Robust Forecasting of Sequences with Periodically Stationary Long Memory Multiplicative Seasonal Increments Observed with Noise and Cointegrated Sequences

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Abstract   The problem of optimal estimation of linear functionals constructed from unobserved values of stochastic sequence with periodically stationary increments based on observations of the sequence with a periodically stationary noise is considered. For sequences with known spectral densities, we obtain formulas for calculating values of the mean square errors and the spectral characteristics of the optimal estimates of the functionals. Formulas that determine the least favorable spectral densities and minimax (robust) spectral characteristics of the optimal linear estimates of functionals are proposed in the case where spectral densities of the sequence are not exactly known while some sets of admissible spectral densities are given.

Keywords   Periodically stationary sequence, SARFIMA, fractional integration, optimal linear estimate, mean square error, least favourable spectral density matrix, minimax spectral characteristic

1. Introduction

Non-stationary and long memory time series models are wildly used in different fields of economics, finance, climatology, air pollution, signal processing etc. (see, for example, papers by Dudek and Hurd [9], Johansen and Nielsen [26], Reisen et al.[54]). A core example – a general multiplicative model, or $SARIMA(p,d,q) \times (P,D,Q)_s$ – was introduced in the book by Box and Jenkins [5]. It includes both integrated and seasonal factors:

$$\Psi(B^s)\psi(B)(1-B)^d(1-B^s)^D x_t = \Theta(B^s)\theta(B)\varepsilon_t, \tag{1}$$

where ε_t, $t \in \mathbb{Z}$, is a sequence of zero mean i.i.d. random variables, and where $\Psi(z)$ and $\Theta(z)$ are two polynomials of degrees of P and Q respectively which have roots outside the unit circle. The parameters d and D are allowed to be fractional. In the case where $|d + D| < 1/2$ and $|D| < 1/2$, the process (1) is stationary and invertible. The paper by Porter-Hudak [53] illustrates an application of a seasonal ARFIMA model to the analysis of the monetary aggregates used by U.S. Federal Reserve. Another model of fractional integration is GARMA processes described by the equation (see Gray, Cheng and Woodward [19])

$$(1 - 2uB + B^2)^d x_t = \varepsilon_t, \quad |u| \le 1. \tag{2}$$

For the resent results dedicated to the statistical inference for seasonal long-memory sequences, we refer to the paper by Tsai, Rachinger and Lin [58], who developed methods of estimation of parameters in case of measurement errors. In their paper Baillie, Kongcharoen and Kapetanios [3] compared MLE and semiparametric estimation procedures for prediction problems based on ARFIMA models. Based on simulation study, they indicate better performance of MLE predictor than the one based on the two-step local Whittle estimation. Hassler and Pohle [22] (see also Hassler [21]) assess a predictive performance of various methods of forecasting of inflation and return volatility time series and show strong evidences for models with a fractional integration component. One of the fields of interests related to time series analysis is optimal filtering. It aims to remove the unobserved components, such as trends, seasonality or noise signal, from the observed data [1, 6].

Another type of non-stationary processes are periodically correlated, or cyclostationary, processes introduced by Gladyshev [15], which belong to the class of processes with time-dependent spectrum and are widely used in signal processing and communications (see Gardner [11, 12], Hurd and Miamee [24], Napolitano [50] for a review of the works on cyclostationarity and its applications). Periodic time series are considered as an extension of a SARIMA model (see Lund [34] for a test assessing if a PARMA model is preferable to a SARMA one) and are suitable for forecasting stream flows with quarterly, monthly or weekly cycles (see Osborn [51]). Baek, Davis and Pipiras [2] have introduced a periodic dynamic factor model (PDFM) with periodic vector autoregressive (PVAR) factors, in contrast to seasonal VARIMA factors. Basawa, Lund and Shao [4] have investigated first-order seasonal autoregressive processes with periodically varying parameters.

Methods of parameters estimations and filtering usually do not take into account the issues arising from real data, namely, the presence of outliers, measurement errors, incomplete information about the spectral, or model, structure etc. From this point of view, we see an increasing interest to robust methods of estimation that are reasonable in such cases (see Reisen, et al. [55], Solci at al. [57] for the examples of robust estimates of SARIMA and PAR models). The paper by Grenander [18] should be marked as the first one where the minimax (robust) extrapolation problem for stationary processes was formulated as a game of two players and solved. Hosoya [23], Kassam [28], Kassam and Poor [29], Franke [10], Vastola and Poor [59], Moklyachuk [41, 42], Liu et al. [33] studied minimax (robust) extrapolation (prediction), interpolation (missing values estimation) and filtering (smoothing) problems for the stationary sequences and processes. Recent results of minimax extrapolation problems for stationary vector processes and periodically correlated processes belong to Moklyachuk and Masyutka [44, 45, 46] and Moklyachuk and Golichenko (Dubovets'ka) [16, 7, 43] respectively. Stationary sequences associated with a periodically correlated sequence are investigated by Makagon et al. [39, 40]. Processes with stationary and periodically stationary increments are investigated by Luz and Moklyachuk [35, 36, 37, 38]. We also mention works by Moklyachuk and Sidei [48], Moklyachuk, Masyutka and Sidei [49], who derive minimax estimates for stationary processes from observations with missed values. Kozak, Luz and Moklyachuk [31, 32] studied the estimation problems for stochastic sequences with periodically stationary increments.

This article is dedicated to the robust forecasting problem for stochastic sequences with periodically stationary long memory multiple seasonal increments, or sequences with periodically stationary general multiplicative (GM) increments, introduced by Luz and Moklyachuk [37]. Estimates of the unknown values of the sequence with periodically stationary GM increments are based on observations of the sequence with the stationary noise sequence.

The article is organized as follows. In Section 2, we recall definitions of generalized multiple (GM) increment sequence $\chi_{\vec{\mu},\vec{s}}^{(d)}(\vec{\xi}(m))$ and stochastic sequences $\xi(m)$ with periodically stationary (periodically correlated, cyclostationary) GM increments. The spectral theory of vector-valued GM increment sequences is discussed. Section 3 deals with the classical forecasting problem for the linear functionals $A\xi$ which are constructed from unobserved values of the sequence $\xi(m)$ when the spectral densities of the sequence $\xi(m)$ and a noise sequence $\eta(m)$ are known. Estimates are obtained by applying the Hilbert space projection technique to the vector-valued sequence $\vec{\xi}(m) + \vec{\eta}(m)$ with stationary GM increments under the stationary noise sequence $\vec{\eta}(m)$ uncorrelated with $\vec{\xi}(m)$. The case of non-stationary fractional integration is discussed as well. Section 4 is dedicated to the minimax (robust) estimates in cases, where spectral densities of sequences are not exactly known while some sets of admissible spectral densities are specified. We illustrate the proposed technique on the particular types of the

sets, which are generalizations of the sets of admissible spectral densities described in a survey article by Kassam and Poor [29] for stationary stochastic processes.

2. Stochastic sequences with periodically stationary generalized multiple increments

2.1. Definition and spectral representation of a periodically stationary GM increment

In this section, we present definition, justification and a brief review of the spectral theory of stochastic sequences with periodically stationary multiple seasonal increments. This type of stochastic sequences will allow us to deal with a wide range of non-stationarity in time series analysis.

Consider a stochastic sequence $\xi(m)$, $m \in \mathbb{Z}$ defined on a probability space $(\Omega, \mathcal{F}, \mathsf{P})$. Denote by B_μ a backward shift operator with the step $\mu \in \mathbb{Z}$, such that $B_\mu \xi(m) = \xi(m - \mu)$; $B := B_1$. Then $B_\mu^s = B_\mu B_\mu \cdots B_\mu$.

Define the incremental operator

$$\chi_{\overline{\mu},\overline{s}}^{(d)}(B) := (1 - B_{\mu_1}^{s_1})^{d_1}(1 - B_{\mu_2}^{s_2})^{d_2} \cdots (1 - B_{\mu_r}^{s_r})^{d_r} = \sum_{k=0}^{n(\gamma)} e_\gamma(k) B^k,$$

where $d := d_1 + d_2 + \ldots + d_r$, $\overline{d} = (d_1, d_2, \ldots, d_r) \in (\mathbb{N}^*)^r$, $\overline{s} = (s_1, s_2, \ldots, s_r) \in (\mathbb{N}^*)^r$ and $\overline{\mu} = (\mu_1, \mu_2, \ldots, \mu_r) \in (\mathbb{N}^*)^r$ or $\in (\mathbb{Z} \setminus \mathbb{N})^r$, $n(\gamma) := \sum_{i=1}^{r} \mu_i s_i d_i$. Here $\mathbb{N}^* = \mathbb{N} \setminus \{0\}$. The explicit formula for the coefficients $e_\gamma(k)$ is given in [37].

Definition 2.1
For a stochastic sequence $\xi(m)$, $m \in \mathbb{Z}$, the sequence

$$\chi_{\overline{\mu},\overline{s}}^{(d)}(\xi(m)) := \chi_{\overline{\mu},\overline{s}}^{(d)}(B)\xi(m) = (1 - B_{\mu_1}^{s_1})^{d_1}(1 - B_{\mu_2}^{s_2})^{d_2} \cdots (1 - B_{\mu_r}^{s_r})^{d_r} \xi(m) =$$

$$= \sum_{l_1=0}^{d_1} \cdots \sum_{l_r=0}^{d_r} (-1)^{l_1 + \ldots + l_r} \binom{d_1}{l_1} \cdots \binom{d_r}{l_r} \xi(m - \mu_1 s_1 l_1 - \cdots - \mu_r s_r l_r) \tag{3}$$

is called a *stochastic generalized multiple (GM) increment sequence* of differentiation order d with a fixed seasonal vector $\overline{s} \in (\mathbb{N}^*)^r$ and a varying step $\overline{\mu} \in (\mathbb{N}^*)^r$ or $\in (\mathbb{Z} \setminus \mathbb{N})^r$.

Definition 2.2
A stochastic GM increment sequence $\chi_{\overline{\mu},\overline{s}}^{(d)}(\xi(m))$ is called a wide sense stationary if the mathematical expectations

$$\mathsf{E}\chi_{\overline{\mu},\overline{s}}^{(d)}(\xi(m_0)) = c_{\overline{s}}^{(d)}(\overline{\mu}),$$

$$\mathsf{E}\chi_{\overline{\mu}_1,\overline{s}}^{(d)}(\xi(m_0 + m))\chi_{\overline{\mu}_2,\overline{s}}^{(d)}(\xi(m_0)) = D_{\overline{s}}^{(d)}(m; \overline{\mu}_1, \overline{\mu}_2)$$

exist for all $m_0, m, \overline{\mu}, \overline{\mu}_1, \overline{\mu}_2$ and do not depend on m_0. The function $c_{\overline{s}}^{(d)}(\overline{\mu})$ is called a mean value and the function $D_{\overline{s}}^{(d)}(m; \overline{\mu}_1, \overline{\mu}_2)$ is called a structural function of the stationary GM increment sequence (of a stochastic sequence with stationary GM increments).

The stochastic sequence $\xi(m)$, $m \in \mathbb{Z}$ determining the stationary GM increment sequence $\chi_{\overline{\mu},\overline{s}}^{(d)}(\xi(m))$ by (3) is called a stochastic sequence with stationary GM increments (or GM increment sequence of order d).

Remark 2.1
For spectral properties of one-pattern increment sequence $\chi_{\mu,1}^{(n)}(\xi(m)) := \xi^{(n)}(m, \mu) = (1 - B_\mu)^n \xi(m)$ see, e.g., [36], p. 1-8; [61], p. 390–430. The corresponding results for continuous time increment process $\xi^{(n)}(t, \tau) = (1 - B_\tau)^n \xi(t)$ are described in [60], [61].

2.2. Definition and spectral representation of stochastic sequences with periodically stationary GM increment

In this subsection, we present definition, justification and a brief review of the spectral theory of stochastic sequences with periodically stationary GM increments.

Definition 2.3
A stochastic sequence $\xi(m)$, $m \in \mathbb{Z}$ is called a stochastic sequence with periodically stationary (periodically correlated) GM increments with period T if the mathematical expectations

$$
\begin{aligned}
\mathsf{E}\chi_{\overline{\mu},T\overline{s}}^{(d)}(\xi(m+T)) &= \mathsf{E}\chi_{\overline{\mu},T\overline{s}}^{(d)}(\xi(m)) = c_{T\overline{s}}^{(d)}(m,\overline{\mu}), \\
\mathsf{E}\chi_{\overline{\mu}_1,T\overline{s}}^{(d)}(\xi(m+T))\chi_{\overline{\mu}_2,T\overline{s}}^{(d)}(\xi(k+T)) &= D_{T\overline{s}}^{(d)}(m+T,k+T;\overline{\mu}_1,\overline{\mu}_2) = D_{T\overline{s}}^{(d)}(m,k;\overline{\mu}_1,\overline{\mu}_2)
\end{aligned}
$$

exist for every $m, k, \overline{\mu}_1, \overline{\mu}_2$ and $T > 0$ is the least integer for which these equalities hold.

Using Definition 2.3, one can directly check that the sequence

$$
\xi_p(m) = \xi(mT + p - 1), \quad p = 1, 2, \ldots, T; \quad m \in \mathbb{Z} \tag{4}
$$

forms a vector-valued sequence $\vec{\xi}(m) = \{\xi_p(m)\}_{p=1,2,\ldots,T}$, $m \in \mathbb{Z}$ with stationary GM increments by the relation

$$
\chi_{\overline{\mu},\overline{s}}^{(d)}(\xi_p(m)) = \chi_{\overline{\mu},T\overline{s}}^{(d)}(\xi(mT + p - 1)), \quad p = 1, 2, \ldots, T,
$$

where $\chi_{\overline{\mu},\overline{s}}^{(d)}(\xi_p(m))$ is the GM increment of the p-th component of the vector-valued sequence $\vec{\xi}(m)$.

The spectral structure of the GM increment is described in the following theorem [27], [37].

Theorem 2.1
1. The mean value and the structural function of the vector-valued stochastic stationary GM increment sequence $\chi_{\overline{\mu},\overline{s}}^{(d)}(\vec{\xi}(m))$ can be represented in the form

$$
c_{\overline{s}}^{(d)}(\overline{\mu}) = c\prod_{i=1}^{r}\mu_i^{d_i}, \tag{5}
$$

$$
D_{\overline{s}}^{(d)}(m;\overline{\mu}_1,\overline{\mu}_2) = \int_{-\pi}^{\pi} e^{i\lambda m}\chi_{\overline{\mu}_1}^{(d)}(e^{-i\lambda})\chi_{\overline{\mu}_2}^{(d)}(e^{i\lambda})\frac{1}{|\beta^{(d)}(i\lambda)|^2}dF(\lambda), \tag{6}
$$

where

$$
\chi_{\overline{\mu}}^{(d)}(e^{-i\lambda}) = \prod_{j=1}^{r}(1 - e^{-i\lambda\mu_j s_j})^{d_j}, \quad \beta^{(d)}(i\lambda) = \prod_{j=1}^{r}\prod_{k_j=-[s_j/2]}^{[s_j/2]}(i\lambda - 2\pi i k_j/s_j)^{d_j},
$$

c is a vector, $F(\lambda)$ is the matrix-valued spectral function of the stationary stochastic sequence $\chi_{\overline{\mu},\overline{s}}^{(d)}(\vec{\xi}(m))$. The vector c and the matrix-valued function $F(\lambda)$ are determined uniquely by the GM increment sequence $\chi_{\overline{\mu},\overline{s}}^{(d)}(\vec{\xi}(m))$.

2. The stationary GM increment sequence $\chi_{\overline{\mu},\overline{s}}^{(d)}(\vec{\xi}(m))$ admits the spectral representation

$$
\chi_{\overline{\mu},\overline{s}}^{(d)}(\vec{\xi}(m)) = \int_{-\pi}^{\pi} e^{im\lambda}\chi_{\overline{\mu}}^{(d)}(e^{-i\lambda})\frac{1}{\beta^{(d)}(i\lambda)}d\vec{Z}_{\xi^{(d)}}(\lambda), \tag{7}
$$

where $d\vec{Z}_{\xi^{(d)}}(\lambda) = \{Z_p(\lambda)\}_{p=1}^{T}$ is a (vector-valued) stochastic process with uncorrelated increments on $[-\pi, \pi)$ connected with the spectral function $F(\lambda)$ by the relation

$$
\mathsf{E}(Z_p(\lambda_2) - Z_p(\lambda_1))(\overline{Z_q(\lambda_2) - Z_q(\lambda_1)}) = F_{pq}(\lambda_2) - F_{pq}(\lambda_1),
$$

$$
-\pi \leq \lambda_1 < \lambda_2 < \pi, \quad p, q = 1, 2, \ldots, T.
$$

Consider another vector-valued stochastic sequence with the stationary GM increments $\vec{\zeta}(m) = \vec{\xi}(m) + \vec{\eta}(m)$, where $\vec{\eta}(m)$ is a vector-valued stationary stochastic sequence, uncorrelated with $\vec{\xi}(m)$, with a spectral representation

$$\vec{\eta}(m) = \int_{-\pi}^{\pi} e^{i\lambda m} d\vec{Z}_{\eta}(\lambda),$$

where $Z_{\eta}(\lambda) = \{Z_{\eta,p}(\lambda)\}_{p=1}^{T}$, $\lambda \in [-\pi, \pi)$, is a stochastic process with uncorrelated increments, that corresponds to the spectral function $G(\lambda)$ [20]. The stochastic stationary GM increment $\chi_{\overline{\mu},\overline{s}}^{(d)}(\vec{\zeta}(m))$ allows the spectral representation

$$\chi_{\overline{\mu},\overline{s}}^{(d)}(\vec{\zeta}(m)) \quad = \quad \int_{-\pi}^{\pi} e^{i\lambda m} \frac{\chi_{\overline{\mu}}^{(d)}(e^{-i\lambda})}{\beta^{(d)}(i\lambda)} d\vec{Z}_{\xi^{(n)}}(\lambda) + \int_{-\pi}^{\pi} e^{i\lambda m} \chi_{\overline{\mu}}^{(d)}(e^{-i\lambda}) d\vec{Z}_{\eta}(\lambda),$$

while $d\vec{Z}_{\eta}(\lambda) = (\beta^{(d)}(i\lambda))^{-1} d\vec{Z}_{\eta^{(n)}}(\lambda)$, $\lambda \in [-\pi, \pi)$. Therefore, in the case where the spectral functions $F(\lambda)$ and $G(\lambda)$ have the spectral density matrices $f(\lambda) = \{f_{ij}(\lambda)\}_{i,j=1}^{T}$ and $g(\lambda) = \{g_{ij}(\lambda)\}_{i,j=1}^{T}$, the spectral density matrix $p(\lambda) = \{p_{ij}(\lambda)\}_{i,j=1}^{T}$ of the stochastic sequence $\vec{\zeta}(m)$ is determined by the formula

$$p(\lambda) = f(\lambda) + |\beta^{(d)}(i\lambda)|^2 g(\lambda).$$

2.3. Moving average representation of periodically stationary GM increment

Denote by $H = L_2(\Omega, \mathcal{F}, \mathsf{P})$ the Hilbert space of random variables ζ with zero first moment, $\mathsf{E}\zeta = 0$, finite second moment, $\mathsf{E}|\zeta|^2 < \infty$, endowed with the inner product $\langle \zeta, \eta \rangle = \mathsf{E}\zeta\overline{\eta}$. Denote by $H(\vec{\xi}^{(d)})$ the closed linear subspace of the space H generated by components $\{\chi_{\overline{\mu},\overline{s}}^{(d)}(\xi_p(m)), p = 1, \dots, T; m \in \mathbb{Z}\}$ of the stationary stochastic GM increment sequence $\vec{\xi}^{(d)} = \{\chi_{\overline{\mu},\overline{s}}^{(d)}(\xi_p(l))\}_{p=1}^{T}$, $\overline{\mu} > \overline{0}$, and denote by $H^q(\vec{\xi}^{(d)})$ the closed linear subspace generated by components $\{\chi_{\overline{\mu},\overline{s}}^{(d)}(\xi_p(m)), p = 1, \dots, T; m \leqslant q\}$, $q \in \mathbb{Z}$. Define a subspace

$$S(\vec{\xi}^{(d)}) = \bigcap_{q \in \mathbb{Z}} H^q(\vec{\xi}^{(d)})$$

of the Hilbert space $H(\vec{\xi}^{(d)})$. Then the space $H(\vec{\xi}^{(d)})$ admits a decomposition $H(\vec{\xi}^{(d)}) = S(\vec{\xi}^{(d)}) \oplus R(\vec{\xi}^{(d)})$ where $R(\vec{\xi}^{(d)})$ is the orthogonal complement of the subspace $S(\vec{\xi}^{(d)})$ in the space $H(\vec{\xi}^{(d)})$.

Definition 2.4
A stationary (wide sense) stochastic GM increment sequence $\chi_{\overline{\mu},\overline{s}}^{(d)}(\vec{\xi}(m)) = \{\chi_{\overline{\mu},\overline{s}}^{(d)}(\xi_p(m))\}_{p=1}^{T}$ is called regular if $H(\vec{\xi}^{(d)}) = R(\vec{\xi}^{(d)})$, and it is called singular if $H(\vec{\xi}^{(d)}) = S(\vec{\xi}^{(d)})$.

Theorem 2.2
A stationary stochastic GM increment sequence $\chi_{\overline{\mu},\overline{s}}^{(d)}(\vec{\xi}(m)) = \{\chi_{\overline{\mu},\overline{s}}^{(d)}(\xi_p(m))\}_{p=1}^{T}$ is uniquely represented in the form

$$\chi_{\overline{\mu},\overline{s}}^{(d)}(\xi_p(m)) = \chi_{\overline{\mu},\overline{s}}^{(d)}(\xi_{S,p}(m)) + \chi_{\overline{\mu},\overline{s}}^{(d)}(\xi_{R,p}(m)) \tag{8}$$

where $\chi_{\overline{\mu},\overline{s}}^{(d)}(\xi_{R,p}(m))$, $p = 1, \dots, T$ is a regular stationary GM increment sequence and $\chi_{\overline{\mu},\overline{s}}^{(d)}(\xi_{S,p}(m))$, $p = 1, \dots, T$ is a singular stationary GM increment sequence. The GM increment sequences $\chi_{\overline{\mu},\overline{s}}^{(d)}(\xi_{R,p}(m))$, $p = 1, \dots, T$ and $\chi_{\overline{\mu},\overline{s}}^{(d)}(\xi_{S,p}(m))$, $p = 1, \dots, T$ are orthogonal for all $m, k \in \mathbb{Z}$. They are defined by the formulas

$$\begin{aligned}
\chi_{\overline{\mu},\overline{s}}^{(d)}(\xi_{S,p}(m)) &= \mathsf{E}[\chi_{\overline{\mu},\overline{s}}^{(d)}(\xi_p(m))|S(\vec{\xi}^{(d)})], \\
\chi_{\overline{\mu},\overline{s}}^{(d)}(\xi_{R,p}(m)) &= \chi_{\overline{\mu},\overline{s}}^{(d)}(\xi_p(m)) - \chi_{\overline{\mu},\overline{s}}^{(d)}(\xi_{S,p}(m)), \quad p = 1, \dots, T.
\end{aligned}$$

Consider an innovation sequence $\vec{\varepsilon}(u) = \{\varepsilon_k(u)\}_{k=1}^q, u \in \mathbb{Z}$ for a regular stationary GM increment, namely, the sequence of uncorrelated random variables such that $\mathsf{E}\varepsilon_k(u)\overline{\varepsilon}_j(v) = \delta_{kj}\delta_{uv}$, $\mathsf{E}|\varepsilon_k(u)|^2 = 1, k, j = 1, \ldots, q; u \in \mathbb{Z}$, and $H^r(\vec{\xi}^{(d)}) = H^r(\vec{\varepsilon})$ holds true for all $r \in \mathbb{Z}$, where $H^r(\vec{\varepsilon})$ is the Hilbert space generated by elements $\{\varepsilon_k(u) : k = 1, \ldots, q; u \leq r\}$, δ_{kj} and δ_{uv} are Kronecker symbols.

Theorem 2.3
A stationary GM increment sequence $\chi_{\overline{\mu},\overline{s}}^{(d)}(\vec{\xi}(m))$ is regular if and only if there exists an innovation sequence $\vec{\varepsilon}(u) = \{\varepsilon_k(u)\}_{k=1}^q, u \in \mathbb{Z}$ and a sequence of matrix-valued functions $\varphi^{(d)}(k,\overline{\mu}) = \{\varphi_{ij}^{(d)}(k,\overline{\mu})\}_{i=\overline{1,T}}^{j=\overline{1,q}}, k \geq 0$, such that

$$\sum_{k=0}^{\infty}\sum_{i=1}^{T}\sum_{j=1}^{q}|\varphi_{ij}^{(d)}(k,\overline{\mu})|^2 < \infty, \quad \chi_{\overline{\mu},\overline{s}}^{(d)}(\vec{\xi}(m)) = \sum_{k=0}^{\infty}\varphi^{(d)}(k,\overline{\mu})\vec{\varepsilon}(m-k). \tag{9}$$

Representation (9) is called the canonical moving average representation of the stochastic stationary GM increment sequence $\chi_{\overline{\mu},\overline{s}}^{(d)}(\vec{\xi}(m))$.

The spectral function $F(\lambda)$ of a stationary GM increment sequence $\chi_{\overline{\mu},\overline{s}}^{(d)}(\vec{\xi}(m))$ which admits the canonical representation (9) has the spectral density $f(\lambda) = \{f_{ij}(\lambda)\}_{i,j=1}^T$ admitting the canonical factorization

$$f(\lambda) = \varphi(e^{-i\lambda})\varphi^*(e^{-i\lambda}), \tag{10}$$

where the function $\varphi(z) = \sum_{k=0}^{\infty}\varphi(k)z^k$ has analytic in the unit circle $\{z : |z| \leq 1\}$ components $\varphi_{ij}(z) = \sum_{k=0}^{\infty}\varphi_{ij}(k)z^k; i = 1, \ldots, T; j = 1, \ldots, q$. Based on moving average representation (9) define

$$\varphi_{\overline{\mu}}(z) = \sum_{k=0}^{\infty}\varphi^{(d)}(k,\overline{\mu})z^k = \sum_{k=0}^{\infty}\varphi_{\overline{\mu}}(k)z^k.$$

Then the following relation holds true:

$$\varphi_{\overline{\mu}}(e^{-i\lambda})\varphi_{\overline{\mu}}^*(e^{-i\lambda}) = \frac{|\chi_{\overline{\mu}}^{(d)}(e^{-i\lambda})|^2}{|\beta^{(d)}(i\lambda)|^2}f(\lambda) = \prod_{j=1}^{r}\frac{\left|1 - e^{-i\lambda\mu_j s_j}\right|^{2d_j}}{\prod_{k_j=-[s_j/2]}^{[s_j/2]}|\lambda - 2\pi k_j/s_j|^{2d_j}}f(\lambda). \tag{11}$$

The one-sided moving average representation (9) and relation (11) are used for finding the mean square optimal estimates of unobserved values of vector-valued sequences with stationary GM increments.

2.4. Stochastic sequences with GM fractional increments

Now we extend the definition of the GM increment sequence $\chi_{\overline{\mu},\overline{s}}^{(d)}(\vec{\xi}(m))$ of the positive integer orders $(d_1, \ldots, d_r)$ to the fractional ones. Within the subsection, we put the step $\overline{\mu} = (1, 1, \ldots, 1)$. Following the results of [37], represent the increment operator $\chi_{\overline{s}}^{(d)}(B)$ in the form

$$\chi_{\overline{s}}^{(R+D)}(B) = (1-B)^{R_0+D_0}\prod_{j=1}^{r}(1-B^{s_j})^{R_j+D_j}, \tag{12}$$

where $(1-B)^{R_0+D_0}$ is an integrating component, $R_j, j = 0, 1, \ldots, r$, are non-negative integer numbers, $1 < s_1 < \ldots < s_r$. Below we describe representations $d_j = R_j + D_j, j = 0, 1, \ldots, r$, of the increment orders d_j by stating conditions on the fractional parts D_j, such that the increment sequence

$$\vec{y}(m) := (1-B)^{R_0}\prod_{j=1}^{r}(1-B^{s_j})^{R_i}\vec{\xi}(m)$$

is a stationary fractionally integrated seasonal stochastic sequence. For example, in case of single increment pattern $(1-B^{s^*})^{R^*+D^*}$, this condition is $|D^*| < 1/2$.

Definition 2.5
A sequence $\chi_{\overline{s}}^{(R+D)}(\vec{\xi}(m))$ is called *a fractional multiple (FM) increment sequence*.

Consider the generating function of the Gegenbauer polynomial:

$$(1 - 2uB + B^2)^{-d} = \sum_{n=0}^{\infty} C_n^{(d)}(u)B^n, \quad C_n^{(d)}(u) = \sum_{k=0}^{[n/2]} \frac{(-1)^k(2u)^{n-2k}\Gamma(d-k+n)}{k!(n-2k)!\Gamma(d)}.$$

The following lemma and theorem hold true [37].

Lemma 2.1
Define the sets $\mathcal{M}_j = \{\nu_{k_j} = 2\pi k_j/s_j : k_j = 0, 1, \ldots, [s_j/2]\}$, $j = 0, 1, \ldots, r$, and the set $\mathcal{M} = \bigcup_{j=0}^{r} \mathcal{M}_j$. Then the multiple seasonal increment operator admits the following representation:

$$
\begin{aligned}
\chi_{\overline{s}}^{(D)}(B) &:= (1-B)^{D_0} \prod_{j=1}^{r} (1-B^{s_j})^{D_j} = \prod_{\nu \in \mathcal{M}} (1 - 2\cos\nu B + B^2)^{\widetilde{D}_\nu} \\
&= (1-B)^{D_0+D_1+\ldots+D_r}(1+B)^{D_\pi} \prod_{\nu \in \mathcal{M} \setminus \{0,\pi\}} (1 - 2\cos\nu B + B^2)^{D_\nu} \\
&= \left(\sum_{m=0}^{\infty} G_{k^*}^+(m)B^m \right)^{-1} = \sum_{m=0}^{\infty} G_{k^*}^-(m)B^m,
\end{aligned}
$$

where

$$G_{k^*}^+(m) = \sum_{0 \le n_1, \ldots, n_{k^*} \le m, n_1+\ldots+n_{k^*}=m} \prod_{\nu \in \mathcal{M}} C_{n_\nu}^{(\widetilde{D}_\nu)}(\cos\nu), \tag{13}$$

$$G_{k^*}^-(m) = \sum_{0 \le n_1, \ldots, n_{k^*} \le m, n_1+\ldots+n_{k^*}=m} \prod_{\nu \in \mathcal{M}} C_{n_\nu}^{(-\widetilde{D}_\nu)}(\cos\nu). \tag{14}$$

$k^* = |\mathcal{M}|$, $D_\nu = \sum_{j=0}^{r} D_j \mathbb{I}\{\nu \in \mathcal{M}_j\}$, $\widetilde{D}_\nu = D_\nu$ for $\nu \in \mathcal{M} \setminus \{0, \pi\}$, $\widetilde{D}_\nu = D_\nu/2$ for $\nu = 0$ and $\nu = \pi$.

Theorem 2.4
Assume that for a stochastic vector-valued sequence $\vec{\xi}(m)$ and fractional differencing orders $d_j = R_j + D_j$, $j = 0, 1, \ldots, r$, the FM increment sequence $\chi_{\overline{1},\overline{s}}^{(R+D)}(\vec{\xi}(m))$ generated by increment operator (12) is a stationary sequence with a bounded from zero and infinity spectral density $\widetilde{f}_{\overline{1}}(\lambda)$. Then for the non-negative integer numbers R_j, $j = 0, 1, \ldots, r$, the GM increment sequence $\chi_{\overline{1},\overline{s}}^{(R)}(\vec{\xi}(m))$ is stationary if $-1/2 < D_\nu < 1/2$ for all $\nu \in \mathcal{M}$, where D_ν are defined by real numbers D_j, $j = 0, 1, \ldots, r$ in Lemma 2.1 and it is long memory if $0 < D_\nu < 1/2$ for at least one $\nu \in \mathcal{M}$, and invertible if $-1/2 < D_\nu < 0$. The spectral density $f(\lambda)$ of the stationary GM increment sequence $\chi_{\overline{1},\overline{s}}^{(R)}(\vec{\xi}(m))$ admits a representation

$$f(\lambda) = |\beta^{(R)}(i\lambda)|^2 \left| \chi_{\overline{1}}^{(R)}(e^{-i\lambda}) \right|^{-2} \left| \chi_{\overline{1}}^{(D)}(e^{-i\lambda}) \right|^{-2} \widetilde{f}_{\overline{1}}(\lambda) =: \left| \chi_{\overline{1}}^{(D)}(e^{-i\lambda}) \right|^{-2} \widetilde{f}(\lambda),$$

where

$$\left| \chi_{\overline{1}}^{(D)}(e^{-i\lambda}) \right|^{-2} = \left| \sum_{m=0}^{\infty} G_{k^*}^+(m)e^{-i\lambda m} \right|^2 = \left| \sum_{m=0}^{\infty} G_{k^*}^-(m)e^{-i\lambda m} \right|^{-2}.$$

The further properties of the spectral density $f(\lambda)$ and the structural function $D_{\overline{s}}^{(R)}(m, \overline{1}, \overline{1})$ of a stationary GM increment vector sequence $\chi_{\overline{1},\overline{s}}^{(R)}(\vec{\xi}(m))$ as well as for examples of an application of Theorem 2.4, can be found in the works by Palma and Bondon [52], Giraitis and Leipus [14], Luz and Moklyachuk [37].

3. Hilbert space projection method of forecasting

3.1. *Forecasting of vector-valued stochastic sequences with stationary GM increments*

Consider a vector-valued stochastic sequence $\vec{\xi}(m)$ with stationary GM increments constructed from the sequence $\xi(m) = \{\xi_p(m)\}_{p=1}^{T}$ with the help of transformation (4). Let the stationary GM increment sequence $\chi_{\overline{\mu},\vec{s}}^{(d)}(\vec{\xi}(m)) = \{\chi_{\overline{\mu},\vec{s}}^{(d)}(\xi_p(m))\}_{p=1}^{T}$ has an absolutely continuous spectral function $F(\lambda)$ and the spectral density $f(\lambda) = \{f_{ij}(\lambda)\}_{i,j=1}^{T}$.

Let $\vec{\eta}(m) = \{\eta_p(m)\}_{p=1}^{T}$ be an uncorrelated with the sequence $\vec{\xi}(m)$ stationary stochastic sequence with absolutely continuous spectral function $G(\lambda)$ and spectral density $g(\lambda) = \{g_{ij}(\lambda)\}_{i,j=1}^{T}$.

Without loss of generality assume that the mean values of the increment sequence $\chi_{\overline{\mu},\vec{s}}^{(d)}(\vec{\xi}(m))$ and the stationary sequence $\vec{\eta}(m)$ equal to $\vec{0}$. We will also consider the increment step $\overline{\mu} > \overline{0}$.

Extrapolation (forecasting) problem. Consider the problem of mean square optimal linear estimation of the functionals

$$A\vec{\xi} = \sum_{k=0}^{\infty}(\vec{a}(k))^{\top}\vec{\xi}(k), \quad A_N\vec{\xi} = \sum_{k=0}^{N}(\vec{a}(k))^{\top}\vec{\xi}(k), \tag{15}$$

which depend on unobserved values of the stochastic sequence $\vec{\xi}(k)$ with stationary GM increments. Estimates are based on observations of the sequence $\vec{\zeta}(m) = \vec{\xi}(m) + \vec{\eta}(m)$ at points $m = -1, -2, \ldots$.

First of all we indicate some conditions which are necessary for solving the considered problem. Assume that coefficients $\vec{a}(k) = \{a_p(k)\}_{p=1}^{T}$, $k \geq 0$, and the linear transformation D^{μ} which is defined in the following part of the section satisfy the conditions

$$\sum_{k=0}^{\infty}\|\vec{a}(k)\| < \infty, \quad \sum_{k=0}^{\infty}(k+1)\|\vec{a}(k)\|^2 < \infty, \tag{16}$$

$$\sum_{k=0}^{\infty}\|(D^{\mu}\mathbf{a})_k\| < \infty, \quad \sum_{k=0}^{\infty}(k+1)\|(D^{\mu}\mathbf{a})_k\|^2 < \infty. \tag{17}$$

Assume also that spectral densities $f(\lambda)$ and $g(\lambda)$ satisfy the minimality condition

$$\int_{-\pi}^{\pi} \text{Tr}\left[\frac{|\beta^{(d)}(i\lambda)|^2}{|\chi_{\overline{\mu}}^{(d)}(e^{-i\lambda})|^2}\left(f(\lambda) + |\beta^{(d)}(i\lambda)|^2 g(\lambda)\right)^{-1}\right]d\lambda < \infty. \tag{18}$$

This is the necessary and sufficient condition under which the mean square errors of the optimal estimates of the functionals $A\vec{\xi}$ and $A_N\vec{\xi}$ are not equal to 0.

We apply the Hilbert space estimation technique proposed by Kolmogorov [30] which can be described as a 3-stage procedure: (i) define a target element (to be estimated) of the space $H = L_2(\Omega, \mathcal{F}, \mathsf{P})$ of random variables γ which have zero mean values and finite variances, $\mathsf{E}\gamma = 0$, $\mathsf{E}|\gamma|^2 < \infty$, endowed with the inner product $\langle \gamma_1; \gamma_2 \rangle = \mathsf{E}\gamma_1\overline{\gamma_2}$, (ii) define a subspace of H generated by observations, (iii) find an estimate of the target element as an orthogonal projection on the defined subspace.

Stage i. The functional $A\vec{\xi}$ does not belong to the space $H = L_2(\Omega, \mathcal{F}, \mathsf{P})$. With the help of the following lemma we describe representations of the functional as a sum of a functional with finite second moments belonging to H and a functional depending on the observed values of the sequence $\vec{\zeta}(k)$ ("initial values") (for more details see [36, 37]).

Lemma 3.1
The functional $A\vec{\xi}$ admits the representation

$$A\vec{\xi} = A\vec{\zeta} - A\vec{\eta} = H\vec{\xi} - V\vec{\zeta}, \tag{19}$$

where

$$H\vec{\xi} := B\chi\vec{\zeta} - A\vec{\eta},$$

$$A\vec{\zeta} = \sum_{k=0}^{\infty} (\vec{a}(k))^{\top}\vec{\zeta}(k), \quad A\vec{\eta} = \sum_{k=0}^{\infty} (\vec{a}(k))^{\top}\vec{\eta}(k),$$

$$B\chi\vec{\zeta} = \sum_{k=0}^{\infty} (\vec{b}_N(k))^{\top}\chi_{\overline{\mu},\overline{s}}^{(d)}(\vec{\zeta}(k)), \quad V\vec{\zeta} = \sum_{k=-n(\gamma)}^{-1} (\vec{v}_N(k))^{\top}\vec{\zeta}(k),$$

the coefficients $\vec{b}(k) = \{b_p(k)\}_{p=1}^{T}, k = 0, 1, \ldots$ and $\vec{v}(k) = \{v_p(k)\}_{p=1}^{T}, k = -1, -2, \ldots, -n(\gamma)$ are calculated by the formulas

$$\vec{v}(k) = \sum_{l=0}^{k+n(\gamma)} \mathsf{diag}_T(e_\nu(l-k))\vec{b}(l), \; k = -1, -2, \ldots, -n(\gamma), \tag{20}$$

$$\vec{b}(k) = \sum_{m=k}^{\infty} \mathsf{diag}_T(d_{\overline{\mu}}(m-k))\vec{a}(m) = (D^{\overline{\mu}}\mathbf{a})_k, \; k = 0, 1, \ldots, \tag{21}$$

$D^{\overline{\mu}}$ is the linear transformation determined by a matrix with the entries $(D^{\overline{\mu}})(k,j) = \mathsf{diag}_T(d_{\overline{\mu}}(j-k))$ if $0 \le k \le j$, and $(D^{\overline{\mu}})(k,j) = 0$ if $0 \le j < k$, $\mathsf{diag}_T(x)$ denotes a $T \times T$ diagonal matrix with the entry x on its diagonal, $\mathbf{a} = ((\vec{a}(0))^{\top}, (\vec{a}(1))^{\top}, \ldots)^{\top}$, coefficients $\{d_{\overline{\mu}}(k) : k \ge 0\}$ are determined by the relationship

$$\sum_{k=0}^{\infty} d_{\overline{\mu}}(k)x^k = \prod_{i=1}^{r} \left(\sum_{j_i=0}^{\infty} x^{\mu_i s_i j_i}\right)^{d_i}.$$

Corollary 3.1
The functional $A_N\vec{\xi}$ admits the representation

$$A_N\vec{\xi} = A_N\vec{\zeta} - A_N\vec{\eta} = H_N\vec{\xi} - V_N\vec{\zeta}, \tag{22}$$

where

$$H_N\vec{\xi} := B_N\chi\vec{\zeta} - A_N\vec{\eta},$$

$$A_N\vec{\zeta} = \sum_{k=0}^{N} (\vec{a}(k))^{\top}\vec{\zeta}(k), \quad A_N\vec{\eta} = \sum_{k=0}^{N} (\vec{a}(k))^{\top}\vec{\eta}(k),$$

$$B_N\chi\vec{\zeta} = \sum_{k=0}^{N} (\vec{b}_N(k))^{\top}\chi_{\overline{\mu},\overline{s}}^{(d)}(\vec{\zeta}(k)), \quad V_N\vec{\zeta} = \sum_{k=-n(\gamma)}^{-1} (\vec{v}_N(k))^{\top}\vec{\zeta}(k),$$

the coefficients $\vec{b}_N(k) = \{b_{N,p}(k)\}_{p=1}^{T}, k = 0, 1, \ldots, N$ and $\vec{v}_N(k) = \{v_{N,p}(k)\}_{p=1}^{T}, k = -1, -2, \ldots, -n(\gamma)$ are calculated by the formulas

$$\vec{v}_N(k) = \sum_{l=0}^{N \wedge k+n(\gamma)} \mathsf{diag}_T(e_\nu(l-k))\vec{b}_N(l), \; k = -1, -2, \ldots, -n(\gamma), \tag{23}$$

$$\vec{b}_N(k) = \sum_{m=k}^{N} \mathsf{diag}_T(d_{\overline{\mu}}(m-k))\vec{a}(m) = (D_N^{\overline{\mu}}\mathbf{a}_N)_k, \; k = 0, 1, \ldots, N, \tag{24}$$

$D_N^{\overline{\mu}}$ is the linear transformation determined by an infinite matrix with the entries $(D_N^{\overline{\mu}})(k,j) = \mathsf{diag}_T(d_{\overline{\mu}}(j-k))$ if $0 \le k \le j \le N$, and $(D_N^{\overline{\mu}})(k,j) = 0$ if $j < k$ or $j, k > N$; $\mathbf{a}_N = ((\vec{a}(0))^{\top}, (\vec{a}(1))^{\top}, \ldots, (\vec{a}(N))^{\top}, \vec{0} \ldots)^{\top}$.

The functional $H\vec{\xi}$ from representation (22) has finite variance and the functional $V\vec{\zeta}$ depends on the known observations of the stochastic sequence $\vec{\zeta}(k)$ at points $k = -n(\gamma), -n(\gamma) + 1, \ldots, -1$. Therefore, estimates $\widehat{A}\vec{\xi}$ and $\widehat{H}\vec{\xi}$ of the functionals $A\vec{\xi}$ and $H\vec{\xi}$ and the mean-square errors $\Delta(f, g; \widehat{A}\vec{\xi}) = \mathsf{E}|A\vec{\xi} - \widehat{A}\vec{\xi}|^2$ and $\Delta(f, g; \widehat{H}\vec{\xi}) = \mathsf{E}|H\vec{\xi} - \widehat{H}\vec{\xi}|^2$ of the estimates $\widehat{A}\vec{\xi}$ and $\widehat{H}\vec{\xi}$ satisfy the following relations

$$\widehat{A}\vec{\xi} = \widehat{H}\vec{\xi} - V\vec{\zeta}, \tag{25}$$

$$\Delta(f, g; \widehat{A}\vec{\xi}) = \mathsf{E}|A\vec{\xi} - \widehat{A}\vec{\xi}|^2 = \mathsf{E}|H\vec{\xi} - \widehat{H}\vec{\xi}|^2 = \Delta(f, g; \widehat{H}\vec{\xi}). \tag{26}$$

Therefore, the estimation problem for the functional $A\vec{\xi}$ is equivalent to the one for the functional $H\vec{\xi}$. This problem can be solved by applying the Hilbert space projection method proposed by Kolmogorov [30].

The functional $H\vec{\xi}$ admits the spectral representation

$$H\vec{\xi} = \int_{-\pi}^{\pi} \left(\vec{B}_{\overline{\mu}}(e^{i\lambda})\right)^{\top} \frac{\chi_{\overline{\mu}}^{(d)}(e^{-i\lambda})}{\beta^{(d)}(i\lambda)} d\vec{Z}_{\xi^{(d)}+\eta^{(d)}}(\lambda) - \int_{-\pi}^{\pi} \left(\vec{A}(e^{i\lambda})\right)^{\top} d\vec{Z}_{\eta}(\lambda),$$

where

$$\vec{B}_{\overline{\mu}}(e^{i\lambda}) = \sum_{k=0}^{\infty} \vec{b}_{\overline{\mu}}(k) e^{i\lambda k} = \sum_{k=0}^{\infty} (D^{\overline{\mu}}\mathbf{a})_k e^{i\lambda k}, \quad \vec{A}(e^{i\lambda}) = \sum_{k=0}^{\infty} \vec{a}(k) e^{i\lambda k}.$$

Stage (ii). Introduce the following notations. Denote by $H^{0-}(\xi_{\overline{\mu},\vec{s}}^{(d)} + \eta_{\overline{\mu},\vec{s}}^{(d)})$ the closed linear subspace generated by values $\{\chi_{\overline{\mu},\vec{s}}^{(d)}(\vec{\xi}(k)) + \chi_{\overline{\mu},\vec{s}}^{(d)}(\vec{\eta}(k)) : k = -1, -2, -3, \ldots\}, \overline{\mu} > \vec{0}$ of the observed GM increments in the Hilbert space $H = L_2(\Omega, \mathcal{F}, \mathsf{P})$ of random variables γ with zero mean value, $\mathsf{E}\gamma = 0$, finite variance, $\mathsf{E}|\gamma|^2 < \infty$, and the inner product $(\gamma_1, \gamma_2) = \mathsf{E}\gamma_1\overline{\gamma_2}$.

Denote by $L_2^{0-}(f(\lambda) + |\beta^{(d)}(i\lambda)|^2 g(\lambda))$ the closed linear subspace of the Hilbert space $L_2(f(\lambda) + |\beta^{(d)}(i\lambda)|^2 g(\lambda))$ of vector-valued functions with the inner product $\langle g_1, g_2 \rangle = \int_{-\pi}^{\pi} (g_1(\lambda))^{\top}(f(\lambda) + |\beta^{(d)}(i\lambda)|^2 g(\lambda))\overline{g_2(\lambda)} d\lambda$ which is generated by the functions

$$e^{i\lambda k} \chi_{\overline{\mu}}^{(d)}(e^{-i\lambda}) \frac{1}{\beta^{(d)}(i\lambda)} \vec{\delta}_l, \quad \vec{\delta}_l = \{\delta_{lp}\}_{p=1}^T, l = 1, \ldots, T; \; k \le -1,$$

where δ_{lp} are Kronecker symbols.

The representation

$$\chi_{\overline{\mu},\vec{s}}^{(d)}(\vec{\xi}(k)) + \chi_{\overline{\mu},\vec{s}}^{(d)}(\vec{\eta}(k)) = \int_{-\pi}^{\pi} e^{i\lambda k} \chi_{\overline{\mu}}^{(d)}(e^{-i\lambda}) \frac{1}{\beta^{(d)}(i\lambda)} dZ_{\xi^{(d)}+\eta^{(d)}}(\lambda)$$

yields a one-to-one correspondence between elements

$$e^{i\lambda k} \chi_{\overline{\mu}}^{(d)}(e^{-i\lambda}) \frac{1}{\beta^{(d)}(i\lambda)} \vec{\delta}_l$$

from the space $L_2^{0-}(f(\lambda) + |\beta^{(d)}(i\lambda)|^2 g(\lambda))$ and elements $\chi_{\overline{\mu},\vec{s}}^{(d)}(\vec{\xi}(k)) + \chi_{\overline{\mu},\vec{s}}^{(d)}(\vec{\eta}(k))$ from the space $H^{0-}(\xi_{\overline{\mu},\vec{s}}^{(d)} + \eta_{\overline{\mu},\vec{s}}^{(d)})$.

Relation (25) implies that every linear estimate $\widehat{A}\vec{\xi}$ of the functional $A\vec{\xi}$ can be represented in the form

$$\widehat{A}\vec{\xi} = \int_{-\pi}^{\pi} (\vec{h}_{\overline{\mu}}(\lambda))^{\top} d\vec{Z}_{\xi^{(d)}+\eta^{(d)}}(\lambda) - \sum_{k=-n(\gamma)}^{-1} (\vec{v}_{\mu}(k))^{\top} (\vec{\xi}(k) + \vec{\eta}(k)), \tag{27}$$

where $\vec{h}_{\overline{\mu}}(\lambda) = \{h_p(\lambda)\}_{p=1}^T$ is the spectral characteristic of the optimal estimate $\widehat{H}\vec{\xi}$.

Stage (iii). At this stage we find the mean square optimal estimate $\widehat{H}\vec{\xi}$ as a projection of the element $H\vec{\xi}$ on the subspace $H^{0-}(\xi_{\bar{\mu},\bar{s}}^{(d)} + \eta_{\bar{\mu},\bar{s}}^{(d)})$. This projection is determined by two conditions:

1) $\widehat{H}\vec{\xi} \in H^{0-}(\xi_{\bar{\mu},\bar{s}}^{(d)} + \eta_{\bar{\mu},\bar{s}}^{(d)})$;

2) $(H\vec{\xi} - \widehat{H}\vec{\xi}) \perp H^{0-}(\xi_{\bar{\mu},\bar{s}}^{(d)} + \eta_{\bar{\mu},\bar{s}}^{(d)})$.

The second condition implies the following relation which holds true for all $k \leq -1$

$$\int_{-\pi}^{\pi} \left[\left(\vec{B}_{\bar{\mu}}(e^{i\lambda})^{\top} \frac{\chi_{\bar{\mu}}^{(d)}(e^{-i\lambda})}{\beta^{(d)}(i\lambda)} - \vec{h}_{\bar{\mu}}(\lambda) \right)^{\top} \left(f(\lambda) + |\beta^{(d)}(i\lambda)|^2 g(\lambda) \right) - \right.$$

$$\left. - (\vec{A}(e^{i\lambda}))^{\top} g(\lambda) \overline{\beta^{(d)}(i\lambda)} \right] \frac{\overline{\chi_{\bar{\mu}}^{(d)}(e^{-i\lambda})}}{\beta^{(d)}(i\lambda)} e^{-i\lambda k} d\lambda = \vec{0}.$$

This relation allows us to derive the spectral characteristic $\vec{h}_{\bar{\mu}}(\lambda)$ of the estimate $\widehat{H}\vec{\mu}$ which can be represented in the form

$$(\vec{h}_{\bar{\mu}}(\lambda))^{\top} = (\vec{B}_{\bar{\mu}}(e^{i\lambda}))^{\top} \frac{\chi_{\bar{\mu}}^{(d)}(e^{-i\lambda})}{\beta^{(d)}(i\lambda)} - (\vec{A}_{\bar{\mu}}(e^{i\lambda}))^{\top} g(\lambda) \frac{\overline{\beta^{(d)}(i\lambda)}}{\chi_{\bar{\mu}}^{(d)}(e^{-i\lambda})} \left(f(\lambda) + |\beta^{(d)}(i\lambda)|^2 g(\lambda) \right)^{-1} -$$

$$- (\vec{C}_{\bar{\mu}}(e^{i\lambda}))^{\top} \frac{\overline{\beta^{(d)}(i\lambda)}}{\chi_{\bar{\mu}}^{(d)}(e^{-i\lambda})} \left(f(\lambda) + |\beta^{(d)}(i\lambda)|^2 g(\lambda) \right)^{-1}, \quad (28)$$

where

$$\vec{A}_{\bar{\mu}}(e^{i\lambda}) = \vec{A}(e^{i\lambda}) \overline{\chi_{\bar{\mu}}^{(d)}(e^{-i\lambda})} = \sum_{k=0}^{\infty} \vec{a}_{\bar{\mu}}(k) e^{ik\lambda},$$

$$\vec{a}_{\bar{\mu}}(m) = \sum_{l=\max\{m-n(\gamma),0\}}^{m} e_{\gamma}(m-l) \vec{a}(l), \quad m \geq 0, \quad (29)$$

$$\vec{C}_{\bar{\mu}}(e^{i\lambda}) = \sum_{k=0}^{\infty} \vec{c}_{\bar{\mu}}(k) e^{ik\lambda},$$

coefficients $\vec{c}_{\bar{\mu}}(k) = \{c_{\bar{\mu},p}(k)\}_{p=1}^{T}, k = 0, 1, \dots,$ are unknown and have to be found.

It follows from condition 1) that the following equations should be satisfied for $j \geq 0$

$$\int_{-\pi}^{\pi} \left[(\vec{B}_{\bar{\mu}}(e^{i\lambda}))^{\top} - (\vec{A}_{\bar{\mu}}(e^{i\lambda}))^{\top} \frac{|\beta^{(d)}(i\lambda)|^2}{|\chi_{\bar{\mu}}^{(d)}(e^{-i\lambda})|^2} g(\lambda) \left(f(\lambda) + |\beta^{(d)}(i\lambda)|^2 g(\lambda) \right)^{-1} - \right.$$

$$\left. - (\vec{C}_{\bar{\mu}}(e^{i\lambda}))^{\top} \frac{|\beta^{(d)}(i\lambda)|^2}{|\chi_{\bar{\mu}}^{(d)}(e^{-i\lambda})|^2} \left(f(\lambda) + |\beta^{(d)}(i\lambda)|^2 g(\lambda) \right)^{-1} \right] e^{-ij\lambda} d\lambda = 0. \quad (30)$$

Define for $k, j \geq 0$ the Fourier coefficients of the corresponding functions

$$T_{k,j}^{\bar{\mu}} = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-i\lambda(k-j)} \frac{|\beta^{(d)}(i\lambda)|^2}{|\chi_{\bar{\mu}}^{(d)}(e^{-i\lambda})|^2} \left[g(\lambda) \left(f(\lambda) + |\beta^{(d)}(i\lambda)|^2 g(\lambda) \right)^{-1} \right]^{\top} d\lambda;$$

$$P_{k,j}^{\bar{\mu}} = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-i\lambda(k-j)} \frac{|\beta^{(d)}(i\lambda)|^2}{|\chi_{\bar{\mu}}^{(d)}(e^{-i\lambda})|^2} \left[\left(f(\lambda) + |\beta^{(d)}(i\lambda)|^2 g(\lambda) \right)^{-1} \right]^{\top} d\lambda;$$

$$Q_{k,j} = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-i\lambda(k-j)} \left[f(\lambda) \left(f(\lambda) + |\beta^{(d)}(i\lambda)|^2 g(\lambda) \right)^{-1} g(\lambda) \right]^{\top} d\lambda.$$

Making use of the defined Fourier coefficients, relation (30) can be presented as a system linear equations

$$\vec{b}_{\overline{\mu}}(j) - \sum_{m=0}^{\infty} T_{j,m}^{\overline{\mu}} \vec{a}_{\overline{\mu}}(m) = \sum_{k=0}^{\infty} P_{j,k}^{\overline{\mu}} \vec{c}_{\overline{\mu}}(k), \ j \geq 0, \tag{31}$$

determining the unknown coefficients $\vec{c}_{\overline{\mu}}(k)$, $k \geq 0$. This system of equations can be written in the form

$$D^{\overline{\mu}} \mathbf{a} - \mathbf{T}_{\overline{\mu}} \mathbf{a}_{\overline{\mu}} = \mathbf{P}_{\overline{\mu}} \mathbf{c}_{\overline{\mu}},$$

where

$$\mathbf{a}_{\overline{\mu}} = ((\vec{a}_{\overline{\mu}}(0))^{\top}, (\vec{a}_{\overline{\mu}}(1))^{\top}, (\vec{a}_{\overline{\mu}}(2))^{\top}, \ldots)^{\top},$$
$$\mathbf{c}_{\overline{\mu}} = ((\vec{c}_{\overline{\mu}}(0))^{\top}, (\vec{c}_{\overline{\mu}}(1))^{\top}, (\vec{c}_{\overline{\mu}}(2))^{\top}, \ldots)^{\top},$$
$$\mathbf{a} = ((\vec{a}(0))^{\top}, (\vec{a}(1))^{\top}, (\vec{a}(2))^{\top}, \ldots)^{\top},$$

$\mathbf{P}_{\overline{\mu}}$ and $\mathbf{T}_{\overline{\mu}}$ are linear operators in the space ℓ_2 defined by matrices with the $T \times T$ matrix entries $(\mathbf{P}_{\overline{\mu}})_{l,k} = P_{l,k}^{\overline{\mu}}$, $l, k \geq 0$ and $(\mathbf{T}_{\overline{\mu}})_{l,k} = T_{l,k}^{\overline{\mu}}$, $l, k \geq 0$; the linear transformation $D^{\overline{\mu}}$ is defined in Lemma 3.1.

Consequently, the unknown coefficients $\vec{c}_{\overline{\mu}}(k)$, $k \geq 0$, which determine the spectral characteristic $\vec{h}_{\overline{\mu}}(\lambda)$ are calculated by the formula

$$\vec{c}_{\overline{\mu}}(k) = (\mathbf{P}_{\overline{\mu}}^{-1} D^{\overline{\mu}} \mathbf{a} - \mathbf{P}_{\overline{\mu}}^{-1} \mathbf{T}_{\overline{\mu}} \mathbf{a}_{\overline{\mu}})_k, \quad k \geq 0, \tag{32}$$

where $(\mathbf{P}_{\overline{\mu}}^{-1} D^{\overline{\mu}} \mathbf{a} - \mathbf{P}_{\overline{\mu}}^{-1} \mathbf{T}_{\overline{\mu}} \mathbf{a}_{\overline{\mu}})_k$, $k \geq 0$, is the kth T-dimension vector element of the vector $\mathbf{P}_{\overline{\mu}}^{-1} D^{\overline{\mu}} \mathbf{a} - \mathbf{P}_{\overline{\mu}}^{-1} \mathbf{T}_{\overline{\mu}} \mathbf{a}_{\overline{\mu}}$ and the function $\vec{C}_{\overline{\mu}}(e^{i\lambda})$ is of the form

$$\vec{C}_{\overline{\mu}}(e^{i\lambda}) = \sum_{k=0}^{\infty} (\mathbf{P}_{\overline{\mu}}^{-1} D^{\overline{\mu}} \mathbf{a} - \mathbf{P}_{\overline{\mu}}^{-1} \mathbf{T}_{\overline{\mu}} \mathbf{a}_{\overline{\mu}})_k e^{ik\lambda}.$$

Remark 3.1
The problem of projection of the element $H\vec{\xi}$ of the Hilbert space H on the closed convex set $H^{0-}(\xi_{\overline{\mu}}^{(n)} + \eta_{\overline{\mu}}^{(n)})$ has a unique solution for each non-zero coefficients $\{\vec{a}(0), \vec{a}(1)), \vec{a}(2), \ldots\}$, satisfying conditions (16) – (17). Therefore, equation (32) has a unique solution for each vector $D^{\overline{\mu}} \mathbf{a}$, which implies an existence of the inverse operator $\mathbf{P}_{\overline{\mu}}^{-1}$.

The spectral characteristic $\vec{h}_{\overline{\mu}}(\lambda)$ of the optimal estimate $\widehat{H\vec{\xi}}$ of the functional $H\vec{\xi}$ can be calculated by the formula

$$(\vec{h}_{\overline{\mu}}(\lambda))^{\top} = (\vec{B}_{\overline{\mu}}(e^{i\lambda}))^{\top} \frac{\chi_{\overline{\mu}}^{(d)}(e^{-i\lambda})}{\beta^{(d)}(i\lambda)} - \frac{\overline{\beta^{(d)}(i\lambda)}}{\chi_{\overline{\mu}}^{(d)}(e^{-i\lambda})} (\vec{A}_{\overline{\mu}}(e^{i\lambda}))^{\top} g(\lambda)(f(\lambda) + |\beta^{(d)}(i\lambda)|^2 g(\lambda))^{-1} -$$

$$- \frac{\overline{\beta^{(d)}(i\lambda)}}{\chi_{\overline{\mu}}^{(d)}(e^{-i\lambda})} \left(\sum_{k=0}^{\infty} (\mathbf{P}_{\overline{\mu}}^{-1} D^{\overline{\mu}} \mathbf{a} - \mathbf{P}_{\overline{\mu}}^{-1} \mathbf{T}_{\overline{\mu}} \mathbf{a}_{\overline{\mu}})_k e^{ik\lambda} \right)^{\top} (f(\lambda) + |\beta^{(d)}(i\lambda)|^2 g(\lambda))^{-1}, \tag{33}$$

The value of the mean square error of the estimate $\widehat{A\vec{\xi}}$ is calculated by the formula

$$\Delta(f, g; \widehat{A\vec{\xi}}) = \Delta(f, g; \widehat{H\vec{\xi}}) = \mathsf{E}|H\vec{\xi} - \widehat{H\vec{\xi}}|^2 =$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} \left[(\vec{A}_{\overline{\mu}}(e^{i\lambda}))^{\top} g(\lambda) + \left(\sum_{k=0}^{\infty} (\mathbf{P}_{\overline{\mu}}^{-1} D^{\overline{\mu}} \mathbf{a} - \mathbf{P}_{\overline{\mu}}^{-1} \mathbf{T}_{\overline{\mu}} \mathbf{a}_{\overline{\mu}})_k e^{ik\lambda} \right)^{\top} \right] \times$$

$$\times \frac{|\beta^{(d)}(i\lambda)|^2}{|\chi_{\overline{\mu}}^{(d)}(e^{-i\lambda})|^2} (f(\lambda) + |\beta^{(d)}(i\lambda)|^2 g(\lambda))^{-1} f(\lambda) (f(\lambda) + |\beta^{(d)}(i\lambda)|^2 g(\lambda))^{-1} \times$$

$$\times \left[g(\lambda)\overline{\vec{A}_{\overline{\mu}}(e^{i\lambda})} + \overline{\sum_{k=0}^{\infty}(\mathbf{P}_{\overline{\mu}}^{-1}D^{\overline{\mu}}\mathbf{a} - \mathbf{P}_{\overline{\mu}}^{-1}\mathbf{T}_{\overline{\mu}}\mathbf{a}_{\overline{\mu}})_k e^{ik\lambda}} \right] d\lambda +$$

$$+ \frac{1}{2\pi}\int_{-\pi}^{\pi} \left[\overline{\chi_{\overline{\mu}}^{(d)}(e^{-i\lambda})}(\vec{A}(e^{i\lambda}))^{\top}f(\lambda) - |\beta^{(d)}(i\lambda)|^2 \left(\overline{\sum_{k=0}^{\infty}(\mathbf{P}_{\overline{\mu}}^{-1}D^{\overline{\mu}}\mathbf{a} - \mathbf{P}_{\overline{\mu}}^{-1}\mathbf{T}_{\overline{\mu}}\mathbf{a}_{\overline{\mu}})_k e^{ik\lambda}} \right)^{\top} \right] \times$$

$$\times \frac{1}{|\chi_{\overline{\mu}}^{(d)}(e^{-i\lambda})|^2}(f(\lambda) + |\beta^{(d)}(i\lambda)|^2 g(\lambda))^{-1}g(\lambda)(f(\lambda) + |\beta^{(d)}(i\lambda)|^2 g(\lambda))^{-1} \times$$

$$\times \left[f(\lambda)\overline{\vec{A}_{\overline{\mu}}(e^{i\lambda})} - |\beta^{(d)}(i\lambda)|^2 \overline{\sum_{k=0}^{\infty}(\mathbf{P}_{\overline{\mu}}^{-1}D^{\overline{\mu}}\mathbf{a} - \mathbf{P}_{\overline{\mu}}^{-1}\mathbf{T}_{\overline{\mu}}\mathbf{a}_{\overline{\mu}})_k e^{ik\lambda}} \right] d\lambda =$$

$$= \left\langle D^{\overline{\mu}}\mathbf{a} - \mathbf{T}_{\overline{\mu}}\mathbf{a}_{\overline{\mu}}, \mathbf{P}_{\overline{\mu}}^{-1}D^{\overline{\mu}}\mathbf{a} - \mathbf{P}_{\overline{\mu}}^{-1}\mathbf{T}_{\overline{\mu}}\mathbf{a}_{\overline{\mu}} \right\rangle + \left\langle \mathbf{Q}\mathbf{a}, \mathbf{a} \right\rangle, \tag{34}$$

where $\mathbf{Q}$ is a linear operator in the space ℓ_2 defined by the matrix with the $T \times T$ matrix elements $(\mathbf{Q})_{l,k} = Q_{l,k}$, $l, k \geq 0$; $\langle \vec{x}, \vec{y} \rangle = \sum_{k=0}^{\infty}(\vec{x}(k))^{\top}\overline{\vec{y}}(k)$ for vectors $\vec{x} = ((\vec{x}(0))^{\top}, (\vec{x}(1))^{\top}, (\vec{x}(2))^{\top}, \dots)^{\top}$, $\vec{y} = ((\vec{y}(0))^{\top}, (\vec{y}(1))^{\top}, (\vec{y}(2))^{\top}, \dots)^{\top}$.

Theorem 3.1
Let $\vec{\xi}(m)$, $m \in \mathbb{Z}$, be a stochastic sequence which defines stationary nth increment sequence $\chi_{\overline{\mu},\overline{s}}^{(d)}(\vec{\xi}(m))$ with absolutely continuous spectral function $F(\lambda)$ which has spectral density $f(\lambda)$. Let $\vec{\eta}(m)$, $m \in \mathbb{Z}$, be an uncorrelated with the sequence $\vec{\xi}(m)$ stationary stochastic sequence with absolutely continuous spectral function $G(\lambda)$ which has spectral density $g(\lambda)$. Let the minimality condition (18) be satisfied. Let coefficients $\vec{a}(k)$, $k \geq 0$, satisfy conditions (16) – (17). The optimal linear estimate $\widehat{A\vec{\xi}}$ of the functional $A\vec{\xi}$ which depend on the unknown values of elements $\vec{\xi}(m)$, $m \geq 0$, based on observations of the sequence $\vec{\xi}(m) + \vec{\eta}(m)$ at points $m = -1, -2, \dots$ is calculated by formula (27). The spectral characteristic $\vec{h}_{\overline{\mu}}(\lambda)$ of the optimal estimate $\widehat{A\vec{\xi}}$ is calculated by formula (33). The value of the mean-square error $\Delta(f, g; \widehat{A\vec{\xi}})$ is calculated by formula (34).

Corollary 3.2
The spectral characteristic $\vec{h}_{\overline{\mu}}(\lambda)$ admits the representation $\vec{h}_{\overline{\mu}}(\lambda) = \vec{h}_{\overline{\mu}}^1(\lambda) - \vec{h}_{\overline{\mu}}^2(\lambda)$, where

$$(\vec{h}_{\overline{\mu}}^1(\lambda))^{\top} = (\vec{B}_{\overline{\mu}}(e^{i\lambda}))^{\top}\frac{\chi_{\overline{\mu}}^{(d)}(e^{-i\lambda})}{\beta^{(d)}(i\lambda)} -$$

$$- \frac{\overline{\beta^{(d)}(i\lambda)}}{\chi_{\overline{\mu}}^{(d)}(e^{-i\lambda})}\left(\sum_{k=0}^{\infty}((P_{\overline{\mu}}^{-1}D^{\overline{\mu}}\mathbf{a})_k e^{ik\lambda}) \right)^{\top}(f(\lambda) + |\beta^{(d)}(i\lambda)|^2 g(\lambda))^{-1}, \tag{35}$$

$$(\vec{h}_{\overline{\mu}}^2(\lambda))^{\top} = \frac{\overline{\beta^{(d)}(i\lambda)}}{\chi_{\overline{\mu}}^{(d)}(e^{-i\lambda})}(\vec{A}_{\overline{\mu}}(e^{i\lambda}))^{\top}g(\lambda)(f(\lambda) + |\beta^{(d)}(i\lambda)|^2 g(\lambda))^{-1} -$$

$$- \frac{\overline{\beta^{(d)}(i\lambda)}}{\chi_{\overline{\mu}}^{(d)}(e^{-i\lambda})}\left(\sum_{k=0}^{\infty}(\mathbf{P}_{\overline{\mu}}^{-1}\mathbf{T}_{\overline{\mu}}\mathbf{a}_{\overline{\mu}})_k e^{ik\lambda} \right)^{\top}(f(\lambda) + |\beta^{(d)}(i\lambda)|^2 g(\lambda))^{-1}. \tag{36}$$

Here $\vec{h}_{\overline{\mu}}^1(\lambda)$ and $\vec{h}_{\overline{\mu}}^2(\lambda)$ are spectral characteristics of the optimal estimates $\widehat{B\vec{\zeta}}$ and $\widehat{A\vec{\eta}}$ of the functionals $B\vec{\zeta}$ and $A\vec{\eta}$ respectively based on observations $\vec{\xi}(m) + \vec{\eta}(m)$ at points $m = -1, -2, \dots$.

From Theorem 3.1 obtain the optimal estimate $\widehat{A_N\vec{\xi}}$ of the functional $A_N\vec{\xi}$ of the unknown values of elements $\vec{\xi}(m)$, $m = 0, 1, 2, \dots, N$, based on observations of the sequence $\vec{\xi}(m) + \vec{\eta}(m)$ at points $m = -1, -2, \dots$. Let

$\vec{a}(k) = 0$, $k > N$. Then the spectral characteristic $\vec{h}_{\overline{\mu},N}(\lambda)$ of the linear estimate

$$\widehat{A}_N \vec{\xi} = \int_{-\pi}^{\pi} (\vec{h}_{\overline{\mu},N}(\lambda))^\top d\vec{Z}_{\xi^{(n)}+\eta^{(n)}}(\lambda) - \sum_{k=-n(\gamma)}^{-1} (\vec{v}_{\overline{\mu},N}(k))^\top (\vec{\xi}(k) + \vec{\eta}(k)), \tag{37}$$

is calculated by the formula

$$(\vec{h}_{\overline{\mu},N}(\lambda))^\top = (\vec{B}_{\overline{\mu},N}(e^{i\lambda}))^\top \frac{\chi_{\overline{\mu}}^{(d)}(e^{-i\lambda})}{\beta^{(d)}(i\lambda)} - \frac{\overline{\beta^{(d)}(i\lambda)}}{\chi_{\overline{\mu}}^{(d)}(e^{-i\lambda})} (\vec{A}_{\overline{\mu},N}(e^{i\lambda}))^\top g(\lambda)(f(\lambda) + |\beta^{(d)}(i\lambda)|^2 g(\lambda))^{-1} -$$

$$- \frac{\overline{\beta^{(d)}(i\lambda)}}{\chi_{\overline{\mu}}^{(d)}(e^{-i\lambda})} \left(\sum_{k=0}^{\infty} (\mathbf{P}_{\overline{\mu}}^{-1} D_N^{\overline{\mu}} \mathbf{a}_N - \mathbf{P}_{\overline{\mu}}^{-1} \mathbf{T}_{\overline{\mu},N} \mathbf{a}_{\overline{\mu},N})_k e^{ik\lambda} \right)^\top (f(\lambda) + |\beta^{(d)}(i\lambda)|^2 g(\lambda))^{-1}, \tag{38}$$

where

$$\vec{B}_{\overline{\mu},N}(e^{i\lambda}) = \sum_{k=0}^{N} (D_N^{\overline{\mu}} \mathbf{a}_N)_k e^{i\lambda k}, \quad \vec{A}_N(e^{i\lambda}) = \sum_{k=0}^{N} \vec{a}(k) e^{i\lambda k}, \quad \vec{A}_{\overline{\mu},N}(e^{i\lambda}) = \sum_{k=0}^{N} \vec{a}_{\overline{\mu},N}(k) e^{i\lambda k},$$

$$\mathbf{a}_N = ((\vec{a}(0))^\top, (\vec{a}(1))^\top, \ldots, (\vec{a}(N))^\top, 0, \ldots)^\top,$$

$$\mathbf{a}_{\overline{\mu},N} = ((\vec{a}_{\overline{\mu},N}(0))^\top, (\vec{a}_{\overline{\mu},N}(1))^\top, \ldots, (\vec{a}_{\overline{\mu},N}(N+n(\gamma)))^\top, 0, \ldots)^\top,$$

$$\vec{a}_{\overline{\mu},N}(m) = \sum_{l=\max\{m-n(\gamma),0\}}^{\min\{m,N\}} e_\gamma(m-l)\vec{a}(l), \quad 0 \le m \le N+n(\gamma), \tag{39}$$

operator $\mathbf{T}_{\overline{\mu},N}$ is a linear operator in the space ℓ_2 defined by the matrix with the $T \times T$ matrix entries $(\mathbf{T}_{\overline{\mu},N})_{l,m} = T_{l,m}^{\overline{\mu}}$, $l \ge 0$, $0 \le m \le N+n(\gamma)$, and $(\mathbf{T}_{\overline{\mu},N})_{l,m} = 0$, $l \ge 0$, $m > N+n(\gamma)$.

The value of the mean square error of the estimate $\widehat{A}_N \vec{\xi}$ is calculated by the formula

$$\Delta(f, g; \widehat{A}_N \vec{\xi}) = \Delta(f, g; \widehat{H}_N \vec{\xi}) = \mathsf{E}|H_N \vec{\xi} - \widehat{H}_N \vec{\xi}|^2 =$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{|\beta^{(d)}(i\lambda)|^2}{|\chi_{\overline{\mu}}^{(d)}(e^{-i\lambda})|^2} \Bigg[(\vec{A}_{\overline{\mu},N}(e^{i\lambda}))^\top g(\lambda) +$$

$$+ \left(\sum_{k=0}^{\infty} (\mathbf{P}_{\overline{\mu}}^{-1} D_N^{\overline{\mu}} \mathbf{a}_N - \mathbf{P}_{\overline{\mu}}^{-1} \mathbf{T}_{\overline{\mu},N} \mathbf{a}_{\overline{\mu},N})_k e^{ik\lambda} \right)^\top \Bigg] \times$$

$$\times (f(\lambda) + |\beta^{(d)}(i\lambda)|^2 g(\lambda))^{-1} f(\lambda) (f(\lambda) + |\beta^{(d)}(i\lambda)|^2 g(\lambda))^{-1} \times$$

$$\times \Bigg[g(\lambda) \overline{\vec{A}_{\overline{\mu},N}(e^{i\lambda})} + \overline{\sum_{k=0}^{\infty} (\mathbf{P}_{\overline{\mu}}^{-1} D_N^{\overline{\mu}} \mathbf{a}_N - \mathbf{P}_{\overline{\mu}}^{-1} \mathbf{T}_{\overline{\mu},N} \mathbf{a}_{\overline{\mu},N})_k e^{ik\lambda}} \Bigg] d\lambda$$

$$+ \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{1}{|\chi_{\overline{\mu}}^{(d)}(e^{-i\lambda})|^2} \left[(\vec{A}_{\overline{\mu},N}(e^{i\lambda}))^\top f(\lambda) - \right.$$

$$- |\beta^{(d)}(i\lambda)|^2 \left(\sum_{k=0}^{\infty} (\mathbf{P}_{\overline{\mu}}^{-1} D_N^{\overline{\mu}} \mathbf{a}_N - \mathbf{P}_{\overline{\mu}}^{-1} \mathbf{T}_{\overline{\mu},N} \mathbf{a}_{\overline{\mu},N})_k e^{ik\lambda} \right)^\top \right] \times$$

$$\times (f(\lambda) + |\beta^{(d)}(i\lambda)|^2 g(\lambda))^{-1} g(\lambda) (f(\lambda) + |\beta^{(d)}(i\lambda)|^2 g(\lambda))^{-1} \times$$

$$\times \left[f(\lambda) \overline{\vec{A}_N(e^{i\lambda})} - \right.$$

$$\left. - |\beta^{(d)}(i\lambda)|^2 \overline{\sum_{k=0}^{\infty} (\mathbf{P}_{\overline{\mu}}^{-1} D_N^{\overline{\mu}} \mathbf{a}_N - \mathbf{P}_{\overline{\mu}}^{-1} \mathbf{T}_{\overline{\mu},N} \mathbf{a}_{\overline{\mu},N})_k e^{ik\lambda}} \right] d\lambda =$$

$$= \langle D_N^{\overline{\mu}} \mathbf{a}_N - \mathbf{T}_{\overline{\mu},N} \mathbf{a}_{\overline{\mu},N}, \mathbf{P}_{\overline{\mu}}^{-1} D_N^{\overline{\mu}} \mathbf{a}_N - \mathbf{P}_{\overline{\mu}}^{-1} \mathbf{T}_{\overline{\mu},N} \mathbf{a}_{\overline{\mu},N} \rangle + \langle \mathbf{Q}_N \mathbf{a}_N, \mathbf{a}_N \rangle, \quad (40)$$

where $\mathbf{Q}_N$ is a linear operator in the space ℓ_2 defined by the matrix with the $T \times T$ matrix elements $(\mathbf{Q}_N)_{l,k} = Q_{l,k}$, $0 \le l, k \le N$, and $(\mathbf{Q}_N)_{l,k} = 0$ otherwise.

The following theorem holds true.

Theorem 3.2
Let $\vec{\xi}(m), m \in \mathbb{Z}\}$, be a stochastic sequence which defines stationary nth increment sequence $\chi_{\overline{\mu},s}^{(d)}(\vec{\xi}(m))$ with an absolutely continuous spectral function $F(\lambda)$ which has spectral density $f(\lambda)$. Let $\vec{\eta}(m), m \in \mathbb{Z}$, be an uncorrelated with the sequence $\vec{\xi}(m)$ stationary stochastic sequence with an absolutely continuous spectral function $G(\lambda)$ which has spectral density $g(\lambda)$. Let the minimality condition (18) be satisfied. The optimal linear estimate $\widehat{A_N \vec{\xi}}$ of the functional $A_N \vec{\xi}$ which depend on the unknown values of elements $\vec{\xi}(k), k = 0, 1, 2, \ldots, N$, from observations of the sequence $\vec{\xi}(m) + \vec{\eta}(m)$ at points $m = -1, -2, \ldots$ is calculated by formula (37). The spectral characteristic $\vec{h}_{\overline{\mu},N}(\lambda)$ of the optimal estimate $\widehat{A_N \vec{\xi}}$ is calculated by formula (38). The value of the mean-square error $\Delta(f, g; \widehat{A_N \vec{\xi}})$ is calculated by formula (40).

As a corollary from the proposed theorem, one can obtain the mean square optimal estimate of the unobserved value $A_{N,p} \vec{\xi} = \xi_p(N) = (\vec{\xi}(N))^\top \boldsymbol{\delta}_p$, $p = 1, 2, \ldots, T$, $N \ge 0$ of the stochastic sequence with nth stationary increments based on observations of the sequence $\vec{\xi}(m) + \vec{\eta}(m)$ at points $m = -1, -2, \ldots$

Corollary 3.3
The optimal linear estimate $\widehat{\xi}_p(N)$ of the unobserved value $\xi_p(N)$, $p = 1, 2, \ldots, T$, $N \ge 0$, of the stochastic sequence with stationary GM increments from observations of the sequence $\vec{\xi}(m) + \vec{\eta}(m)$ at points $m = -1, -2, \ldots$ is calculated by formula

$$\widehat{\xi}_p(N) = \int_{-\pi}^{\pi} (\vec{h}_{\overline{\mu},N,p}(\lambda))^\top d\vec{Z}_{\xi^{(n)} + \eta^{(n)}}(\lambda) - \sum_{k=-n(\gamma)}^{-1} (\vec{v}_{\overline{\mu},N,p}(k))^\top (\vec{\xi}(k) + \vec{\eta}(k)). \quad (41)$$

The spectral characteristic $\vec{h}_{\overline{\mu},N,p}(\lambda)$ of the estimate is calculated by the formula

$$(\vec{h}_{\overline{\mu},N,p}(\lambda))^\top = \frac{\chi_{\overline{\mu}}^{(d)}(e^{-i\lambda})}{\beta^{(d)}(i\lambda)} \left(\boldsymbol{\delta}_p \sum_{k=0}^{N} d_{\overline{\mu}}(N-k) e^{i\lambda k} \right)^\top -$$

$$- (e^{i\lambda N} \boldsymbol{\delta}_p)^\top g(\lambda) \overline{\beta^{(d)}(i\lambda)} (f(\lambda) + |\beta^{(d)}(i\lambda)|^2 g(\lambda))^{-1}$$

$$- \frac{\overline{\beta^{(d)}(i\lambda)} \left(\sum_{k=0}^{\infty} (\mathbf{P}_{\overline{\mu}}^{-1} \mathbf{d}_{\overline{\mu},N,p} - \mathbf{P}_{\overline{\mu}}^{-1} \mathbf{T}_{\overline{\mu},N} \vec{\mathbf{a}}_{\overline{\mu},N,p})_k e^{i\lambda k} \right)^\top}{\chi_{\overline{\mu}}^{(d)}(e^{-i\lambda})} (f(\lambda) + |\beta^{(d)}(i\lambda)|^2 g(\lambda))^{-1}. \quad (42)$$

where

$$
\begin{aligned}
\mathbf{d}_{\overline{\mu},N,p} &= ((d_{\overline{\mu}}(N)\boldsymbol{\delta}_p)^\top, (d_{\overline{\mu}}(N-1)\boldsymbol{\delta}_p)^\top, (d_{\overline{\mu}}(N-2)\boldsymbol{\delta}_p)^\top, \ldots, (d_{\overline{\mu}}(0)\boldsymbol{\delta}_p)^\top, 0, \ldots)^\top, \\
\vec{\mathbf{a}}_{\overline{\mu},N,p} &= (0,\ldots,0,(\vec{a}_{\overline{\mu},N,p}(N))^\top, (\vec{a}_{\overline{\mu},N,p}(N+1))^\top, \ldots, (\vec{a}_{\overline{\mu},N,p}(N+n(\gamma)))^\top, 0, \ldots)^\top, \\
\vec{a}_{\overline{\mu},N,p}(m) &= e_\gamma(m-N)\vec{\delta}_p, \quad N \le m \le N + n(\gamma).
\end{aligned}
$$

The value of the mean square error of the optimal estimate is calculated by the formula

$$
\Delta(f,g;\widehat{\widehat{\xi}}_p(N)) = \mathsf{E}|\xi_p(N) - \widehat{\widehat{\xi}}_p(N)|^2 =
$$

$$
= \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{|\beta^{(d)}(i\lambda)|^2}{|\chi_{\overline{\mu}}^{(d)}(e^{-i\lambda})|^2} \left[\overline{\chi_{\overline{\mu}}^{(d)}(e^{-i\lambda})}\left(e^{i\lambda N}\boldsymbol{\delta}_p\right)^\top g(\lambda) + \right.
$$

$$
+ \left.\left(\sum_{k=0}^{\infty}(\mathbf{P}_{\overline{\mu}}^{-1}\mathbf{d}_{\overline{\mu},N,p} - \mathbf{P}_{\overline{\mu}}^{-1}\mathbf{T}_{\overline{\mu},N}\vec{\mathbf{a}}_{\overline{\mu},N,p})_k e^{i\lambda k}\right)^\top \right] \times
$$

$$
\times (f(\lambda) + |\beta^{(d)}(i\lambda)|^2 g(\lambda))^{-1} f(\lambda) (f(\lambda) + |\beta^{(d)}(i\lambda)|^2 g(\lambda))^{-1} \times
$$

$$
\times \left[g(\lambda)\chi_{\overline{\mu}}^{(d)}(e^{-i\lambda})\left(e^{-i\lambda N}\boldsymbol{\delta}_p\right) + \overline{\sum_{k=0}^{\infty}(\mathbf{P}_{\overline{\mu}}^{-1}\mathbf{d}_{\overline{\mu},N,p} - \mathbf{P}_{\overline{\mu}}^{-1}\mathbf{T}_{\overline{\mu},N}\vec{\mathbf{a}}_{\overline{\mu},N,p})_k e^{i\lambda k}} \right] d\lambda +
$$

$$
+ \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{1}{|\chi_{\overline{\mu}}^{(d)}(e^{-i\lambda})|^2} \left[\overline{\chi_{\overline{\mu}}^{(d)}(e^{-i\lambda})}\left(e^{i\lambda N}\boldsymbol{\delta}_p\right)^\top f(\lambda) - \right.
$$

$$
- \left.|\beta^{(d)}(i\lambda)|^2 \left(\sum_{k=0}^{\infty}(\mathbf{P}_{\overline{\mu}}^{-1}\mathbf{d}_{\overline{\mu},N,p} - \mathbf{P}_{\overline{\mu}}^{-1}\mathbf{T}_{\overline{\mu},N}\vec{\mathbf{a}}_{\overline{\mu},N,p})_k e^{i\lambda k}\right)^\top \right] \times
$$

$$
\times (f(\lambda) + |\beta^{(d)}(i\lambda)|^2 g(\lambda))^{-1} g(\lambda) (f(\lambda) + |\beta^{(d)}(i\lambda)|^2 g(\lambda))^{-1} \times
$$

$$
\times \left[f(\lambda)\chi_{\overline{\mu}}^{(d)}(e^{-i\lambda})\left(e^{-i\lambda N}\boldsymbol{\delta}_p\right) - |\beta^{(d)}(i\lambda)|^2\overline{\sum_{k=0}^{\infty}(\mathbf{P}_{\overline{\mu}}^{-1}\mathbf{d}_{\overline{\mu},N,p} - \mathbf{P}_{\overline{\mu}}^{-1}\mathbf{T}_{\overline{\mu},N}\vec{\mathbf{a}}_{\overline{\mu},N,p})_k e^{i\lambda k}} \right] d\lambda =
$$

$$
= \langle \mathbf{d}_{\overline{\mu},N,p} - \mathbf{T}_{\overline{\mu},N}\vec{\mathbf{a}}_{\overline{\mu},N,p}, \mathbf{P}_{\overline{\mu}}^{-1}\mathbf{d}_{\overline{\mu},N,p} - \mathbf{P}_{\overline{\mu}}^{-1}\mathbf{T}_{\overline{\mu},N}\vec{\mathbf{a}}_{\overline{\mu},N,p} \rangle + \langle \mathbf{Q}_{0,0}\boldsymbol{\delta}_p, \boldsymbol{\delta}_p \rangle. \tag{43}
$$

Remark 3.2
The filtering problem in the presence of fractional integration can be solved using Theorem 3.1, Theorem 3.2 and Corollary 3.3 under conditions of Theorem 2.4 on the increment orders d_i.

3.2. *Forecasting based on factorizations of the spectral densities*

In Theorem 3.1, Theorem 3.2 and Corollary 3.3, formulas for finding forecasts of the linear functionals $A\xi$, $A_N\xi$ and the value $\xi(p)$, $p \ge 0$, are derived using the Fourier coefficients of the functions

$$
\frac{|\beta^{(d)}(i\lambda)|^2}{|\chi_{\overline{\mu}}^{(d)}(e^{-i\lambda})|^2} \left[g(\lambda)(f(\lambda) + |\beta^{(d)}(i\lambda)|^2 g(\lambda))^{-1} \right]^\top,
$$

$$
\frac{|\beta^{(d)}(i\lambda)|^2}{|\chi_{\overline{\mu}}^{(d)}(e^{-i\lambda})|^2} \left[(f(\lambda) + |\beta^{(d)}(i\lambda)|^2 g(\lambda))^{-1} \right]^\top.
$$

Assume that the following canonical factorizations take place

$$\frac{|\chi_{\overline{\mu}}^{(d)}(e^{-i\lambda})|^2}{|\beta^{(d)}(i\lambda)|^2}(f(\lambda) + |\beta^{(d)}(i\lambda)|^2 g(\lambda)) = \Theta_{\overline{\mu}}(e^{-i\lambda})\Theta_{\overline{\mu}}^*(e^{-i\lambda}),$$

$$\Theta_{\overline{\mu}}(e^{-i\lambda}) = \sum_{k=0}^{\infty} \theta_{\overline{\mu}}(k)e^{-i\lambda k}, \quad (44)$$

$$g(\lambda) = \sum_{k=-\infty}^{\infty} g(k)e^{i\lambda k} = \Phi(e^{-i\lambda})\Phi^*(e^{-i\lambda}), \quad \Phi(e^{-i\lambda}) = \sum_{k=0}^{\infty} \phi(k)e^{-i\lambda k}. \quad (45)$$

Define the matrix-valued function $\Psi_{\overline{\mu}}(e^{-i\lambda}) = \{\Psi_{ij}(e^{-i\lambda})\}_{i=\overline{1,q}}^{j=\overline{1,T}}$ by the equation

$$\Psi_{\overline{\mu}}(e^{-i\lambda})\Theta_{\overline{\mu}}(e^{-i\lambda}) = E_q,$$

where E_q is an identity $q \times q$ matrix. One can check that the following factorization takes place

$$\frac{|\beta^{(d)}(i\lambda)|^2}{|\chi_{\overline{\mu}}^{(d)}(e^{-i\lambda})|^2}(f(\lambda) + |\beta^{(d)}(i\lambda)|^2 g(\lambda))^{-1} = \Psi_{\overline{\mu}}^*(e^{-i\lambda})\Psi_{\overline{\mu}}(e^{-i\lambda}),$$

$$\Psi_{\overline{\mu}}(e^{-i\lambda}) = \sum_{k=0}^{\infty} \psi_{\overline{\mu}}(k)e^{-i\lambda k}, \quad (46)$$

Remark 3.3
Any spectral density matrix $f(\lambda)$ is self-adjoint: $f(\lambda) = f^*(\lambda)$. Thus, $(f(\lambda))^\top = \overline{f(\lambda)}$. One can check that an inverse spectral density $f^{-1}(\lambda)$ is also self-adjoint $f^{-1}(\lambda) = (f^{-1}(\lambda))^*$ and $(f^{-1}(\lambda))^\top = \overline{f^{-1}(\lambda)}$.

The following Lemmas provide factorizations of the operators $\mathbf{P}_{\overline{\mu}}$ and $\mathbf{T}_{\overline{\mu}}$, which contain coefficients of factorizations (44) – (46).

Lemma 3.2
Let factorization (44) takes place and let $q \times T$ matrix function $\Psi_{\overline{\mu}}(e^{-i\lambda})$ satisfy equation $\Psi_{\overline{\mu}}(e^{-i\lambda})\Theta_{\overline{\mu}}(e^{-i\lambda}) = E_q$
Define the linear operators $\Psi_{\overline{\mu}}$ and $\Theta_{\overline{\mu}}$ in the space ℓ_2 by the matrices with the matrix entries $(\Psi_{\overline{\mu}})_{k,j} = \psi_{\overline{\mu}}(k - j)$, $(\Theta_{\overline{\mu}})_{k,j} = \theta_{\overline{\mu}}(k - j)$ for $0 \leq j \leq k$, $(\Psi_{\overline{\mu}})_{k,j} = 0$, $(\Theta_{\overline{\mu}})_{k,j} = 0$ for $0 \leq k < j$. Then:
a) the linear operator $\mathbf{P}_{\overline{\mu}}$ admits the factorization

$$\mathbf{P}_{\overline{\mu}} = (\Psi_{\overline{\mu}})^\top \overline{\Psi}_{\overline{\mu}};$$

b) the inverse operator $(\mathbf{P}_{\overline{\mu}})^{-1}$ admits the factorization

$$(\mathbf{P}_{\overline{\mu}})^{-1} = \overline{\Theta}_{\overline{\mu}}(\Theta_{\overline{\mu}})^\top.$$

Proof. Making use of factorization (46), obtain the relation

$$\frac{|\beta^{(d)}(i\lambda)|^2}{|\chi_{\overline{\mu}}^{(d)}(e^{-i\lambda})|^2}\left[(f(\lambda) + |\beta^{(d)}(i\lambda)|^2 g(\lambda))^{-1}\right]^\top$$

$$= \sum_{m=-\infty}^{\infty} P_{\overline{\mu}}(m)e^{i\lambda m} = (\Psi_{\overline{\mu}}(e^{-i\lambda}))^\top \overline{\Psi_{\overline{\mu}}(e^{-i\lambda})}$$

$$= \sum_{m=-\infty}^{-1} \sum_{k=-m}^{\infty} \psi_{\overline{\mu}}^\top(k)\overline{\psi}_{\overline{\mu}}(k+m)e^{i\lambda m} \sum_{m=0}^{\infty} \sum_{k=0}^{\infty} \psi_{\overline{\mu}}^\top(k)\overline{\psi}_{\overline{\mu}}(k+m)e^{i\lambda m}.$$

Thus, $P_{\overline{\mu}}(m) = \sum_{k=0}^{\infty} \psi_{\overline{\mu}}^{\top}(k)\overline{\psi}_{\overline{\mu}}(k+m)$, $m \geq 0$, and $P_{\overline{\mu}}(-m) = (P_{\overline{\mu}}(m))^*$, $m \geq 0$. In the case $i \geq j$, we have

$$P_{i,j}^{\mu} = P_{\overline{\mu}}(i-j) = \sum_{l=i}^{\infty} \psi_{\overline{\mu}}^{\top}(l-i)\overline{\psi}_{\overline{\mu}}(l-j) = ((\Psi_{\overline{\mu}})^{\top}\overline{\Psi}_{\overline{\mu}})_{i,j},$$

and, in the case $i < j$, we have

$$P_{i,j}^{\mu} = P_{\overline{\mu}}(i-j) = \overline{P_{\overline{\mu}}(j-i)} = \sum_{l=j}^{\infty} \psi_{\overline{\mu}}^{\top}(l-i)\overline{\psi}_{\overline{\mu}}(l-j) = ((\Psi_{\overline{\mu}})^{\top}\overline{\Psi}_{\overline{\mu}})_{i,j},$$

which proves statement a).

From factorizations (44) and (46), obtain

$$E_q = \Psi_{\overline{\mu}}(e^{-i\lambda})\Theta_{\overline{\mu}}(e^{-i\lambda}) = \sum_{j=0}^{\infty} \left(\sum_{k=0}^{j} \psi_{\overline{\mu}}(k)\theta_{\overline{\mu}}(j-k) \right) e^{-i\lambda j}. \tag{47}$$

Thus, we derive the relations

$$\mathsf{diag}(\delta_{i,j}) = \sum_{k=0}^{i-j} \psi_{\overline{\mu}}(k)\theta_{\overline{\mu}}(i-j-k) = \sum_{p=i}^{j} \psi_{\overline{\mu}}(i-p)\theta_{\overline{\mu}}(p-j) = (\Psi_{\overline{\mu}}\Theta_{\overline{\mu}})_{i,j}, \quad \square$$

which imply $\Psi_{\overline{\mu}}\Theta_{\overline{\mu}} = E_q$. Using $\mathbf{P}_{\overline{\mu}} = (\Psi_{\overline{\mu}})^{\top}\overline{\Psi}_{\overline{\mu}}$ we get $(\Theta_{\overline{\mu}})^{\top}\mathbf{P}_{\overline{\mu}} = \overline{\Psi}_{\overline{\mu}}$ and $\mathbf{P}_{\overline{\mu}}\overline{\Theta}_{\overline{\mu}} = (\Psi_{\overline{\mu}})^{\top}$, or $(\Theta_{\overline{\mu}})^{\top} = \overline{\Psi}_{\overline{\mu}}(\mathbf{P}_{\overline{\mu}})^{-1}$ and $\overline{\Theta}_{\overline{\mu}} = (\mathbf{P}_{\overline{\mu}})^{-1}(\Psi_{\overline{\mu}})^{\top}$. The last two relations imply

$$\overline{\Theta}_{\overline{\mu}}(\Theta_{\overline{\mu}})^{\top} = (\mathbf{P}_{\overline{\mu}})^{-1}(\Psi_{\overline{\mu}})^{\top}\overline{\Psi}_{\overline{\mu}}(\mathbf{P}_{\overline{\mu}})^{-1} = (\mathbf{P}_{\overline{\mu}})^{-1}\mathbf{P}_{\overline{\mu}}(\mathbf{P}_{\overline{\mu}})^{-1} = (\mathbf{P}_{\overline{\mu}})^{-1},$$

which proves statement b). $\square$

Lemma 3.3
Let factorizations (44) and (45) take place. Then the operator $\mathbf{T}_{\overline{\mu}}$ admits the representation

$$\mathbf{T}_{\overline{\mu}} = (\Psi_{\overline{\mu}})^{\top}\mathbf{Z}_{\overline{\mu}},$$

where $\mathbf{Z}_{\overline{\mu}}$ is a linear operator in the space ℓ_2 defined by a matrix with the entries

$$(\mathbf{Z}_{\overline{\mu}})_{k,j} = \sum_{l=j}^{\infty} \overline{\psi}_{\overline{\mu}}(l-j)\overline{g}(l-k), \quad g(k) = \sum_{m=\max\{0,-k\}}^{\infty} \phi(m)\phi^*(k+m), \quad k,j \geq 0.$$

Proof. Factorizations (45), (46) and Remark 3.3 imply

$$\frac{|\beta^{(d)}(i\lambda)|^2}{|\chi_{\overline{\mu}}^{(d)}(e^{-i\lambda})|^2} \left[g(\lambda)(f(\lambda) + |\beta^{(d)}(i\lambda)|^2 g(\lambda))^{-1} \right]^{\top} = (\Psi_{\overline{\mu}}(e^{-i\lambda}))^{\top}\overline{\Psi}_{\overline{\mu}}(e^{-i\lambda})\overline{g}(\lambda)$$

$$= \sum_{l=0}^{\infty} \Psi_{\overline{\mu}}^{\top}(l)e^{-i\lambda l} \sum_{j=0}^{\infty} Z_{\overline{\mu}}(j)e^{i\lambda j} = \sum_{k\in\mathbb{Z}} \sum_{j=0}^{\infty} \Psi_{\overline{\mu}}^{\top}(l)Z_{\overline{\mu}}(l+k)e^{i\lambda k}.$$

Then

$$(\mathbf{T}_{\overline{\mu}})_{k,j} = \mathbf{T}_{\overline{\mu}}(k-j) = \sum_{m=k}^{\infty} \Psi_{\overline{\mu}}^{\top}(m-k)Z_{\overline{\mu}}(m-j) = (\Psi_{\overline{\mu}}^{\top}\mathbf{Z}_{\overline{\mu}})_{k,j}$$

The representation for the entries $(\mathbf{Z}_{\overline{\mu}})_{k,j} = Z_{\overline{\mu}}(k-j)$ follows from

$$\sum_{j=0}^{\infty} Z_{\overline{\mu}}(j)e^{i\lambda j} = \overline{\Psi}_{\overline{\mu}}(e^{-i\lambda})\overline{g}(\lambda) = \sum_{k\in\mathbb{Z}} \sum_{l=0}^{\infty} \overline{\Psi}_{\overline{\mu}}(l)\overline{g}_{\overline{\mu}}(l-k)e^{i\lambda k}. \quad \square$$

Remark 3.4
Lemma 3.2 and Lemma 3.3 imply the factorization

$$(\mathbf{P}_{\overline{\mu}})^{-1}\mathbf{T}_{\overline{\mu}}\mathbf{a}_{\overline{\mu}} = \overline{\Theta}_{\overline{\mu}}(\Theta_{\overline{\mu}})^{\top}(\Psi_{\overline{\mu}})^{\top}\mathbf{Z}_{\overline{\mu}}\mathbf{a}_{\overline{\mu}} = \overline{\Theta}_{\overline{\mu}}\mathbf{Z}_{\overline{\mu}}\mathbf{a}_{\overline{\mu}} = \overline{\Theta}_{\overline{\mu}}\mathbf{e}_{\overline{\mu}},$$

where $\mathbf{e}_{\overline{\mu}} := \mathbf{Z}_{\overline{\mu}}\mathbf{a}_{\overline{\mu}}$.

Assuming that factorizations (44), (45), (46) take place and making use of Remark 3.4, spectral characteristic (33) and mean-squarer error (34) can be presented in terms of the coefficients of the mentioned factorizations. Make the following transformations:

$$
\begin{aligned}
&\frac{|\beta^{(d)}(i\lambda)|^2}{|\chi_{\overline{\mu}}^{(d)}(e^{-i\lambda})|^2}\left[(f(\lambda)+|\beta^{(d)}(i\lambda)|^2 g(\lambda))^{-1}\right]^{\top}\left(\sum_{k=0}^{\infty}(\mathbf{P}_{\overline{\mu}}^{-1}\mathbf{T}_{\overline{\mu}}\mathbf{a}_{\overline{\mu}})_k e^{ik\lambda}\right)\\
&= \left(\sum_{k=0}^{\infty}\psi_{\overline{\mu}}^{\top}(k)e^{-i\lambda k}\right)\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\overline{\psi}_{\overline{\mu}}(j)(\overline{\Theta}_{\overline{\mu}}\mathbf{e}_{\overline{\mu}})_k e^{i\lambda(k+j)}\\
&= \left(\sum_{k=0}^{\infty}\psi_{\overline{\mu}}^{\top}(k)e^{-i\lambda k}\right)\sum_{m=0}^{\infty}\sum_{p=0}^{m}\sum_{k=p}^{m}\overline{\psi}_{\overline{\mu}}(m-k)\overline{\theta}_{\overline{\mu}}(k-p)e_{\overline{\mu}}(p)e^{i\lambda m}\\
&= \left(\sum_{k=0}^{\infty}\psi_{\overline{\mu}}^{\top}(k)e^{-i\lambda k}\right)\sum_{m=0}^{\infty}\sum_{p=0}^{\infty}\mathsf{diag}(\delta_{m,p})e_{\overline{\mu}}(p)e^{i\lambda m}\\
&= \left(\sum_{k=0}^{\infty}\psi_{\overline{\mu}}^{\top}(k)e^{-i\lambda k}\right)\sum_{m=0}^{\infty}e_{\overline{\mu}}(m)e^{i\lambda m},
\end{aligned}
\tag{48}
$$

where $e_{\overline{\mu}}(m) = (\mathbf{Z}_{\overline{\mu}}\mathbf{a}_{\overline{\mu}})_m$, $m \geq 0$, is the m-th vector entry of the vector $\mathbf{e}_{\overline{\mu}} = \mathbf{Z}_{\overline{\mu}}\mathbf{a}_{\overline{\mu}}$.
Using factorizations (45) and (46) conclude the following transformations:

$$
\begin{aligned}
&\frac{|\beta^{(d)}(i\lambda)|^2}{\chi_{\overline{\mu}}^{(d)}(e^{-i\lambda})|^2}\left[(f(\lambda)+|\beta^{(d)}(i\lambda)|^2 g(\lambda))^{-1}\right]^{\top}(g(\lambda))^{\top}\vec{A}_{\overline{\mu}}(e^{i\lambda})\\
&= \left(\sum_{k=0}^{\infty}\psi_{\overline{\mu}}^{\top}(k)e^{-i\lambda k}\right)\left(\sum_{k=-\infty}^{\infty}Z_{\overline{\mu}}(k)e^{i\lambda k}\right)\sum_{j=0}^{\infty}a_{\overline{\mu}}(j)e^{i\lambda j}\\
&= \left(\sum_{k=0}^{\infty}\psi_{\overline{\mu}}^{\top}(k)e^{-i\lambda k}\right)\sum_{m=-\infty}^{\infty}\sum_{j=0}^{\infty}Z_{\overline{\mu}}(m-j)a_{\overline{\mu}}(j)e^{i\lambda m}\\
&= \left(\sum_{k=0}^{\infty}\psi_{\overline{\mu}}^{\top}(k)e^{-i\lambda k}\right)\sum_{m=-\infty}^{\infty}e_{\overline{\mu}}(m)e^{i\lambda m}.
\end{aligned}
\tag{49}
$$

Making use of (48) and (49), formula (36) for the spectral characteristic $\overrightarrow{h}\,\frac{2}{\mu}(\lambda)$ of the optimal estimate $\widehat{\overrightarrow{A}}\eta$ can be presented as

$$
\begin{aligned}
\vec{h}_{\overline{\mu}}^{2}(\lambda) &= \frac{\chi_{\overline{\mu}}^{(d)}(e^{-i\lambda})}{\beta^{(d)}(i\lambda)} \left(\sum_{k=0}^{\infty} \psi_{\overline{\mu}}^{\top}(k) e^{-i\lambda k} \right) \sum_{m=1}^{\infty} e_{\overline{\mu}}(-m) e^{-i\lambda m} \\
&= \frac{\chi_{\overline{\mu}}^{(d)}(e^{-i\lambda})}{\beta^{(d)}(i\lambda)} \left(\sum_{k=0}^{\infty} \psi_{\overline{\mu}}^{\top}(k) e^{-i\lambda k} \right) \sum_{m=1}^{\infty} \sum_{j=0}^{\infty} \sum_{l=0}^{\infty} \overline{\psi}_{\overline{\mu}}(l) \overline{g}(m+j+l) a_{\overline{\mu}}(j) e^{-i\lambda m} \\
&= \frac{\chi_{\overline{\mu}}^{(d)}(e^{-i\lambda})}{\beta^{(d)}(i\lambda)} \left(\sum_{k=0}^{\infty} \psi_{\overline{\mu}}^{\top}(k) e^{-i\lambda k} \right) \sum_{m=1}^{\infty} (\overline{\psi}_{\overline{\mu}} \mathbf{C}_{\mu,g})_m e^{-i\lambda m} \\
&= \frac{\chi_{\overline{\mu}}^{(d)}(e^{-i\lambda})}{\beta^{(d)}(i\lambda)} \Psi_{\overline{\mu}}^{\top}(e^{-i\lambda}) C_{\mu,g}(e^{-i\lambda}),
\end{aligned}
$$

where $\overline{\psi}_{\overline{\mu}} = (\overline{\psi}_{\overline{\mu}}(0), \overline{\psi}_{\overline{\mu}}(1), \overline{\psi}_{\overline{\mu}}(2), \dots)$,

$$
(\overline{\psi}_{\overline{\mu}} \mathbf{C}_{\mu,g})_m = \sum_{k=0}^{\infty} \overline{\psi}_{\overline{\mu}}(k) \mathbf{c}_{\mu,g}(k+m),
$$

$$
\mathbf{c}_{\mu,g}(m) = \sum_{k=0}^{\infty} \overline{g}(m+k) a_{\overline{\mu}}(k) = \sum_{l=0}^{\infty} \overline{\phi}(l) \sum_{k=0}^{\infty} \phi^{\top}(l+m+k) a_{\overline{\mu}}(k) = \sum_{l=0}^{\infty} \overline{\phi}(l) (\widetilde{\Phi} \mathbf{a}_{\overline{\mu}})_{l+m},
$$

$$
(\widetilde{\Phi} \mathbf{a}_{\overline{\mu}})_m = \sum_{k=0}^{\infty} \phi^{\top}(m+k) a_{\overline{\mu}}(k).
$$

Making use of the inverse operator $(\mathbf{P}_{\overline{\mu}})^{-1}$ factorization and following the transformation steps of (48) obtain the formula for the spectral characteristic $\vec{h}_{\overline{\mu}}^{1}(\lambda)$ of the optimal estimate $\widehat{\overrightarrow{B}}\xi$:

$$
\begin{aligned}
\vec{h}_{\overline{\mu}}^{1}(\lambda) &= \frac{\chi_{\overline{\mu}}^{(d)}(e^{-i\lambda})}{\beta^{(d)}(i\lambda)} \left(\vec{B}_{\overline{\mu}}(e^{i\lambda}) - \frac{|\beta^{(d)}(i\lambda)|^2}{|\chi_{\overline{\mu}}^{(d)}(e^{-i\lambda})|^2} \left[(f(\lambda) + |\beta^{(d)}(i\lambda)|^2 g(\lambda))^{-1} \right]^{\top} \right. \\
&\qquad \times \left. \left(\sum_{k=0}^{\infty} ((\mathbf{P}_{\overline{\mu}}^{-1} D^{\overline{\mu}} \mathbf{a})_k e^{ik\lambda} \right) \right) \\
&= \frac{\chi_{\overline{\mu}}^{(d)}(e^{-i\lambda})}{\beta^{(d)}(i\lambda)} \left(\vec{B}_{\overline{\mu}}(e^{i\lambda}) - \left(\sum_{k=0}^{\infty} \psi_{\overline{\mu}}^{\top}(k) e^{-i\lambda k} \right) \sum_{m=0}^{\infty} \sum_{p=0}^{\infty} \theta_{\overline{\mu}}^{\top}(p) \vec{b}_{\overline{\mu}}(b+m) e^{i\lambda m} \right) \\
&= \frac{\chi_{\overline{\mu}}^{(d)}(e^{-i\lambda})}{\beta^{(d)}(i\lambda)} \left(\vec{B}_{\overline{\mu}}(e^{i\lambda}) - \Psi_{\overline{\mu}}^{\top}(e^{-i\lambda}) \vec{r}_{\overline{\mu}}(e^{i\lambda}) \right) \\
&= \frac{\chi_{\overline{\mu}}^{(d)}(e^{-i\lambda})}{\beta^{(d)}(i\lambda)} \Psi_{\overline{\mu}}^{\top}(e^{-i\lambda}) \vec{C}_{\mu,1}(e^{-i\lambda}),
\end{aligned}
$$

where

$$
\begin{aligned}
\vec{r}_{\overline{\mu}}(e^{i\lambda}) &= \sum_{m=0}^{\infty}(\theta_{\overline{\mu}}^{\top}D^{\mu}\mathbf{A})_m e^{i\lambda m} = \sum_{m=0}^{\infty}\sum_{p=0}^{\infty}\theta_{\overline{\mu}}^{\top}(p)\vec{b}_{\overline{\mu}}(b+m)e^{i\lambda m}, \\
\vec{C}_{\mu,1}(e^{-i\lambda}) &= \sum_{m=1}^{\infty}(\theta_{\overline{\mu}}^{\top}\widetilde{\mathbf{B}}_{\overline{\mu}})_m e^{-i\lambda m} = \sum_{m=1}^{\infty}\sum_{p=m}^{\infty}\theta_{\overline{\mu}}^{\top}(p)\vec{b}_{\overline{\mu}}(p-m)e^{-i\lambda m} \\
&= \sum_{m=1}^{\infty}\sum_{p=0}^{\infty}\theta_{\overline{\mu}}^{\top}(m+p)\vec{b}_{\overline{\mu}}(p)e^{-i\lambda m},
\end{aligned}
$$

vector $\theta_{\overline{\mu}}^{\top} = ((\theta_{\overline{\mu}}(0))^{\top},(\theta_{\overline{\mu}}(1))^{\top},(\theta_{\overline{\mu}}(2))^{\top},\ldots)$; $\mathbf{A}$ is a linear symmetric operator determined by the matrix with the vector entries $(\mathbf{A})_{k,j} = \vec{a}(k+j)$, $k,j \geq 0$; $\widetilde{\mathbf{B}}_{\overline{\mu}}$ is a linear operator, which is determined by a matrix with the vector entries $(\widetilde{\mathbf{B}}_{\overline{\mu}})_{k,j} = \vec{b}_{\overline{\mu}}(k-j)$ for $0 \leq j \leq k$, $(\widetilde{\mathbf{B}}_{\overline{\mu}})_{k,j} = 0$ for $0 \leq k < j$.

Then the spectral characteristic $\vec{h}_{\overline{\mu}}(\lambda)$ of the estimate $\widehat{A}\xi$ can be calculated by the formula

$$
\begin{aligned}
\vec{h}_{\overline{\mu}}(\lambda) &= \frac{\chi_{\overline{\mu}}^{(d)}(e^{-i\lambda})}{\beta^{(d)}(i\lambda)}\left(\sum_{k=0}^{\infty}\psi_{\overline{\mu}}^{\top}(k)e^{-i\lambda k}\right)\sum_{m=1}^{\infty}\left(\theta_{\overline{\mu}}^{\top}\widetilde{\mathbf{B}}_{\overline{\mu}} - \overline{\psi}_{\overline{\mu}}\mathbf{C}_{\mu,g}\right)_m e^{-i\lambda m} \\
&= \frac{\chi_{\overline{\mu}}^{(d)}(e^{-i\lambda})}{\beta^{(d)}(i\lambda)}\Psi_{\overline{\mu}}^{\top}(e^{-i\lambda})\left(\vec{C}_{\mu,1}(e^{-i\lambda}) - \vec{C}_{\mu,g}(e^{-i\lambda})\right) \\
&= \vec{B}_{\overline{\mu}}(e^{i\lambda})\frac{\chi_{\overline{\mu}}^{(d)}(e^{-i\lambda})}{\beta^{(d)}(i\lambda)} - \widetilde{h}_{\overline{\mu}}(\lambda), \quad (50)
\end{aligned}
$$

$$
\begin{aligned}
\widetilde{h}_{\overline{\mu}}(\lambda) &= \frac{\chi_{\overline{\mu}}^{(d)}(e^{-i\lambda})}{\beta^{(d)}(i\lambda)}\left(\sum_{k=0}^{\infty}\psi_{\overline{\mu}}^{\top}(k)e^{-i\lambda k}\right) \\
&\quad \times \left(\sum_{m=0}^{\infty}(\theta_{\overline{\mu}}^{\top}D^{\mu}\mathbf{A})_m e^{i\lambda m} + \sum_{m=1}^{\infty}(\overline{\psi}_{\overline{\mu}}\mathbf{C}_{\mu,g})_m e^{-i\lambda m}\right) \\
&= \frac{\chi_{\overline{\mu}}^{(d)}(e^{-i\lambda})}{\beta^{(d)}(i\lambda)}\Psi_{\overline{\mu}}^{\top}(e^{-i\lambda})\left(\vec{r}_{\overline{\mu}}(e^{i\lambda}) + \vec{C}_{\mu,g}(e^{-i\lambda})\right),
\end{aligned}
$$

The value of the mean square error of the estimate $\widehat{A}\xi$ is calculated by the formula

$$
\begin{aligned}
\Delta\left(f,g;\widehat{A}\xi\right) &= \Delta\left(f,g;\widehat{H}\xi\right) = \mathsf{E}\left|H\xi - \widehat{H}\xi\right|^2 \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi}(\vec{A}(e^{i\lambda}))^{\top}g(\lambda)\overline{\vec{A}(e^{i\lambda})}d\lambda \\
&\quad + \frac{1}{2\pi}\int_{-\pi}^{\pi}(\widetilde{\vec{h}}_{\overline{\mu}}(e^{i\lambda}))^{\top}(f(\lambda) + |\beta^{(d)}(i\lambda)|^2 g(\lambda))\overline{\widetilde{\vec{h}}_{\overline{\mu}}(e^{i\lambda})}d\lambda \\
&\quad - \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{\beta^{(d)}(i\lambda)}{\chi_{\overline{\mu}}^{(d)}(e^{-i\lambda})}(\widetilde{h}_{\overline{\mu}}(e^{i\lambda}))^{\top}g(\lambda)\overline{A_{\overline{\mu}}(e^{i\lambda})}d\lambda \\
&\quad - \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{\overline{\beta^{(d)}(i\lambda)}}{\chi_{\overline{\mu}}^{(d)}(e^{-i\lambda})}(A_{\overline{\mu}}(e^{i\lambda}))^{\top}g(\lambda)\overline{\widetilde{\vec{h}}_{\overline{\mu}}(e^{i\lambda})}d\lambda \\
&= \|\Phi^{\top}\mathbf{a}_{\overline{\mu}}\|^2 + \|\widetilde{\Phi}\mathbf{a}_{\overline{\mu}}\|_1^2 + \left\langle\theta_{\overline{\mu}}^{\top}D^{\mu}\mathbf{A} - \overline{\psi}_{\overline{\mu}}\mathbf{C}_{\mu,g},\theta_{\overline{\mu}}^{\top}D^{\mu}\mathbf{A}\right\rangle \\
&\quad - \left\langle\theta_{\overline{\mu}}^{\top}D^{\mu}\mathbf{A},\mathbf{Z}_{\overline{\mu}}\mathbf{a}_{\overline{\mu}}\right\rangle - \left\langle\mathbf{Z}_{\overline{\mu}}\mathbf{a}_{\overline{\mu}},\overline{\psi}_{\overline{\mu}}\mathbf{C}_{\mu,g}\right\rangle_1, \quad (51)
\end{aligned}
$$

where $\|\vec{x}\|_1^2 = \langle \vec{x}, \vec{x} \rangle_1$, $\langle \vec{x}, \vec{y} \rangle_1 = \sum_{k=1}^{\infty} (\vec{x}(k))^{\top} \overline{\vec{y}}(k)$ for the vectors $\vec{x} = ((\vec{x}(0))^{\top}, (\vec{x}(1))^{\top}, (\vec{x}(2))^{\top}, \ldots)^{\top}$, $\vec{y} = ((\vec{y}(0))^{\top}, (\vec{y}(1))^{\top}, (\vec{y}(2))^{\top}, \ldots)^{\top}$.

The obtained results are summarized in the form of the following theorem.

Theorem 3.3

Let $\vec{\xi}(m)$, $m \in \mathbb{Z}$, be a vector stochastic sequence which determines the stationary GM increment sequence $\chi_{\overline{\mu}, \overline{s}}^{(d)}(\vec{\xi}(m))$ and let $\vec{\eta}(m)$, $m \in \mathbb{Z}$, be vector stationary stochastic sequence, uncorrelated with $\vec{\xi}(m)$. Suppose that the coefficients $\vec{a}(k)$, $k \geq 0$, satisfy conditions (16) – (17) and the spectral densities $f(\lambda)$ and $g(\lambda)$ of the stochastic sequences $\vec{\xi}(m)$ and $\vec{\eta}(m)$ admit the canonical factorizations (44) – (46). Then the spectral characteristic $\vec{h}_{\overline{\mu}}(\lambda)$ and the value of the mean square error $\Delta(f, g; \widehat{A\vec{\xi}})$ of the optimal estimate $\widehat{A\vec{\xi}}$ of the functional $A\vec{\xi}$ based on observations of the sequence $\vec{\xi}(m) + \vec{\eta}(m)$ at points $m = -1, -2, \ldots$ can be calculated by formulas (50) and (51) respectively.

3.3. Forecasting of stochastic sequences with periodically stationary increment

Consider the problem of mean square optimal linear estimation of the functionals

$$A\vartheta = \sum_{k=0}^{\infty} a^{(\vartheta)}(k)\vartheta(k), \quad A_M\vartheta = \sum_{k=0}^{N} a^{(\vartheta)}(k)\vartheta(k) \tag{52}$$

which depend on unobserved values of the stochastic sequence $\vartheta(m)$ with periodically stationary increments. Estimates are based on observations of the sequence $\zeta(m) = \vartheta(m) + \eta(m)$ at points $m = -1, -2, \ldots$.

The functional $A\vartheta$ can be represented in the form

$$
\begin{aligned}
A\vartheta &= \sum_{k=0}^{\infty} a^{(\vartheta)}(k)\vartheta(k) = \sum_{m=0}^{\infty} \sum_{p=1}^{T} a^{(\vartheta)}(mT + p - 1)\vartheta(mT + p - 1) \\
&= \sum_{m=0}^{\infty} \sum_{p=1}^{T} a_p(m)\xi_p(m) = \sum_{m=0}^{\infty} (\vec{a}(m))^{\top} \vec{\xi}(m) = A\vec{\xi},
\end{aligned}
$$

where

$$\vec{\xi}(m) = (\xi_1(m), \xi_2(m), \ldots, \xi_T(m))^{\top},$$

$$\xi_p(m) = \vartheta(mT + p - 1); \ p = 1, 2, \ldots, T; \tag{53}$$

$$\vec{a}(m) = (a_1(m), a_2(m), \ldots, a_T(m))^{\top},$$

$$a_p(m) = a^{(\vartheta)}(mT + p - 1); \ p = 1, 2, \ldots, T. \tag{54}$$

Theorem 3.4

Let a stochastic sequence $\vartheta(k)$ with periodically stationary increments generate by formula (53) a vector-valued stochastic sequence $\vec{\xi}(m)$ which determine a stationary stochastic GM increment sequence $\chi_{\overline{\mu}, \overline{s}}^{(d)}(\vec{\xi}(m))$ with the spectral density matrix $f(\lambda)$. Let $\vec{\eta}(m)$, $m \in \mathbb{Z}$, $\vec{\eta}(m) = (\eta_1(m), \eta_2(m), \ldots, \eta_T(m))^{\top}$, $\eta_p(m) = \eta(mT + p - 1)$; $p = 1, 2, \ldots, T$, be uncorrelated with the sequence $\vec{\xi}(m)$ stationary stochastic sequence with an absolutely continuous spectral function $G(\lambda)$ which has spectral density matrix $g(\lambda)$. Let the minimality condition (18) be satisfied. Let coefficients $\vec{a}(k)$, $k \geqslant 0$ determined by formula (54) satisfy conditions (16) – (17). Then the optimal linear estimate $\widehat{A\vartheta}$ of the functional $A\vartheta$ based on observations of the sequence $\zeta(m) = \vartheta(m) + \eta(m)$ at points $m = -1, -2, \ldots$ is calculated by formula (27). The spectral characteristic $\vec{h}_{\overline{\mu}}(\lambda) = \{h_p(\lambda)\}_{p=1}^{T}$ and the value of the mean square error $\Delta(f; \widehat{A\zeta})$ of the optimal estimate $\widehat{A\zeta}$ are calculated by formulas (33) and (34) respectively.

The functional $A_M \vartheta$ can be represented in the form

$$
\begin{aligned}
A_M \vartheta &= \sum_{k=0}^{M} a^{(\vartheta)}(k)\vartheta(k) = \sum_{m=0}^{N} \sum_{p=1}^{T} a^{(\vartheta)}(mT+p-1)\vartheta(mT+p-1) \\
&= \sum_{m=0}^{N} \sum_{p=1}^{T} a_p(m)\xi_p(m) = \sum_{m=0}^{N} (\vec{a}(m))^{\top} \vec{\xi}(m) = A_N \vec{\xi},
\end{aligned}
$$

where $N = [\frac{M}{T}]$, the sequence $\vec{\xi}(m)$ is determined by formula (53),

$$
\begin{aligned}
(\vec{a}(m))^{\top} &= (a_1(m), a_2(m), \dots, a_T(m))^{\top}, \\
a_p(m) &= a^{\vartheta}(mT+p-1); \ 0 \le m \le N; 1 \le p \le T; mT+p-1 \le M; \\
a_p(N) &= 0; \quad M+1 \le NT+p-1 \le (N+1)T-1; 1 \le p \le T.
\end{aligned}
\tag{55}
$$

Making use of the introduced notations and statements of Theorem 3.2 we can claim that the following theorem holds true.

Theorem 3.5
Let a stochastic sequence $\vartheta(k)$ with periodically stationary increments generate by formula (53) a vector-valued stochastic sequence $\vec{\xi}(m)$ which determine a stationary stochastic GM increment sequence $\chi_{\vec{\mu},\vec{s}}^{(d)}(\vec{\xi}(m))$ with the spectral density matrix $f(\lambda)$. Let $\{\vec{\eta}(m), m \in \mathbb{Z}\}$, $\vec{\eta}(m) = (\eta_1(m), \eta_2(m), \dots, \eta_T(m))^{\top}$, $\eta_p(m) = \eta(mT+p-1)$; $p = 1, 2, \dots, T$, be uncorrelated with the sequence $\vec{\xi}(m)$ stationary stochastic sequence with an absolutely continuous spectral function $G(\lambda)$ which has spectral density matrix $g(\lambda)$. Let the minimality condition (18) be satisfied. Let coefficients $\vec{a}(k), k \geqslant 0$ be determined by formula (55). The optimal linear estimate $\widehat{A}_M \zeta$ of the functional $A_M \zeta = A_N \vec{\xi}$ based on observations of the sequence $\zeta(m) = \vartheta(m) + \eta(m)$ at points $m = -1, -2, \dots$ is calculated by formula (37). The spectral characteristic $\vec{h}_{\overline{\mu},N}(\lambda) = \{h_{\overline{\mu},N,p}(\lambda)\}_{p=1}^{T}$ and the value of the mean square error $\Delta(f; \widehat{A}_M \zeta)$ are calculated by formulas (38) and (40) respectively.

As a corollary from the proposed theorem, one can obtain the mean square optimal estimate of the unobserved value $\vartheta(M)$, $M \ge 0$ of a stochastic sequence $\vartheta(m)$ with periodically stationary increments based on observations of the sequence $\zeta(m) = \vartheta(m) + \eta(m)$ at points $m = -1, -2, \dots$ Making use of the notations $\vartheta(M) = \vartheta_p(N) = (\vec{\xi}(N))^{\top} \boldsymbol{\delta}_p$, $N = [\frac{M}{T}]$, $p = M + 1 - NT$, and the obtained results we can conclude that the following corollary holds true.

Corollary 3.4
Let a stochastic sequence $\vartheta(k)$ with periodically stationary increments generate by formula (53) a vector-valued stochastic sequence $\vec{\xi}(m)$ which determine the stationary stochastic GM increment sequence $\chi_{\vec{\mu},\vec{s}}^{(d)}(\vec{\xi}(m))$ with the spectral density matrix $f(\lambda)$. Let $\vec{\eta}(m), m \in \mathbb{Z}$, $\vec{\eta}(m) = (\eta_1(m), \eta_2(m), \dots, \eta_T(m))^{\top}$, $\eta_p(m) = \eta(mT+p-1)$; $p = 1, 2, \dots, T$, be uncorrelated with the sequence $\vec{\xi}(m)$ stationary stochastic sequence with an absolutely continuous spectral function $G(\lambda)$ which has spectral density matrix $g(\lambda)$. Let the minimality condition (18) be satisfied. The optimal linear estimate $\widehat{\vartheta}(M)$ of the unobserved value $\vartheta(M)$, $M \ge 0$ of a stochastic sequence $\vartheta(m)$ with periodically stationary increments based on observations of the sequence $\zeta(m) = \vartheta(m) + \eta(m)$ at points $m = -1, -2, \dots$ is calculated by formula (41). The spectral characteristic $\vec{h}_{\overline{\mu},N,p}(\lambda)$ of the estimate is calculated by the formula (42). The value of the mean square error of the optimal estimate is calculated by the formula (43).

3.4. Forecasting of one class of cointegrated vector stochastic sequences

Consider two seasonal vector stochastic sequences $\{\vec{\xi}(m), m \in \mathbb{Z}\}$ and $\{\vec{\zeta}(m), m \in \mathbb{Z}\}$ with absolutely continuous spectral functions $F(\lambda)$ and $P(\lambda)$ and spectral densities $f(\lambda)$ and $p(\lambda)$ respectively. Assume that both of them have the same order d and seasonal vector $\overline{s}$.

Definition 3.1
Within this subsection, a pair of seasonal vector stochastic sequences $\{\vec{\xi}(m), m \in \mathbb{Z}\}$ and $\{\vec{\zeta}(m), m \in \mathbb{Z}\}$ are called seasonally cointegrated if there exists a constant $\alpha \neq 0$ such that the linear combination sequence $\vec{\zeta}(m) - \alpha\vec{\xi}(m)$ is a stationary vector stochastic sequence.

Under the forecast of two seasonally cointegrated stochastic sequences we understand the mean-square optimal linear estimates of the functionals

$$A\vec{\xi} = \sum_{k=0}^{\infty} (\vec{a}(k))^{\top} \vec{\xi}(k), \quad A_N\vec{\xi} = \sum_{k=0}^{N} (\vec{a}(k))^{\top} \vec{\xi}(k),$$

which depend on the unobserved values of the stochastic sequence $\vec{\xi}(m)$ based on observations of the stochastic sequence $\vec{\zeta}(m)$ at points $m = -1, -2, \ldots$. Applying the results of Subsection 3, the forecasts can be found under an assumption that the vector sequences $\vec{\xi}(m)$ and $\vec{\zeta}(m) - \alpha\vec{\xi}(m)$ are uncorrelated.

Let the minimality condition holds true:

$$\int_{-\pi}^{\pi} \text{Tr}\left[\frac{|\beta^{(d)}(i\lambda)|^2}{|\chi_{\overline{\mu}}^{(d)}(e^{-i\lambda})|^2} p(\lambda)^{-1} \right] d\lambda < \infty. \tag{56}$$

Determine operators $\mathbf{P}_{\overline{\mu}}^{\alpha}$, $\mathbf{T}_{\overline{\mu}}^{\alpha}$, $\mathbf{Q}^{\alpha}$ with the help of the Fourier coefficients

$$P_{k,j}^{\overline{\mu},\alpha} = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-i\lambda(k-j)} \frac{|\beta^{(d)}(i\lambda)|^2}{|\chi_{\overline{\mu}}^{(d)}(e^{-i\lambda})|^2} \left[(p(\lambda))^{-1} \right]^{\top} d\lambda;$$

$$T_{k,j}^{\overline{\mu},\beta} = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-i\lambda(k-j)} \frac{1}{|\chi_{\overline{\mu}}^{(d)}(e^{-i\lambda})|^2} \left[(p(\lambda) - \alpha^2 f(\lambda)) (p(\lambda))^{-1} \right]^{\top} d\lambda;$$

$$Q_{k,j}^{;\alpha} = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-i\lambda(k-j)} \frac{1}{|\beta^{(d)}(i\lambda)|^2} \left[f(\lambda) (p(\lambda))^{-1} (p(\lambda) - \alpha^2 f(\lambda)) \right]^{\top} d\lambda.$$

of the functions

$$\frac{|\beta^{(d)}(i\lambda)|^2}{|\chi_{\overline{\mu}}^{(d)}(e^{-i\lambda})|^2} \left[(p(\lambda))^{-1} \right]^{\top}, \quad \frac{1}{|\chi_{\overline{\mu}}^{(d)}(e^{-i\lambda})|^2} \left[(p(\lambda) - \alpha^2 f(\lambda)) (p(\lambda))^{-1} \right]^{\top}, \tag{57}$$

$$\frac{1}{|\chi_{\overline{\mu}}^{(d)}(e^{-i\lambda})|^2} \left[f(\lambda) (p(\lambda))^{-1} (p(\lambda) - \alpha^2 f(\lambda)) \right]^{\top}$$

in the same way as we defined operators $\mathbf{P}_{\overline{\mu}}$, $\mathbf{T}_{\overline{\mu}}$, $\mathbf{Q}$. Theorem 3.1 implies that the spectral characteristic $h_{\overline{\mu}}^{\alpha}(\lambda)$ of the optimal estimate

$$\widehat{A\vec{\xi}} = \int_{-\pi}^{\pi} (h_{\overline{\mu}}^{\alpha}(\lambda))^{\top} d\vec{Z}_{\zeta^{(d)}}(\lambda) - \sum_{k=-n(\nu)}^{-1} (\vec{v}_{\overline{\mu}}(k))^{\top} \zeta(k), \tag{58}$$

of the functional $A\vec{\xi}$ is calculated by the formula

$$(\vec{h}_{\overline{\mu}}^{\alpha}(\lambda))^{\top} = (\vec{B}_{\overline{\mu}}(e^{i\lambda}))^{\top} \frac{\chi_{\overline{\mu}}^{(d)}(e^{-i\lambda})}{\beta^{(d)}(i\lambda)} -$$

$$- \frac{1}{\chi_{\overline{\mu}}^{(d)}(e^{-i\lambda})\beta^{(d)}(i\lambda)} (\vec{A}_{\overline{\mu}}(e^{i\lambda}))^{\top} (p(\lambda) - \alpha^2 f(\lambda))(p(\lambda))^{-1} -$$

$$- \frac{\overline{\beta^{(d)}(i\lambda)}}{\chi_{\overline{\mu}}^{(d)}(e^{-i\lambda})} \left(\sum_{k=0}^{\infty} ((\mathbf{P}_{\overline{\mu}}^{\alpha})^{-1} D^{\overline{\mu}}\mathbf{a} - (\mathbf{P}_{\overline{\mu}}^{\alpha})^{-1} \mathbf{T}_{\overline{\mu}}^{\alpha}\mathbf{a}_{\overline{\mu}})_k e^{ik\lambda} \right)^{\top} (p(\lambda))^{-1}. \tag{59}$$

The value of the mean square error of the estimate $\widehat{A}\vec{\xi}$ is calculated by the formula

$$
\Delta(f, g; \widehat{A}\vec{\xi}) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left[\frac{1}{|\beta^{(d)}(i\lambda)|^2} (\vec{A}_{\overline{\mu}}(e^{i\lambda}))^{\top} (p(\lambda) - \alpha^2 f(\lambda)) + \right.
$$

$$
+ \left(\sum_{k=0}^{\infty} ((\mathbf{P}_{\overline{\mu}}^{\alpha})^{-1} D^{\overline{\mu}} \mathbf{a} - (\mathbf{P}_{\overline{\mu}}^{\alpha})^{-1} \mathbf{T}_{\overline{\mu}}^{\alpha} \mathbf{a}_{\overline{\mu}})_k e^{ik\lambda} \right)^{\top} \Bigg]
$$

$$
\times \frac{|\beta^{(d)}(i\lambda)|^2}{|\chi_{\overline{\mu}}^{(d)}(e^{-i\lambda})|^2} (p(\lambda))^{-1} f(\lambda) (p(\lambda))^{-1} \times
$$

$$
\times \left[\frac{1}{|\beta^{(d)}(i\lambda)|^2} (p(\lambda) - \alpha^2 f(\lambda)) \overline{\vec{A}(e^{i\lambda})} + \overline{\sum_{k=0}^{\infty} ((\mathbf{P}_{\overline{\mu}}^{\alpha})^{-1} D^{\overline{\mu}} \mathbf{a} - (\mathbf{P}_{\overline{\mu}}^{\alpha})^{-1} \mathbf{T}_{\overline{\mu}}^{\alpha} \mathbf{a}_{\overline{\mu}})_k e^{ik\lambda}} \right] d\lambda +
$$

$$
+ \frac{1}{2\pi} \int_{-\pi}^{\pi} \left[(\vec{A}_{\overline{\mu}}(e^{i\lambda}))^{\top} f(\lambda) - |\beta^{(d)}(i\lambda)|^2 \left(\sum_{k=0}^{\infty} ((\mathbf{P}_{\overline{\mu}}^{\alpha})^{-1} D^{\overline{\mu}} \mathbf{a} - (\mathbf{P}_{\overline{\mu}}^{\alpha})^{-1} \mathbf{T}_{\overline{\mu}}^{\alpha} \mathbf{a}_{\overline{\mu}})_k e^{ik\lambda} \right)^{\top} \right]
$$

$$
\times \frac{1}{|\chi_{\overline{\mu}}^{(d)}(e^{-i\lambda})|^2} \frac{1}{|\beta^{(d)}(i\lambda)|^2} (p(\lambda))^{-1} (p(\lambda) - \alpha^2 f(\lambda)) (p(\lambda))^{-1} \times
$$

$$
\times \left[f(\lambda) \overline{\vec{A}_{\overline{\mu}}(e^{i\lambda})} - |\beta^{(d)}(i\lambda)|^2 \overline{\sum_{k=0}^{\infty} ((\mathbf{P}_{\overline{\mu}}^{\alpha})^{-1} D^{\overline{\mu}} \mathbf{a} - (\mathbf{P}_{\overline{\mu}}^{\alpha})^{-1} \mathbf{T}_{\overline{\mu}}^{\alpha} \mathbf{a}_{\overline{\mu}})_k e^{ik\lambda}} \right] d\lambda =
$$

$$
= \left\langle D^{\overline{\mu}} \mathbf{a} - \mathbf{T}_{\overline{\mu}}^{\alpha} \mathbf{a}_{\overline{\mu}}, (\mathbf{P}_{\overline{\mu}}^{\alpha})^{-1} D^{\overline{\mu}} \mathbf{a} - (\mathbf{P}_{\overline{\mu}}^{\alpha})^{-1} \mathbf{T}_{\overline{\mu}}^{\alpha} \mathbf{a}_{\overline{\mu}} \right\rangle + \left\langle \mathbf{Q}^{\alpha} \mathbf{a}, \mathbf{a} \right\rangle. \tag{60}
$$

Theorem 3.6
Let $\vec{\xi}(m)$, $m \in \mathbb{Z}$, and $\vec{\zeta}(m)$, $m \in \mathbb{Z}$ be seasonally cointegrated stochastic sequences with the spectral densities $f(\lambda)$ and $p(\lambda)$ respectively. Suppose that the spectral density $p(\lambda)$ satisfy the minimality condition (56) and the coefficients $\vec{a}(k)$, $k \geq 0$, satisfy conditions (16) – (17). If the stochastic sequences $\xi(m)$ and $\vec{\zeta}(m) - \alpha\vec{\xi}(m)$ are uncorrelated, then the spectral characteristic $\vec{h}_{\overline{\mu}}^{\alpha}(\lambda)$ and the value of the mean square error $\Delta(f, g; \widehat{A}\xi)$ of the optimal estimate $\widehat{A}\vec{\xi}$ (58) of the functional $A\vec{\xi}$ based on observations of the sequence $\vec{\zeta}(m)$ at points $m = -1, -2, \ldots$ are calculated by formulas (59) and (60) respectively.

Let operators $\mathbf{P}_{\overline{\mu}}^{\alpha}$, $\mathbf{T}_{\overline{\mu},N}^{\alpha}$, $\mathbf{Q}_N^{\alpha}$ be defined by the Fourier coefficients of the functions (57) in the same way as we defined operators $\mathbf{P}_{\overline{\mu}}$, $\mathbf{T}_{\overline{\mu},N}$, $\mathbf{Q}_N$. Theorem 3.2 implies that the spectral characteristic $\vec{h}_{\overline{\mu},N}^{\alpha}(\lambda)$ of the optimal estimate

$$
\widehat{A}_N \vec{\xi} = \int_{-\pi}^{\pi} \vec{h}_{\overline{\mu},N}^{\alpha}(\lambda) d\vec{Z}_{\zeta^{(n)}}(\lambda) - \sum_{k=-n(\gamma)}^{-1} (\vec{v}_{\overline{\mu},N}(k))^{\top} \vec{\zeta}(k) \tag{61}
$$

of the functional $A_N \vec{\xi}$ is calculated by the formula

$$
(\vec{h}_{\overline{\mu},N}^{\alpha}(\lambda))^{\top} = (\vec{B}_{\overline{\mu},N}(e^{i\lambda}))^{\top} \frac{\chi_{\overline{\mu}}^{(d)}(e^{-i\lambda})}{\beta^{(d)}(i\lambda)} -
$$

$$
- \frac{1}{\chi_{\overline{\mu}}^{(d)}(e^{-i\lambda}) \beta^{(d)}(i\lambda)} (\vec{A}_{\overline{\mu},N}(e^{i\lambda}))^{\top} (p(\lambda) - \alpha^2 f(\lambda))(p(\lambda))^{-1} -
$$

$$
- \frac{\overline{\beta^{(d)}(i\lambda)}}{\chi_{\overline{\mu}}^{(d)}(e^{-i\lambda})} \left(\sum_{k=0}^{\infty} ((\mathbf{P}_{\overline{\mu}}^{\alpha})^{-1} D^{\overline{\mu}} \mathbf{a}_N - (\mathbf{P}_{\overline{\mu}}^{\alpha})^{-1} \mathbf{T}_{\overline{\mu},N}^{\alpha} \mathbf{a}_{\overline{\mu},N})_k e^{ik\lambda} \right)^{\top} (p(\lambda))^{-1}. \tag{62}
$$

The value of the mean square error of the estimate $\widehat{A}_N\xi$ is calculated by the formula

$$
\Delta(f, g; \widehat{A}_N\vec{\xi}) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\left[\frac{1}{|\beta^{(d)}(i\lambda)|^2}(\vec{A}_{\overline{\mu},N}(e^{i\lambda}))^\top(p(\lambda) - \alpha^2 f(\lambda))+\right.
$$

$$
+\left(\sum_{k=0}^{\infty}((\mathbf{P}_{\overline{\mu}}^{\alpha})^{-1}D_N^{\overline{\mu}}\mathbf{a}_N - (\mathbf{P}_{\overline{\mu}}^{\alpha})^{-1}\mathbf{T}_{\overline{\mu},N}^{\alpha}\mathbf{a}_{\overline{\mu},N})_k e^{ik\lambda}\right)^\top\Bigg]\times
$$

$$
\times\frac{|\beta^{(d)}(i\lambda)|^2}{|\chi_{\overline{\mu}}^{(d)}(e^{-i\lambda})|^2}(p(\lambda))^{-1}f(\lambda)(p(\lambda))^{-1}\times
$$

$$
\times\left[\frac{1}{|\beta^{(d)}(i\lambda)|^2}(p(\lambda) - \alpha^2 f(\lambda))\overline{\vec{A}_{\overline{\mu},N}(e^{i\lambda})}+\right.
$$

$$
\left.+\overline{\sum_{k=0}^{\infty}((\mathbf{P}_{\overline{\mu}}^{\alpha})^{-1}D_N^{\overline{\mu}}\mathbf{a}_N - (\mathbf{P}_{\overline{\mu}}^{\alpha})^{-1}\mathbf{T}_{\overline{\mu},N}^{\alpha}\mathbf{a}_{\overline{\mu},N})_k e^{ik\lambda}}\right]d\lambda+
$$

$$
+\frac{1}{2\pi}\int_{-\pi}^{\pi}\left[\frac{1}{|\beta^{(d)}(i\lambda)|^2}(\vec{A}_{\overline{\mu},N}(e^{i\lambda}))^\top f(\lambda)-\right.
$$

$$
-\left(\sum_{k=0}^{\infty}((\mathbf{P}_{\overline{\mu}}^{\alpha})^{-1}D_N^{\overline{\mu}}\mathbf{a}_N - (\mathbf{P}_{\overline{\mu}}^{\alpha})^{-1}\mathbf{T}_{\overline{\mu},N}^{\alpha}\mathbf{a}_{\overline{\mu},N})_k e^{ik\lambda}\right)^\top\Bigg]\times
$$

$$
\times\frac{|\beta^{(d)}(i\lambda)|^2}{|\chi_{\overline{\mu}}^{(d)}(e^{-i\lambda})|^2}(p(\lambda))^{-1}(p(\lambda) - \alpha^2 f(\lambda))(p(\lambda))^{-1}\times
$$

$$
\times\left[\frac{1}{|\beta^{(d)}(i\lambda)|^2}f(\lambda)\overline{\vec{A}_{\overline{\mu},N}(e^{i\lambda})} - \overline{\sum_{k=0}^{\infty}((\mathbf{P}_{\overline{\mu}}^{\alpha})^{-1}D_N^{\overline{\mu}}\mathbf{a}_N - (\mathbf{P}_{\overline{\mu}}^{\alpha})^{-1}\mathbf{T}_{\overline{\mu},N}^{\alpha}\mathbf{a}_{\overline{\mu},N})_k e^{ik\lambda}}\right]d\lambda =
$$

$$
= \left\langle D_N^{\overline{\mu}}\mathbf{a}_N - \mathbf{T}_{\overline{\mu},N}^{\alpha}\mathbf{a}_{\overline{\mu},N}, (\mathbf{P}_{\overline{\mu}}^{\alpha})^{-1}D_N^{\overline{\mu}}\mathbf{a}_N - (\mathbf{P}_{\overline{\mu}}^{\alpha})^{-1}\mathbf{T}_{\overline{\mu},N}^{\alpha}\mathbf{a}_{\overline{\mu},N}\right\rangle + \left\langle \mathbf{Q}_N^{\alpha}\mathbf{a}_N, \mathbf{a}_N\right\rangle. \tag{63}
$$

Theorem 3.7
Let $\vec{\xi}(m)$, $m \in \mathbb{Z}$ and $\vec{\zeta}(m)$, $m \in \mathbb{Z}$ be seasonally cointegrated stochastic sequences with the spectral densities $f(\lambda)$ and $p(\lambda)$ respectively. Suppose that the spectral density $p(\lambda)$ satisfy the minimality condition (56). If the stochastic sequences $\vec{\xi}(m)$ and $\vec{\zeta}(m) - \vec{\alpha}\xi(m)$ are uncorrelated, then the spectral characteristic $\vec{h}_{\overline{\mu},N}^{\alpha}(\lambda)$ and the value of the mean square error $\Delta(f, g; \widehat{A}_N\vec{\xi})$ of the optimal estimate $\widehat{A}_N\vec{\xi}$ (61) of the functional $A_N\vec{\xi}$ based on observations of the sequence $\vec{\zeta}(m)$ at points $m = -1, -2, \ldots$ are calculated by formulas (62) and (63) respectively.

Let the spectral densities $f(\lambda)$ and $p(\lambda)$ admit the factorizations

$$
\frac{|\chi_{\overline{\mu}}^{(d)}(e^{-i\lambda})|^2}{|\beta^{(d)}(i\lambda)|^2}p(\lambda) = \Theta_{\overline{\mu}}^{\alpha}(e^{-i\lambda})(\Theta_{\overline{\mu}}^{\alpha}(e^{-i\lambda}))^*, \quad \Theta_{\overline{\mu}}^{\alpha}(e^{-i\lambda}) = \sum_{k=0}^{\infty}\theta_{\overline{\mu}}^{\alpha}(k)e^{-i\lambda k}, \tag{64}
$$

$$
\frac{|\beta^{(d)}(i\lambda)|^2}{|\chi_{\overline{\mu}}^{(d)}(e^{-i\lambda})|^2}(p(\lambda))^{-1} = (\Psi_{\overline{\mu}}^{\alpha}(e^{-i\lambda}))^*\Psi_{\overline{\mu}}^{\alpha}(e^{-i\lambda}), \quad \Psi_{\overline{\mu}}^{\alpha}(e^{-i\lambda}) = \sum_{k=0}^{\infty}\psi_{\overline{\mu}}^{\alpha}(k)e^{-i\lambda k}, \tag{65}
$$

$$
|\beta^{(d)}(i\lambda)|^{-2}(p(\lambda) - \alpha^2 f(\lambda)) = \Phi^{\alpha}(e^{-i\lambda})(\Phi^{\alpha}(e^{-i\lambda}))^*, \quad \Phi^{\alpha}(e^{-i\lambda}) = \sum_{k=0}^{\infty}\phi^{\alpha}(k)e^{-i\lambda k}. \tag{66}
$$

Let operators and vectors $\mathbf{C}^\alpha_{\mu,g}$, $\widetilde{\Phi}^\alpha$, Φ^α, $\mathbf{Z}^\alpha_{\overline{\mu}}$, $\theta^\alpha_{\overline{\mu}}$, $\overline{\psi}^\alpha_{\overline{\mu}}$ be defined by coefficients of the canonical factorizations (64) – (66) in the same way as were defined operators and vectors $\mathbf{C}_{\mu,g}$, $\widetilde{\Phi}$, Φ, $\mathbf{Z}_{\overline{\mu}}$, $\theta_{\overline{\mu}}$, $\overline{\psi}_{\overline{\mu}}$. From Theorem 3.3 we obtain that the spectral characteristic $\vec{h}^\alpha_{\overline{\mu}}(\lambda)$ of the optimal estimate $\widehat{A\vec{\xi}}$ of the functional $A\vec{\xi}$ can be calculated by the formula

$$\vec{h}^\alpha_{\overline{\mu}}(\lambda) = \frac{\chi^{(d)}_{\overline{\mu}}(e^{-i\lambda})}{\beta^{(d)}(i\lambda)}\left(\sum_{k=0}^\infty (\psi^\alpha_{\overline{\mu}}(k))^\top e^{-i\lambda k}\right)\sum_{m=1}^\infty \left((\theta^\alpha_{\overline{\mu}})^\top \widetilde{\mathbf{B}}_{\overline{\mu}} - \overline{\psi}^\alpha_{\overline{\mu}}\mathbf{C}^\alpha_{\mu,g}\right)_m e^{-i\lambda m}. \tag{67}$$

The value of the mean square error of the estimate $\widehat{A\vec{\xi}}$ is calculated by the formula

$$\begin{aligned}\Delta(f,g;\widehat{A\vec{\xi}}) &= \|(\Phi^\alpha)^\top \mathbf{a}_{\overline{\mu}}\|^2 + \|\widetilde{\Phi}^\alpha \mathbf{a}_{\overline{\mu}}\|_1^2 + \left\langle (\theta^\alpha_{\overline{\mu}})^\top D^\mu \mathbf{A} - \overline{\psi}^\alpha_{\overline{\mu}}\mathbf{C}^\alpha_{\mu,g}, (\theta^\alpha_{\overline{\mu}})^\top D^\mu \mathbf{A}\right\rangle \\ &\quad - \left\langle (\theta^\alpha_{\overline{\mu}})^\top D^\mu \mathbf{A}, \mathbf{Z}^\alpha_{\overline{\mu}}\mathbf{a}_{\overline{\mu}}\right\rangle - \left\langle \mathbf{Z}^\alpha_{\overline{\mu}}\mathbf{a}_{\overline{\mu}}, \overline{\psi}^\alpha_{\overline{\mu}}\mathbf{C}^\alpha_{\mu,g}\right\rangle_1. \end{aligned} \tag{68}$$

Theorem 3.8
Let $\vec{\xi}(m)$, $m \in \mathbb{Z}\}$, and $\vec{\zeta}(m)$, $m \in \mathbb{Z}$, be seasonally cointegrated stochastic vector sequences with the spectral densities $f(\lambda)$ and $p(\lambda)$ which admit the canonical factorizations (64) – (66). Suppose that the coefficients $\vec{a}(k)$, $k \geq 0$, satisfy conditions (16) – (17). Then the spectral characteristic $h^\beta_{\overline{\mu}}(\lambda)$ and the value of the mean square error $\Delta(f,g;\widehat{A\vec{\xi}})$ of the optimal estimate $\widehat{A\vec{\xi}}$ of the functional $A\vec{\xi}$ based on observations of the sequence $\vec{\zeta}(m)$ at points $m = -1, -2, \ldots$ are calculated by formulas (67) and (68) respectively.

4. Minimax (robust) method of forecasting

Values of the mean square errors and the spectral characteristics of the optimal estimates of the functionals $A\vec{\xi}$ and $A_N\vec{\xi}$ depending on the unobserved values of a stochastic sequence $\vec{\xi}(m)$ which determine a stationary stochastic GM increment sequence $\chi^{(d)}_{\overline{\mu},s}(\vec{\xi}(m))$ with the spectral density matrix $f(\lambda)$ based on observations of the sequence $\vec{\xi}(m) + \vec{\eta}(m)$ at points $m = -1, -2, \ldots$ can be calculated by formulas (33), (34) and (38), (40) respectively, under the condition that spectral densities $f(\lambda)$ and $g(\lambda)$ of stochastic sequences $\vec{\xi}(m)$ and $\vec{\eta}(m)$ are exactly known.

In practical cases, however, spectral densities of sequences usually are not exactly known. If in such cases a set $\mathcal{D} = \mathcal{D}_f \times \mathcal{D}_g$ of admissible spectral densities is defined, the minimax (robust) approach to estimation of linear functionals depending on unobserved values of stochastic sequences with stationary GM increments may be applied. This method consists in finding an estimate that minimizes the maximal values of the mean square errors for all spectral densities from a given class $\mathcal{D} = \mathcal{D}_f \times \mathcal{D}_g$ of admissible spectral densities simultaneously.

To formalize this approach we present the following definitions.

Definition 4.1
For a given class of spectral densities $\mathcal{D} = \mathcal{D}_f \times \mathcal{D}_g$ the spectral densities $f^0(\lambda) \in \mathcal{D}_f$, $g^0(\lambda) \in \mathcal{D}_g$ are called least favorable in the class $\mathcal{D}$ for the optimal linear forecasting of the functional $A\vec{\xi}$ if the following relation holds true:

$$\Delta(f^0, g^0) = \Delta(h(f^0, g^0); f^0, g^0) = \max_{(f,g)\in\mathcal{D}_f\times\mathcal{D}_g} \Delta(h(f,g); f, g).$$

Definition 4.2
For a given class of spectral densities $\mathcal{D} = \mathcal{D}_f \times \mathcal{D}_g$ the spectral characteristic $h^0(\lambda)$ of the optimal linear estimate of the functional $A\vec{\xi}$ is called minimax-robust if there are satisfied the conditions

$$h^0(\lambda) \in H_\mathcal{D} = \bigcap_{(f,g)\in\mathcal{D}_f\times\mathcal{D}_g} L_2^{0-}(f(\lambda) + |\beta^{(d)}(i\lambda)|^2 g(\lambda)),$$

$$\min_{h\in H_\mathcal{D}} \max_{(f,g)\in\mathcal{D}_f\times\mathcal{D}_g} \Delta(h; f, g) = \max_{(f,g)\in\mathcal{D}_f\times\mathcal{D}_g} \Delta(h^0; f, g).$$

Taking into account the introduced definitions and the derived relations we can verify that the following lemmas hold true.

Lemma 4.1
Spectral densities $f^0 \in \mathcal{D}_f$, $g^0 \in \mathcal{D}_g$ which satisfy condition (18) are least favorable in the class $\mathcal{D} = \mathcal{D}_f \times \mathcal{D}_g$ for the optimal linear forecasting of the functional $A\vec{\xi}$ if operators $\mathbf{P}^0_{\overline{\mu}}$, $\mathbf{T}^0_{\overline{\mu}}$, $\mathbf{Q}^0$ determined by the Fourier coefficients of the functions

$$\frac{|\beta^{(d)}(i\lambda)|^2}{|\chi^{(d)}_{\overline{\mu}}(e^{-i\lambda})|^2} \left[g^0(\lambda)(f^0(\lambda) + |\beta^{(d)}(i\lambda)|^2 g^0(\lambda))^{-1} \right]^{\top},$$

$$\frac{|\beta^{(d)}(i\lambda)|^2}{|\chi^{(d)}_{\overline{\mu}}(e^{-i\lambda})|^2} \left[f^0(\lambda) + |\beta^{(d)}(i\lambda)|^2 g^0(\lambda))^{-1} \right]^{\top},$$

$$\left[f^0(\lambda)(f^0(\lambda) + |\beta^{(d)}(i\lambda)|^2 g^0(\lambda))^{-1} g^0(\lambda) \right]^{\top}$$

determine a solution of the constrained optimisation problem

$$\max_{(f,g)\in\mathcal{D}_f\times\mathcal{D}_g} (\langle D^{\overline{\mu}}\mathbf{a} - \mathbf{T}_{\overline{\mu}}\mathbf{a}_{\overline{\mu}}, \mathbf{P}^{-1}_{\overline{\mu}} D^{\overline{\mu}}\mathbf{a} - \mathbf{P}^{-1}_{\overline{\mu}}\mathbf{T}_{\overline{\mu}}\mathbf{a}_{\overline{\mu}}\rangle + \langle \mathbf{Q}\mathbf{a}, \mathbf{a}\rangle)$$

$$= \langle D^{\overline{\mu}}\mathbf{a} - \mathbf{T}^0_{\overline{\mu}}\mathbf{a}_{\overline{\mu}}, (\mathbf{P}^0_{\overline{\mu}})^{-1} D^{\overline{\mu}}\mathbf{a} - (\mathbf{P}^0_{\overline{\mu}})^{-1}\mathbf{T}^0_{\overline{\mu}}\mathbf{a}_{\overline{\mu}}\rangle + \langle \mathbf{Q}^0\mathbf{a}, \mathbf{a}\rangle. \qquad (69)$$

The minimax spectral characteristic $h^0 = h_{\overline{\mu}}(f^0, g^0)$ is calculated by formula (33) if $h_{\overline{\mu}}(f^0, g^0) \in H_{\mathcal{D}}$.

Lemma 4.2
The spectral densities $f^0 \in \mathcal{D}_f$, $g^0 \in \mathcal{D}_g$ which admit canonical factorizations (11), (44) and (45) are least favourable densities in the class $\mathcal{D}$ for the optimal linear forecasting of the functional $A\vec{\xi}$ based on observations of the sequence $\vec{\xi}(m) + \vec{\eta}(m)$ at points $m = -1, -2, \ldots$ if the matrix coefficients of canonical factorizations (44) and (45) determine a solution to the constrained optimization problem

$$\|\Phi^{\top}\mathbf{a}_{\overline{\mu}}\|^2 \quad + \quad \|\widetilde{\Phi}\mathbf{a}_{\overline{\mu}}\|^2_1 + \langle \theta^{\top}_{\overline{\mu}} D^{\mu}\mathbf{A} - \overline{\psi}_{\overline{\mu}}\mathbf{C}_{\mu,g}, \theta^{\top}_{\overline{\mu}} D^{\mu}\mathbf{A}\rangle$$
$$- \quad \langle \theta^{\top}_{\overline{\mu}} D^{\mu}\mathbf{A}, \mathbf{Z}_{\overline{\mu}}\mathbf{a}_{\overline{\mu}}\rangle - \langle \mathbf{Z}_{\overline{\mu}}\mathbf{a}_{\overline{\mu}}, \overline{\psi}_{\overline{\mu}}\mathbf{C}_{\mu,g}\rangle_1 \to \sup, \qquad (70)$$

$$f(\lambda) \quad = \quad \frac{|\beta^{(d)}(i\lambda)|^2}{|\chi^{(d)}_{\overline{\mu}}(e^{-i\lambda})|^2}\Theta_{\overline{\mu}}(e^{-i\lambda})\Theta^*_{\overline{\mu}}(e^{-i\lambda}) - |\beta^{(d)}(i\lambda)|^2\Phi(e^{-i\lambda})\Phi^*(e^{-i\lambda}) \in \mathcal{D}_f,$$

$$g(\lambda) = \Phi(e^{-i\lambda})\Phi^*(e^{-i\lambda}) \in \mathcal{D}_g.$$

The minimax spectral characteristic $\vec{h}^0 = \vec{h}_{\overline{\mu}}(f^0, g^0)$ is calculated by formula (50) if $\vec{h}_{\overline{\mu}}(f^0, g^0) \in H_{\mathcal{D}}$.

Lemma 4.3
The spectral density $g^0 \in \mathcal{D}_g$ which admits canonical factorizations (44), (45) with the known spectral density $f(\lambda)$ is the least favourable in the class $\mathcal{D}_g$ for the optimal linear forecasting of the functional $A\xi$ based on observations of the sequence $\vec{\xi}(m) + \vec{\eta}(m)$ at points $m = -1, -2, \ldots$ if the matrix coefficients of the canonical factorizations

$$f(\lambda) + |\beta^{(d)}(i\lambda)|^2 g^0(\lambda) = \frac{|\beta^{(d)}(i\lambda)|^2}{|\chi^{(d)}_{\overline{\mu}}(e^{-i\lambda})|^2} \left(\sum_{k=0}^{\infty} \theta^0_{\overline{\mu}}(k)e^{-i\lambda k} \right) \left(\sum_{k=0}^{\infty} \theta^0_{\overline{\mu}}(k)e^{-i\lambda k} \right)^*, \qquad (71)$$

$$g^0(\lambda) = \left(\sum_{k=0}^{\infty} \phi^0(k)e^{-i\lambda k} \right) \left(\sum_{k=0}^{\infty} \phi^0(k)e^{-i\lambda k} \right)^* \qquad (72)$$

determine a solution to the constrained optimization problem

$$
\begin{aligned}
\|\Phi^\top \mathbf{a}_{\overline{\mu}}\|^2 \quad &+ \quad \|\widetilde{\Phi}\mathbf{a}_{\overline{\mu}}\|_1^2 + \left\langle \theta_{\overline{\mu}}^\top D^\mu \mathbf{A} - \overline{\psi}_{\overline{\mu}} \mathbf{C}_{\mu,g}, \theta_{\overline{\mu}}^\top D^\mu \mathbf{A} \right\rangle \\
&- \quad \left\langle \theta_{\overline{\mu}}^\top D^\mu \mathbf{A}, \mathbf{Z}_{\overline{\mu}}\mathbf{a}_{\overline{\mu}} \right\rangle - \left\langle \mathbf{Z}_{\overline{\mu}}\mathbf{a}_{\overline{\mu}}, \overline{\psi}_{\overline{\mu}} \mathbf{C}_{\mu,g} \right\rangle_1 \to \sup, \\
&\quad g(\lambda) = \Phi(e^{-i\lambda})\Phi^*(e^{-i\lambda}) \in \mathcal{D}_g.
\end{aligned}
\tag{73}
$$

The minimax spectral characteristic $\vec{h}^0 = \vec{h}_{\overline{\mu}}(f, g^0)$ is calculated by formula (50) if $\vec{h}_{\overline{\mu}}(f, g^0) \in H_{\mathcal{D}}$.

Lemma 4.4

The spectral density $f^0 \in \mathcal{D}_f$ which admits canonical factorizations (11), (44) with the known spectral density $g(\lambda)$ is the least favourable spectral density in the class $\mathcal{D}_f$ for the optimal linear forecasting of the functional $A\vec{\xi}$ based on observations of the sequence $\vec{\xi}(m) + \vec{\eta}(m)$ at points $m = -1, -2, \dots$ if matrix coefficients of the canonical factorization

$$
f^0(\lambda) + |\beta^{(d)}(i\lambda)|^2 g(\lambda) = \frac{|\beta^{(d)}(i\lambda)|^2}{|\chi_{\overline{\mu}}^{(d)}(e^{-i\lambda})|^2} \left(\sum_{k=0}^{\infty} \theta_{\overline{\mu}}^0(k) e^{-i\lambda k} \right) \left(\sum_{k=0}^{\infty} \theta_{\overline{\mu}}^0(k) e^{-i\lambda k} \right)^*,
\tag{74}
$$

and the equastion $\Psi_{\overline{\mu}}^0(e^{-i\lambda})\Theta_{\overline{\mu}}^0(e^{-i\lambda}) = E_q$ determine a solution to the constrained optimization problem

$$
\left\langle \theta_{\overline{\mu}}^\top D^\mu \mathbf{A} - \overline{\psi}_{\overline{\mu}} \mathbf{C}_{\mu,g}, \theta_{\overline{\mu}}^\top D^\mu \mathbf{A} \right\rangle - \left\langle \theta_{\overline{\mu}}^\top D^\mu \mathbf{A}, \mathbf{Z}_{\overline{\mu}}\mathbf{a}_{\overline{\mu}} \right\rangle - \left\langle \mathbf{Z}_{\overline{\mu}}\mathbf{a}_{\overline{\mu}}, \overline{\psi}_{\overline{\mu}} \mathbf{C}_{\mu,g} \right\rangle_1 \to \sup,
\tag{75}
$$

$$
f(\lambda) = \frac{|\beta^{(d)}(i\lambda)|^2}{|\chi_{\overline{\mu}}^{(d)}(e^{-i\lambda})|^2} \Theta_{\overline{\mu}}(e^{-i\lambda})\Theta_{\overline{\mu}}^*(e^{-i\lambda}) - |\beta^{(d)}(i\lambda)|^2 \Phi(e^{-i\lambda})\Phi^*(e^{-i\lambda}) \in \mathcal{D}_f
$$

for the fixed matrix coefficients $\{\phi(k) : k \geq 0\}$. The minimax spectral characteristic $\vec{h}^0 = \vec{h}_{\overline{\mu}}(f^0, g)$ is calculated by formula (50) if $\vec{h}_{\overline{\mu}}(f^0, g) \in H_{\mathcal{D}}$.

For more detailed analysis of properties of the least favorable spectral densities and minimax-robust spectral characteristics we observe that the minimax spectral characteristic h^0 and the least favourable spectral densities (f^0, g^0) form a saddle point of the function $\Delta(h; f, g)$ on the set $H_{\mathcal{D}} \times \mathcal{D}$.

The saddle point inequalities

$$
\Delta(h; f^0, g^0) \geq \Delta(h^0; f^0, g^0) \geq \Delta(h^0; f, g) \quad \forall f \in \mathcal{D}_f, \forall g \in \mathcal{D}_g, \forall h \in H_{\mathcal{D}}
$$

hold true if $h^0 = h_{\overline{\mu}}(f^0, g^0)$ and $h_{\overline{\mu}}(f^0, g^0) \in H_{\mathcal{D}}$, where (f^0, g^0) is a solution of the constrained optimization problem

$$
\widetilde{\Delta}(f, g) = -\Delta(h_{\overline{\mu}}(f^0, g^0); f, g) \to \inf, \quad (f, g) \in \mathcal{D},
\tag{76}
$$

where the functional $\Delta(h_{\overline{\mu}}(f^0, g^0); f, g)$ is calculated by the formula

$$
\begin{aligned}
&\Delta(h_{\overline{\mu}}(f^0, g^0); f, g) = \\
&\quad = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{|\beta^{(d)}(i\lambda)|^2}{|\chi_{\overline{\mu}}^{(d)}(e^{-i\lambda})|^2} \left[(\vec{A}_{\overline{\mu}}(e^{i\lambda}))^\top g^0(\lambda) + (\vec{C}_{\overline{\mu}}^0(e^{i\lambda}))^\top \right] \times \\
&\quad\quad \times (f^0(\lambda) + |\beta^{(d)}(i\lambda)|^2 g^0(\lambda))^{-1} f(\lambda) (f^0(\lambda) + |\beta^{(d)}(i\lambda)|^2 g^0(\lambda))^{-1} \times \\
&\quad\quad\quad \times \left[g^0(\lambda)\overline{\vec{A}_{\overline{\mu}}(e^{i\lambda})} + \overline{\vec{C}_{\overline{\mu}}^0(e^{i\lambda})} \right] d\lambda + \\
&\quad + \frac{1}{2\pi} \int_{-\pi}^{\pi} \left(\frac{|\beta^{(d)}(i\lambda)|^2}{|\chi_{\overline{\mu}}^{(d)}(e^{-i\lambda})|^2} \right)^2 \left[\frac{\chi_{\overline{\mu}}^{(d)}(e^{-i\lambda})}{|\beta^{(d)}(i\lambda)|^2} (\vec{A}_{\overline{\mu}}(e^{i\lambda}))^\top f^0(\lambda) - \chi_{\overline{\mu}}^{(d)}(e^{-i\lambda})(\vec{C}_{\overline{\mu}}^0(e^{i\lambda}))^\top \right] \times \\
&\quad\quad \times (f^0(\lambda) + |\beta^{(d)}(i\lambda)|^2 g^0(\lambda))^{-1} g(\lambda) (f^0(\lambda) + |\beta^{(d)}(i\lambda)|^2 g^0(\lambda))^{-1} \times \\
&\quad\quad\quad \times \left[\overline{\frac{\chi_{\overline{\mu}}^{(d)}(e^{-i\lambda})}{|\beta^{(d)}(i\lambda)|^2}} f^0(\lambda)\overline{\vec{A}_{\overline{\mu}}(e^{i\lambda})} - \overline{\chi_{\overline{\mu}}^{(d)}(e^{-i\lambda})}\overline{\vec{C}_{\overline{\mu}}^0(e^{i\lambda})} \right] d\lambda,
\end{aligned}
$$

where

$$\vec{C}_{\overline{\mu}}^0(e^{i\lambda}) = \sum_{k=0}^{\infty} (\mathbf{P}_{\overline{\mu}}^0)^{-1} D^{\overline{\mu}} \mathbf{a} - (\mathbf{P}_{\overline{\mu}}^0)^{-1} \mathbf{T}_{\overline{\mu}}^0 \mathbf{a}_{\overline{\mu}})_k e^{ik\lambda},$$

or it is calculated by the formula

$$\Delta\left(h_\mu(f^0, g^0); f, g\right) =$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{|\chi_{\overline{\mu}}^{(d)}(e^{-i\lambda})|^2}{|\beta^{(d)}(i\lambda)|^2} (\mathbf{r}_{\overline{\mu},f}^0(e^{-i\lambda}))^\top \Psi_{\overline{\mu}}^0(e^{-i\lambda})) f(\lambda) (\Psi_{\overline{\mu}}^0(e^{-i\lambda}))^* \overline{\mathbf{r}_{\overline{\mu},f}^0(e^{-i\lambda})} d\lambda +$$

$$+ \frac{1}{2\pi} \int_{-\pi}^{\pi} (\mathbf{r}_{\overline{\mu},g}^0(e^{-i\lambda}))^\top \Psi_{\overline{\mu}}^0(e^{-i\lambda}) g(\lambda) (\Psi_{\overline{\mu}}^0(e^{-i\lambda}))^* \overline{\mathbf{r}_{\overline{\mu},g}^0(e^{-i\lambda})} d\lambda,$$

where

$$\mathbf{r}_{\overline{\mu},f}^0(e^{-i\lambda}) = \sum_{m=0}^{\infty} ((\theta^0)_{\overline{\mu}}^\top D^\mu \mathbf{A})_m e^{i\lambda m} + \sum_{m=1}^{\infty} (\overline{\psi}_{\overline{\mu}}^0 \mathbf{C}_{\mu,g}^0)_m e^{-i\lambda m},$$

$$\mathbf{r}_{\overline{\mu},g}^0(e^{-i\lambda}) = \chi_{\overline{\mu}}^{(d)}(e^{-i\lambda}) \left(\sum_{m=0}^{\infty} ((\theta^0)_{\overline{\mu}}^\top D^\mu \mathbf{A})_m e^{i\lambda m} + \sum_{m=1}^{\infty} (\overline{\psi}_{\overline{\mu}}^0 \mathbf{C}_{\mu,g}^0)_m e^{-i\lambda m} \right) - (\Theta_{\overline{\mu}}^0(e^{-i\lambda}))^\top A(e^{i\lambda}).$$

The constrained optimization problem (76) is equivalent to the unconstrained optimisation problem

$$\Delta_{\mathcal{D}}(f,g) = \widetilde{\Delta}(f,g) + \delta(f,g|\mathcal{D}_f \times \mathcal{D}_g) \to \inf, \tag{77}$$

where $\delta(f,g|\mathcal{D}_f \times \mathcal{D}_g)$ is the indicator function of the set $\mathcal{D} = \mathcal{D}_f \times \mathcal{D}_g$. Solution (f^0, g^0) to this unconstrained optimization problem is characterized by the condition $0 \in \partial\Delta_{\mathcal{D}}(f^0, g^0)$, where $\partial\Delta_{\mathcal{D}}(f^0, g^0)$ is the subdifferential of the functional $\Delta_{\mathcal{D}}(f,g)$ at point $(f^0, g^0) \in \mathcal{D} = \mathcal{D}_f \times \mathcal{D}_g$, that is the set of all continuous linear functionals Λ on $L_1 \times L_1$ which satisfy the inequality $\Delta_{\mathcal{D}}(f,g) - \Delta_{\mathcal{D}}(f^0, g^0) \geq \Lambda((f,g) - (f^0, g^0)), (f,g) \in \mathcal{D}$ (see [42, 56] for more details). This condition makes it possible to find the least favourable spectral densities in some special classes of spectral densities $\mathcal{D} = \mathcal{D}_f \times \mathcal{D}_g$.

In the case of cointegrated vector sequences (in terms of Subsection 3.4) we have the following optimization problem for determining the least favourable spectral densities:

$$\Delta_{\mathcal{D}}(f,p) = \widetilde{\Delta}(f,p) + \delta(f,p|\mathcal{D}_f \times \mathcal{D}_p) \to \inf, \tag{78}$$

$$\widetilde{\Delta}(f,p) = -\Delta\left(h_{\overline{\mu}}^\alpha(f^0, p^0); f, p\right),$$

$$\Delta\left(h_{\overline{\mu}}^\alpha(f^0, p^0); f, p\right) =$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} \left[\frac{1}{|\beta^{(d)}(i\lambda)|^2} (\vec{A}_{\overline{\mu}}(e^{i\lambda}))^\top (p^0(\lambda) - \alpha^2 f^0(\lambda)) + \left(C_{\overline{\mu}}^{\alpha,0}(e^{i\lambda})\right)^\top \right] \times$$

$$\times \frac{|\beta^{(d)}(i\lambda)|^2}{|\chi_{\overline{\mu}}^{(d)}(e^{-i\lambda})|^2} (p^0(\lambda))^{-1} f(\lambda) (p^0(\lambda))^{-1} \times$$

$$\times \left[\frac{1}{|\beta^{(d)}(i\lambda)|^2} (p^0(\lambda) - \alpha^2 f^0(\lambda)) \overline{\vec{A}_{\overline{\mu}}(e^{i\lambda})} + \overline{C_{\overline{\mu}}^{\alpha,0}(e^{i\lambda})} \right] d\lambda +$$

$$+ \frac{1}{2\pi} \int_{-\pi}^{\pi} \left[\frac{1}{|\beta^{(d)}(i\lambda)|^2} (\vec{A}_{\overline{\mu}}(e^{i\lambda}))^\top f^0(\lambda) - \left(C_{\overline{\mu}}^{\alpha,0}(e^{i\lambda})\right)^\top \right] \times$$

$$\times \frac{|\beta^{(d)}(i\lambda)|^2}{|\chi_{\overline{\mu}}^{(d)}(e^{-i\lambda})|^2} (p^0(\lambda))^{-1} (p(\lambda) - \alpha^2 f(\lambda)) (p^0(\lambda))^{-1} \times$$

$$\times \left[\frac{1}{|\beta^{(d)}(i\lambda)|^2} f^0(\lambda) \overrightarrow{\vec{A}_{\overline{\mu}}(e^{i\lambda})} - \frac{|\beta^{(d)}(i\lambda)|^2}{|\chi_{\overline{\mu}}^{(d)}(e^{-i\lambda})|^2} \overline{C_{\overline{\mu}}^{\alpha,0}(e^{i\lambda})} \right] d\lambda,$$

$$C_{\overline{\mu}}^{\alpha,0}(e^{i\lambda}) = \sum_{k=0}^{\infty} \left(\left((\mathbf{P}_{\overline{\mu}}^{\alpha})^0\right)^{-1} \left(D^{\overline{\mu}} \mathbf{a} - (\mathbf{T}_{\overline{\mu}}^{\alpha})^0 \mathbf{a}_{\overline{\mu}} \right) \right)_k e^{i\lambda k}.$$

A solution (f^0, p^0) of this optimization problem is characterized by the condition $0 \in \partial \Delta_{\mathcal{D}}(f^0, p^0)$.

The form of the functionals $\Delta(h_{\overline{\mu}}(f^0, g^0); f, g)$, $\Delta(h_{\overline{\mu}}(f^0, p^0); f, p)$ is convenient for application the Lagrange method of indefinite multipliers for finding solution to the problem (77). Making use of the method of Lagrange multipliers and the form of subdifferentials of the indicator functions $\delta(f, g | \mathcal{D}_f \times \mathcal{D}_g)$, $\delta(f, g | \mathcal{D}_f \times \mathcal{D}_p)$ of the sets $\mathcal{D}_f \times \mathcal{D}_g$, $\mathcal{D}_f \times \mathcal{D}_p$ of spectral densities, we describe relations that determine least favourable spectral densities in some special classes of spectral densities (see [36, 42] for additional details).

4.1. Least favorable spectral density in classes $\mathcal{D}_0 \times \mathcal{D}_V^U$

Consider the forecasting problem for the functional $A\vec{\xi}$ which depends on unobserved values of a sequence $\vec{\xi}(m)$ with stationary increments based on observations of the sequence $\vec{\xi}(m) + \vec{\eta}(m)$ at points $m = -1, -2, \ldots$ under the condition that the sets of admissible spectral densities $\mathcal{D}_{f0}^k, \mathcal{D}_{Vg}^{Uk}, k = 1, 2, 3, 4$ are defined as follows:

$$\mathcal{D}_{f0}^1 = \left\{ f(\lambda) \left| \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{|\chi_{\overline{\mu}}^{(d)}(e^{-i\lambda})|^2}{|\beta^{(d)}(i\lambda)|^2} f(\lambda) d\lambda = P \right. \right\},$$

$$\mathcal{D}_{f0}^2 = \left\{ f(\lambda) \left| \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{|\chi_{\overline{\mu}}^{(d)}(e^{-i\lambda})|^2}{|\beta^{(d)}(i\lambda)|^2} \mathrm{Tr}\,[f(\lambda)] d\lambda = p \right. \right\},$$

$$\mathcal{D}_{f0}^3 = \left\{ f(\lambda) \left| \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{|\chi_{\overline{\mu}}^{(d)}(e^{-i\lambda})|^2}{|\beta^{(d)}(i\lambda)|^2} f_{kk}(\lambda) d\lambda = p_k, k = \overline{1,T} \right. \right\},$$

$$\mathcal{D}_{f0}^4 = \left\{ f(\lambda) \left| \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{|\chi_{\overline{\mu}}^{(d)}(e^{-i\lambda})|^2}{|\beta^{(d)}(i\lambda)|^2} \langle B_1, f(\lambda) \rangle \, d\lambda = p \right. \right\},$$

$$\mathcal{D}_{Vg}^{U1} = \left\{ g(\lambda) \left| V(\lambda) \le g(\lambda) \le U(\lambda), \frac{1}{2\pi} \int_{-\pi}^{\pi} g(\lambda) d\lambda = Q \right. \right\},$$

$$\mathcal{D}_{Vg}^{U\,2} = \left\{ g(\lambda) \left| \mathrm{Tr}\,[V(\lambda)] \le \mathrm{Tr}\,[g(\lambda)] \le \mathrm{Tr}\,[U(\lambda)], \frac{1}{2\pi} \int_{-\pi}^{\pi} \mathrm{Tr}\,[g(\lambda)] d\lambda = q \right. \right\},$$

$$\mathcal{D}_{Vg}^{U3} = \left\{ g(\lambda) \left| v_{kk}(\lambda) \le g_{kk}(\lambda) \le u_{kk}(\lambda), \frac{1}{2\pi} \int_{-\pi}^{\pi} g_{kk}(\lambda) d\lambda = q_k, k = \overline{1,T} \right. \right\},$$

$$\mathcal{D}_{Vg}^{U4} = \left\{ g(\lambda) \left| \langle B_2, V(\lambda) \rangle \le \langle B_2, g(\lambda) \rangle \le \langle B_2, U(\lambda) \rangle, \frac{1}{2\pi} \int_{-\pi}^{\pi} \langle B_2, g(\lambda) \rangle \, d\lambda = q \right. \right\}.$$

Here spectral densities $V(\lambda), U(\lambda)$ are known and fixed, $p, p_k, q, q_k, k = \overline{1,T}$ are given numbers, P, B_1, Q, B_2 are given positive-definite Hermitian matrices.

Define

$$\mathbf{C}_{\overline{\mu}}^{f0}(e^{i\lambda}) := g^0(\lambda) \vec{A}_{\overline{\mu}}(e^{i\lambda}) + \sum_{k=0}^{\infty} ((\mathbf{P}_{\overline{\mu}}^0)^{-1} D^{\overline{\mu}} \mathbf{a} - (\mathbf{P}_{\overline{\mu}}^0)^{-1} \mathbf{T}_{\overline{\mu}}^0 \mathbf{a}_{\overline{\mu}})_k e^{ik\lambda},$$

$$\mathbf{C}_{\overline{\mu}}^{g0}(e^{i\lambda}) := \chi_{\overline{\mu}}^{(d)}(e^{-i\lambda}) |\beta^{(d)}(i\lambda)|^{-2} f^0(\lambda) \vec{A}_{\overline{\mu}}(e^{i\lambda}) - - \chi_{\overline{\mu}}^{(d)}(e^{-i\lambda}) \sum_{k=0}^{\infty} ((\mathbf{P}_{\overline{\mu}}^0)^{-1} D^{\overline{\mu}} \mathbf{a} - (\mathbf{P}_{\overline{\mu}}^0)^{-1} \mathbf{T}_{\overline{\mu}}^0 \mathbf{a}_{\overline{\mu}})_k e^{ik\lambda}.$$

From the condition $0 \in \partial \Delta_{\mathcal{D}}(f^0, g^0)$ we find the following equations which determine the least favourable spectral densities for these given sets of admissible spectral densities.

For the first set of admissible spectral densities $\mathcal{D}_{f0}^1 \times \mathcal{D}_{Vg}^{U1}$ we have equations

$$
\left(\mathbf{C}_{\frac{f0}{\mu}}(e^{i\lambda})\right)\left(\mathbf{C}_{\frac{f0}{\mu}}(e^{i\lambda})\right)^* = \left(\frac{|\chi_{\overline{\mu}}^{(d)}(e^{-i\lambda})|^2}{|\beta^{(d)}(i\lambda)|^2}(f^0(\lambda) + |\beta^{(d)}(i\lambda)|^2 g^0(\lambda))\right) \vec{\alpha}_f \cdot \vec{\alpha}_f^* \times
$$
$$
\times \left(\frac{|\chi_{\overline{\mu}}^{(d)}(e^{-i\lambda})|^2}{|\beta^{(d)}(i\lambda)|^2}(f^0(\lambda) + |\beta^{(d)}(i\lambda)|^2 g^0(\lambda))\right), \quad (79)
$$

$$
\left(\mathbf{C}_{\frac{g0}{\mu}}(e^{i\lambda})\right)\left(\mathbf{C}_{\frac{g0}{\mu}}(e^{i\lambda})\right)^* =
$$
$$
= \left(\frac{|\chi_{\overline{\mu}}^{(d)}(e^{-i\lambda})|^2}{|\beta^{(d)}(i\lambda)|^2}(f^0(\lambda) + |\beta^{(d)}(i\lambda)|^2 g^0(\lambda))\right)(\vec{\beta} \cdot \vec{\beta}^* + \Gamma_1(\lambda) + \Gamma_2(\lambda)) \times
$$
$$
\times \left(\frac{|\chi_{\overline{\mu}}^{(d)}(e^{-i\lambda})|^2}{|\beta^{(d)}(i\lambda)|^2}(f^0(\lambda) + |\beta^{(d)}(i\lambda)|^2 g^0(\lambda))\right), \quad (80)
$$

where $\vec{\alpha}_f$ and $\vec{\beta}$ are vectors of Lagrange multipliers, the matrix $\Gamma_1(\lambda) \leq 0$ and $\Gamma_1(\lambda) = 0$ if $g_0(\lambda) > V(\lambda)$, the matrix $\Gamma_2(\lambda) \geq 0$ and $\Gamma_2(\lambda) = 0$ if $g_0(\lambda) < U(\lambda)$.

For the second set of admissible spectral densities $\mathcal{D}_{f0}^2 \times \mathcal{D}_{Vg}^{U2}$ we have equations

$$
\left(\mathbf{C}_{\frac{f0}{\mu}}(e^{i\lambda})\right)\left(\mathbf{C}_{\frac{f0}{\mu}}(e^{i\lambda})\right)^* = \alpha_f^2 \left(\frac{|\chi_{\overline{\mu}}^{(d)}(e^{-i\lambda})|^2}{|\beta^{(d)}(i\lambda)|^2}(f^0(\lambda) + |\beta^{(d)}(i\lambda)|^2 g^0(\lambda))\right)^2, \quad (81)
$$

$$
\left(\mathbf{C}_{\frac{g0}{\mu}}(e^{i\lambda})\right)\left(\mathbf{C}_{\frac{g0}{\mu}}(e^{i\lambda})\right)^* = (\beta^2 + \gamma_1(\lambda) + \gamma_2(\lambda)) \left(\frac{|\chi_{\overline{\mu}}^{(d)}(e^{-i\lambda})|^2}{|\beta^{(d)}(i\lambda)|^2}(f^0(\lambda) + |\beta^{(d)}(i\lambda)|^2 g^0(\lambda))\right)^2, \quad (82)
$$

where α_f^2, β^2 are Lagrange multipliers, the function $\gamma_1(\lambda) \leq 0$ and $\gamma_1(\lambda) = 0$ if $\mathrm{Tr}\,[g_0(\lambda)] > \mathrm{Tr}\,[V(\lambda)]$, the function $\gamma_2(\lambda) \geq 0$ and $\gamma_2(\lambda) = 0$ if $\mathrm{Tr}\,[g_0(\lambda)] < \mathrm{Tr}\,[U(\lambda)]$.

For the third set of admissible spectral densities $\mathcal{D}_{f0}^3 \times \mathcal{D}_{Vg}^{U3}$ we have equations

$$
\left(\mathbf{C}_{\frac{f0}{\mu}}(e^{i\lambda})\right)\left(\mathbf{C}_{\frac{f0}{\mu}}(e^{i\lambda})\right)^* = \left(\frac{|\chi_{\overline{\mu}}^{(d)}(e^{-i\lambda})|^2}{|\beta^{(d)}(i\lambda)|^2}(f^0(\lambda) + |\beta^{(d)}(i\lambda)|^2 g^0(\lambda))\right)\left\{\alpha_{fk}^2 \delta_{kl}\right\}_{k,l=1}^T \times
$$
$$
\times \left(\frac{|\chi_{\overline{\mu}}^{(d)}(e^{-i\lambda})|^2}{|\beta^{(d)}(i\lambda)|^2}(f^0(\lambda) + |\beta^{(d)}(i\lambda)|^2 g^0(\lambda))\right), \quad (83)
$$

$$
\left(\mathbf{C}_{\frac{g0}{\mu}}(e^{i\lambda})\right)\left(\mathbf{C}_{\frac{g0}{\mu}}(e^{i\lambda})\right)^* =
$$
$$
= \left(\frac{|\chi_{\overline{\mu}}^{(d)}(e^{-i\lambda})|^2}{|\beta^{(d)}(i\lambda)|^2}(f^0(\lambda) + |\beta^{(d)}(i\lambda)|^2 g^0(\lambda))\right)\left\{(\beta_k^2 + \gamma_{1k}(\lambda) + \gamma_{2k}(\lambda))\delta_{kl}\right\}_{k,l=1}^T \times
$$
$$
\times \left(\frac{|\chi_{\overline{\mu}}^{(d)}(e^{-i\lambda})|^2}{|\beta^{(d)}(i\lambda)|^2}(f^0(\lambda) + |\beta^{(d)}(i\lambda)|^2 g^0(\lambda))\right), \quad (84)
$$

where α_{fk}^2, β_k^2 are Lagrange multipliers, δ_{kl} are Kronecker symbols, functions $\gamma_{1k}(\lambda) \leq 0$ and $\gamma_{1k}(\lambda) = 0$ if $g_{kk}^0(\lambda) > v_{kk}(\lambda)$, functions $\gamma_{2k}(\lambda) \geq 0$ and $\gamma_{2k}(\lambda) = 0$ if $g_{kk}^0(\lambda) < u_{kk}(\lambda)$.

For the fourth set of admissible spectral densities $\mathcal{D}_{f0}^4 \times \mathcal{D}_{Vg}^{U4}$ we have equations

$$\left(\mathbf{C}_{\overline{\mu}}^{f0}(e^{i\lambda})\right)\left(\mathbf{C}_{\overline{\mu}}^{f0}(e^{i\lambda})\right)^* = \alpha_f^2 \left(\frac{|\chi_{\overline{\mu}}^{(d)}(e^{-i\lambda})|^2}{|\beta^{(d)}(i\lambda)|^2}(f^0(\lambda) + |\beta^{(d)}(i\lambda)|^2 g^0(\lambda))\right) B_1^\top \times$$
$$\times \left(\frac{|\chi_{\overline{\mu}}^{(d)}(e^{-i\lambda})|^2}{|\beta^{(d)}(i\lambda)|^2}(f^0(\lambda) + |\beta^{(d)}(i\lambda)|^2 g^0(\lambda))\right), \quad (85)$$

$$\left(\mathbf{C}_{\overline{\mu}}^{g0}(e^{i\lambda})\right)\left(\mathbf{C}_{\overline{\mu}}^{g0}(e^{i\lambda})\right)^* = (\beta^2 + \gamma_1'(\lambda) + \gamma_2'(\lambda))\left(\frac{|\chi_{\overline{\mu}}^{(d)}(e^{-i\lambda})|^2}{|\beta^{(d)}(i\lambda)|^2}(f^0(\lambda) + |\beta^{(d)}(i\lambda)|^2 g^0(\lambda))\right) B_2^\top \times$$
$$\times \left(\frac{|\chi_{\overline{\mu}}^{(d)}(e^{-i\lambda})|^2}{|\beta^{(d)}(i\lambda)|^2}(f^0(\lambda) + |\beta^{(d)}(i\lambda)|^2 g^0(\lambda))\right), \quad (86)$$

where β^2, α_f^2, are Lagrange multipliers, functions $\gamma_1'(\lambda) \leq 0$ and $\gamma_1'(\lambda) = 0$ if $\langle B_2, g_0(\lambda)\rangle > \langle B_2, V(\lambda)\rangle$, functions $\gamma_2'(\lambda) \geq 0$ and $\gamma_2'(\lambda) = 0$ if $\langle B_2, g_0(\lambda)\rangle < \langle B_2, U(\lambda)\rangle$.

The following theorem holds true.

Theorem 4.1
Let the minimality condition (18) hold true. The least favorable spectral densities $f_0(\lambda)$, $g_0(\lambda)$, in the classes $\mathcal{D}_{f0}^k \times \mathcal{D}_{Vg}^{Uk}$, $k = 1, 2, 3, 4$ for the optimal linear extrapolation of the functional $A\vec{\xi}$ from observations of the sequence $\vec{\xi}(m) + \vec{\eta}(m)$ at points $m = -1, -2, \ldots$ are determined by equations (79)–(80), (81)–(82), (83)–(84), (85)–(86), respectively, the constrained optimization problem (69) and restrictions on densities from the corresponding classes $\mathcal{D}_{f0}^k, \mathcal{D}_{Vg}^{Uk}$, $k = 1, 2, 3, 4$. The minimax-robust spectral characteristic of the optimal estimate of the functional $A\vec{\xi}$ is determined by the formula (33).

If the spectral densities $f(\lambda)$ and $g(\lambda)$ admit canonical factorizations (11), (44) and (45), we can derive the following equation for the least favourable spectral densities.

For the first set of admissible spectral densities $\mathcal{D}_{f0}^1 \times \mathcal{D}_{Vg}^{U1}$:

$$\left(\mathbf{r}_{\overline{\mu},f}^0(e^{i\lambda})\right)\left(\mathbf{r}_{\overline{\mu},f}^0(e^{i\lambda})\right)^* = (\Theta_{\overline{\mu}}(e^{-i\lambda}))^\top \vec{\alpha}_f \cdot \vec{\alpha}_f^* \overline{\Theta_{\overline{\mu}}(e^{-i\lambda})}, \quad (87)$$

$$\left(\mathbf{r}_{\overline{\mu},g}^0(e^{i\lambda})\right)\left(\mathbf{r}_{\overline{\mu},g}^0(e^{i\lambda})\right)^* = (\Theta_{\overline{\mu}}(e^{-i\lambda}))^\top (\vec{\beta} \cdot \vec{\beta}^* + \Gamma_1(\lambda) + \Gamma_2(\lambda))\overline{\Theta_{\overline{\mu}}(e^{-i\lambda})} \quad (88)$$

where $\vec{\alpha}_f$ and $\vec{\beta}$ are vectors of Lagrange multipliers, the matrix $\Gamma_1(\lambda) \leq 0$ and $\Gamma_1(\lambda) = 0$ if $g_0(\lambda) > V(\lambda)$, the matrix $\Gamma_2(\lambda) \geq 0$ and $\Gamma_2(\lambda) = 0$ if $g_0(\lambda) < U(\lambda)$.

For the second set of admissible spectral densities $\mathcal{D}_{f0}^2 \times \mathcal{D}_{Vg}^{U2}$ we have equations

$$\left(\mathbf{r}_{\overline{\mu},f}^0(e^{i\lambda})\right)\left(\mathbf{r}_{\overline{\mu},f}^0(e^{i\lambda})\right)^* = \alpha_f^2 (\Theta_{\overline{\mu}}(e^{-i\lambda}))^\top \overline{\Theta_{\overline{\mu}}(e^{-i\lambda})}, \quad (89)$$

$$\left(\mathbf{r}_{\overline{\mu},g}^0(e^{i\lambda})\right)\left(\mathbf{r}_{\overline{\mu},g}^0(e^{i\lambda})\right)^* = (\beta^2 + \gamma_1(\lambda) + \gamma_2(\lambda))(\Theta_{\overline{\mu}}(e^{-i\lambda}))^\top \overline{\Theta_{\overline{\mu}}(e^{-i\lambda})}, \quad (90)$$

where α_f^2, β^2 are Lagrange multipliers, the function $\gamma_1(\lambda) \leq 0$ and $\gamma_1(\lambda) = 0$ if $\mathrm{Tr}\,[g_0(\lambda)] > \mathrm{Tr}\,[V(\lambda)]$, the function $\gamma_2(\lambda) \geq 0$ and $\gamma_2(\lambda) = 0$ if $\mathrm{Tr}\,[g_0(\lambda)] < \mathrm{Tr}\,[U(\lambda)]$.

For the third set of admissible spectral densities $\mathcal{D}_{f0}^3 \times \mathcal{D}_{Vg}^{U3}$ we have equations

$$\left(\mathbf{r}_{\overline{\mu},f}^0(e^{i\lambda})\right)\left(\mathbf{r}_{\overline{\mu},f}^0(e^{i\lambda})\right)^* = (\Theta_{\overline{\mu}}(e^{-i\lambda}))^\top \left\{\alpha_{fk}^2 \delta_{kl}\right\}_{k,l=1}^T \overline{\Theta_{\overline{\mu}}(e^{-i\lambda})}, \quad (91)$$

$$\left(\mathbf{r}_{\overline{\mu},g}^0(e^{i\lambda})\right)\left(\mathbf{r}_{\overline{\mu},g}^0(e^{i\lambda})\right)^* = (\Theta_{\overline{\mu}}(e^{-i\lambda}))^\top \left\{(\beta_k^2 + \gamma_{1k}(\lambda) + \gamma_{2k}(\lambda))\delta_{kl}\right\}_{k,l=1}^T \overline{\Theta_{\overline{\mu}}(e^{-i\lambda})}, \quad (92)$$

where α_{fk}^2, β_k^2 are Lagrange multipliers, δ_{kl} are Kronecker symbols, functions $\gamma_{1k}(\lambda) \le 0$ and $\gamma_{1k}(\lambda) = 0$ if $g_{kk}^0(\lambda) > v_{kk}(\lambda)$, functions $\gamma_{2k}(\lambda) \ge 0$ and $\gamma_{2k}(\lambda) = 0$ if $g_{kk}^0(\lambda) < u_{kk}(\lambda)$.

For the fourth set of admissible spectral densities $\mathcal{D}_{f0}^4 \times \mathcal{D}_{Vg}^{U4}$ we have equations

$$\left(\mathbf{r}_{\overline{\mu},f}^0(e^{i\lambda})\right)\left(\mathbf{r}_{\overline{\mu},f}^0(e^{i\lambda})\right)^* = \alpha_f^2 (\Theta_{\overline{\mu}}(e^{-i\lambda}))^\top B_1 \overline{\Theta_{\overline{\mu}}(e^{-i\lambda})}, \tag{93}$$

$$\left(\mathbf{r}_{\overline{\mu},g}^0(e^{i\lambda})\right)\left(\mathbf{r}_{\overline{\mu},g}^0(e^{i\lambda})\right)^* = (\beta^2 + \gamma_1'(\lambda) + \gamma_2'(\lambda))(\Theta_{\overline{\mu}}(e^{-i\lambda}))^\top B_2 \overline{\Theta_{\overline{\mu}}(e^{-i\lambda})}, \tag{94}$$

where β^2, α_f^2, are Lagrange multipliers, functions $\gamma_1'(\lambda) \le 0$ and $\gamma_1'(\lambda) = 0$ if $\langle B_2, g_0(\lambda)\rangle > \langle B_2, V(\lambda)\rangle$, functions $\gamma_2'(\lambda) \ge 0$ and $\gamma_2'(\lambda) = 0$ if $\langle B_2, g_0(\lambda)\rangle < \langle B_2, U(\lambda)\rangle$.

The following theorem holds true.

Theorem 4.2
The least favorable spectral densities $f_0(\lambda)$, $g_0(\lambda)$ in the classes $\mathcal{D}_{f0}^k \times \mathcal{D}_{Vg}^{Uk}$, $k = 1, 2, 3, 4$ for the optimal linear forecasting of the functional $A\vec{\xi}$ from observations of the sequence $\vec{\xi}(m) + \vec{\eta}(m)$ at points $m = -1, -2, \ldots$ are determined by canonical factorizations (11), (44) and (45), equations (87)–(88), (89)–(90), (91)–(92), (93)–(94), respectively, constrained optimization problem (70) and restrictions on densities from the corresponding classes $\mathcal{D}_{f0}^k, \mathcal{D}_{Vg}^{Uk}, k = 1, 2, 3, 4$. The minimax-robust spectral characteristic of the optimal estimate of the functional $A\vec{\xi}$ is determined by the formula (33).

Theorem 4.3
If the spectral density $g(\lambda)$ is known, the least favorable spectral density $f_0(\lambda)$ in the classes $\mathcal{D}_{f0}^k$, $k = 1, 2, 3, 4$ for the optimal linear forecasting of the functional $A\vec{\xi}$ from observations of the sequence $\vec{\xi}(m) + \vec{\eta}(m)$ at points $m = -1, -2, \ldots$ is determined by canonical factorizations (11) and (44) , equations (87), (89), (91), (93), respectively, constrained optimization problem (75) and restrictions on density from the corresponding classes $\mathcal{D}_{f0}^k$, $k = 1, 2, 3, 4$. The minimax-robust spectral characteristic of the optimal estimate of the functional $A\vec{\xi}$ is determined by the formula (33).

4.2. *Least favorable spectral density in classes $\mathcal{D}_0 \times \mathcal{D}_V^U$ for cointegrated vector sequences*

Consider the forecasting problem for the functional $A\vec{\xi}$ which depends on unobserved values of a sequence $\vec{\xi}(m)$ with stationary increments based on observations of the vector sequence $\vec{\zeta}(m)$, cointegrated with $\vec{\xi}(m)$ in terms of Definition 3.1, at points $m = -1, -2, \ldots$ for the sets of admissible spectral densities $\mathcal{D}_{f0}^k, \mathcal{D}_{Vp}^{Uk}, k = 1, 2, 3, 4$, where the sets $\mathcal{D}_{f0}^k$ are defined in Subsection 4.1, the sets $\mathcal{D}_{Vp}^{Uk}$ are the following:

$$\mathcal{D}_{Vp}^{U1} = \left\{ p(\lambda) \middle| V(\lambda) \le p(\lambda) \le U(\lambda), \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{|\chi_{\overline{\mu}}^{(d)}(e^{-i\lambda})|^2}{|\beta^{(d)}(i\lambda)|^2} p(\lambda)d\lambda = Q \right\},$$

$$\mathcal{D}_{Vp}^{U}{}^2 = \left\{ p(\lambda) \middle| \text{Tr}\,[V(\lambda)] \le \text{Tr}\,[p(\lambda)] \le \text{Tr}\,[U(\lambda)], \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{|\chi_{\overline{\mu}}^{(d)}(e^{-i\lambda})|^2}{|\beta^{(d)}(i\lambda)|^2} \text{Tr}\,[p(\lambda)]d\lambda = q \right\},$$

$$\mathcal{D}_{Vp}^{U3} = \left\{ p(\lambda) \middle| v_{kk}(\lambda) \le p_{kk}(\lambda) \le u_{kk}(\lambda), \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{|\chi_{\overline{\mu}}^{(d)}(e^{-i\lambda})|^2}{|\beta^{(d)}(i\lambda)|^2} p_{kk}(\lambda)d\lambda = q_k, k = \overline{1, T} \right\},$$

$$\mathcal{D}_{Vp}^{U4} = \left\{ p(\lambda) \middle| \langle B_2, V(\lambda)\rangle \le \langle B_2, p(\lambda)\rangle \le \langle B_2, U(\lambda)\rangle, \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{|\chi_{\overline{\mu}}^{(d)}(e^{-i\lambda})|^2}{|\beta^{(d)}(i\lambda)|^2} \langle B_2, p(\lambda)\rangle d\lambda = q \right\}.$$

Here spectral densities $V(\lambda), U(\lambda)$ are known and fixed, $q, q_k, k = \overline{1, T}$ are given numbers, Q, B_2 are given positive-definite Hermitian matrices.

Define

$$\mathbf{C}_{\overline{\mu},\alpha}^{f0}(e^{i\lambda}) := \frac{\overline{\chi_{\overline{\mu}}^{(d)}(e^{-i\lambda})}}{|\beta^{(d)}(i\lambda)|^2} \vec{A}(e^{i\lambda})(p^0(\lambda) - \alpha^2 f^0(\lambda)) + \sum_{k=0}^{\infty} \left(\left((\mathbf{P}_{\overline{\mu}}^{\alpha})^0\right)^{-1}\left(D^{\overline{\mu}}\mathbf{a} - (\mathbf{T}_{\overline{\mu}}^{\alpha})^0 \mathbf{a}_{\overline{\mu}}\right)\right)_k e^{i\lambda k},$$

$$\mathbf{C}_{\overline{\mu},\alpha}^{p0}(e^{i\lambda}) := \frac{\overline{\chi_{\overline{\mu}}^{(d)}(e^{-i\lambda})}}{|\beta^{(d)}(i\lambda)|^2}\vec{A}(e^{i\lambda})f^0(\lambda) - \sum_{k=0}^{\infty}\left(\left((\mathbf{P}_{\overline{\mu}}^{\alpha})^0\right)^{-1}\left(D^{\overline{\mu}}\mathbf{a} - (\mathbf{T}_{\overline{\mu}}^{\alpha})^0\mathbf{a}_{\overline{\mu}}\right)\right)_k e^{i\lambda k}.$$

The condition $0 \in \partial\Delta_{\mathcal{D}}(f^0, p^0)$ implies the following equations which determine the least favourable spectral densities for these given sets of admissible spectral densities.

For the first set of admissible spectral densities $\mathcal{D}_{f0}^1 \times \mathcal{D}_{Vp}^{U1}$ we have equations

$$\left(\mathbf{C}_{\overline{\mu},\alpha}^{f0}(e^{i\lambda})\right)\left(\mathbf{C}_{\overline{\mu},\alpha}^{f0}(e^{i\lambda})\right)^* - \alpha^2\left(\mathbf{C}_{\overline{\mu},\alpha}^{p0}(e^{i\lambda})\right)\left(\mathbf{C}_{\overline{\mu},\alpha}^{p0}(e^{i\lambda})\right)^* =$$
$$= \left(\frac{|\chi_{\overline{\mu}}^{(d)}(e^{-i\lambda})|^2}{|\beta^{(d)}(i\lambda)|^2}p^0(\lambda)\right)\vec{\alpha}_f\cdot\vec{\alpha}_f^*\left(\frac{|\chi_{\overline{\mu}}^{(d)}(e^{-i\lambda})|^2}{|\beta^{(d)}(i\lambda)|^2}p^0(\lambda)\right), \quad (95)$$

$$(\mathbf{C}_{\overline{\mu},\alpha}^{p0}(e^{i\lambda}))(\mathbf{C}_{\overline{\mu},\alpha}^{p0}(e^{i\lambda}))^* = \left(\frac{|\chi_{\overline{\mu}}^{(d)}(e^{-i\lambda})|^2}{|\beta^{(d)}(i\lambda)|^2}p^0(\lambda)\right)(\vec{\beta}\cdot\vec{\beta}^* + \Gamma_1(\lambda) + \Gamma_2(\lambda))\left(\frac{|\chi_{\overline{\mu}}^{(d)}(e^{-i\lambda})|^2}{|\beta^{(d)}(i\lambda)|^2}p^0(\lambda)\right), \quad (96)$$

where $\vec{\alpha}_f$ and $\vec{\beta}$ are vectors of Lagrange multipliers, the matrix $\Gamma_1(\lambda) \leq 0$ and $\Gamma_1(\lambda) = 0$ if $g_0(\lambda) > V(\lambda)$, the matrix $\Gamma_2(\lambda) \geq 0$ and $\Gamma_2(\lambda) = 0$ if $p_0(\lambda) < U(\lambda)$.

For the second set of admissible spectral densities $\mathcal{D}_{f0}^2 \times \mathcal{D}_{Vp}^{U2}$ we have equations

$$\left(\mathbf{C}_{\overline{\mu},\alpha}^{f0}(e^{i\lambda})\right)\left(\mathbf{C}_{\overline{\mu},\alpha}^{f0}(e^{i\lambda})\right)^* - \alpha^2\left(\mathbf{C}_{\overline{\mu},\alpha}^{p0}(e^{i\lambda})\right)\left(\mathbf{C}_{\overline{\mu},\alpha}^{p0}(e^{i\lambda})\right)^* = \alpha_f^2\left(\frac{|\chi_{\overline{\mu}}^{(d)}(e^{-i\lambda})|^2}{|\beta^{(d)}(i\lambda)|^2}p^0(\lambda)\right)^2, \quad (97)$$

$$\left(\mathbf{C}_{\overline{\mu},\alpha}^{p0}(e^{i\lambda})\right)\left(\mathbf{C}_{\overline{\mu},\alpha}^{p0}(e^{i\lambda})\right)^* = (\beta^2 + \gamma_1(\lambda) + \gamma_2(\lambda))\left(\frac{|\chi_{\overline{\mu}}^{(d)}(e^{-i\lambda})|^2}{|\beta^{(d)}(i\lambda)|^2}p^0(\lambda)\right)^2, \quad (98)$$

where α_f^2, β^2 are Lagrange multipliers, the function $\gamma_1(\lambda) \leq 0$ and $\gamma_1(\lambda) = 0$ if $\mathrm{Tr}\,[p_0(\lambda)] > \mathrm{Tr}\,[V(\lambda)]$, the function $\gamma_2(\lambda) \geq 0$ and $\gamma_2(\lambda) = 0$ if $\mathrm{Tr}\,[p_0(\lambda)] < \mathrm{Tr}\,[U(\lambda)]$.

For the third set of admissible spectral densities $\mathcal{D}_{f0}^3 \times \mathcal{D}_{Vp}^{U3}$ we have equations

$$\left(\mathbf{C}_{\overline{\mu},\alpha}^{f0}(e^{i\lambda})\right)\left(\mathbf{C}_{\overline{\mu},\alpha}^{f0}(e^{i\lambda})\right)^* - \alpha^2\left(\mathbf{C}_{\overline{\mu},\alpha}^{p0}(e^{i\lambda})\right)\left(\mathbf{C}_{\overline{\mu},\alpha}^{p0}(e^{i\lambda})\right)^* =$$
$$= \left(\frac{|\chi_{\overline{\mu}}^{(d)}(e^{-i\lambda})|^2}{|\beta^{(d)}(i\lambda)|^2}p^0(\lambda)\right)\left\{\alpha_{fk}^2\delta_{kl}\right\}_{k,l=1}^T\left(\frac{|\chi_{\overline{\mu}}^{(d)}(e^{-i\lambda})|^2}{|\beta^{(d)}(i\lambda)|^2}p^0(\lambda)\right), \quad (99)$$

$$\left(\mathbf{C}_{\overline{\mu},\alpha}^{p0}(e^{i\lambda})\right)\left(\mathbf{C}_{\overline{\mu},\alpha}^{p0}(e^{i\lambda})\right)^* = \left(\frac{|\chi_{\overline{\mu}}^{(d)}(e^{-i\lambda})|^2}{|\beta^{(d)}(i\lambda)|^2}p^0(\lambda)\right)\times$$
$$\times\left\{(\beta_k^2 + \gamma_{1k}(\lambda) + \gamma_{2k}(\lambda))\delta_{kl}\right\}_{k,l=1}^T\left(\frac{|\chi_{\overline{\mu}}^{(d)}(e^{-i\lambda})|^2}{|\beta^{(d)}(i\lambda)|^2}p^0(\lambda)\right), \quad (100)$$

where α_{fk}^2, β_k^2 are Lagrange multipliers, δ_{kl} are Kronecker symbols, functions $\gamma_{1k}(\lambda) \leq 0$ and $\gamma_{1k}(\lambda) = 0$ if $p_{kk}^0(\lambda) > v_{kk}(\lambda)$, functions $\gamma_{2k}(\lambda) \geq 0$ and $\gamma_{2k}(\lambda) = 0$ if $p_{kk}^0(\lambda) < u_{kk}(\lambda)$.

For the fourth set of admissible spectral densities $\mathcal{D}_{f0}^4 \times \mathcal{D}_{Vp}^{U4}$ we have equations

$$\left(\mathbf{C}_{\overline{\mu},\alpha}^{f0}(e^{i\lambda})\right)\left(\mathbf{C}_{\overline{\mu},\alpha}^{f0}(e^{i\lambda})\right)^* - \alpha^2\left(\mathbf{C}_{\overline{\mu},\alpha}^{p0}(e^{i\lambda})\right)\left(\mathbf{C}_{\overline{\mu},\alpha}^{p0}(e^{i\lambda})\right)^* =$$
$$= \alpha_f^2\left(\frac{|\chi_{\overline{\mu}}^{(d)}(e^{-i\lambda})|^2}{|\beta^{(d)}(i\lambda)|^2}p^0(\lambda)\right)B_1^\top\left(\frac{|\chi_{\overline{\mu}}^{(d)}(e^{-i\lambda})|^2}{|\beta^{(d)}(i\lambda)|^2}p^0(\lambda)\right), \quad (101)$$

$$\left(\mathbf{C}^{p0}_{\overline{\mu},\alpha}(e^{i\lambda})\right)\left(\mathbf{C}^{p0}_{\overline{\mu},\alpha}(e^{i\lambda})\right)^{*} = (\beta^2 + \gamma'_1(\lambda) + \gamma'_2(\lambda))\left(\frac{|\chi^{(d)}_{\overline{\mu}}(e^{-i\lambda})|^2}{|\beta^{(d)}(i\lambda)|^2}p^0(\lambda)\right)B_2^{\top} \times$$

$$\times \left(\frac{|\chi^{(d)}_{\overline{\mu}}(e^{-i\lambda})|^2}{|\beta^{(d)}(i\lambda)|^2}p^0(\lambda)\right), \quad (102)$$

where α_f^2, β^2, are Lagrange multipliers, functions $\gamma'_1(\lambda) \leq 0$ and $\gamma'_1(\lambda) = 0$ if $\langle B_2, p_0(\lambda)\rangle > \langle B_2, V(\lambda)\rangle$, functions $\gamma'_2(\lambda) \geq 0$ and $\gamma'_2(\lambda) = 0$ if $\langle B_2, p_0(\lambda)\rangle < \langle B_2, U(\lambda)\rangle$.

The following theorem holds true.

Theorem 4.4
Let the minimality condition (56) hold true. The least favorable spectral densities $f_0(\lambda)$, $p_0(\lambda)$, in the classes $\mathcal{D}^k_{f0} \times \mathcal{D}^{Uk}_{Vp}, k = 1, 2, 3, 4$ for the optimal linear forecasting of the functional $A\vec{\xi}$ from observations of the vector sequence $\vec{\zeta}(m)$, cointegrated with $\vec{\xi}(m)$ in terms of Definition 3.1, at points $m = -1, -2, \dots$ are determined by equations (95)–(96), (97)–(98), (99)–(100), (101)–(102), respectively, the constrained optimization problem (69) with $g(\lambda) := |\beta^{(d)}(i\lambda)|^{-2}(p(\lambda) - \alpha^2 f(\lambda))$ and restrictions on densities from the corresponding classes $\mathcal{D}^k_{f0}, \mathcal{D}^{Uk}_{Vp}, k = 1, 2, 3, 4$. The minimax-robust spectral characteristic of the optimal estimate of the functional $A\vec{\xi}$ is determined by the formula (59).

4.3. *Least favorable spectral density in classes* $\mathcal{D}_{\varepsilon} \times \mathcal{D}_{1\delta}$

Consider the prediction problem for the functional $A\vec{\xi}$ which depends on unobserved values of a sequence $\vec{\xi}(m)$ with stationary increments based on observations of the sequence $\vec{\xi}(m) + \vec{\eta}(m)$ at points $m = -1, -2, \dots$ under the condition that the sets of admissible spectral densities $\mathcal{D}^k_{f\varepsilon}, \mathcal{D}^k_{g1\delta}, k = 1, 2, 3, 4$ are defined as follows:

$$\mathcal{D}^1_{f\varepsilon} = \left\{f(\lambda)\middle|\mathrm{Tr}\,[f(\lambda)] = (1-\varepsilon)\mathrm{Tr}\,[f_1(\lambda)] + \varepsilon\mathrm{Tr}\,[W(\lambda)], \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{|\chi^{(d)}_{\overline{\mu}}(e^{-i\lambda})|^2}{|\beta^{(d)}(i\lambda)|^2}\mathrm{Tr}\right.$$

$$\mathcal{D}^2_{f\varepsilon} = \left\{f(\lambda)\middle|f_{kk}(\lambda) = (1-\varepsilon)f^1_{kk}(\lambda) + \varepsilon w_{kk}(\lambda), \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{|\chi^{(d)}_{\overline{\mu}}(e^{-i\lambda})|^2}{|\beta^{(d)}(i\lambda)|^2}f_{kk}(\lambda)d\lambda = p_k, k = \overline{1,T}\right\};$$

$$\mathcal{D}^3_{f\varepsilon} = \left\{f(\lambda)\middle|\langle B_1, f(\lambda)\rangle = (1-\varepsilon)\langle B_1, f_1(\lambda)\rangle + \varepsilon\langle B_1, W(\lambda)\rangle, \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{|\chi^{(d)}_{\overline{\mu}}(e^{-i\lambda})|^2}{|\beta^{(d)}(i\lambda)|^2}\langle B_1, f(\lambda)\rangle\,d\lambda = p\right\};$$

$$\mathcal{D}^4_{f\varepsilon} = \left\{f(\lambda)\middle|f(\lambda) = (1-\varepsilon)f_1(\lambda) + \varepsilon W(\lambda), \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{|\chi^{(d)}_{\overline{\mu}}(e^{-i\lambda})|^2}{|\beta^{(d)}(i\lambda)|^2}f(\lambda)d\lambda = P\right\}.$$

$$\mathcal{D}^1_{g1\delta} = \left\{g(\lambda)\middle|\frac{1}{2\pi}\int_{-\pi}^{\pi}|\mathrm{Tr}(g(\lambda) - g_1(\lambda))|\,d\lambda \leq \delta\right\};$$

$$\mathcal{D}^2_{g1\delta} = \left\{g(\lambda)\middle|\frac{1}{2\pi}\int_{-\pi}^{\pi}\left|g_{kk}(\lambda) - g^1_{kk}(\lambda)\right|\,d\lambda \leq \delta_k, k = \overline{1,T}\right\};$$

$$\mathcal{D}^3_{g1\delta} = \left\{g(\lambda)\middle|\frac{1}{2\pi}\int_{-\pi}^{\pi}|\langle B_2, g(\lambda) - g_1(\lambda)\rangle|\,d\lambda \leq \delta\right\};$$

$$\mathcal{D}^4_{g1\delta} = \left\{g(\lambda)\middle|\frac{1}{2\pi}\int_{-\pi}^{\pi}\left|g_{ij}(\lambda) - g^1_{ij}(\lambda)\right|\,d\lambda \leq \delta^j_i, i, j = \overline{1,T}\right\}.$$

Here $f_1(\lambda)$, $g_1(\lambda)$ are fixed spectral densities, $W(\lambda)$ is an unknown spectral density, $p, p_k, k = \overline{1,T}$, are given numbers, P is a given positive-definite Hermitian matrices, $\delta, \delta_k, k = \overline{1,T}, \delta^j_i, i, j = \overline{1,T}$, are given numbers.

From the condition $0 \in \partial\Delta_{\mathcal{D}}(f^0, g^0)$ we find the following equations which determine the least favourable spectral densities for these given sets of admissible spectral densities.

For the first set of admissible spectral densities $\mathcal{D}^1_{f\varepsilon} \times \mathcal{D}^1_{g1\delta}$ we have equations

$$\left(\mathbf{C}^{f0}_{\overline{\mu}}(e^{i\lambda})\right)\left(\mathbf{C}^{f0}_{\overline{\mu}}(e^{i\lambda})\right)^* = (\alpha_f^2 + \gamma_1(\lambda))\left(\frac{|\chi^{(d)}_{\overline{\mu}}(e^{-i\lambda})|^2}{|\beta^{(d)}(i\lambda)|^2}(f^0(\lambda) + |\beta^{(d)}(i\lambda)|^2 g^0(\lambda))\right)^2, \qquad (103)$$

$$\left(\mathbf{C}^{g0}_{\overline{\mu}}(e^{i\lambda})\right)\left(\mathbf{C}^{g0}_{\overline{\mu}}(e^{i\lambda})\right)^* = \beta^2 \gamma_2(\lambda)\left(\frac{|\chi^{(d)}_{\overline{\mu}}(e^{-i\lambda})|^2}{|\beta^{(d)}(i\lambda)|^2}(f^0(\lambda) + |\beta^{(d)}(i\lambda)|^2 g^0(\lambda))\right)^2, \qquad (104)$$

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}|\mathrm{Tr}\,(g_0(\lambda) - g_1(\lambda))|\,d\lambda = \delta, \qquad (105)$$

where α_f^2, β^2 are Lagrange multipliers, the function $\gamma_1(\lambda) \le 0$ and $\gamma_1(\lambda) = 0$ if $\mathrm{Tr}\,[f_0(\lambda)] > (1 - \varepsilon)\mathrm{Tr}\,[f_1(\lambda)]$, the function $|\gamma_2(\lambda)| \le 1$ and

$$\gamma_2(\lambda) = \mathrm{sign}\,(\mathrm{Tr}\,(g_0(\lambda) - g_1(\lambda))): \ \mathrm{Tr}\,(g_0(\lambda) - g_1(\lambda)) \neq 0.$$

For the second set of admissible spectral densities $\mathcal{D}^2_{f\varepsilon} \times \mathcal{D}^2_{g1\delta}$ we have equation

$$\left(\mathbf{C}^{f0}_{\overline{\mu}}(e^{i\lambda})\right)\left(\mathbf{C}^{f0}_{\overline{\mu}}(e^{i\lambda})\right)^* =$$
$$= \left(\frac{|\chi^{(d)}_{\overline{\mu}}(e^{-i\lambda})|^2}{|\beta^{(d)}(i\lambda)|^2}(f^0(\lambda) + |\beta^{(d)}(i\lambda)|^2 g^0(\lambda))\right)\left\{(\alpha_{fk}^2 + \gamma_k^1(\lambda))\delta_{kl}\right\}_{k,l=1}^T \times$$
$$\times \left(\frac{|\chi^{(d)}_{\overline{\mu}}(e^{-i\lambda})|^2}{|\beta^{(d)}(i\lambda)|^2}(f^0(\lambda) + |\beta^{(d)}(i\lambda)|^2 g^0(\lambda))\right), \quad (106)$$

$$\left(\mathbf{C}^{g0}_{\overline{\mu}}(e^{i\lambda})\right)\left(\mathbf{C}^{g0}_{\overline{\mu}}(e^{i\lambda})\right)^* =$$
$$= \left(\frac{|\chi^{(d)}_{\overline{\mu}}(e^{-i\lambda})|^2}{|\beta^{(d)}(i\lambda)|^2}(f^0(\lambda) + |\beta^{(d)}(i\lambda)|^2 g^0(\lambda))\right)\left\{\beta_k^2 \gamma_k^2(\lambda)\delta_{kl}\right\}_{k,l=1}^T \times$$
$$\times \left(\frac{|\chi^{(d)}_{\overline{\mu}}(e^{-i\lambda})|^2}{|\beta^{(d)}(i\lambda)|^2}(f^0(\lambda) + |\beta^{(d)}(i\lambda)|^2 g^0(\lambda))\right), \quad (107)$$

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\left|g_{kk}^0(\lambda) - g_{kk}^1(\lambda)\right|\,d\lambda = \delta_k, \qquad (108)$$

where α_{fk}^2, β_k^2 are Lagrange multipliers, functions $\gamma_k^1(\lambda) \le 0$ and $\gamma_k^1(\lambda) = 0$ if $f_{kk}^0(\lambda) > (1 - \varepsilon)f_{kk}^1(\lambda)$, functions $\left|\gamma_k^2(\lambda)\right| \le 1$ and

$$\gamma_k^2(\lambda) = \mathrm{sign}\,(g_{kk}^0(\lambda) - g_{kk}^1(\lambda)): \ g_{kk}^0(\lambda) - g_{kk}^1(\lambda) \neq 0, \ k = \overline{1,T}.$$

For the third set of admissible spectral densities $\mathcal{D}^3_{f\varepsilon} \times \mathcal{D}^3_{g1\delta}$ we have equation

$$\left(\mathbf{C}^{f0}_{\overline{\mu}}(e^{i\lambda})\right)\left(\mathbf{C}^{f0}_{\overline{\mu}}(e^{i\lambda})\right)^* =$$
$$= (\alpha_f^2 + \gamma_1'(\lambda))\left(\frac{|\chi^{(d)}_{\overline{\mu}}(e^{-i\lambda})|^2}{|\beta^{(d)}(i\lambda)|^2}(f^0(\lambda) + |\beta^{(d)}(i\lambda)|^2 g^0(\lambda))\right) B_1^\top \times$$
$$\times \left(\frac{|\chi^{(d)}_{\overline{\mu}}(e^{-i\lambda})|^2}{|\beta^{(d)}(i\lambda)|^2}(f^0(\lambda) + |\beta^{(d)}(i\lambda)|^2 g^0(\lambda))\right), \quad (109)$$

$$
\left(\mathbf{C}_{\overline{\mu}}^{g0}(e^{i\lambda})\right)\left(\mathbf{C}_{\overline{\mu}}^{g0}(e^{i\lambda})\right)^{*} =
$$

$$
= \beta^2 \gamma_2'(\lambda) \left(\frac{|\chi_{\overline{\mu}}^{(d)}(e^{-i\lambda})|^2}{|\beta^{(d)}(i\lambda)|^2}(f^0(\lambda) + |\beta^{(d)}(i\lambda)|^2 g^0(\lambda)) \right) B_2^\top \times
$$

$$
\times \left(\frac{|\chi_{\overline{\mu}}^{(d)}(e^{-i\lambda})|^2}{|\beta^{(d)}(i\lambda)|^2}(f^0(\lambda) + |\beta^{(d)}(i\lambda)|^2 g^0(\lambda)) \right), \quad (110)
$$

$$
\frac{1}{2\pi} \int_{-\pi}^{\pi} |\langle B_2, g_0(\lambda) - g_1(\lambda)\rangle| \, d\lambda = \delta, \tag{111}
$$

where α_f^2, β^2 are Lagrange multipliers, function $\gamma_1'(\lambda) \le 0$ and $\gamma_1'(\lambda) = 0$ if $\langle B_1, f_0(\lambda)\rangle > (1-\varepsilon)\langle B_1, f_1(\lambda)\rangle$, function $|\gamma_2'(\lambda)| \le 1$ and

$$
\gamma_2'(\lambda) = \text{sign } \langle B_2, g_0(\lambda) - g_1(\lambda)\rangle : \ \langle B_2, g_0(\lambda) - g_1(\lambda)\rangle \neq 0.
$$

For the fourth set of admissible spectral densities $\mathcal{D}_{f\varepsilon}^4 \times \mathcal{D}_{g1\delta}^4$ we have equation

$$
\left(\mathbf{C}_{\overline{\mu}}^{f0}(e^{i\lambda})\right)\left(\mathbf{C}_{\overline{\mu}}^{f0}(e^{i\lambda})\right)^{*} =
$$

$$
= \left(\frac{|\chi_{\overline{\mu}}^{(d)}(e^{-i\lambda})|^2}{|\beta^{(d)}(i\lambda)|^2}(f^0(\lambda) + |\beta^{(d)}(i\lambda)|^2 g^0(\lambda)) \right)(\vec{\alpha}_f \cdot \vec{\alpha}_f^{*} + \Gamma(\lambda)) \times
$$

$$
\times \left(\frac{|\chi_{\overline{\mu}}^{(d)}(e^{-i\lambda})|^2}{|\beta^{(d)}(i\lambda)|^2}(f^0(\lambda) + |\beta^{(d)}(i\lambda)|^2 g^0(\lambda)) \right), \quad (112)
$$

$$
\left(\mathbf{C}_{\overline{\mu}}^{g0}(e^{i\lambda})\right)\left(\mathbf{C}_{\overline{\mu}}^{g0}(e^{i\lambda})\right)^{*} =
$$

$$
= \left(\frac{|\chi_{\overline{\mu}}^{(d)}(e^{-i\lambda})|^2}{|\beta^{(d)}(i\lambda)|^2}(f^0(\lambda) + |\beta^{(d)}(i\lambda)|^2 g^0(\lambda)) \right) \{\beta_{ij}(\lambda)\gamma_{ij}(\lambda)\}_{i,j=1}^{T} \times
$$

$$
\times \left(\frac{|\chi_{\overline{\mu}}^{(d)}(e^{-i\lambda})|^2}{|\beta^{(d)}(i\lambda)|^2}(f^0(\lambda) + |\beta^{(d)}(i\lambda)|^2 g^0(\lambda)) \right), \quad (113)
$$

$$
\frac{1}{2\pi} \int_{-\pi}^{\pi} |g_{ij}^0(\lambda) - g_{ij}^1(\lambda)| \, d\lambda = \delta_i^j, \tag{114}
$$

where $\vec{\alpha}_f$, β_{ij} are Lagrange multipliers, function $\Gamma(\lambda) \le 0$ and $\Gamma(\lambda) = 0$ if $f_0(\lambda) > (1-\varepsilon)f_1(\lambda)$, functions $|\gamma_{ij}(\lambda)| \le 1$ and

$$
\gamma_{ij}(\lambda) = \frac{g_{ij}^0(\lambda) - g_{ij}^1(\lambda)}{|g_{ij}^0(\lambda) - g_{ij}^1(\lambda)|} : \ g_{ij}^0(\lambda) - g_{ij}^1(\lambda) \neq 0, \ i,j = \overline{1,T}.
$$

The following theorem holds true.

Theorem 4.5
Let the minimality condition (18) hold true. The least favorable spectral densities $f_0(\lambda)$, $g_0(\lambda)$ in classes $\mathcal{D}_{f\varepsilon}^k \times \mathcal{D}_{g1\delta}^k, k = 1, 2, 3, 4$ for the optimal linear extrapolation of the functional $A\vec{\xi}$ from observations of the vector sequence $\vec{\xi}(m) + \vec{\eta}(m)$ at points $m = -1, -2, \ldots$ are determined by equations (103) – (105), (106) – (108), (109) – (111), (112) – (114), respectively, the constrained optimization problem (69) and restrictions on densities from the corresponding classes $\mathcal{D}_{f\varepsilon}^k, \mathcal{D}_{g1\delta}^k, k = 1, 2, 3, 4$. The minimax-robust spectral characteristic of the optimal estimate of the functional $A\vec{\xi}$ is determined by the formula (33).

Let the spectral densities $f(\lambda)$ and $g(\lambda)$ admit canonical factorizations (11), (44) and (45). Then we derive the following equation for the least favourable spectral densities.

For the first set of admissible spectral densities $\mathcal{D}^1_{f\varepsilon} \times \mathcal{D}^1_{g1\delta}$ we have equations

$$\left(\mathbf{r}^0_{\overline{\mu},f}(e^{i\lambda})\right)\left(\mathbf{r}^0_{\overline{\mu},f}(e^{i\lambda})\right)^* = (\alpha^2_f + \gamma_1(\lambda))(\Theta_{\overline{\mu}}(e^{-i\lambda}))^\top \overline{\Theta_{\overline{\mu}}(e^{-i\lambda})}, \tag{115}$$

$$\left(\mathbf{r}^0_{\overline{\mu},g}(e^{i\lambda})\right)\left(\mathbf{r}^0_{\overline{\mu},g}(e^{i\lambda})\right)^* = \beta^2\gamma_2(\lambda)(\Theta_{\overline{\mu}}(e^{-i\lambda}))^\top \overline{\Theta_{\overline{\mu}}(e^{-i\lambda})}, \tag{116}$$

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}|\mathrm{Tr}\,(g_0(\lambda) - g_1(\lambda))|\,d\lambda = \delta, \tag{117}$$

where α^2_f, β^2 are Lagrange multipliers, the function $\gamma_1(\lambda) \le 0$ and $\gamma_1(\lambda) = 0$ if $\mathrm{Tr}\,[f_0(\lambda)] > (1-\varepsilon)\mathrm{Tr}\,[f_1(\lambda)]$, the function $|\gamma_2(\lambda)| \le 1$ and

$$\gamma_2(\lambda) = \mathrm{sign}\,(\mathrm{Tr}\,(g_0(\lambda) - g_1(\lambda))) : \ \mathrm{Tr}\,(g_0(\lambda) - g_1(\lambda)) \neq 0.$$

For the second set of admissible spectral densities $\mathcal{D}^2_{f\varepsilon} \times \mathcal{D}^2_{g1\delta}$ we have equation

$$\left(\mathbf{r}^0_{\overline{\mu},f}(e^{i\lambda})\right)\left(\mathbf{r}^0_{\overline{\mu},f}(e^{i\lambda})\right)^* = (\Theta_{\overline{\mu}}(e^{-i\lambda}))^\top \left\{(\alpha^2_{fk} + \gamma^1_k(\lambda))\delta_{kl}\right\}^T_{k,l=1} \overline{\Theta_{\overline{\mu}}(e^{-i\lambda})}, \tag{118}$$

$$\left(\mathbf{r}^0_{\overline{\mu},g}(e^{i\lambda})\right)\left(\mathbf{r}^0_{\overline{\mu},g}(e^{i\lambda})\right)^* = (\Theta_{\overline{\mu}}(e^{-i\lambda}))^\top \left\{\beta^2_k\gamma^2_k(\lambda)\delta_{kl}\right\}^T_{k,l=1} \overline{\Theta_{\overline{\mu}}(e^{-i\lambda})}, \tag{119}$$

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\left|g^0_{kk}(\lambda) - g^1_{kk}(\lambda)\right|d\lambda = \delta_k, \tag{120}$$

where α^2_{fk}, β^2_k are Lagrange multipliers, functions $\gamma^1_k(\lambda) \le 0$ and $\gamma^1_k(\lambda) = 0$ if $f^0_{kk}(\lambda) > (1-\varepsilon)f^1_{kk}(\lambda)$, functions $\left|\gamma^2_k(\lambda)\right| \le 1$ and

$$\gamma^2_k(\lambda) = \mathrm{sign}\,(g^0_{kk}(\lambda) - g^1_{kk}(\lambda)) : \ g^0_{kk}(\lambda) - g^1_{kk}(\lambda) \neq 0, \ k = \overline{1, T}.$$

For the third set of admissible spectral densities $\mathcal{D}^3_{f\varepsilon} \times \mathcal{D}^3_{g1\delta}$ we have equation

$$\left(\mathbf{r}^0_{\overline{\mu},f}(e^{i\lambda})\right)\left(\mathbf{r}^0_{\overline{\mu},f}(e^{i\lambda})\right)^* = (\alpha^2_f + \gamma'_1(\lambda))(\Theta_{\overline{\mu}}(e^{-i\lambda}))^\top B_1 \overline{\Theta_{\overline{\mu}}(e^{-i\lambda})}, \tag{121}$$

$$\left(\mathbf{r}^0_{\overline{\mu},g}(e^{i\lambda})\right)\left(\mathbf{r}^0_{\overline{\mu},g}(e^{i\lambda})\right)^* = \beta^2\gamma'_2(\lambda)(\Theta_{\overline{\mu}}(e^{-i\lambda}))^\top B_2 \overline{\Theta_{\overline{\mu}}(e^{-i\lambda})}, \tag{122}$$

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}|\langle B_2, g_0(\lambda) - g_1(\lambda)\rangle|\,d\lambda = \delta, \tag{123}$$

where α^2_f, β^2 are Lagrange multipliers, function $\gamma'_1(\lambda) \le 0$ and $\gamma'_1(\lambda) = 0$ if $\langle B_1, f_0(\lambda)\rangle > (1-\varepsilon)\langle B_1, f_1(\lambda)\rangle$, function $|\gamma'_2(\lambda)| \le 1$ and

$$\gamma'_2(\lambda) = \mathrm{sign}\,\langle B_2, g_0(\lambda) - g_1(\lambda)\rangle : \ \langle B_2, g_0(\lambda) - g_1(\lambda)\rangle \neq 0.$$

For the fourth set of admissible spectral densities $\mathcal{D}^4_{f\varepsilon} \times \mathcal{D}^4_{g1\delta}$ we have equation

$$\left(\mathbf{r}^0_{\overline{\mu},f}(e^{i\lambda})\right)\left(\mathbf{r}^0_{\overline{\mu},f}(e^{i\lambda})\right)^* = (\Theta_{\overline{\mu}}(e^{-i\lambda}))^\top (\vec{\alpha}_f \cdot \vec{\alpha}^*_f + \Gamma(\lambda))\overline{\Theta_{\overline{\mu}}(e^{-i\lambda})}, \tag{124}$$

$$\left(\mathbf{r}^0_{\overline{\mu},g}(e^{i\lambda})\right)\left(\mathbf{r}^0_{\overline{\mu},g}(e^{i\lambda})\right)^* = (\Theta_{\overline{\mu}}(e^{-i\lambda}))^\top \{\beta_{ij}(\lambda)\gamma_{ij}(\lambda)\}^T_{i,j=1} \overline{\Theta_{\overline{\mu}}(e^{-i\lambda})}, \tag{125}$$

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\left|g^0_{ij}(\lambda) - g^1_{ij}(\lambda)\right|d\lambda = \delta^j_i, \tag{126}$$

where $\vec{\alpha}_f$, β_{ij} are Lagrange multipliers, function $\Gamma(\lambda) \le 0$ and $\Gamma(\lambda) = 0$ if $f_0(\lambda) > (1-\varepsilon)f_1(\lambda)$, functions $|\gamma_{ij}(\lambda)| \le 1$ and

$$\gamma_{ij}(\lambda) = \frac{g^0_{ij}(\lambda) - g^1_{ij}(\lambda)}{\left|g^0_{ij}(\lambda) - g^1_{ij}(\lambda)\right|} : \ g^0_{ij}(\lambda) - g^1_{ij}(\lambda) \neq 0, \ i, j = \overline{1, T}.$$

The following theorem holds true.

Theorem 4.6

The least favorable spectral densities $f_0(\lambda)$, $g_0(\lambda)$ in the classes $\mathcal{D}^k_{f\varepsilon} \times \mathcal{D}^k_{g1\delta}, k = 1, 2, 3, 4$ for the optimal linear forecasting of the functional $A\vec{\xi}$ from observations of the vector sequence $\vec{\xi}(m) + \vec{\eta}(m)$ at points $m = -1, -2, \ldots$ by canonical factorizations (11), (44) and (45), equations (115) – (117), (118) – (120), (121) – (123), (124) – (126), respectively, the constrained optimization problem (70) and restrictions on densities from the corresponding classes $\mathcal{D}^k_{f\varepsilon}, \mathcal{D}^k_{g1\delta}, k = 1, 2, 3, 4$. The minimax-robust spectral characteristic of the optimal estimate of the functional $A\vec{\xi}$ is determined by the formula (33).

Theorem 4.7

If the spectral density $g(\lambda)$ is known, the least favorable spectral density $f_0(\lambda)$ in the classes $\mathcal{D}^k_{f\varepsilon}$, $k = 1, 2, 3, 4$ for the optimal linear forecasting of the functional $A\vec{\xi}$ from observations of the sequence $\vec{\xi}(m) + \vec{\eta}(m)$ at points $m = -1, -2, \ldots$ is determined by canonical factorizations (11) and (44) , equations (115), (118), (121), (124), respectively, constrained optimization problem (75) and restrictions on density from the corresponding classes $\mathcal{D}^k_{f\varepsilon}$, $k = 1, 2, 3, 4$. The minimax-robust spectral characteristic of the optimal estimate of the functional $A\vec{\xi}$ is determined by the formula (33).

4.4. Least favorable spectral density for cointegrated vector sequences in classes $\mathcal{D}_\varepsilon \times \mathcal{D}_{1\delta}$

Consider the minimax forecasting problem for the functional $A\vec{\xi}$ which depends on unobserved values of a vector sequence $\vec{\xi}(m)$ with GM increments based on observations of the Vector sequence $\vec{\zeta}(m)$, cointegrated with $\vec{\xi}(m)$ in terms of Definition 3.1, at points $m = -1, -2, \ldots$ for the sets of admissible spectral densities $\mathcal{D}^k_{f\varepsilon}$, $k = 1, 2, 3, 4$, defined in Subsection 4.3 and $\mathcal{D}^k_{p1\delta}$, $k = 1, 2, 3, 4$, defined as follows:

$$\mathcal{D}^1_{p1\delta} = \left\{ g(\lambda) \left| \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{|\chi^{(d)}_{\overline{\mu}}(e^{-i\lambda})|^2}{|\beta^{(d)}(i\lambda)|^2} \left| \mathrm{Tr}(p(\lambda) - p_1(\lambda)) \right| d\lambda \leq \delta \right. \right\};$$

$$\mathcal{D}^2_{p1\delta} = \left\{ p(\lambda) \left| \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{|\chi^{(d)}_{\overline{\mu}}(e^{-i\lambda})|^2}{|\beta^{(d)}(i\lambda)|^2} \left| p_{kk}(\lambda) - p^1_{kk}(\lambda) \right| d\lambda \leq \delta_k, k = \overline{1, T} \right. \right\};$$

$$\mathcal{D}^3_{p1\delta} = \left\{ p(\lambda) \left| \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{|\chi^{(d)}_{\overline{\mu}}(e^{-i\lambda})|^2}{|\beta^{(d)}(i\lambda)|^2} \left| \langle B_2, p(\lambda) - p_1(\lambda) \rangle \right| d\lambda \leq \delta \right. \right\};$$

$$\mathcal{D}^4_{p1\delta} = \left\{ p(\lambda) \left| \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{|\chi^{(d)}_{\overline{\mu}}(e^{-i\lambda})|^2}{|\beta^{(d)}(i\lambda)|^2} \left| p_{ij}(\lambda) - p^1_{ij}(\lambda) \right| d\lambda \leq \delta^j_i, i, j = \overline{1, T} \right. \right\}.$$

Here $p_1(\lambda)$ is a fixed spectral density, $\delta, \delta_k, k = \overline{1, T}, \delta^j_i, i, j = \overline{1, T}$, are given numbers.

The condition $0 \in \partial\Delta_\mathcal{D}(f^0, g^0)$ implies the following equations which determine the least favourable spectral densities for these given sets of admissible spectral densities.

For the first set of admissible spectral densities $\mathcal{D}^1_{f\varepsilon} \times \mathcal{D}^1_{p1\delta}$ we have equations

$$\left(\mathbf{C}^{f0}_{\overline{\mu},\alpha}(e^{i\lambda}) \right) \left(\mathbf{C}^{f0}_{\overline{\mu},\alpha}(e^{i\lambda}) \right)^* - \alpha^2 \left(\mathbf{C}^{p0}_{\overline{\mu},\alpha}(e^{i\lambda}) \right) \left(\mathbf{C}^{p0}_{\overline{\mu},\alpha}(e^{i\lambda}) \right)^* = (\alpha^2_f + \gamma_1(\lambda)) \left(\frac{|\chi^{(d)}_{\overline{\mu}}(e^{-i\lambda})|^2}{|\beta^{(d)}(i\lambda)|^2} p^0(\lambda) \right)^2, \quad (127)$$

$$\left(\mathbf{C}^{p0}_{\overline{\mu},\alpha}(e^{i\lambda}) \right) \left(\mathbf{C}^{p0}_{\overline{\mu},\alpha}(e^{i\lambda}) \right)^* = \beta^2 \gamma_2(\lambda) \left(\frac{|\chi^{(d)}_{\overline{\mu}}(e^{-i\lambda})|^2}{|\beta^{(d)}(i\lambda)|^2} p^0(\lambda) \right)^2, \quad (128)$$

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{|\chi^{(d)}_{\overline{\mu}}(e^{-i\lambda})|^2}{|\beta^{(d)}(i\lambda)|^2} \left| \mathrm{Tr}\left(p_0(\lambda) - p_1(\lambda) \right) \right| d\lambda = \delta, \quad (129)$$

where α_f^2, β^2 are Lagrange multipliers, the function $\gamma_1(\lambda) \leq 0$ and $\gamma_1(\lambda) = 0$ if $\mathrm{Tr}\,[f_0(\lambda)] > (1-\varepsilon)\mathrm{Tr}\,[f_1(\lambda)]$, the function $|\gamma_2(\lambda)| \leq 1$ and

$$\gamma_2(\lambda) = \mathrm{sign}\,(\mathrm{Tr}\,(p_0(\lambda) - p_1(\lambda))) : \ \mathrm{Tr}\,(p_0(\lambda) - p_1(\lambda)) \neq 0.$$

For the second set of admissible spectral densities $\mathcal{D}_{f\varepsilon}^2 \times \mathcal{D}_{p1\delta}^2$ we have equation

$$\left(\mathbf{C}_{\overline{\mu},\alpha}^{f0}(e^{i\lambda})\right)\left(\mathbf{C}_{\overline{\mu},\alpha}^{f0}(e^{i\lambda})\right)^* - \alpha^2 \left(\mathbf{C}_{\overline{\mu},\alpha}^{p0}(e^{i\lambda})\right)\left(\mathbf{C}_{\overline{\mu},\alpha}^{p0}(e^{i\lambda})\right)^* =$$

$$= \left(\frac{|\chi_{\overline{\mu}}^{(d)}(e^{-i\lambda})|^2}{|\beta^{(d)}(i\lambda)|^2}p^0(\lambda)\right)\left\{(\alpha_{fk}^2 + \gamma_k^1(\lambda))\delta_{kl}\right\}_{k,l=1}^T \left(\frac{|\chi_{\overline{\mu}}^{(d)}(e^{-i\lambda})|^2}{|\beta^{(d)}(i\lambda)|^2}p^0(\lambda)\right), \quad (130)$$

$$\left(\mathbf{C}_{\overline{\mu},\alpha}^{p0}(e^{i\lambda})\right)\left(\mathbf{C}_{\overline{\mu},\alpha}^{p0}(e^{i\lambda})\right)^* = \left(\frac{|\chi_{\overline{\mu}}^{(d)}(e^{-i\lambda})|^2}{|\beta^{(d)}(i\lambda)|^2}p^0(\lambda)\right)\left\{\beta_k^2\gamma_k^2(\lambda)\delta_{kl}\right\}_{k,l=1}^T \left(\frac{|\chi_{\overline{\mu}}^{(d)}(e^{-i\lambda})|^2}{|\beta^{(d)}(i\lambda)|^2}p^0(\lambda)\right), \quad (131)$$

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{|\chi_{\overline{\mu}}^{(d)}(e^{-i\lambda})|^2}{|\beta^{(d)}(i\lambda)|^2}\left|p_{kk}^0(\lambda) - p_{kk}^1(\lambda)\right|d\lambda = \delta_k, \quad (132)$$

where α_{fk}^2, β_k^2 are Lagrange multipliers, functions $\gamma_k^1(\lambda) \leq 0$ and $\gamma_k^1(\lambda) = 0$ if $f_{kk}^0(\lambda) > (1-\varepsilon)f_{kk}^1(\lambda)$, functions $|\gamma_k^2(\lambda)| \leq 1$ and

$$\gamma_k^2(\lambda) = \mathrm{sign}\,(p_{kk}^0(\lambda) - p_{kk}^1(\lambda)) : \ p_{kk}^0(\lambda) - p_{kk}^1(\lambda) \neq 0, \ k = \overline{1,T}.$$

For the third set of admissible spectral densities $\mathcal{D}_{f\varepsilon}^3 \times \mathcal{D}_{p1\delta}^3$ we have equation

$$\left(\mathbf{C}_{\overline{\mu},\alpha}^{f0}(e^{i\lambda})\right)\left(\mathbf{C}_{\overline{\mu},\alpha}^{f0}(e^{i\lambda})\right)^* - \alpha_f^2 \left(\mathbf{C}_{\overline{\mu},\alpha}^{p0}(e^{i\lambda})\right)\left(\mathbf{C}_{\overline{\mu},\alpha}^{p0}(e^{i\lambda})\right)^* =$$

$$= (\alpha^2 + \gamma_1'(\lambda))\left(\frac{|\chi_{\overline{\mu}}^{(d)}(e^{-i\lambda})|^2}{|\beta^{(d)}(i\lambda)|^2}p^0(\lambda)\right)B_1^\top\left(\frac{|\chi_{\overline{\mu}}^{(d)}(e^{-i\lambda})|^2}{|\beta^{(d)}(i\lambda)|^2}p^0(\lambda)\right), \quad (133)$$

$$\left(\mathbf{C}_{\overline{\mu},\alpha}^{p0}(e^{i\lambda})\right)\left(\mathbf{C}_{\overline{\mu},\alpha}^{p0}(e^{i\lambda})\right)^* = \beta^2\gamma_2'(\lambda)\left(\frac{|\chi_{\overline{\mu}}^{(d)}(e^{-i\lambda})|^2}{|\beta^{(d)}(i\lambda)|^2}p^0(\lambda)\right)B_2^\top\left(\frac{|\chi_{\overline{\mu}}^{(d)}(e^{-i\lambda})|^2}{|\beta^{(d)}(i\lambda)|^2}p^0(\lambda)\right), \quad (134)$$

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{|\chi_{\overline{\mu}}^{(d)}(e^{-i\lambda})|^2}{|\beta^{(d)}(i\lambda)|^2}\left|\langle B_2, p_0(\lambda) - p_1(\lambda)\rangle\right|d\lambda = \delta, \quad (135)$$

where α_f^2, β^2 are Lagrange multipliers, function $\gamma_1'(\lambda) \leq 0$ and $\gamma_1'(\lambda) = 0$ if $\langle B_1, f_0(\lambda)\rangle > (1-\varepsilon)\langle B_1, f_1(\lambda)\rangle$, function $|\gamma_2'(\lambda)| \leq 1$ and

$$\gamma_2'(\lambda) = \mathrm{sign}\,\langle B_2, p_0(\lambda) - p_1(\lambda)\rangle : \ \langle B_2, p_0(\lambda) - p_1(\lambda)\rangle \neq 0.$$

For the fourth set of admissible spectral densities $\mathcal{D}_{f\varepsilon}^4 \times \mathcal{D}_{p1\delta}^4$ we have equation

$$\left(\mathbf{C}_{\overline{\mu},\alpha}^{f0}(e^{i\lambda})\right)\left(\mathbf{C}_{\overline{\mu},\alpha}^{f0}(e^{i\lambda})\right)^* - \alpha^2 \left(\mathbf{C}_{\overline{\mu},\alpha}^{p0}(e^{i\lambda})\right)\left(\mathbf{C}_{\overline{\mu},\alpha}^{p0}(e^{i\lambda})\right)^* =$$

$$= \left(\frac{|\chi_{\overline{\mu}}^{(d)}(e^{-i\lambda})|^2}{|\beta^{(d)}(i\lambda)|^2}p^0(\lambda)\right)(\vec{\alpha}_f \cdot \vec{\alpha}_f^* + \Gamma(\lambda))\left(\frac{|\chi_{\overline{\mu}}^{(d)}(e^{-i\lambda})|^2}{|\beta^{(d)}(i\lambda)|^2}p^0(\lambda)\right), \quad (136)$$

$$\left(\mathbf{C}_{\overline{\mu},\alpha}^{p0}(e^{i\lambda})\right)\left(\mathbf{C}_{\overline{\mu},\alpha}^{p0}(e^{i\lambda})\right)^* = \left(\frac{|\chi_{\overline{\mu}}^{(d)}(e^{-i\lambda})|^2}{|\beta^{(d)}(i\lambda)|^2}p^0(\lambda)\right)\left\{\beta_{ij}(\lambda)\gamma_{ij}(\lambda)\right\}_{i,j=1}^T \left(\frac{|\chi_{\overline{\mu}}^{(d)}(e^{-i\lambda})|^2}{|\beta^{(d)}(i\lambda)|^2}p^0(\lambda)\right), \quad (137)$$

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{|\chi_{\overline{\mu}}^{(d)}(e^{-i\lambda})|^2}{|\beta^{(d)}(i\lambda)|^2} \left| p_{ij}^0(\lambda) - p_{ij}^1(\lambda) \right| d\lambda = \delta_i^j, \tag{138}$$

where $\vec{\alpha}_f$, β_{ij} are Lagrange multipliers, function $\Gamma(\lambda) \leq 0$ and $\Gamma(\lambda) = 0$ if $f_0(\lambda) > (1 - \varepsilon)f_1(\lambda)$, functions $|\gamma_{ij}(\lambda)| \leq 1$ and

$$\gamma_{ij}(\lambda) = \frac{p_{ij}^0(\lambda) - p_{ij}^1(\lambda)}{\left| p_{ij}^0(\lambda) - p_{ij}^1(\lambda) \right|} : \ p_{ij}^0(\lambda) - p_{ij}^1(\lambda) \neq 0, \ i, j = \overline{1, T}.$$

The following theorem holds true.

Theorem 4.8
Let the minimality condition (56) hold true. The least favorable spectral densities $f_0(\lambda)$, $p_0(\lambda)$ in classes $\mathcal{D}_{f\varepsilon}^k \times \mathcal{D}_{p1\delta}^k, k = 1, 2, 3, 4$ for the optimal linear filtering of the functional $A\vec{\xi}$ from observations of the vector sequence $\vec{\zeta}(m)$ cointegrated with $\vec{\xi}(m)$ in terms of Definition 3.1, at points $m = -1, -2, \ldots$ are determined by equations (127) – (129), (130)–(132), (133) – (135), (136) – (138), respectively, the constrained optimization problem (69) with $g(\lambda) := |\beta^{(d)}(i\lambda)|^{-2}(p(\lambda) - \alpha^2 f(\lambda))$ and restrictions on densities from the corresponding classes $\mathcal{D}_{f\varepsilon}^k, \mathcal{D}_{p1\delta}^k, k = 1, 2, 3, 4$. The minimax-robust spectral characteristic of the optimal estimate of the functional $A\vec{\xi}$ is determined by the formula (59).

5. Conclusions

In this article, we dealt with stochastic sequences with periodically stationary GM increments introduced in [37]. We give a definition of one class of vector seasonally cointegrated sequences related to stationary GM increment. These non-stationary stochastic sequences combine periodic structure of covariation functions of sequences as well as integrating one.

We derived solutions of the forecasting problem for the linear functionals constructed from the unobserved values of a sequence with periodically stationary GM increments. Estimates are based on observations of the sequence with a periodically stationary noise. We obtained the estimates by representing the sequence under investigation as a vector-valued sequence with stationary GM increments. Based on the solutions for these type of sequences, we solved the corresponding problem for the defined class of seasonally cointegrated vector sequences. The problem is investigated in the case of spectral certainty, where spectral densities of sequences are exactly known. In this case we propose an approach based on the Hilbert space projection method. We derive formulas for calculating the spectral characteristics and the mean-square errors of the optimal estimates of the functionals. In the case of spectral uncertainty where the spectral densities are not exactly known while, instead, some sets of admissible spectral densities are specified, the minimax-robust method is applied. We propose a representation of the mean square error in the form of a linear functional in L_1 with respect to spectral densities, which allows us to solve the corresponding constrained optimization problem and describe the minimax (robust) estimates of the functionals. Formulas that determine the least favorable spectral densities and minimax (robust) spectral characteristic of the optimal linear estimates of the functionals are derived for a wide list of specific classes of admissible spectral densities.

These least favourable spectral density matrices are solutions of the optimization problem $\Delta_{\mathcal{D}}(f, g) = \widetilde{\Delta}(f, g) + \delta(f, g | \mathcal{D}_f \times \mathcal{D}_g) \to \inf$, where $\delta(f, g | \mathcal{D}_f \times \mathcal{D}_g)$ is the indicator function of the set $\mathcal{D} = \mathcal{D}_f \times \mathcal{D}_g$. Solution (f^0, g^0) to this unconstrained optimisation problem is characterized by the condition $0 \in \partial \Delta_{\mathcal{D}}(f^0, g^0)$, where $\partial \Delta_{\mathcal{D}}(f^0, g^0)$ is the subdifferential of the functional $\Delta_{\mathcal{D}}(f, g)$ at point $(f^0, g^0) \in \mathcal{D} = \mathcal{D}_f \times \mathcal{D}_g$. This condition makes it possible to find the least favourable spectral densities in some special classes of spectral densities. These are: classes D_0 of densities with the moment restrictions, classes $D_{1\delta}$ which describe the "δ-neighborhood" models in the space L_1 of a fixed bounded spectral density, classes D_ε which describe the "ε-contaminated" models of a fixed bounded spectral density, classes D_V^U which describe the "strip" models of spectral densities.

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