(X_1, \ldots, X_n) \in B) = \int_{\mathcal{F}} \mathbb{P}_F(f(X_1, \ldots, X_n) \in B) d\mu(F) \quad (1)$$

for any Borel set B and any Borel function $f : \mathcal{R}^n \to \mathcal{R}$, $n \geq 1$. Moreover, $\mathbb{P}_F(f(X_1, \ldots, X_n) \in B)$ is comuted under the assumption that the random variables $\{X_n, n \geq 1\}$ are i.i.d. with common distribution F. Two trivial examples are i.i.d random variables and totally determined random variables $(\{X, X, \ldots\})$. Two non-trivial but simple examples are $\{X + \varepsilon_i, i \geq 1\}$ and $\{Y.\varepsilon_i, i \geq 1\}$ where the ε_i's i.i.d and independent of X and Y, respectively. For more detail on exchangeable sequence random variables see Chow and Teicher [1] .

Limit theorems, in particular laws of large numbers, play exceedingly important role in probability theory and its applications in mathematical statistics. In this context, law of large numbers for exchangeable random variables are still an attractive area of research. Taylor and Hu [9] obtained a necessary and sufficient condition for the strong law

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ISSN 2310-5070 (online) ISSN 2311-004X (print)
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of large numbers for sequences of exchangeable random variables. Also, Hong and Lee [2] and, Stocia and Li [8] studied weak law of large numbers for sequences of exchangeable random variables, Naderi et al. [5] surveyed an extension of the Jajte weak law of large numbers for exchangeable random variables and Huang [10] extended the Marcinkiewicz type theorem to the case of exchangeable random variables. In this paper we will focus on strong laws of large numbers (SLLN) for exchangeable random variables.

The approach to the weighted law of large numbers follow the idea of Jajte [4] and we extend his result to the case of certain exchangeable random variables. Let us recall some notation. Jajte studied a large class of summability method defined as follows: it is said that a sequence $\{X_n, n \geq 1\}$ of r.v.'s is almost surely summable to a r.v. X by the method (h, g) if

$$\frac{1}{g(n)} \sum_{k=1}^{n} \frac{1}{h(k)} X_k \to X, \text{ almost surely, as } n \to \infty.$$

For a sequence $\{X_n, n \geq 1\}$ of i.i.d. random variables Jajte proved that $\{X_n - \mathbb{E}X_n \mathbb{I}[|X_n| \leq \phi(n)], n \geq 1\}$ is almost surely summable to 0 by the method (h, g) iff $\mathbb{E}\phi^{-1}(|X_1|) < \infty$, where g, h and $\phi(y) = g(y)h(y)$ are functions satisfying the following additional conditions.

(A1) Let g be a positive, increasing function with $\lim_{y\to\infty} g(y) = \infty$ and let h be a positive function.
(A2) For some $d \geq 0$, ϕ is strictly increasing on the interval $[d, \infty)$ with range $[0, \infty)$ and $\lim_{y\to\infty} \phi(y) = \infty$, there exists a constant C such that $\phi(y + 1)/\phi(y) \leq C$, for all $y \geq d$.
(A3) There exist constants a and b such that

$$\phi^2(s) \int_s^\infty \frac{dx}{\phi^2(x)} \leq as + b, \quad \text{for all } s > d.$$

For further extensions and improvements of the result of Jajte, we refer the reader to Naderi et al. [7], Naderi et al. [6].

In Section 2 we will present our main results which are devoted to the SLLN in the Jajte version for weighted sums of exchangeable random variables. We will study sequences summable by the method (h, g) described above, and we will make a simulation study in Section 3.

2. Main results

In the following we will also use the notation $m(n) = \mathbb{E}X\mathbb{I}[|X| \leq \phi(n)]$, for $n \geq 1$. Let us state a extension of Theorem 1 in [4] in the exchangeable setting.

Theorem 1
Let g, h and ϕ, satisfy the conditions (A1) – (A3), and consider $\left(X_n\right)_{n\geq1}$ be sequence of exchangeable random variables. If $\mathbb{E}[\phi^{-1}(|X|)] < \infty$ then

$$\frac{1}{g(n)} \sum_{k=1}^{n} \frac{X_k - m(k)}{h(k)} \to 0 \;\; a.s. \; as \;\; n \to \infty, \tag{2}$$

where ϕ^{-1} is the inverse of ϕ.

Proof
For $1 \leq k \leq n$ define $S_n = \frac{1}{g(n)} \sum_{k=1}^{n} \frac{X_k - m(k)}{h(k)}$, by de Finetti's theorem, dominated convergence theorem and

Theorem 1 in [4] lead to for $\varepsilon > 0$

$$\mathbb{P}\left(|S_n| > \varepsilon\ i.o.\right) = \mathbb{P}\left(\bigcap_{n=1}^{\infty}\bigcup_{l\geq n}[|S_l > \varepsilon|]\right) = \lim_{n\to\infty}\mathbb{P}\left(\bigcup_{l\geq n}[|S_l > \varepsilon|]\right)$$

$$= \lim_{n\to\infty}\int_{\mathcal{F}}\mathbb{P}_F\left(\bigcup_{l\geq n}[|S_l > \varepsilon|]\right)d\mu(F) = \int_{\mathcal{F}}\lim_{n\to\infty}\mathbb{P}_F\left(\bigcup_{l\geq n}[|S_l > \varepsilon|]\right)d\mu(F)$$

$$= \int_{\mathcal{F}}\mathbb{P}_F\left(|S_n| > \varepsilon\ i.o.\right)d\mu(F) = 0.$$

Which implies (2). □

In next theorem we consider a converse of Theorem 1. In this situation we need strong condition.

Theorem 2
Let g, h and ϕ, satisfy the conditions (A1) and (A2), and consider $\left(X_n\right)_{n\geq 1}$ be sequence of exchangeable random variables. If

$$\frac{1}{g(n)}\sum_{k=1}^{n}\frac{X_k - m(k)}{h(k)} \to 0 \quad \mu - a.s. \tag{3}$$

then

$$\mathbb{E}[\phi^{-1}(|X|)] < \infty. \tag{4}$$

Proof
Since $\lim_{k\to\infty}(m(k)/\phi(k)) = 0$ then by (3) we have $\lim_{k\to\infty}(X_k/\phi(k)) = 0 \ \ \mu - a.s.$ in result using the Borel Cantelli lemma we have

$$\sum_{k=1}^{\infty}\mathbb{P}_F(|X_k| \geq \phi(k)) = \sum_{k=1}^{\infty}\mathbb{P}_F(\phi^{-1}(|X|) \geq k) < \infty,$$

now we can write

$$\sum_{k=1}^{\infty}\mathbb{P}(|X_k| \geq \phi(k)) = \sum_{k=1}^{\infty}\int_{\mathcal{F}}\mathbb{P}_F(|X_k| \geq \phi(k))d\mu(F) = \int_{\mathcal{F}}\sum_{k=1}^{\infty}\mathbb{P}_F(|X_k| \geq \phi(k))d\mu(F)$$

$$= \int_{\mathcal{F}}\sum_{k=1}^{\infty}\mathbb{P}_F(\phi^{-1}(|X|) \geq k) < \infty.$$

that shows (4). □

Remark 1
Let $\left(X_n\right)_{n\geq 1}$ be sequence of i.i.d. random variables then all of the above theorems are true in this case.

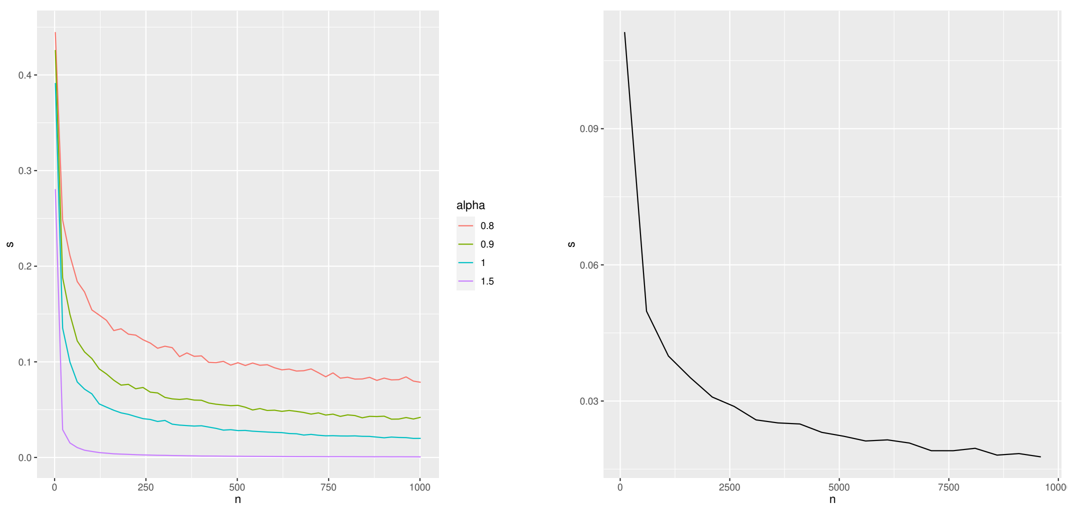
Remark 2
By considering $h(x) = 1$, $g(x) = x^{1/p}$ $\left[\phi(x) = x^{1/p}\right]$ for $0 \leq p \leq 2$ and $h(x) = x$, $g(x) = \log(x)$ $[\phi(x) = x\log(x)]$, Theorem 1 implies a version of Marcinkiewica-Zygmund and Logaritmic mean SLLN respectivly for exchangeable random variables.

3. Simulation

In this section we may to check the efficiency of convergence in Theorem 1 by some numeric example. To have sequences of exchangeable random variables, we take $\{X_n = X.\varepsilon_n, n \geq 1\}$ and $\{Y_n = Y + \xi_n, n \geq 1\}$ where $\{\varepsilon_n, n \geq 1\}$ and $\{\xi_n, n \geq 1\}$ are two sequences of i.i.d., independent of X and Y respectively, random variables such that for each $i \geq 1$, $P(\varepsilon_i = 1) = P(\varepsilon_i = -1) = \frac{1}{2}$, $P(\xi_i = 0) = P(\xi_i = 1) = \frac{1}{2}$, $X \sim N(\mu, \sigma^2)$ and $Y \sim Poi(\lambda)$. Also based on the rest condition of the Theorem 1 and to show two classical theorems that links in some sense the SLLN of Kolmogorov and that of Marcinkiewicz, we put $g_1(n) = n^{\alpha}$, $\alpha > \frac{1}{2}$, and $h_1(n) = 1$ in case $\{X_n, n \geq 1\}$ (it is as Cesàro means if $\alpha = 1$) and to show logarithmic means we put $g_2(n) = n$ and $h_2(n) = 1og(n)$ in case $\{Y_n, n \geq 1\}$ (it is clear that for $i = 1, 2$, $E[\phi_i^{-1}|X|] < \infty$ where $\phi_i(|X|) = g_i(|X|)h_i(|X|)$).

In the first numeric example $X \sim N(0, 1)$, $n = 2(10)1000$ and $\alpha = \{0.8, 0.9, 1, 1.5\}$. For each α and each n, we simulate $X_1 = x_1, ..., X_n = x_n$ and compute $s_{n,\alpha} = \left| \frac{1}{n^{\alpha}} \sum_{j=1}^{n} x_i \right|$. By repeating this procedure $B = 5000$ times, the vector $\left\{s^1_{n,\alpha}, ..., s^{B=5000}_{n,\alpha}\right\}$ will be observed and finally we estimate $\widehat{S}_{n,\alpha} = \frac{\sum_{i=1}^{B=5000} s^i_{n,\alpha}}{B=5000}$. The results are shown in figure1 (a) that exhibits the scatter plot of $(n, \widehat{S}_{n,\alpha})$ for each $\alpha = \{0.8, 0.9, 1, 1.5\}$.

At the second numeric example $Y \sim Poi(1)$, $n = 2(50)1000$. For each n, we simulate $Y_1 = y_1, ..., Y_n = y_n$ and compute $s_n = \left| \frac{1}{\log(n)} \sum_{j=1}^{n} \frac{y_j}{j} \right|$. By repeating this procedure $B = 5000$ times, the vector $\left\{s^1_n, ..., s^{B=5000}_n\right\}$ will be observed and finally we estimate $\widehat{S}_n = \frac{\sum_{i=1}^{5000} s^i_n}{5000}$. The result is shown in figure 1 (b) that exhibits the scatter plot of $(n, \widehat{S}_n)$. It is observed from Figure 1 that S_n is a decreasing function of n and tending to 0 in all cases. Moreover Figure 1 (a) indicates that the rate of convergence increases as α increases.



(a) (b)

Figure 1. (a)The plot of S_n versus n in case X_n, (b)The plot of S_n versus n in case Y_n.

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