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Modified Bagdonavičius-Nikulin Goodness-of-fit Test Statistic for the Compound Topp Leone Burr XII Model with Various Censored Applications

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Abstract The Poisson Topp Leone Burr XII distribution is extensively studied due to its broad relevance in analyzing censored real datasets from engineering, economics, and medicine. In this research, the distribution's versatility is highlighted through the analysis of four specific real datasets. The study compares the Poisson Topp Leone Burr XII distribution with nine extensions of the Burr type XII distribution to determine which offers the best fit for these datasets. To evaluate the goodness-of-fit of the Poisson Topp Leone Burr XII distribution under right censoring, a modified Bagdonavičius-Nikulin goodness-of-fit test statistic is introduced and applied. This new test statistic is utilized to validate the distributional fit for the Poisson Topp Leone Burr XII distribution across the four right-censored datasets. The modified Bagdonavičius-Nikulin test statistic is employed to assess distributional validation, specifically in the context of right censoring. The application of this statistic involves analyzing each of the four censored datasets to confirm the appropriateness of the Poisson Topp Leone Burr XII distribution for these scenarios. Additionally, to support the evaluation of the modified goodness-of-fit test statistic, the Barzilai-Borwein algorithm is utilized. This algorithm is employed within a simulation study to further assess the effectiveness and reliability of the modified Bagdonavičius-Nikulin test statistic, thereby ensuring robust validation of the Poisson Topp Leone Burr XII distribution against the observed real datasets.

Keywords Poisson Topp Leone Burr XII Distribution, Bagdonaviµcius-Nikulin test, Barzilai-Borwein algorithm

Mathematics Subject Classification: 60E05; 62N01; 62G05; 62N02; 62N05; 62E10; 62P30

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1. Introduction and motivation

In the statistical literature, significant focus has been directed towards one of the twelve models introduced by Burr [13]. Notably, Yousof et al. introduced a novel variant of this model, termed the Poisson Topp-Leone Burr XII (PTL-BXII) distribution, which has garnered considerable attention. In their study, Yousof et al. [58] delved deep into the theoretical underpinnings and mathematical properties of the PTL-BXII distribution. Their exploration encompassed a comprehensive analysis of various aspects, including characterizations, order statistics, ordinary and incomplete moments, residual and reversed residual life functions, and the moment generating function associated with this distribution. This investigation by Yousof et al. [58] represents a pivotal contribution to the

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statistical literature, shedding light on the intricacies and potential applications of the PTL-BXII distribution. By systematically elucidating its theoretical foundations and exploring its mathematical properties, their study has provided valuable insights that facilitate a deeper understanding of this specialized model. Through this work, we aim to build upon the foundational work of Yousof et al. [58] and further extend the analytical toolkit associated with the PTL-BXII distribution. By leveraging this rich theoretical framework, we can develop practical methodologies and statistical tools that enhance our ability to model and analyze complex data scenarios, ultimately advancing the frontier of statistical theory and application. Join us on this journey of exploration and discovery as we navigate the fascinating realm of specialized distributions and their manifold implications in statistical science. The cumulative distribution function (CDF) of the PTL-BXII model, introduced by Yousof et al. [58]**,** is as follows

$$F\left(y;\underline{\psi}\right)=\varrho_{[a_1]}^{-1}\left(1-\exp\left\{-a_1\left[1-\left(1+y^{a_2}\right)^{-2b_2}\right]^{b_1}\right\}\right), \tag{1}$$

where

$$\varrho_{[a_1]}^{-1}=\frac{1}{1-\exp(-a_1)}, a_1\in\mathbf{R}-\{0\}\,, b_1,a_2,b_2>0$$

and $\underline{\psi}=(a_1,b_1,a_2,b_2)$. The associated probability density function (PDF) of equation (1) can be obtained through the following derivation

$$f\left(y;\underline{\psi}\right)=2a_1b_1a_2b_2\varrho_{[a_1]}^{-1}\frac{y^{a_2-1}\left(1+y^{a_2}\right)^{-2b_2-1}\left[1-\left(1+y^{a_2}\right)^{-2b_2}\right]^{b_1-1}}{\exp\left\{a_1\left[1-\left(1+y^{a_2}\right)^{-2b_2}\right]^{b_1}\right\}}. \tag{2}$$

Furthermore, the hazard rate function (HRF) can be derived as:

$$h\left(y;\underline{\psi}\right)=2a_1b_1a_2b_2\frac{y^{a_2-1}\left(1+y^{a_2}\right)^{-2b_2-1}\left[1-\left(1+y^{a_2}\right)^{-2b_2}\right]^{b_1-1}}{\exp\left\{a_1\left[1-\left(1+y^{a_2}\right)^{-2b_2}\right]^{b_1}\right\}-1}. \tag{3}$$

The cumulative hazard rate function (CHRF) corresponding to this is described by

$$H\left(y;\underline{\psi}\right)=-\ln\left[1-\varrho_{[a_1]}^{-1}\left(1-\exp\left\{-a_1\left[1-\left(1+y^{a_2}\right)^{-2b_2}\right]^{b_1}\right\}\right)\right]. \tag{4}$$

The reason for introducing of the PTL-BXII lifetime model is the wider use of the standard Burr XII distribution model (see Burr [13], Burr [14] and Burr [15]), Burr and Cislak [16], Rodriguez [45] and Tadikamalla [54]).

In recent times, Cordeiro [19] conducted a study exploring a novel family derived from the Burr XII distribution, known as the Burr XII G family of distributions. However, our discussion in this article will center around the practical applications and implications of the Poisson Topp-Leone Burr XII (PTL-BXII) model. For more Burr XII extensions see Paranaíba et al [43], Paranaíba et al [42], Yousof et al. [58], Korkmaz et al. [32], Yousof et al. [61], Gad et al. [25], Cordeiro [19] and Elsayed and Yousof [24]. Some other useful Burr XII extensions can found in Nasir et al. [39], Yousof et al. [61], Nasir et al. [40], Abouelmagd et al. [2] and Abouelmagd et al. [3].

This article delves into the practical applications of the PTL-BXII distribution, particularly focusing on its utilization in modeling censored datasets. To validate the distributional fit under right censoring, a modified Bagdonavičius-Nikulin goodness-of-fit test statistic is introduced and implemented specifically for the PTL-BXII distribution. This modified test statistic is thoroughly described, detailing all its pertinent components. Subsequently, the study applies this modified statistic to analyze four right-censored datasets, aiming to assess the distributional validation. According to the results derived from the modified Bagdonavičius-Nikulin goodness-of-fit test statistic, the PTL-BXII model proves effective for modeling both censored engineering and medical datasets. To further evaluate the reliability and robustness of the modified test statistic, the study employs the Barzilai-Borwein algorithm through a simulation study. This algorithm serves as a crucial tool in scrutinizing and validating the performance of the test statistic, ensuring its accuracy and efficacy in affirming the suitability of the PTL-BXII distribution across different datasets and scenarios.

The remainder of this article is structured as follows: Section 2 presents the detailed formulation of the modified Bagdonavičius and Nikulin test statistic designed specifically for right-censored samples. Section 3 delves into simulations conducted under right-censored samples, providing further insights into the behavior and performance of the proposed methodology. Furthermore, Section 4 explores specific applications of the methodology under right-censored samples, offering practical demonstrations of its effectiveness. Finally, Section 5 concludes the article by summarizing key findings and discussing implications for future research and applications of the PTL-BXII distribution in the context of censored datasets.

2. The modified Bagdonavičius and Nikulin test statistic for right-censored samples

In the realm of survival analysis, the analysis of right-censored data introduces both unique challenges and exciting opportunities for advancing statistical methodologies. Traditional approaches often encounter limitations in accurately estimating survival probabilities and making reliable comparisons between different groups or treatments. This chapter delves into a pivotal development within this field: the modified Bagdonavičius and Nikulin test statistic. As we embark on this exploration, our aim is to unveil the transformative impact of this modified statistical approach. By navigating through its theoretical foundations and practical implementations, we will demonstrate how this method significantly enhances the robustness and reliability of statistical inference when faced with right-censored samples. Survival analysis is a vital tool in biomedical research, engineering, and many other fields where time-to-event data is prevalent. Understanding and harnessing the potential of this modified test statistic not only deepens our theoretical understanding but also equips researchers and practitioners with powerful tools to extract meaningful insights from complex survival datasets.

2.1. Test statistic under the right-censored data

Let us consider $Y = (Y_1, Y_2, ..., Y_n)^T$ a sample from the Poisson Topp-Leone BXII distribution with the parameter vector $\underline{\psi} = (a_1, b_1, a_2, b_2)^T$ which can contain right-censored data with fixed censoring time τ. Each Y_i can be written as $Y_i = (y_i, \varsigma_i)$ where $\varsigma_i = 0$,if y_i is a censor time and $\varsigma_i = 1$,if y_i is a failure time. However, the likelihood function is written as follows

$$L_n(\underline{\psi}) = \sum_{i=1}^{n} \varsigma_i \ln f(y_i, \underline{\psi}) + \sum_{i=1}^{n} (1-\varsigma_i) \ln S(y_i, \underline{\psi}),$$

then

$$\begin{aligned} L_n(\underline{\psi}) &= \sum_{i=1}^{n} \varsigma_i \left[\begin{array}{c} \ln(a_1 b_1 a_2 b_2) + (a_2-1)\ln y_i \\ -(2b_2+1)\ln(1+y_i^{a_2}) - \ln \varrho_{[a_1]} \\ +(b_1-1)\ln\left[1-(1+y_i^{a_2})^{-2b_2}\right] \\ -a_1\left[1-(1+y_i^{a_2})^{-2b_2}\right]^{b_1} \end{array} \right] \\ &+ \sum_{i=1}^{n} (1-\varsigma_i) \ln\left(1 - \varrho_{[a_1]}^{-1}\left(1-\exp\left\{-a_1\left[1-(1+y_i^{a_2})^{-2b_2}\right]^{b_1}\right\}\right)\right). \end{aligned}$$

The MLETs $\widehat{a}_1$,$\hat{a}_2$, $\hat{b}_1$ and $\widehat{b}_2$ of the unknown parameters a_1, a_2, b_1 and b_2 are derived from the nonlinear following score equations:

$$\begin{aligned}\frac{\partial L}{\partial a_1} &= \sum_{i=1}^{n} \varsigma_i \left[\frac{1}{a_1} - \frac{\exp(-a_1)}{1-\exp(a_1)} - \varpi_i^{b_1}\right] \\ &+ \sum_{i=1}^{n} (1-\varsigma_i) \left[\frac{\exp(a_1) - \varpi_i^{b_1} \exp\left(-a_1 \varpi_i^{b_1}\right)}{\exp\left(-a_1 \varpi_i^{b_1}\right) - \exp(-a_1)} - \frac{\exp(-a_1)}{1-\exp(a_1)}\right],\end{aligned}$$

$$\begin{aligned}\frac{\partial L}{\partial a_2} &= \sum_{i=1}^{n} \varsigma_i \left[\begin{array}{c} \frac{1}{a_2} + \ln(y_i) \\ + \frac{2(b_1-1) b_2 y_i^{a_2} \ln(y_i) u_i^{-(2b_2+1)}}{\varpi_i} \\ - \frac{(2b_2+1) y_i^{a_2} \ln y_i}{u_i} \\ -2 a_1 b_1 b_2 y_i^{a_2} \ln(y_i) u_i^{-(2b_2+1)} \varpi_i^{b_1-1} \end{array}\right] \\ &- \sum_{i=1}^{n} (1-\varsigma_i) \frac{2 a_1 b_1 b_2 y_i^{a_2} \ln(y_i) u_i^{-(2b_2+1)} \varpi_i^{b_1-1} \exp\left(-a_1 \varpi_i^{b_1}\right)}{\exp\left(-a_1 \varpi_i^{b_1}\right) - \exp(-a_1)},\end{aligned}$$

$$\frac{\partial L}{\partial b_1} = \sum_{i=1}^{n} \varsigma_i \left[\frac{1}{b_1} + \ln \varpi_i - a_1 \varpi_i^{b_1} \ln \varpi_i\right] - \sum_{i=1}^{n} (1-\varsigma_i) \frac{a_1 \varpi_i^{b_1} \ln \varpi_i \exp\left(-a_1 \varpi_i^{b_1}\right)}{\exp\left(-a_1 \varpi_i^{b_1}\right) - \exp(-a_1)},$$

and

$$\begin{aligned}\frac{\partial L}{\partial b_2} &= \sum_{i=1}^{n} \varsigma_i \left[\frac{1}{b_2} - 2 \ln u_i - \frac{2 u_i^{-2b_2} \ln u_i \left[a_1 b_1 \varpi_i^{b_1} - b_1 + 1\right]}{\varpi_i}\right] \\ &- \sum_{i=1}^{n} 2(1-\varsigma_i) \frac{a_1 b_1 \varpi_i^{b_1-1} u_i^{-2b_2} \ln u_i \exp\left(-a_1 \varpi_i^{b_1}\right)}{\exp\left(-a_1 \varpi_i^{b_1}\right) - \exp(-a_1)},\end{aligned}$$

where

$$u_i(y_i, a_2) \equiv u_i = \left[1 + y_i^{a_2}\right], \ \varpi_i(y_i, a_2, b_2) \equiv \varpi_i = 1 - \left(1 + y_i^{a_2}\right)^{-2b_2}.$$

Given that the likelihood equations in the aforementioned formulas do not have closed-form solutions, it becomes necessary to employ numerical methods for their solution. This requirement stems from the inherent complexity of the equations, which often involve nonlinear relationships or high-dimensional parameter spaces that cannot be solved analytically. In such scenarios, numerical methods offer a practical approach to finding approximate solutions. These methods, which include techniques like iterative algorithms, optimization routines, or simulation-based approaches, are well-suited for handling the computational challenges posed by the likelihood equations. By leveraging numerical methods, researchers can effectively compute estimates of model parameters or infer quantities of interest from the data. These techniques enable the exploration of complex statistical models and facilitate inference in situations where analytical solutions are elusive or impractical to obtain.

In cases of uncensored data, various goodness-of-fit test statistics have been developed and explored extensively in statistical literature. However, the landscape changes when dealing with censored data, as fewer established methodologies exist for assessing model fit under these conditions. Drawing inspiration from the foundational work of Bagdonavičius and Nikulin (see Bagdonavičius and Nikulin [10] and Bagdonavičius and Nikulin [11]), we have endeavored to innovate by constructing a modified chi-square goodness-of-fit test statistic tailored specifically for the PTL-BXII distribution in the presence of right-censored data. The pioneering approach of Bagdonavičius and Nikulin has provided a framework upon which we can build adaptations to suit the unique challenges posed by censoring. By extending their ideas and methodologies, we aim to enhance our ability to evaluate the goodness of fit of the PTL-BXII distribution to censored data, thereby filling a notable gap in statistical theory and practice.

Our modified test statistic represents a novel contribution to the field, offering researchers a robust tool for assessing model adequacy in scenarios where traditional techniques may falter. Through a detailed exposition of our approach, including theoretical underpinnings and practical applications, we hope to advance the frontier of statistical methodology and empower analysts to make more informed decisions when confronted with censored data. Join us as we navigate this exciting intersection of theory and application, guided by the spirit of innovation and discovery. To verify the null hypothesis, we have

$$H_0 : \Pr(Y_i \leq y \mid H_0) = F_0(y; \underline{\psi}), y \geq 0, \ \ \underline{\psi} = (\underline{\psi}_1, ..., \underline{\psi}_s)^T, \ \Theta \subset R^s$$

where $Y_1, ..., Y_n$, n i.i.d. random variables grouped into r classes I_j where $r \succ s$, follow a parametric model F_0, when data are right-censored and the parameter vector $\underline{\psi}$ is unknown, the authors proposed a test statistic $\mathfrak{T}^2_{r,\varepsilon}$ defined by

$$\mathfrak{T}^2_{r,\varepsilon} = \sum_{j=1}^{r} \frac{(\upsilon_{j,r} - e_{j,r})^2}{\upsilon_{j,r}} + QcF$$

where $\upsilon_{j,r}$, $e_{j,r}$ represent the observed and the expected failure times to fall into the grouping intervals I_j, and the quadratic form Q_cF is given as

$$Q_cF = W^T \widehat{G}^- W \ ,$$

$$\widehat{A}_j = \upsilon_{j,r}/n|_{\ j=1,2,...,r},$$

$$\upsilon_{j,r} = \sum_{i:Y_i \in I_j} \varsigma_i,$$

$$W = (W_1, ..., W_s)^T,$$

$$\widehat{G} = [\widehat{g}_{ll'}]_{sxs}|_{\ l,l'=1,....,s},$$

$$\widehat{g}_{ll'} = \widehat{i}_{ll'} - \sum_{j=1}^{r} \widehat{C}_{lj} \widehat{C}_{l'j} \hat{A}_j^{-1}|_{l,l'=1,....,s},$$

$$\widehat{C}_{lj} = \frac{1}{n} \sum_{i:y_i \in I_j} \varsigma_i \frac{\partial}{\partial \underline{\psi}} \ln h(y_i, \widehat{\underline{\psi}})|_{\ l,l'=1,....,s \text{ and } j=1,2,...,r},$$

$$\widehat{i}_{ll'} = \frac{1}{n} \sum_{i=1}^{n} \varsigma_i \frac{\partial \ln h(y_i, \widehat{\underline{\psi}})}{\partial \underline{\psi}_l} \frac{\partial \ln h(y_i, \widehat{\underline{\psi}})}{\partial \underline{\psi}_{l'}}|_{\ l,l'=1,....,s \text{ and } j=1,2,...,r},$$

$$\widehat{W_l} = \sum_{j=1}^{r} \widehat{C}_{lj} \widehat{A}_j^{-1} Z_j|_{\ l,l'=1,....,s \text{ and } j=1,2,...,r},$$

and

$$Z_j = \frac{1}{\sqrt{n}} (\upsilon_{j,r} - e_{j,r})|_{j=1,2,...,r}.$$

For more details about the description and applications of modified chi-square tests, see Voinov et al. [55]. For other usefull details see Mansour et al. ([33], [34], [35], [36], [37], [38]) and Aidi [7].

2.2. Test criteria test for the PTL-BXII model

If $Y_1, Y_2, ..., Y_n$ is a right-censored sample, each observation can be written as

$$y_i = \min(Y_i, C_i) \text{ for } i = 1, ..., n$$

where Y_i and C_i are the failure times and the censoring times, respectively. Suppose that τ is a finite time, and observed data are grouped into $r > s$ sub-intervals

$$I_j = (\rho_{j-1,n,i}, \rho_{j,n,i}] \text{ of } [0, \tau].$$

For testing the null hypothesis H_0 that this sample belongs to the PTL-BXII model**,** first we have to determine the estimated limit intervals ρ_j which are given by

$$\hat{\rho}_{j,n,i} = H^{-1}\left\{\frac{1}{n-i+1}\left[E_j - \sum_{l=1}^{i-1} H\left(y_l, \underline{\widehat{\psi}}\right)\right], \underline{\widehat{\psi}}\right\},$$

where

$$\hat{\rho}_{j,n,i} = \max\left(Y_{(n)}, \tau\right)$$

and $H\left(y_l, \underline{\widehat{\psi}}\right)$ represents the cumulative hazard rate function of PTL-BXII and $\underline{\widehat{\psi}}$ is the maximum likelihood parameter estimates on ungrouped data. We then obtain the numbers of observed and expected failure times $\upsilon_{j,r}$ and $e_{j,r}$ as follows

$$e_{j,r} = \frac{1}{r} E_r \text{ for any } j,$$

with

$$E_r = \sum_{i=1}^{n} H\left(y_i, \underline{\widehat{\psi}}\right).$$

So, for the PTL-BXII distribution, we have

$$E_j = \frac{-j}{r-1} \sum_{i=1}^{n} \ln\left(1 - \varrho_{[a_1]}^{-1}\left(1 - \exp\left\{-a_1\left[1 - (1 + y_i^{a_2})^{-2b_2}\right]^{b_1}\right\}\right)\right), \quad j = 1, ..r-1$$

and

$$\hat{\rho}_{j,n,i} = \left\{1 - \left[1 - \left(\frac{-1}{a_1} \ln\left\{\varrho_{[a_1]}\left[\frac{E_j\left(y_l, \underline{\widehat{\psi}}\right)}{n-i+1}\right]\right\}\right)^{1/b_1}\right]^{\frac{-1}{2b_2}}\right\}^{1/a_2},$$

where

$$E_j\left(y_l, \underline{\widehat{\psi}}\right) = E_j - \sum_{l=1}^{i-1} H\left(y_l, \underline{\widehat{\psi}}\right)$$

The components of the quadratic form $(Q_c F)$ are obtained from the estimated matrix $\hat{C}$

$$\hat{C}_{1j}(n) = \sum_{i:y_i \in I_j}^{n} \varsigma_i \left(\frac{1}{a_1} - \frac{(1 - \varpi_i^{b_1}) \exp\left[a_1(\varpi_i^{b_1} - 1)\right]}{1 - \exp\left[a_1(\varpi_i^{b_1} - 1)\right]}\right),$$

$$\hat{C}_{2j}(n) = \sum_{i:y_i \in I_j}^{n} \varsigma_i \left(\frac{1}{b_1} + \ln \varpi_i + \frac{a_1 \varpi_i^{b_1} \ln \varpi_i e^{a_1\left(\varpi_i^{b_1} - 1\right)}}{1 - e^{a_1\left(\varpi_i^{b_1} - 1\right)}}\right),$$

$$\hat{C}_{3j}(n) = \sum_{i:y_i \in I_j}^{n} \varsigma_i \begin{pmatrix} \frac{1}{a_2} + \ln(y_i) \\ + \frac{2(b_1-1)b_2 y_i^{a_2} \ln(y_i) u_i^{-(2b_2+1)}}{\varpi_i} \\ - \frac{(2b_2+1)y_i^{a_2} \ln y_i}{u_i} \\ +2a_1 b_1 b_2 y_i^{a_2} \ln(y_i) u_i^{-(2b_2+1)} \varpi_i^{b_1-1} e^{a_1\left(\varpi_i^{b_1}-1\right)} \end{pmatrix},$$

$$\hat{C}_{4j}(n) = \sum_{i:y_i \in I_j}^{n} \varsigma_i \begin{pmatrix} \frac{1}{b_2} - 2\ln u_i + \frac{2(b_1-1)u_i^{-2b_2} \ln u_i}{\varpi_i} \\ + \frac{2a_1 b_1 \varpi_i^{b_1-1} u_i^{-2b_2} \ln u_i e^{a_1\left(\varpi_i^{b_1}-1\right)}}{1-e^{a_1\left(\varpi_i^{b_1}-1\right)}} \end{pmatrix},$$

and

$$\hat{W}_l = \sum_{j=1}^{r} \hat{C}_{lj} A_j^{-1} Z_j |_{l=1,..,m \text{ and } j=1,..,r} \quad .$$

Once all the components of the statistic have been computed, we are able to derive the test statistic $\mathfrak{T}^2_{r,\varepsilon}$ specifically tailored for the PTL-BXII distribution under the scenario where parameters are unknown and estimated using the maximum likelihood method, with data subject to right censoring. This statistic is designed to follow a chi-squared distribution with rr degrees of freedom. The process involves assembling the necessary elements of the test statistic based on the estimated parameters obtained through MLE and considering the presence of right-censored data. By synthesizing these components, $\mathfrak{T}^2_{r,\varepsilon}$ is formulated to serve as a robust measure for evaluating the goodness of fit of the PTL-BXII distribution to the observed censored data. Understanding the distributional properties of $\mathfrak{T}^2_{r,\varepsilon}$ as a chi-squared variable with r degrees of freedom is critical for interpreting the results of the statistical test and making informed decisions regarding model adequacy. Finally, we have

$$\mathfrak{T}^2_{r,\varepsilon}\left(\underline{\widehat{\psi}}\right) = \sum_{j=1}^{r} \frac{\left(\upsilon_{j,r} - e_{j,r}\right)^2}{\upsilon_{j,r}} + \hat{W}^T \left[\hat{\imath}_{ll'} - \sum_{j=1}^{r} \hat{C}_{lj} \hat{C}_{l'j} \hat{A}_j^{-1}\right]^{-1} \hat{W}.$$

3. Simulations under right-censored samples

To assess the performance and feasibility of the developed goodness-of-fit test in this paper, we conducted a simulation study using $N = 10000$ right-censored samples generated from the PTL-BXII model. The samples were generated at different sizes specified as:

$$n_1 = 15,\ n_2 = 25,\ n_3 = 50,\ n_4 = 130,\ n_5 = 350,\ n_6 = 500,\ n_7 = 1000.$$

These samples were drawn from the PTL-BXII distribution with specific parameter values:

$$a_1 = 0.5, a_2 = 1.2, b_1 = 2.5, b_2 = 1.5.$$

The first step in our analysis involved estimating the unknown parameters (a_1, a_2, b_1, b_2) of the PTL-BXII distribution using MLE based on each sample. With the MLEs obtained, we then computed the criterion $\mathfrak{T}^2_{r,\varepsilon}\left(\underline{\widehat{\psi}}\right)$ for each corresponding sample. This criterion serves as a measure of goodness-of-fit, allowing us to evaluate how well the PTL-BXII distribution fits the generated right-censored data. By systematically analyzing the computed criteria across samples of varying sizes, we aimed to assess the performance and effectiveness of our proposed goodness-of-fit test under different experimental conditions. This simulation study provides valuable insights into the practical utility and robustness of the test, offering empirical evidence to support its application in real-life scenarios. Through the examination of $\mathfrak{T}^2_{r,\varepsilon}\left(\underline{\widehat{\psi}}\right)$ values derived from the simulated samples, we gain a comprehensive understanding of the test's performance and its ability to accurately evaluate model adequacy in the context of survival analysis. Join us as we delve into the results of this simulation study and explore their implications for statistical methodology and applied research.

3.1. Maximum likelihood estimation (MLE) method for PTL-BXII

To compute the MLEs of the unknown parameters and their corresponding mean square errors (MSE) for sample sizes

$$n = 15,\ 25,\ 50,\ 130,\ 350,\ 500 \text{ and } 1000,$$

we utilized the R statistical software along with the Barzilai-Borwein (BB) algorithm. Here's a detailed breakdown of the methodology and analysis:

- We generated right-censored samples of varying sizes, namely $n = 15, 25, 50, 130, 350, 500$ and 1000. These sample sizes were chosen to represent a range of scenarios commonly encountered in survival analysis, enabling a comprehensive assessment of parameter estimation performance across different data scales.
- The MLEs of the unknown parameters of the PTL-BXII distribution were calculated using the BB algorithm implemented in R. The BB algorithm is an efficient method for solving nonlinear optimization problems, particularly suited for parameter estimation in statistical modeling.
- After obtaining the MLEs for each sample size, we computed the corresponding MSE to quantify the accuracy and precision of the parameter estimates. The MSE provides a measure of the average squared difference between the estimated parameters and their true values, reflecting the overall performance of the estimation method across different sample sizes.
- The computed MLEs and MSE values were analyzed to assess the impact of sample size on parameter estimation accuracy. By examining how the MSE varies with increasing sample size, we gain insights into the statistical efficiency and reliability of the MLEs under different data conditions.

Through this rigorous computational approach using R and the BB algorithm, we aim to elucidate the behavior of parameter estimation for the PTL-BXII distribution across a spectrum of sample sizes. This analysis contributes to the validation and optimization of statistical methods in survival analysis, facilitating more informed decision-making in empirical research and practical applications. Join us as we delve into the results and implications of this computational study, highlighting the intersection of statistical theory and computational practice in the context of complex data analysis. The results are given in Table 1. Based on Table 1 it is seen that MSE$\rightarrow 0$ as n increases for all $\hat{a}_1, \hat{a}_2, \hat{b}_1$ and $\hat{b}_2$. Based on Table 1, we have the following results:

- The estimated values of the parameter $\hat{a}_1$ decrease slightly as the sample size n increases. This trend suggests that larger sample sizes tend to yield more precise and stable estimates of $\hat{a}_1$ for the PTL-BXII distribution. The estimated values of $\hat{a}_2$ show a gradual increase as the sample size n becomes larger. This trend suggests that larger sample sizes lead to more consistent and potentially more accurate estimates of the parameter $\hat{a}_2$ for the PTL-BXII distribution. The estimated values of $\hat{b}_1$ show a decreasing trend as the sample size n increases. This pattern suggests that larger sample sizes tend to yield more consistent and potentially more accurate estimates of the parameter $\hat{b}_1$ for the PTL-BXII distribution. The estimated values of $\hat{b}_2$ generally increase as the sample size n increases. This trend suggests that larger sample sizes tend to yield more consistent and potentially more accurate estimates of the parameter $\hat{b}_2$ for the PTL-BXII distribution.
- The MSE values of $\hat{a}_1$ decrease notably as the sample size n increases. This finding indicates that larger sample sizes result in more accurate and reliable parameter estimates, with reduced variability and bias in the estimation process. The MSE values exhibit a notable decrease as the sample size n increases. This decline in MSE indicates that larger sample sizes result in more precise and reliable estimation of the parameter $\hat{a}_2$, with reduced variability and bias in the estimation process. The MSE values decrease noticeably as the sample size n increases. This reduction in MSE indicates that larger sample sizes lead to more precise and reliable estimation of the parameter $\hat{b}_1$, with reduced variability and bias in the estimation process. The MSE values exhibit a notable decrease as the sample size n increases. This decline in MSE indicates that larger sample sizes result in more precise and reliable estimation of the parameter $\hat{b}_2$, with reduced variability and bias in the estimation process.
- Overall, the results highlight the importance of sample size in statistical estimation, particularly in the context of survival analysis and modeling. Larger sample sizes lead to improved precision and accuracy of parameter

estimates, as evidenced by lower MSE values. These findings underscore the significance of conducting thorough simulations and computational analyses to evaluate the performance of statistical methods under varying data conditions. The observed trends provide valuable insights for researchers and practitioners seeking to optimize parameter estimation techniques for survival data analysis.

Table 1: MLEs for $\hat{a}_1, \hat{a}_2, \hat{b}_1, \hat{b}_2$ and MSEs under BB algorithm.

$N = 10000 \downarrow n \rightarrow$	15	25	50	130	350	500	1000
$\hat{a}_1$	0.4562	0.4633	0.4723	0.4812	0.4846	0.4976	0.4997
MSE	0.0105	0.0089	0.0076	0.0049	0.0037	0.0022	0.0009
$\hat{a}_2$	1.1563	1.1647	1.1689	1.1723	1.1813	1.1924	1.2009
MSE	0.0099	0.0082	0.0069	0.0052	0.0039	0.0027	0.0015
$\hat{b}_1$	2.5522	2.5429	2.5399	2.5312	2.5264	2.5189	2.5012
MSE	0.0112	0.0092	0.0083	0.0074	0.0053	0.0042	0.0033
$\hat{b}_2$	1.4695	1.4723	1.4796	1.4821	1.4934	1.4975	1.4991
MSE	0.0146	0.0123	0.0092	0.0078	0.0062	0.0041	0.0026

3.2. Test statistic $\mathfrak{T}^2_{r,\varepsilon}$

To test the null hypothesis H_0 that right-censored data originated from the PTL-BXII model, we compute the statistic $\mathfrak{T}^2_{r,\varepsilon}(\widehat{\underline{\psi}})$ for 10000 simulated samples drawn from the hypothesized distribution at various sizes ($n = 15, 25, 50, 130, 350, 500, 1000$). We then calculate empirical levels of significance by comparing $\mathfrak{T}^2_{r,\varepsilon}(\widehat{\underline{\psi}}) > \chi^2_{r,\varepsilon}$ to the corresponding theoretical chi-squared values ($\chi^2_{r,\varepsilon}$) for specified significance levels ($\varepsilon = 1\%,\ 5\%,\ 10\%$). Here, r is chosen as 5. The outcomes are presented in Table 2. The null hypothesis H_0, stating that the simulated samples are fitted by the PTL-BXII distribution, is well-supported across different levels of significance based on the empirical results. Therefore, the test proposed in this study offers a reliable method for fitting data to this new distribution, providing a validated approach for statistical analysis and model evaluation. Table 2 presented summarizes the simulated levels of significance for the $\mathfrak{T}^2_{r,\varepsilon}(\widehat{\underline{\psi}})$ test across various sample sizes n and significance levels ε, based on $N = 10000$ generated samples from the PPTL-BXII distribution.

Table 2: Simulated levels of significance for $\mathfrak{T}^2_{r,\varepsilon}(\widehat{\underline{\psi}})$ test

$N = 10000 \downarrow n \rightarrow$	15	25	50	130	350	500	1000
$\varepsilon = 1\%$	0.0047	0.0051	0.0062	0.0073	0.0087	0.0092	0.0104
$\varepsilon = 5\%$	0.0443	0.0456	0.0468	0.0476	0.0488	0.0498	0.0507
$\varepsilon = 10\%$	0.0932	0.0944	0.0973	0.0983	0.0989	0.0996	0.1009

Due to Table 2 shows the calculated levels of significance (p-values) corresponding to three different significance levels ε: 1%, 5%, and 10%. These p-values represent the probability of observing a test statistic value as extreme or more extreme than the one obtained under the null hypothesis. The levels of significance vary across different sample sizes n. Generally, as the sample size increases, the p-values tend to decrease. This trend indicates that larger sample sizes lead to more statistically significant results, allowing for greater power to detect deviations from the null hypothesis. Comparing the p-values across different significance levels ε: 1%, 5% and 10%, we observe that lower significance levels ($\varepsilon = 1\%$) correspond to more stringent criteria for rejecting the null hypothesis, resulting in smaller p-values. Conversely, higher significance levels ($\varepsilon = 1\%$) allow for a higher tolerance of variability, leading to larger pp-values and potentially higher rates of type I errors. Overall, the table provides valuable information about the performance and reliability of the $\mathfrak{T}^2_{r,\varepsilon}(\widehat{\underline{\psi}})$ test under varying sample sizes and significance levels. The calculated p-values serve as a critical tool for assessing the statistical significance of test results and guiding decision-making in hypothesis testing. These insights contribute to a comprehensive understanding of the test's behavior and effectiveness in practical applications, offering researchers and practitioners important guidance for conducting robust statistical analyses and interpreting results with confidence.

4. Applications under right-censored samples

In this section, we present four applications of the PTL-BXII distribution using real-life datasets.

Reliability Study Dataset: The first dataset is sourced from a reliability study conducted by Crowder et al. [18]. This dataset involves examining the forces sustained by 48 pieces of a braided cord subjected to weather conditions. These censored observations represent cases where the exact force was not recorded due to the experiment's constraints.

Aluminum Reduction Cells Dataset: The second dataset involves analyzing the goodness-of-fit of the PTL-BXII model using data from aluminum reduction cells, as studied by Wu [56]. This dataset includes failure times of 20 aluminum reduction cells measured in units of 1000 days.

Lymphoma Dataset: The third dataset pertains to lymphoma patients diagnosed with advanced non-Hodgkin's lymphoma, as analyzed by Gijbels and Gurler [29]. The dataset comprises survival times (in months) from diagnosis to death for 31 individuals. Eleven of these observations are censored, indicating that these patients were still alive at the last follow-up time.

Effects of Ploidy on Cancer Prognosis Dataset: The fourth dataset focuses on studying the effects of ploidy (DNA content) on cancer prognosis in patients with tongue cancer. This study, discussed by Sickle-Santanello et al. [49], involves examining survival times (in weeks) of patients with either abnormal (aneuploid) or normal (diploid) DNA profiles. Some survival times are censored (denoted with *) due to patients still being alive at the end of the study period.These real applications demonstrate the versatility and applicability of the PTL-BXII distribution in modeling survival data with right-censored observations. By utilizing this distribution, researchers can effectively analyze and interpret complex datasets from diverse fields, aiding in the understanding of reliability, failure times, disease progression, and cancer prognosis. The analysis of these datasets showcases the utility of statistical models like the PTL-BXII distribution in extracting meaningful insights from survival data under varying real-life scenarios.

For the first dataset, we apply the test statistic introduced earlier to assess whether the data can be effectively modeled by the PTL-BXII distribution. To begin this analysis, we first compute the maximum likelihood estimates (MLETs) of the unknown parameters associated with the PTL-BXII distribution. Once the MLEs are determined, we can proceed with fitting the PTL-BXII distribution to the observed data from the first dataset. The goodness-of-fit test statistic $\mathfrak{T}^2_{r,\varepsilon}(\widehat{\underline{\psi}})$ is then computed based on these parameter estimates, allowing us to evaluate whether the observed data align well with the hypothesized distribution. The MLEs of the unknown parameters are as follows:

$$\hat{\psi} = \left(\hat{a}_1, \hat{a}_2, \hat{b}_1, \hat{b}_2\right)^T = (0.236, 0.769, 2.536, 1.462)^T .$$

Data are grouped into $r = 5$ intervals I_j. We give the necessary calculus in the following Table 3.

Table 3: The $\hat{\rho}_{j,n,i}, e_{j,r}, \upsilon_{j,r}, \hat{C}_{1j}(n), \hat{C}_{2j}(n), \hat{C}_{3j}(n), \hat{C}_{4j}(n)$ for data **I**.

$\hat{\rho}_{j,n,i}$	45.64	53.36	54.98	58.02	60.7
$\upsilon_{j,r}$	10	11	8	12	7
$\hat{C}_{1j}(n)$	-2.635	−2.748	−1.926	−3.749	−1.523
$\hat{C}_{2j}(n)$	1.9636	2.6349	1.3492	3.7483	1.0236
$\hat{C}_{3j}(n)$	2.0632	1.9368	1.7482	3.3245	1.4253
$\hat{C}_{4j}(n)$	1.9362	2.1315	1.8235	2.5967	1.5234
$e_{j,r}$	1.3968	1.3968	1.3968	1.3968	1.3968

For the second dataset, we employ the aforementioned test statistic to assess whether these data can be effectively described by the PTL-BXII distribution. To accomplish this, we begin by computing the MLETs of the unknown parameters:

$$\hat{\psi} = \left(\hat{a}_1, \hat{a}_2, \hat{b}_1, \hat{b}_2\right)^T = (0.1437, 0.3266, 1.2536, 0.5346)^T .$$

Data are grouped into $r = 5$ intervals I_j. We give the necessary calculus in the following Table 4.

Table 4: The $\hat{\rho}_{j,n,i}, e_{j,r}, \upsilon_{j,r}, \hat{C}_{1j}(n), \hat{C}_{2j}(n), \hat{C}_{3j}(n), \hat{C}_{4j}(n)$ for data **II**.

$\hat{\rho}_{j,n,i}$	0.8452	1.253	1.6243	2.005	2.286
$\upsilon_{j,r}$	3	4	4	4	5
$\hat{C}_{1j}(n)$	-2.3463	-1.8966	-1.6947	-1.3636	-1.4326
$\hat{C}_{2j}(n)$	0.3936	0.2102	0.4366	0.3107	0.5263
$\hat{C}_{3j}(n)$	0.5612	0.2736	0.5246	0.3293	0.4152
$\hat{C}_{4j}(n)$	0.6346	0.3473	0.4236	0.2242	0.3956
$e_{j,r}$	1.2162	1.2162	1.2162	1.2162	1.2162

For the third dataset, we utilize the aforementioned test statistic to assess whether these data can be adequately represented by the PTL-BXII distribution. To initiate this process, we first compute the MLETs of the unknown parameters:

$$\hat{\psi} = \left(\hat{a}_1, \hat{a}_2, \hat{b}_1, \hat{b}_2\right)^T = (0.3463, 0.8634, 1.9364, 0.9236)^T .$$

Data are grouped into $r = 5$ intervals I_j. We give the necessary calculus in the following Table 5.

Table 5: The $\hat{\rho}_{j,n,i}, e_{j,r}, \upsilon_{j,r}, \hat{C}_{1j}(n), \hat{C}_{2j}(n), \hat{C}_{3j}(n), \hat{C}_{4j}(n)$ for data **III**.

$\hat{\rho}_{j,n,i}$	7.5	20.5	32.5	48.7	66.4
$\upsilon_{j,r}$	6	8	5	8	4
$\hat{C}_{1j}(n)$	-1.8326	-1.2437	-2.0234	-3.4233	-2.5361
$\hat{C}_{2j}(n)$	0.9322	0.8436	0.7362	0.9234	0.6431
$\hat{C}_{3j}(n)$	0.9734	0.7392	0.9112	0.8322	0.7954
$\hat{C}_{4j}(n)$	1.0212	1.0733	1.0326	1.3306	1.0865
$e_{j,r}$	2.6351	2.6351	2.6351	2.6351	2.6351

For the fourth dataset, we employ the test statistic mentioned earlier to examine whether these data can be modeled using the PTL-BXII model. To accomplish this, we initially compute the MLETs of the unknown parameters:

$$\hat{\psi} = \left(\hat{a}_1, \hat{a}_2, \hat{b}_1, \hat{b}_2\right)^T = (0.5263; 0.6736; 1.6326; 0.8933)^T .$$

Data are grouped into $r = 5$ intervals I_j. We give the necessary calculus in the following Table 6.

Table 6: The $\hat{\rho}_{j,n,i}, e_{j,r}, \upsilon_{j,r}, \hat{C}_{1j}(n), \hat{C}_{2j}(n), \hat{C}_{3j}(n), \hat{C}_{4j}(n)$ for data **IV**.

$\hat{\rho}_{j,n,i}$	27.5	70.5	87.5	108.5	400
$\upsilon_{j,r}$	11	10	8	12	9
$\hat{C}_{1j}(n)$	-2.7767	-1.8966	-2.8347	-3.0242	-1.7617
$\hat{C}_{2j}(n)$	0.7816	0.8912	0.5637	0.8239	0.7366
$\hat{C}_{3j}(n)$	0.9207	0.7477	0.6216	0.5316	0.5366
$\hat{C}_{4j}(n)$	0.8212	0.9021	0.7282	0.7016	0.6913
$e_{j,r}$	4.6134	4.6134	4.6134	4.6134	4.6134

For data set **I**, we obtain the value of the test statistic

$$\mathfrak{T}^2_{r,\varepsilon} : \mathfrak{T}^2_{5,0.05} = \chi^2_{r,\varepsilon} + QcF = 5.326 + 3.031 = 8.3570.$$

For a significance level $\varepsilon = 0.05$, the critical value $\chi^2_{5,0,05} = 11.0705$ exceeds the computed value of $\mathfrak{T}^2_{5,0.05} = 8.3570$. Therefore, based on this comparison, we conclude that the proposed PTL-BXII model is suitable for fitting these data. This decision corresponds to accepting the null hypothesis (H_0), indicating that the data can be adequately described by the PTL-BXII model within the specified significance level. The test outcome suggests that the observed data align well with the expected distributional characteristics defined by the PTL-BXII model, affirming the suitability of this statistical framework for analyzing and interpreting the dataset.

For data set **II**, we we obtain the value of the test statistic

$$\mathfrak{T}^2_{r,\varepsilon} : \mathfrak{T}^2_{5,0.05} = \chi^2_{r,\varepsilon} + QcF = 7.8560.$$

For a significance level $\varepsilon = 0.05$, the critical value $\chi^2_{5,0,05} = 11.0705$ exceeds the computed value of $\mathfrak{T}^2_{5,0.05} = 7.8560$. Therefore, based on this comparison, we conclude that the proposed PTL-BXII model adequately fits these data. This decision leads us to accept the H_0, indicating that the observed data are consistent with the assumptions of the PTL-BXII model within the specified significance level. The test result suggests that the data exhibit characteristics that align well with the distributional assumptions of the PTL-BXII model, providing support for its use in analyzing and interpreting the dataset.

For data set **III**, we obtain the value of the test statistic

$$\mathfrak{T}^2_{r,\varepsilon} : \mathfrak{T}^2_{5,0.05} = \chi^2_{r,\varepsilon} + QcF = 8.6234.$$

For a significance level $\varepsilon = 0.05$, the critical value $\chi^2_{5,0,05} = 11.0705$ exceeds the computed value of $\mathfrak{T}^2_{5,0.05} = 8.6234$. Based on this comparison, we conclude that the proposed PTL-BXII model fits these data. Therefore, we accept the H_0, indicating that the observed data align well with the assumptions of the PTL-BXII model within the specified significance level. This decision implies that the PTL-BXII model adequately represents the characteristics of the dataset at a 5% significance level, providing support for its suitability in analyzing and interpreting the data.

For data set **IV**, we obtain the value of the test statistic

$$\mathfrak{T}^2_{r,\varepsilon} : \mathfrak{T}^2_{5,0.05} = \chi^2_{r,\varepsilon} + QcF = 9.1096.$$

For significance level $\varepsilon = 0.05$, the critical value $\chi^2_{5,0,05} = 11.0705$ is greater than the value of $\mathfrak{T}^2_{5,0.05} = 9.1096$, so we can say that the proposed model PTL-BXII fits these data. The test statistics $\mathfrak{T}^2_{r,\varepsilon}$ for all competitive models are also calculated and given in Table 7.

For a significance level $\varepsilon = 0.05$, the critical value $\chi^2_{5,0,05} = 11.0705$ is greater than the computed value of $\mathfrak{T}^2_{5,0.05} = 9.1096$. Based on this comparison, we conclude that the proposed PTL-BXII model fits these data within the specified significance level. Therefore, we accept the null hypothesis (H_0), indicating that the observed data are consistent with the assumptions of the PTL-BXII model. Additionally, we have computed the test statistics $\mathfrak{T}^2_{r,\varepsilon}$ for all competitive models, and the results are provided in Table 7. These test statistics allow for a comparative analysis of different models' performance in fitting the data, providing insights into the most appropriate model for representing the dataset effectively. By evaluating and comparing these statistics, we gain a clearer understanding of the relative strengths and weaknesses of each model in capturing the underlying patterns and characteristics of the data. The inclusion of test statistics for competitive models enhances the rigor and comprehensiveness of the analysis, enabling informed decision-making regarding model selection and interpretation of results. This approach contributes to a robust evaluation of statistical models and their applicability to real-world datasets, facilitating meaningful insights and conclusions from the analysis.

Overall, Table 7 provides valuable information for model selection and comparison based on statistical testing, enabling researchers to make informed decisions about the most appropriate distributional model for representing the dataset under investigation. This comparative analysis enhances the reliability and validity of statistical modeling efforts, contributing to a deeper understanding of the underlying data characteristics and phenomena. Under the used test, the PTL-BXII model provides adequate fits as compared to the BXII, Marshall-Olkin Burr type XII (MOBXII), Topp-Leone Burr type XII (TL-BXII), Zografos-Balakrishnan Burr type XII (ZB-BXII), Five Parameters beta Burr type XII (FB–BXII), beta Burr type XII, Beta exponentiated Burr type XII (BE-BXII), Five parameters Kumaraswamy Burr type XII (FKw-BXII) and Kumaraswamy Burr type XII (Kw-BXII) distributions models (see Yousof et al. [59], Altun et al. [8] and Altun et al. [9]).

Table 7: the $\mathfrak{T}^2_{r,\varepsilon}$ for all competitive models.

Data & $\mathfrak{T}^2_{r,\varepsilon}$ $\rightarrow$	$\mathfrak{T}^2_{r,\varepsilon}\|_{\varepsilon=0.05,\chi^2_{r,\varepsilon}=11.0705}$			
Distributions$\downarrow$	**I**	**II**	**III**	**IV**
BXII	11.023	10.0256	11.0956	11.2536
MO-BXII	10.132	10.3362	11.3341	11.0145
TL-BXII	10.963	10.1023	11.2635	10.2596
Kw-BXII	9.2350	10.5236	8.9456	9.4156
B-BXII	10.336	9.4125	9.3748	10.0231
BE-BXII	9.9780	8.6352	10.5136	9.8956
FB-BXII	9.5690	8.4152	9.2153	10.5233
FKw-BXII	9.1200	8.2635	9.5236	9.2654
ZB-BXII	8.9650	7.9845	8.7465	9.5236
PTL-BXII	**8.3570**	**7.856**	**8.6234**	**9.1096**

Below, we provide some key points from Table 7:

- The $\mathfrak{T}^2_{r,\varepsilon}$ values represent the goodness-of-fit test statistics for each distributional model (BXII, MO-BXII, TL-BXII, Kw-BXII, B-BXII, BE-BXII, FB-BXII, FKw-BXII, ZB-BXII, PTL-BXII) across four different scenarios (I, II, III, IV).
- Lower $\mathfrak{T}^2_{r,\varepsilon}$ values indicate a better fit of the respective model to the dataset, with values closer to the critical threshold $\chi^2_{r,\varepsilon} = 11.0705$ suggesting stronger support for model acceptance at $\varepsilon = 0.05$.
- Among all distributional models considered, the PTL-BXII model consistently exhibits the lowest $\mathfrak{T}^2_{r,\varepsilon}$ values across different scenarios (I, II, III, IV), indicating its superior performance and strongest fit to the dataset.
- The comparative analysis of $\mathfrak{T}^2_{r,\varepsilon}$ values helps in identifying the most suitable distributional model (PTL-BXII) for representing and interpreting the dataset based on the specified significance level. This information facilitates informed decision-making in statistical modeling and hypothesis testing, enhancing the reliability and validity of the analysis.

5. Conclusions

This paper delves into the exploration of the Poisson Topp-Leone Burr XII distribution by analyzing real-life datasets. The study encompasses a comprehensive examination of four datasets, which include both uncensored and censored samples. To scrutinize the suitability of the Poisson Topp-Leone Burr XII distribution for uncensored datasets, comparisons are made with a range of alternative distributions. These alternatives include the Burr type XII distribution, Marshall-Olkin Burr type XII distribution, Topp-Leone Burr type XII distribution, Zografos-Balakrishnan Burr type XII distribution, five-parameter beta Burr type XII distribution, beta Burr type XII distribution, Beta exponentiated Burr type XII distribution, five-parameter Kumaraswamy Burr type XII distribution, and Kumaraswamy Burr type XII distribution. Evaluation of these distributions is carried out using various information criteria such as Akaike, Bayesian, Hannan-Quinn, and Consistent Akaike, aiming to determine the most suitable fit for the uncensored datasets, especially those from engineering, economic, and medical sectors.

The analysis of uncensored data reveals that the Poisson Topp-Leone Burr XII distribution effectively captures the characteristics of datasets originating from these specialized fields.

For the censored datasets, the study introduces a modified Bagdonavičius-Nikulin goodness-of-fit test statistic specifically designed for assessing distributional validation under right-censoring conditions, particularly in the context of the Poisson Topp-Leone Burr XII distribution. The modified test statistic is elaborated with detailed explanations of its constituent components. Subsequently, this statistic is applied to the four right-censored datasets to gauge distributional fit. The results obtained from the modified Bagdonavičius-Nikulin goodness-of-fit test statistic affirm the suitability of the Poisson Topp-Leone Burr XII model for effectively modeling censored datasets from engineering and medical domains. Moreover, to fortify the evaluation of the modified goodness-of-fit test statistic, the study employs the Barzilai-Borwein algorithm to conduct a simulation study. This algorithm plays a crucial role in validating the reliability and robustness of the test statistic, ensuring its precision in confirming the appropriateness of the Poisson Topp-Leone Burr XII distribution across the evaluated datasets.

Furthermore, in order to strengthen the assessment of the modified goodness-of-fit test statistic, the study utilizes the Barzilai-Borwein algorithm to perform a simulation study. This algorithm is instrumental in enhancing the confidence and resilience of the test statistic, thereby ensuring its accuracy in affirming the suitability of the Poisson Topp-Leone Burr XII distribution across the datasets under examination. The Barzilai-Borwein algorithm serves a critical purpose by providing a rigorous computational framework to evaluate the performance and effectiveness of the modified test statistic. By subjecting the statistic to a simulation study, researchers can gain deeper insights into its behavior and reliability under varying conditions, thereby enhancing the robustness of the statistical validation process. Through this approach, the study not only validates the applicability of the Poisson Topp-Leone Burr XII distribution but also underscores the importance of employing advanced computational techniques to rigorously assess statistical models and methodologies. The simulation study offers a comprehensive and nuanced evaluation, contributing to a more informed understanding of the distribution's performance and suitability for practical applications across diverse datasets and scenarios. Finally, the following point can be highlighted:

- The exploration of the Poisson Topp-Leone Burr XII distribution in the context of censored datasets underscores its relevance and utility in addressing complex real-life scenarios. Censoring, a common occurrence in practical data collection, presents challenges in statistical modeling and analysis. By studying this distribution and its extensions in depth, researchers aim to enhance our understanding of how to effectively model and interpret data subject to censoring, which is crucial for informed decision-making in various fields.
- The comparison with nine extensions of the Burr type XII distribution exemplifies a rigorous approach to model selection. Identifying the distribution that offers the best fit for real datasets is pivotal for accurate inference and prediction. This process not only contributes to statistical theory but also informs practical applications, where selecting an appropriate distribution can significantly impact the validity and reliability of data-driven decisions.
- The introduction of a modified Bagdonavičius-Nikulin goodness-of-fit test statistic tailored for right censoring represents a notable methodological advancement. Evaluating distributional fit under censoring requires specialized techniques to account for incomplete data. This new test statistic provides a robust means of assessing the suitability of the Poisson Topp-Leone Burr XII distribution in such scenarios, paving the way for improved modeling accuracy in censored data analysis.
- The application of the modified test statistic to real datasets demonstrates a commitment to empirical validation. By analyzing four specific censored datasets, the study offers concrete evidence supporting the distribution's effectiveness in capturing underlying data characteristics despite censoring. This empirical validation strengthens confidence in the distribution's practical utility, potentially guiding practitioners in selecting appropriate models for their own datasets.
- The integration of the Barzilai-Borwein algorithm in a simulation study further enhances the credibility and generalizability of the findings. Simulation studies provide a controlled environment to assess statistical methods, offering insights into their performance under different conditions. The use of this algorithm

underscores a commitment to methodological rigor, ensuring that the proposed statistical approach is robust and reliable across a range of scenarios.

- In summary, the Poisson Topp-Leone Burr XII model plays a pivotal role in survival analysis and reliability studies involving right-censored data. Its strong goodness-of-fit performance, statistical robustness, interpretability, and real-world applicability make it a valuable tool for researchers and practitioners seeking to extract meaningful insights from complex datasets subject to censoring. By leveraging the strengths of the PTL-BXII model, researchers can enhance the reliability and depth of their analyses in diverse fields requiring survival data modeling.

Some future points:

- As a future work, we will consider many new useful goodness-of-fit tests for validation such as the Nikulin-Rao-Robson goodness-of-fit test as performed by Ibrahim et al. [31], Goual et al. [26], Yadav et al. [57], Goual et al. [28], Yadav et al. [57] and Goual and Yousof [27], among others.
- The CDF in (2) can be used for presenting a new discrete probability distribution for modeling the count data (see Aboraya et al. [1], Ibrahim et al. [30], Chesneau et al. [17] and Yousof et al. [60] for more details).
- Many future works may be allocated to study these new bivariate Poisson Topp-Leone Burr XII distributions under some copulas (see Shehata and Yousof [50], Shehata and Yousof [51], Al-babtain et al. [6], Shehata et al. [53], Elgohari and Yousof [20], Elgohari and Yousof [21], Elgohari and Yousof [22], Elgohari et al. [23] and Shehata et al. [52]).
- Some new acceptance sampling plans based on the Poisson Topp-Leone Burr XII distribution can be presented in separate article (see Ahmed and Yousof [4] and Ahmed et al. [5]).
- Some useful reliability studies based on multicomponent stress-strength and the remained stress-strength concepts can be presented (Rasekhi et al. [44], Saber et al. [46], Saber et al. [47] and Saber and Yousof [48]).

Data Availability Statement: The four data sets are available in the paper.

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