STATISTICS, OPTIMIZATION AND INFORMATION COMPUTING
Stat., Optim. Inf. Comput., Vol. **10**, September 2022, pp 1250–1266.
Published online in International Academic Press (www.IAPress.org)

Best Linear Unbiased Estimation and Prediction of Record Values Based on Kumaraswamy Distributed Data

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Abstract
To predict a future upper record value based on Kumaraswamy distributed data, an explicit expression for single and product moments has been established along with some enhanced expressions that makes the applying process on mathematical softwares easier. The best linear unbiased estimator approach for estimating the parameters and the prediction of future record values have been considered and some important tables have been created to help in the calculation processes. Two illustrative examples based on a simulation study and a real-life data are provided to assess the performance of the introduced results.

Keywords upper record values, Kumaraswamy distribution, best linear unbiased estimation, best linear unbiased prediction, moments and product moments.

AMS 2010 subject classifications 62G30, 62F10, 62M20

DOI: 10.19139/soic-2310-5070-1397

1. Introduction

Let $X_1, X_2, X_3, ...$ be a sequence of independent and identically distributed (iid) random variables. If X_j is greater than $X_1, X_2, ..., X_{j-1}$, then X_j is an upper record value. Thus $X_j > X_i$ for all $i < j$.
The record times $U(n), n \geq 1$ at which the records appear, is define as

$$U(n) = min\{j : j > U(n-1), X_j > X_{U(n-1)}\}, \qquad n > 1.$$

$U(1) = 1$ with probability 1 and $X_{U(1)}$ is the first upper (lower) record. The sequence $X_{U(n)}, n \geq 1$ defines a sequence of upper record values.

The theory of record values was first introduced by Chandler [8]. Feller [10] gave some examples of record values in gambling. For more reading about the details of the record value and its applications from the point view of the statisticians, the reader can refer to Arnold et al. [3], [1], Khan et al. [13], and Nevzorov [21]. Record values appears in many real-life events, such as sports, economics, weather, pollution levels, industry and so on. Like in earthquakes, it might be interesting to know if there is an upcoming harder than the last greater earthquake, while the amount of rainfall that is greater than or smaller than the previous ones is important to people studying hydrology and climatology. In some industrial experiment, it's important to observe the greatest number of defective products in an operation run and expect if there is a greater number in the next runs and compare both

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ISSN 2310-5070 (online) ISSN 2311-004X (print)

with the allowable limits.

Prediction of future records has a great interest among authors. Many of them introduced some methods to do so like Arnold et al. [3], Ahsanullah [2] and Balakrishnan and Cohen [4] by estimating the parameters then predicting a future record value. Some other authors apply these methods to different distributions such as Kumar [15], Singh et al. [22], Chacko and Shy Mary ([6], [7]) and Barakat et al.[5].

The probability density function (pdf) of $X_{U(n)}$ is given by (see, Arnold et al. [3])

$$f_{U(n)}(x) = \frac{[-ln(\bar{F}(x))]^{n-1} f(x)}{\Gamma(n)}, \tag{1}$$

where $f(x)$ and $F(x)$ are the pdf and the cumulative distribution function (cdf) of the original iid random variables. Also, $\bar{F}(x) = 1 - F(x)$, $\Gamma(n) = (n-1)!$ and $n \geq 1$.

While the joint pdf of mth and nth upper record values for $m < n$ is given by

$$f_{U(n),U(m)}(x,y) = \frac{[-ln(\bar{F}(x))]^{m-1}[-ln(\bar{F}(y)) + ln(\bar{F}(x))]^{n-m-1} f(x)f(y)}{\Gamma(m)\Gamma(m-n)\bar{F}(x)}. \tag{2}$$

In 1980, Kumaraswamy [16] established a new distribution by the name double-bounded distribution which has been known later by his name (Kumaraswamy distribution). Because he found out that the following probability distributions such as beta, normal and log-normal along with empirical distributions such as polynomial-transformed-normal and Johnson's do not fit well with the hydrological data such as daily rainfall and daily stream flow. Nadarajah [20] has mentioned that many papers in the hydrological literature have used this distribution because it is deemed as a better alternative to beta distribution. The similarities and differences between the Kumaraswamy and beta distributions along with the background and genesis of the Kumaraswamy distribution was studied by Jones [12]. Also, this distribution shows to be applicable to natural phenomena whose outcomes have lower and upper bounds, such as atmosphere temperatures, scores obtained on a test, the heights of individuals, etc. For cases when scientists use probability distributions which have infinite lower and/or upper bounds to fit data while in the reality the bounds are finite, this distribution could be more appropriate for these situations. The Kumaraswamy distribution opens the road for new types of distributions. For example, the exponentiated Kumaraswamy which has been introduced by Lemonte et al. [18] and its general mathematical and statistical properties were studied. Also, Corderio et al. [9] done the same for a new type called Kumaraswamy Weibull. Meanwhile, on the choice of the parameters of the Kumaraswamy distributions (a and b), it can be employed to approximate many distributions such as triangular, uniform or almost any single model distribution. And can also reproduce results of the beta or the truncated normal distributions (see, Kumaraswamy [16] and Kumaraswamy et al. [17]).

And since in the above-mentioned real-life examples, records come as a great interest because it will help in making series and important decisions and since these data by most follows the Kumaraswamy distribution, its prediction comes as great interest to be founded. Our focus in this paper is to find that by estimating the parameters then predicting the nth future record value when it follows Kumaraswamy distribution by using best linear unbiased estimation (BLUE) and best linear unbiased prediction (BLUP) methods. Using BLUE and BLUP methods in the estimation and prediction processes give much simpler, directly substituted, and easier applicable formulas.

Statistical analysis of record values arising from the Kumaraswamy distribution was considered by Nadar et al. [19]. Kızılaslan and Nadar [14] obtained the parameter estimates based on lower record values from Kumaraswamy distributed data and their corresponding inter-record times under the non-Bayesian and then Bayesian frameworks. While Wang [23] studies the point estimates using maximum likelihood estimation (MLE) and proposed pivotal quantity-based estimation for the parameters then found the exact confidence intervals and confidence regions for them based on k-records.

Kumaraswamy distribution has two parameters a and b and according to their values the shape of the distribution will change between 9 different shapes which can help in studding some life events. For example, the daily rainfall follows a pattern when $a = 1$ and $b > 1$. And on many occasions, it happens for values of $a > 1$ and $a < 1$. While for $a = 1$ and $b < 1$, a water levels in a cistern from which outflow occurs only through a crest spillway will follow that pattern.

A random variable x is considered to be Kumaraswamy distributed if:

$$x = \frac{z - z_{min}}{z_{max} - z_{min}},$$

where z is a random variable of process $[z]$, z_{min} is the lower bound of the random variable and z_{max} is the upper bound.

The pdf and the cdf of the Kumaraswamy distribution are given by

$$f(x) = a\,b\,x^{a-1}(1 - x^a)^{b-1}, \tag{3}$$

where $0 < x < 1$, $a > 0$ and $b > 0$, and

$$F(x) = 1 - [1 - x^a]^b. \tag{4}$$

In this paper, record values based on Kumaraswamy distribution are considered. In section2, the single moment from which we can find the mean and variance of record values based on Kumaraswamy distribution also the product moment to find covariance between two records has introduced. In section 3, the BLUE of the parameters along with some important tables that will make the calculations easier has established along with the BLUP method for future record values. In section 4, to check the efficiency of the study a simulation study and a real data example has performed.

2. Moments of the Record Values

Let $X_{U(1)}, X_{U(2)}, ..., X_{U(n)}$ be the first n upper record values that comes from a sequence of iid Kumaraswamy distributed random variables. We will denote $E(X^j_{U(n)})$ by α^j_n, $Var(X_{U(n)})$ by σ^2_n, $E(X_{U(m)}, X_{U(n)})$ by $\alpha_{m,n}$ and $Cov(X_{U(m)}, X_{U(n)})$ by $\sigma_{m,n}$, where $j \geq 1$.

Theorem 1. *The jth moment of the nth upper record value for $n \geq 1$ can be calculated from the following formula*

$$\alpha^j_n = \frac{b^n}{\Gamma(n)} \sum_{p=0}^{\infty} a_p(n-1) \frac{\Gamma(n + p + \frac{j}{a})\Gamma(b)}{\Gamma(b + n + p + \frac{j}{a})}. \tag{5}$$

For $n = 1$

$$\alpha^j_1 = \frac{b\,\Gamma(1 + \frac{j}{a})\Gamma(b)}{\Gamma(1 + b + \frac{j}{a})}.$$

While the product moments of the mth and nth upper record values for $m < n$ will be given by

$$\alpha_{m,n} = \frac{a^2 b^n}{\Gamma(m)\Gamma(n-m)} \sum_{k=0}^{n-m-1} \sum_{v=0}^{b-1} \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} \sum_{w=0}^{\infty} (-1)^{k+v+m+1-n} \binom{n-m-1}{k} \binom{b-1}{v}$$
$$a_p(k) a_q(n-k-2) \frac{1}{(a(w + n - k - 1 + q) + 1)(a(w + n + q + v + p) + 2)}. \tag{6}$$

Proof

$$\alpha_n^j = \int_{-\infty}^{\infty} x^j f_{U(n)}(x)dx. \tag{7}$$

Now upon using (1), (3) and (4) in (7) we get the following

$$\alpha_n^j = \frac{ab^n}{\Gamma(n)} \int_0^1 x^{a+j-1}(1-x^a)^{b-1}[-ln(1-x^a)]^{n-1}dx.$$

We will take the transformation $x^a = u$, we get

$$\alpha_n^j = \frac{b^n}{\Gamma(n)} \int_0^1 (u^{\frac{1}{a}})^j(1-u)^{b-1}[-ln(1-u)]^{n-1}du.$$

Then we will use the logarithmic expansion introduced by Balakrishnan and Cohen [4]

$$[-ln(1-t)]^i = \left(\sum_{p=1}^{\infty} \frac{t^p}{p}\right)^i = \sum_{p=0}^{\infty} a_p(i)t^{i+p}, \quad |t| < 1, \tag{8}$$

where $a_p(i)$ is the coefficient of t^{i+p} in the expansion $\left(\sum_{p=1}^{\infty} \frac{t^p}{p}\right)^i$, we get

$$\alpha_n^j = \frac{b^n}{\Gamma(n)} \sum_{p=0}^{\infty} a_p(n-1) \int_0^1 u^{\frac{j}{a}+p+n-1}(1-u)^{b-1}du.$$

By integrating the integal part we reach (5). Now for the product moments, we have

$$\alpha_{m,n} = \int_{-\infty}^{\infty}\int_x^{\infty} x\, y\, f_{U(m),U(n)}(x,y)dy\, dx. \tag{9}$$

By substituting (2), (3) and (4) in (9) we get

$$\alpha_{m,n} = \frac{a^2b^n}{\Gamma(m)\Gamma(n-m)} \int_0^1 \frac{x^a}{1-x^a}[-ln(1-x^a)]^{m-1}I(x)\, dx,$$

where

$$I(x) = \int_x^1 y^a(1-y^a)^{b-1}[ln(1-x^a) - ln(1-y^a)]^{n-m-1}dy.$$

Using the binomial expansion, we get

$$I(x) = ab^{k+1} \sum_{k=0}^{n-m-1} \binom{n-m-1}{k} [ln(1-x^a)]^{n-m-k-1} \int_x^1 y^a(1-y^a)^{b-1}[-ln(1-y^a)]^k dy.$$

After using the logarithm expansion on the term $[-ln(1-y^a)]^k$ and binomial expansion on $(1-y^a)^{b-1}$, we reach the following

$$I(x) = ab^{k+1} \sum_{k=0}^{n-m-1}\sum_{v=0}^{b-1}\sum_{p=0}^{\infty} \binom{n-m-1}{k}\binom{b-1}{v}(-1)^v a_p(k)[ln(1-x^a)]^{n-m-k-1}\frac{1-x^{a(v+p+k+1)+1}}{a(v+p+k+1)+1}.$$

When substituting $I(x)$ in $\alpha_{m,n}$ and by taking the same two expansions on the relative terms we get to final expression (6).

□

Remark 1. Another form of α_n^j could be conclude if after we took the logarithmic expansion, we used the binomial expansion on the term $(1-u)^{b-1}$ which will lead to the following expression

$$\alpha_n^j = \frac{b^n}{\Gamma(n)} \sum_{v=0}^{b-1} \sum_{p=0}^{\infty} (-1)^v \binom{b-1}{v} a_p(n-1) \frac{1}{n+p+v+\frac{j}{a}}.$$

Remark 2. To calculate the variance of the nth upper record for $n \geq 1$, we will use the will known rule

$$\sigma_n^2 = E(X_{U(n)}^2) - (E(X_{U(n)}))^2.$$

By substituting in (5) once for $j=1$ and another for $j=2$ we reached the following variance expression

$$\sigma_n^2 = \frac{b^n}{\Gamma(n)} \sum_{p=0}^{\infty} a_p(n-1) \frac{\Gamma(n+p+\frac{2}{a})\Gamma(b)}{\Gamma(b+n+p+\frac{2}{a})} - \left(\frac{b^n}{\Gamma(n)} \sum_{p=0}^{\infty} a_p(n-1) \frac{\Gamma(n+p+\frac{1}{a})\Gamma(b)}{\Gamma(b+n+p+\frac{1}{a})} \right)^2. \tag{10}$$

Corollary 1. *The followings are another easier applicable forms on mathematical softwares for formula (6)*
1. When $m=1$ and $n=2$

$$\alpha_{m,n} = a^2 b^2 \left[\sum_{v=0}^{b-1} \sum_{w=0}^{\infty} \frac{(-1)^v \binom{b-1}{v}}{(a(w+1)+1)(a(2+v+w)+2)} \right].$$

2. While when $m=1$ and $n>2$

$$\alpha_{m,n} = \frac{a^2 b^n}{\Gamma(m)\Gamma(n-m)} \Bigg[\sum_{k=1}^{n-m-2} \sum_{v=0}^{b-1} \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} \sum_{w=0}^{\infty} (-1)^{k+v+m+1-n} \binom{n-m-1}{k} \binom{b-1}{v}$$
$$a_p(k) a_q(n-k-2) \frac{1}{(a(w+n-k-1+q)+1)(a(w+n+q+v+p)+2)}$$
$$+ \sum_{v=0}^{b-1} \sum_{q=0}^{\infty} \sum_{w=0}^{\infty} (-1)^{v+m+1-n} \binom{b-1}{v} a_q(n-2) \frac{1}{(a(n+q+w-1)+1)(a(n+q+w+v)+2)}$$
$$+ \sum_{v=0}^{b-1} \sum_{r=0}^{\infty} \sum_{w=0}^{\infty} \binom{b-1}{v} (-1)^v a_r(n-m-1) \frac{1}{(a(w+1)+1)(a(n-m+r+v+1+w)+2)} \Bigg].$$

2. And for $m>1$

$$\alpha_{m,n} = \frac{a^2 b^n}{\Gamma(m)\Gamma(n-m)} \Bigg[\sum_{k=1}^{n-m-1} \sum_{v=0}^{b-1} \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} \sum_{w=0}^{\infty} (-1)^{k+v+m+1-n} \binom{n-m-1}{k} \binom{b-1}{v}$$
$$a_p(k) a_q(n-k-2) \frac{1}{(a(w+n-k-1+q)+1)(a(w+n+q+v+p)+2)}$$
$$+ \sum_{v=0}^{b-1} \sum_{q=0}^{\infty} \sum_{w=0}^{\infty} (-1)^{v+m+1-n} \binom{b-1}{v} a_q(n-2) \frac{1}{(a(n+q+w-1)+1)(a(n+q+w+v)+2)} \Bigg].$$

Note: I applied all previous formulas for different values of n and m to find values of α_n^j or $\alpha_{m,n}$ with summations to ∞ on MATHEMATICA. And found that after 1000 iterations, the values do not change that much.

3. Best Linear Unbiased Estimator (BLUE) and Predictor (BLUP)

3.1. *BLUE of the Parameters*

Suppose that $T_{U(1)}, T_{U(2)}, ..., T_{U(n)}$ be the first n upper record values that comes from a sequence of iid Kumaraswamy distributed random variables on the form

$$g(t;\mu,\sigma) = a\, b \left(\frac{t-\mu}{\sigma}\right)^{a-1}\left(1-\left(\frac{t-\mu}{\sigma}\right)^{a}\right)^{b-1},$$

where $0 \le \mu < t < 1,\ a, b, \sigma > 0.$

Let $X_U = (X_{U(1)}, X_{U(2)}, ..., X_{U(n)})^T$ where $X_{U(i)} = \frac{T_{U(i)}-\mu}{\sigma}, i = 1, 2, ..., n$ be the column vector of n upper records from a population with standard Kumaraswamy distribution which its pdf is given by (1) and its joint pdf is given by (2). Then, the BLUEs of the two parameters μ and σ can be given by the following (see, Balakrishnan and Cohen [4])

$$\begin{aligned}\hat{\mu} &= \frac{\alpha^T\Sigma^{-1}\alpha\mathbf{1}^T\Sigma^{-1} - \alpha^T\Sigma^{-1}\mathbf{1}\alpha^T\Sigma^{-1}}{(\alpha^T\Sigma^{-1}\alpha)(\mathbf{1}^T\Sigma^{-1}\mathbf{1}) - (\alpha^T\Sigma^{-1}\mathbf{1})^2}X_U \\ &= \sum_{i=1}^{n} c_i X_{U(i)}\end{aligned} \tag{11}$$

and

$$\begin{aligned}\hat{\sigma} &= \frac{\mathbf{1}^T\Sigma^{-1}\mathbf{1}\alpha^T\Sigma^{-1} - \mathbf{1}^T\Sigma^{-1}\alpha\mathbf{1}^T\Sigma^{-1}}{(\alpha^T\Sigma^{-1}\alpha)(\mathbf{1}^T\Sigma^{-1}\mathbf{1}) - (\alpha^T\Sigma^{-1}\mathbf{1})^2}X_U \\ &= \sum_{i=1}^{n} d_i X_{U(i)},\end{aligned} \tag{12}$$

where

X_U: represents the column vector of the existed upper records.

α: represents the column vector of the expected values for these upper record values from our distribution.

Σ: represents the variance-covariance matrix of the upper record values from our distribution.

1: a column vector of dimension n with all entries of the number 1.

Table 1. Means of the upper records α_n when $a = 1$.

b	n									
	1	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
1	0.5000	0.7500	0.8750	0.9375	0.9688	0.9844	0.9922	0.9961	0.9980	0.9990
2	0.3333	0.5556	0.7037	0.8025	0.8683	0.9122	0.9415	0.9610	0.9740	0.9827
3	0.2500	0.4375	0.5781	0.6836	0.7627	0.8220	0.8665	0.8999	0.9249	0.9437
4	0.2000	0.3600	0.4880	0.5904	0.6723	0.7379	0.7903	0.8322	0.8658	0.8926
5	0.1667	0.3056	0.4213	0.5177	0.5981	0.6651	0.7209	0.7674	0.8062	0.8385
6	0.1429	0.2653	0.3703	0.4602	0.5373	0.6034	0.6601	0.7086	0.7503	0.7859
7	0.1250	0.2344	0.3301	0.4138	0.4871	0.5512	0.6073	0.6564	0.6993	0.7369
8	0.1111	0.2099	0.2977	0.3757	0.4451	0.5067	0.5615	0.6103	0.6536	0.6921
9	0.1000	0.1900	0.2710	0.3439	0.4095	0.4686	0.5217	0.5695	0.6126	0.6513
10	0.0909	0.1736	0.2487	0.3170	0.3791	0.4355	0.4868	0.5335	0.5759	0.6145

Also, the variances and covariance of these BLUEs was given by (see, Balakrishnan and Cohen [4])

$$Var(\hat{\mu}) = \sigma^2 \frac{(\alpha^T \Sigma^{-1} \alpha)}{(\alpha^T \Sigma^{-1} \alpha)(\mathbf{1}^T \Sigma^{-1} \mathbf{1}) - (\alpha^T \Sigma^{-1} \mathbf{1})^2}, \tag{13}$$

$$Var(\hat{\sigma}) = \sigma^2 \frac{(\mathbf{1}^T \Sigma^{-1} \mathbf{1})}{(\alpha^T \Sigma^{-1} \alpha)(\mathbf{1}^T \Sigma^{-1} \mathbf{1}) - (\alpha^T \Sigma^{-1} \mathbf{1})^2}, \tag{14}$$

$$Cov(\hat{\mu}, \hat{\sigma}) = \sigma^2 \frac{(\alpha^T \Sigma^{-1} \mathbf{1})}{(\alpha^T \Sigma^{-1} \alpha)(\mathbf{1}^T \Sigma^{-1} \mathbf{1}) - (\alpha^T \Sigma^{-1} \mathbf{1})^2}. \tag{15}$$

Other forms have been introduced by Arnold et al. [3], which will make calculating the formulas from (11) to (15) easier. The values of c_i and d_i that we introduced in formulas (11) and (12) to find their values, will be the outcomes of the following matrix

$$V = (A^T \Sigma^{-1} A)^{-1} A^T \Sigma^{-1},$$

where $A = (\mathbf{1} \;\; \alpha)$ is a partitioned matrix.

While, the values of the formulas from (13) to(15) are simply the outcomes of the following matrix

$$O = \begin{bmatrix} O_{11} & O_{12} \\ O_{21} & O_{22} \end{bmatrix} \sigma^2 = (A^T \Sigma^{-1} A)^{-1} \sigma^2,$$

where $Var(\hat{\mu}) = O_{11}\sigma^2$, $Var(\hat{\sigma}) = O_{22}\sigma^2$ and $Cov(\hat{\mu}, \hat{\sigma}) = O_{12}\sigma^2$.

Now to get the values of c_i and d_i for calculating $\hat{\mu}$ and $\hat{\sigma}$ we can use Tables 1 and 2. In Table 1, the mean values of upper records are calculated for $a = 1$ and for values of b from 1 to 10 (since some important Kumaraswamy patterns occurs around these value) and this table can help into finding the values of the column vector α. Table 2 contains the variances (for value of $m = n$) and covariances of upper records at the same values of a and b for different values of m and n to specify the values of the matrix Σ. And Table 3 will introduce the values of c_i and d_i at the previous values of a and b for different values of n to the 10th record which has been calculated using the help of the values from Tables 1 and 2. While in Table 4, the variances of $\hat{\mu}$ and $\hat{\sigma}$ along with their covariances has been calculated in terms of σ^2 where the fisrt and second values will represent the variance of $\hat{\mu}$ and $\hat{\sigma}$ respectively, while the third value will be for the covariance between them.

Table 2. Variances and covariances of the upper records $\sigma_{m,n}$ when $a = 1$.

b	m	n = 1	2	3	4	5	6	7	8	9	10
1	**1**	0.083 33	0.041 67	0.020 83	0.010 42	0.005 21	0.002 60	0.001 30	0.000 65	0.000 33	0.000 16
	2		0.048 61	0.024 31	0.012 15	0.006 08	0.003 04	0.001 52	0.000 76	0.000 38	0.000 19
	3			0.021 41	0.010 71	0.005 35	0.002 68	0.001 34	0.000 67	0.000 33	0.000 17
	4				0.008 44	0.004 22	0.002 11	0.001 05	0.000 53	0.000 26	0.000 13
	5					0.003 14	0.001 57	0.000 78	0.000 39	0.000 20	0.000 10
	6						0.001 13	0.000 56	0.000 28	0.000 14	0.000 07
	7							0.000 40	0.000 20	0.000 10	0.000 05
	8								0.000 14	0.000 07	0.000 03
	9									0.000 05	0.000 02
	10										0.000 02
b	m	n = 1	2	3	4	5	6	7	8	9	10
2	**1**	0.055 56	0.037 04	0.024 69	0.016 46	0.010 97	0.007 32	0.004 88	0.003 25	0.002 17	0.001 45
	2		0.052 47	0.034 98	0.023 32	0.015 55	0.010 36	0.006 91	0.004 61	0.003 07	0.002 05
	3			0.037 21	0.024 81	0.016 54	0.011 02	0.007 35	0.004 90	0.003 27	0.002 18
	4				0.023 48	0.015 65	0.010 44	0.006 96	0.004 64	0.003 09	0.002 06
	5					0.013 91	0.009 27	0.006 18	0.004 12	0.002 75	0.001 83
	6						0.007 92	0.005 28	0.003 52	0.002 35	0.001 56
	7							0.004 39	0.002 92	0.001 95	0.001 30
	8								0.002 38	0.001 59	0.001 06
	9									0.001 28	0.000 85
	10										0.000 68
b	m	n = 1	2	3	4	5	6	7	8	9	10
3	**1**	0.037 50	0.028 13	0.021 09	0.015 82	0.011 87	0.008 90	0.006 67	0.005 01	0.003 75	0.002 82
	2		0.043 59	0.032 70	0.024 52	0.018 39	0.013 79	0.010 35	0.007 76	0.005 82	0.004 36
	3			0.038 02	0.028 52	0.021 39	0.016 04	0.012 03	0.009 02	0.006 77	0.005 08
	4				0.029 49	0.022 12	0.016 59	0.012 44	0.009 33	0.007 00	0.005 25
	5					0.021 45	0.016 08	0.012 06	0.009 05	0.006 79	0.005 09
	6						0.014 98	0.011 23	0.008 43	0.006 32	0.004 74
	7							0.010 18	0.007 63	0.005 72	0.004 29
	8								0.006 77	0.005 08	0.003 81
	9									0.004 44	0.003 33
	10										0.002 88
b	m	n = 1	2	3	4	5	6	7	8	9	10
4	**1**	0.026 67	0.021 33	0.017 07	0.013 65	0.010 92	0.008 74	0.006 99	0.005 59	0.004 47	0.003 58
	2		0.034 84	0.027 88	0.022 30	0.017 84	0.014 27	0.011 42	0.009 13	0.007 31	0.005 85
	3			0.034 15	0.027 32	0.021 86	0.017 49	0.013 99	0.011 19	0.008 95	0.007 16
	4				0.029 76	0.023 81	0.019 05	0.015 24	0.012 19	0.009 75	0.007 80
	5					0.024 31	0.019 45	0.015 56	0.012 45	0.009 96	0.007 97
	6						0.019 07	0.015 26	0.012 21	0.009 76	0.007 81
	7							0.014 55	0.011 64	0.009 31	0.007 45
	8								0.010 87	0.008 70	0.006 96
	9									0.008 00	0.006 40
	10										0.005 81
b	m	n = 1	2	3	4	5	6	7	8	9	10
5	**1**	0.019 84	0.016 53	0.013 78	0.011 48	0.009 57	0.007 97	0.006 64	0.005 54	0.004 61	0.003 85
	2		0.027 95	0.023 29	0.019 41	0.016 18	0.013 48	0.011 23	0.009 36	0.007 80	0.006 50
	3			0.029 53	0.024 61	0.020 51	0.017 09	0.014 24	0.011 87	0.009 89	0.008 24
	4				0.027 74	0.023 12	0.019 26	0.016 05	0.013 38	0.011 15	0.009 29
	5					0.024 43	0.020 36	0.016 96	0.014 14	0.011 78	0.009 82
	6						0.020 65	0.017 21	0.014 34	0.011 95	0.009 96
	7							0.016 98	0.014 15	0.011 79	0.009 83
	8								0.013 67	0.011 39	0.009 49
	9									0.010 84	0.009 03
	10										0.008 49

b	m	n									
		1	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
6	**1**	0.015 31	0.013 12	0.011 25	0.009 64	0.008 26	0.007 08	0.006 07	0.005 20	0.004 46	0.003 82
	2		0.022 72	0.019 48	0.016 70	0.014 31	0.012 27	0.010 51	0.009 01	0.007 72	0.006 62
	3			0.025 31	0.021 69	0.018 59	0.015 94	0.013 66	0.011 71	0.010 04	0.008 60
	4				0.025 05	0.021 47	0.018 40	0.015 77	0.013 52	0.011 59	0.009 93
	5					0.023 25	0.019 93	0.017 08	0.014 64	0.012 55	0.010 76
	6						0.020 71	0.017 75	0.015 22	0.013 04	0.011 18
	7							0.017 94	0.015 38	0.013 18	0.011 30
	8								0.015 22	0.013 05	0.011 18
	9									0.012 72	0.010 90
	10										0.010 49

b	m	n									
		1	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
7	**1**	0.012 15	0.010 63	0.009 30	0.008 14	0.007 12	0.006 23	0.005 45	0.004 77	0.004 18	0.003 65
	2		0.018 76	0.016 41	0.014 36	0.012 57	0.010 99	0.009 62	0.008 42	0.007 37	0.006 44
	3			0.021 71	0.019 00	0.016 62	0.014 55	0.012 73	0.011 14	0.009 74	0.008 53
	4				0.022 34	0.019 55	0.017 11	0.014 97	0.013 10	0.011 46	0.010 03
	5					0.021 55	0.018 86	0.016 50	0.014 44	0.012 63	0.011 05
	6						0.019 96	0.017 47	0.015 28	0.013 37	0.011 70
	7							0.017 97	0.015 73	0.013 76	0.012 04
	8								0.015 85	0.013 87	0.012 14
	9									0.013 76	0.012 04
	10										0.011 80

b	m	n									
		1	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
8	**1**	0.009 88	0.008 78	0.007 80	0.006 94	0.006 17	0.005 48	0.004 87	0.004 33	0.003 85	0.003 42
	2		0.015 70	0.013 96	0.012 41	0.011 03	0.009 80	0.008 72	0.007 75	0.006 89	0.006 12
	3			0.018 73	0.016 65	0.014 80	0.013 15	0.011 69	0.010 39	0.009 24	0.008 21
	4				0.019 86	0.017 65	0.015 69	0.013 95	0.012 40	0.011 02	0.009 79
	5					0.019 73	0.017 54	0.015 59	0.015 28	0.013 22	0.010 95
	6						0.018 83	0.016 74	0.014 88	0.013 22	0.011 75
	7							0.017 47	0.015 53	0.013 80	0.012 27
	8								0.015 87	0.014 11	0.012 54
	9									0.014 20	0.012 62
	10										0.012 54

b	m	n									
		1	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
9	**1**	0.008 18	0.007 36	0.006 63	0.005 96	0.005 37	0.004 83	0.004 35	0.003 91	0.003 52	0.003 17
	2		0.013 32	0.011 99	0.010 79	0.009 71	0.008 74	0.007 87	0.007 08	0.006 37	0.005 73
	3			0.016 27	0.014 64	0.013 18	0.011 86	0.010 67	0.009 61	0.008 65	0.007 78
	4				0.017 66	0.015 89	0.014 30	0.012 87	0.011 59	0.010 43	0.009 38
	5					0.017 97	0.016 17	0.014 56	0.013 10	0.011 79	0.010 61
	6						0.017 56	0.015 80	0.014 22	0.012 80	0.011 52
	7							0.016 67	0.015 01	0.013 51	0.012 16
	8								0.015 51	0.013 96	0.012 57
	9									0.014 21	0.012 79
	10										0.012 85

b	m	n									
		1	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
10	**1**	0.006 89	0.006 26	0.005 69	0.005 17	0.004 70	0.004 28	0.003 89	0.003 53	0.003 21	0.002 92
	2		0.011 43	0.010 39	0.009 45	0.008 59	0.007 81	0.007 10	0.006 45	0.005 87	0.005 33
	3			0.014 23	0.012 94	0.011 76	0.010 69	0.009 72	0.008 84	0.008 03	0.007 30
	4				0.015 75	0.014 31	0.013 01	0.011 83	0.010 75	0.009 78	0.008 89
	5					0.016 33	0.014 85	0.013 50	0.012 27	0.011 16	0.010 14
	6						0.016 27	0.014 79	0.013 44	0.012 22	0.011 11
	7							0.015 75	0.014 32	0.013 02	0.011 83
	8								0.014 94	0.013 58	0.012 35
	9									0.013 95	0.012 68
	10										0.012 86

Table 3. Coefficients of the BLUEs of μ and σ when $a = 1$.

b	n	c_1	c_2	c_3	c_4	c_5	c_6	c_7	c_8	c_9	c_{10}
1	**2**	3.000 00	−2.000 00								
	3	2.500 70	0.249 80	−1.499 86							
	4	2.077 13	0.075 17	0.234 59	−1.386 86						
	5	2.024 75	0.024 44	0.075 87	0.227 90	−1.352 96					
	6	2.008 39	0.007 39	0.028 56	0.078 84	0.191 15	−1.314 33				
	7	2.002 99	0.001 94	0.012 58	0.029 62	0.046 77	0.214 22	−1.308 12			
	8	2.001 12	0.000 07	0.004 91	0.016 98	−0.007 87	0.091 19	0.227 14	−1.333 54		
	9	2.000 76	−0.000 27	0.003 52	0.005 23	−0.008 35	0.048 35	0.100 36	−0.029 46	−1.120 13	
	10	2.000 73	−0.000 32	0.003 56	0.005 20	−0.009 54	0.047 45	0.101 30	−0.043 10	−1.012 13	−0.093 14
	n	d_1	d_2	d_3	d_4	d_5	d_6	d_7	d_8	d_9	d_{10}
	2	−4.000 00	4.000 00								
	3	−2.499 86	−0.500 41	3.000 27							
	4	−2.154 18	−0.151 35	−0.466 70	2.772 23						
	5	−2.049 56	−0.050 01	−0.149 74	−0.453 06	2.702 36					
	6	−2.016 56	−0.015 63	−0.054 32	−0.152 43	−0.411 80	2.650 75				
	7	−2.005 78	−0.004 76	−0.022 46	−0.054 24	−0.123 79	−0.398 59	2.609 61			
	8	−2.002 06	−0.001 02	−0.007 15	−0.029 02	−0.014 80	−0.153 19	−0.452 75	2.659 99		
	9	−2.001 28	−0.000 29	−0.004 18	−0.003 85	−0.013 75	−0.061 42	−0.181 16	−0.133 43	2.399 37	
	10	−2.001 21	−0.000 18	−0.004 26	−0.003 79	−0.011 04	−0.059 37	−0.183 32	−0.102 37	2.153 50	0.212 04
b	n	c_1	c_2	c_3	c_4	c_5	c_6	c_7	c_8	c_9	c_{10}
2	**2**	2.499 33	−1.499 33								
	3	1.833 60	0.165 67	−0.999 27							
	4	1.642 87	0.070 37	0.144 66	−0.857 90						
	5	1.566 85	0.032 78	0.068 38	0.129 64	−0.797 65					
	6	1.532 50	0.015 19	0.032 57	0.063 69	0.132 99	−0.776 95				
	7	1.515 92	0.007 13	0.016 74	0.029 82	0.068 54	0.123 79	−0.761 92			
	8	1.507 89	0.003 40	0.008 75	0.013 67	0.037 35	0.064 15	0.108 13	−0.743 35		
	9	1.504 07	0.000 95	0.005 57	0.004 15	0.022 62	0.030 26	0.046 97	0.149 15	−0.763 75	
	10	1.501 96	0.000 17	0.003 12	0.001 22	0.014 77	0.008 59	0.019 92	0.093 19	0.118 44	−0.761 37
	n	d_1	d_2	d_3	d_4	d_5	d_6	d_7	d_8	d_9	d_{10}
	2	−4.498 43	4.498 43								
	3	−2.500 48	−0.498 46	2.998 94							
	4	−1.928 44	−0.212 65	−0.431 93	2.573 02						
	5	−1.700 10	−0.099 73	−0.202 82	−0.393 25	2.395 90					
	6	−1.597 41	−0.047 15	−0.095 74	−0.196 06	−0.386 71	2.323 06				
	7	−1.547 67	−0.022 96	−0.048 29	−0.094 49	−0.193 43	−0.377 83	2.284 67			
	8	−1.523 37	−0.011 69	−0.024 09	−0.045 62	−0.099 06	−0.197 37	−0.348 46	2.249 66		
	9	−1.512 05	−0.004 41	−0.014 69	−0.017 40	−0.055 37	−0.096 86	−0.167 08	−0.397 14	2.264 99	
	10	−1.505 89	−0.002 12	−0.007 51	−0.008 83	−0.032 46	−0.033 59	−0.088 10	−0.233 80	−0.310 16	2.222 45
b	n	c_1	c_2	c_3	c_4	c_5	c_6	c_7	c_8	c_9	c_{10}
3	**2**	2.333 33	−1.333 33								
	3	1.708 65	0.124 42	−0.833 06							
	4	1.517 30	0.059 88	0.103 92	−0.681 10						
	5	1.432 82	0.031 79	0.056 57	0.092 43	−0.613 61					
	6	1.389 89	0.017 62	0.032 53	0.053 40	0.082 28	−0.575 74				
	7	1.364 26	0.009 74	0.018 35	0.029 15	0.044 98	0.133 99	−0.600 47			
	8	1.352 89	0.005 62	0.011 88	0.018 14	0.027 27	0.107 39	−0.053 67	−0.469 52		
	9	1.345 10	0.003 03	0.007 95	0.010 95	0.016 08	0.085 90	−0.087 43	0.155 15	−0.536 72	
	10	1.340 47	0.000 95	0.005 56	0.006 67	0.007 79	0.073 47	−0.106 70	0.119 76	0.117 33	−0.565 30
	n	d_1	d_2	d_3	d_4	d_5	d_6	d_7	d_8	d_9	d_{10}
	2	−5.333 33	5.333 33								
	3	−2.833 44	−0.500 34	3.333 78							
	4	−2.068 55	−0.242 36	−0.411 69	2.722 60						
	5	−1.730 95	−0.130 09	−0.222 46	−0.368 79	2.452 29					
	6	−1.558 86	−0.073 32	−0.126 13	−0.212 36	−0.337 03	2.307 70				
	7	−1.461 70	−0.043 43	−0.072 36	−0.120 41	−0.195 63	−0.382 77	2.276 29			
	8	1.410 51	−0.024 88	−0.043 24	−0.070 84	−0.115 88	−0.263 06	−0.184 92	2.113 33		
	9	−1.379 09	−0.014 43	−0.027 38	−0.041 88	−0.070 78	−0.176 37	−0.048 77	−0.405 61	2.164 31	
	10	−1.361 34	−0.006 46	−0.018 22	−0.025 43	−0.038 94	−0.128 67	0.025 21	−0.269 71	−0.346 92	2.170 47

b	n	c_1	c_2	c_3	c_4	c_5	c_6	c_7	c_8	c_9	c_{10}
4	**2**	2.250 00	−1.250 00								
	3	1.649 28	0.106 11	−0.750 90							
	4	1.460 26	0.053 89	0.077 13	−0.591 28						
	5	1.372 76	0.031 74	0.044 74	0.070 68	−0.519 91					
	6	1.325 41	0.019 61	0.027 09	0.043 76	0.065 90	−0.481 78				
	7	1.297 73	0.013 11	0.016 45	0.028 22	0.042 17	0.057 68	−0.455 35			
	8	1.280 87	0.008 62	0.010 27	0.018 28	0.028 20	0.036 45	0.058 73	−0.441 43		
	9	1.269 91	0.006 34	0.006 01	0.012 12	0.020 02	0.021 07	0.037 74	0.061 42	−0.434 62	
	10	1.262 94	0.004 49	0.003 39	0.008 07	0.013 86	0.012 63	0.024 86	0.040 42	0.063 28	−0.433 94
	n	d_1	d_2	d_3	d_4	d_5	d_6	c_7	d_8	d_9	d_{10}
	2	−6.250 00	6.250 00								
	3	−3.248 04	−0.504 41	3.752 45							
	4	−2.302 15	−0.265 62	−0.391 04	2.958 81						
	5	−1.864 79	−0.154 91	−0.229 13	−0.349 89	2.598 71					
	6	−1.628 72	−0.094 43	−0.141 15	−0.215 68	−0.322 28	2.402 26				
	7	−1.489 95	−0.061 85	−0.087 79	−0.137 78	−0.203 33	−0.301 49	2.282 18			
	8	−1.405 41	−0.039 31	−0.056 82	−0.087 96	−0.133 31	−0.195 03	−0.295 98	2.213 83		
	9	−1.350 65	−0.027 95	−0.035 51	−0.057 16	−0.092 43	−0.118 16	−0.191 12	−0.298 15	2.171 12	
	10	−1.316 11	−0.018 73	−0.022 52	−0.037 11	−0.061 88	−0.076 31	−0.127 22	−0.194 01	−0.298 15	2.152 04
b	n	c_1	c_2	c_3	c_4	c_5	c_6	c_7	c_8	c_9	c_{10}
5	**2**	2.200 14	−1.200 14								
	3	1.616 61	0.083 93	−0.700 54							
	4	1.429 44	0.046 72	0.062 83	−0.538 99						
	5	1.340 52	0.029 06	0.037 86	0.057 29	−0.464 73					
	6	1.291 21	0.019 10	0.024 51	0.036 94	0.048 86	−0.420 62				
	7	1.261 26	0.012 96	0.016 07	0.025 80	0.030 97	0.047 80	−0.394 86			
	8	1.241 78	0.008 87	0.010 56	0.018 00	0.021 23	0.031 87	0.050 77	−0.383 09		
	9	1.228 95	0.006 58	0.006 91	0.013 27	0.013 70	0.022 91	0.036 50	0.033 78	−0.362 60	
	10	1.220 19	0.004 64	0.004 40	0.009 36	0.009 29	0.015 58	0.028 09	0.020 18	0.055 54	−0.367 27
	n	d_1	d_2	d_3	d_4	d_5	d_6	c_7	d_8	d_9	d_{10}
	2	−7.199 42	7.199 42								
	3	−3.700 39	−0.500 27	4.200 66							
	4	−2.576 69	−0.276 83	−0.382 42	3.235 94						
	5	−2.043 94	−0.171 07	−0.232 82	−0.336 36	2.784 19					
	6	−1.747 52	−0.111 18	−0.152 53	−0.214 03	−0.303 50	2.528 75				
	7	−1.567 77	−0.074 30	−0.101 89	−0.147 16	−0.196 07	−0.282 74	2.369 93			
	8	−1.451 74	−0.049 97	−0.069 10	−0.100 76	−0.138 11	−0.187 87	−0.283 58	2.281 13		
	9	−1.374 04	−0.036 08	−0.046 95	−0.072 07	−0.092 50	−0.133 57	−0.197 19	−0.243 79	2.196 19	
	10	−1.322 47	−0.024 65	−0.032 19	−0.049 07	−0.066 56	−0.090 42	−0.147 67	−0.163 73	−0.265 31	2.162 06
b	n	c_1	c_2	c_3	c_4	c_5	c_6	c_7	c_8	c_9	c_{10}
6	**2**	2.167 48	−1.167 48								
	3	1.594 81	0.072 76	−0.667 57							
	4	1.409 66	0.042 33	0.051 90	−0.503 88						
	5	1.320 93	0.027 06	0.032 61	0.045 67	−0.426 26					
	6	1.270 08	0.018 56	0.021 97	0.029 97	0.042 96	−0.383 54				
	7	1.238 69	0.012 82	0.014 97	0.021 02	0.029 89	0.038 40	−0.355 79			
	8	1.217 84	0.009 51	0.010 23	0.015 22	0.020 89	0.028 34	0.033 94	−0.335 99		
	9	1.203 58	0.006 80	0.007 26	0.010 92	0.015 82	0.019 81	0.022 99	0.039 95	−0.327 13	
	10	1.193 52	0.005 63	0.004 65	0.007 68	0.012 17	0.014 71	0.017 18	0.028 95	0.027 83	−0.312 32
	n	d_1	d_2	d_3	d_4	d_5	d_6	c_7	d_8	d_9	d_{10}
	2	−8.169 93	8.169 93								
	3	−4.165 01	−0.503 58	4.668 59							
	4	−2.868 17	−0.290 40	−0.370 76	3.529 32						
	5	−2.246 80	−0.183 49	−0.235 67	−0.318 99	2.984 96					
	6	−1.891 30	−0.124 03	−0.161 28	−0.209 24	−0.295 92	2.681 78				
	7	−1.672 17	−0.083 97	−0.112 42	−0.146 71	−0.204 67	−0.263 90	2.483 83			
	8	−1.525 69	−0.060 75	−0.079 15	−0.106 01	−0.141 43	−0.193 20	−0.254 11	2.360 35		
	9	−1.426 66	−0.041 90	−0.058 53	−0.076 15	−0.106 22	−0.133 97	−0.178 02	−0.250 08	2.271 53	
	10	−1.355 63	−0.033 65	−0.040 09	−0.053 25	−0.080 46	−0.097 96	−0.137 06	−0.172 41	−0.234 69	2.205 20

b	n	c_1	c_2	c_3	c_4	c_5	c_6	c_7	c_8	c_9	c_{10}
7	**2**	2.142 60	−1.142 60								
	3	1.580 34	0.062 41	−0.642 75							
	4	1.396 68	0.035 98	0.047 60	−0.480 26						
	5	1.307 46	0.024 11	0.030 18	0.038 70	−0.400 45					
	6	1.256 61	0.015 63	0.021 86	0.026 20	0.035 01	−0.355 32				
	7	1.223 99	0.011 50	0.015 59	0.018 56	0.024 47	0.032 57	−0.326 68			
	8	1.202 09	0.008 47	0.011 49	0.013 25	0.018 78	0.021 76	0.032 73	−0.308 56		
	9	1.186 80	0.006 51	0.008 08	0.009 96	0.013 52	0.016 76	0.023 58	0.031 42	−0.296 62	
	10	1.176 13	0.004 34	0.006 98	0.007 31	0.009 97	0.012 86	0.017 57	0.025 40	0.017 02	−0.277 59
	n	d_1	d_2	d_3	d_4	d_5	d_6	c_7	d_8	d_9	d_{10}
	2	−9.140 77	9.140 77								
	3	−4.642 55	−0.499 61	5.142 16							
	4	−3.174 24	−0.288 31	−0.377 01	3.839 56						
	5	−2.459 91	−0.193 24	−0.237 58	−0.315 64	3.206 36					
	6	−2.053 22	−0.125 46	−0.171 05	−0.215 66	−0.276 44	2.841 83				
	7	−1.792 07	−0.092 38	−0.120 80	−0.154 49	−0.192 05	−0.263 25	2.615 05			
	8	−1.617 25	−0.068 15	−0.088 14	−0.112 12	−0.146 64	−0.176 93	−0.253 26	2.462 48		
	9	−1.496 08	−0.052 61	−0.061 05	−0.086 07	−0.104 95	−0.137 27	−0.180 64	−0.233 50	2.352 16	
	10	−1.409 08	−0.034 99	−0.052 13	−0.064 48	−0.076 02	−0.105 56	−0.131 72	−0.184 48	−0.203 40	2.261 86
b	n	c_1	c_2	c_3	c_4	c_5	c_6	c_7	c_8	c_9	c_{10}
8	**2**	2.124 49	−1.124 49								
	3	1.569 39	0.055 26	−0.624 65							
	4	1.387 07	0.032 11	0.043 04	−0.462 22						
	5	1.298 07	0.020 91	0.029 39	0.033 08	−0.381 45					
	6	1.246 90	0.014 66	0.020 87	0.023 31	0.026 95	−0.332 69				
	7	1.213 51	0.011 25	0.015 31	0.017 14	0.018 54	0.029 13	−0.304 89			
	8	1.176 86	0.006 40	0.009 38	−0.130 40	0.323 98	−0.158 41	0.221 23	−0.449 05		
	9	1.170 08	0.005 41	0.008 83	−0.083 41	0.213 70	−0.099 85	0.150 30	−0.167 75	−0.197 32	
	10	1.163 40	0.004 37	0.008 36	0.034 52	−0.055 90	0.047 96	−0.016 87	0.066 77	0.038 81	−0.291 43
	n	d_1	d_2	d_3	d_4	d_5	d_6	c_7	d_8	d_9	d_{10}
	2	−10.121 50	10.121 50								
	3	−5.123 63	−0.500 36	5.623 98							
	4	−3.486 56	−0.292 44	−0.371 42	4.150 42						
	5	−2.685 71	−0.191 66	−0.248 60	−0.306 22	3.432 19					
	6	−2.223 36	−0.135 22	−0.171 55	−0.217 92	−0.258 21	3.006 25				
	7	−1.922 71	−0.104 53	−0.121 54	−0.162 33	−0.182 54	−0.251 60	2.745 25			
	8	−1.593 01	−0.060 92	−0.068 17	1.164 90	−2.930 21	1.435 50	−1.987 69	4.039 60		
	9	−1.532 36	−0.052 08	−0.063 25	0.744 93	−1.944 62	0.912 10	−1.353 75	1.525 54	1.763 50	
	10	−1.472 28	−0.042 73	−0.059 01	−0.316 06	0.480 88	−0.417 69	0.150 25	−0.584 32	−0.360 83	2.621 78
b	n	c_1	c_2	c_3	c_4	c_5	c_6	c_7	c_8	c_9	c_{10}
9	**2**	2.111 11	−1.111 11								
	3	1.560 36	0.051 60	−0.611 95							
	4	1.380 00	0.032 01	0.032 82	−0.444 83						
	5	1.291 21	0.021 24	0.021 92	0.033 11	−0.367 49					
	6	1.239 41	0.015 89	0.014 64	0.024 81	0.022 63	−0.317 39				
	7	1.205 78	0.012 53	0.009 48	0.018 94	0.017 48	0.025 62	−0.289 83			
	8	1.183 07	0.009 51	0.006 99	0.015 92	0.011 65	0.019 61	0.019 19	−0.265 95		
	9	1.166 68	0.007 44	0.004 99	0.012 89	0.008 31	0.015 84	0.013 85	0.023 68	−0.253 67	
	10	1.154 61	0.006 04	0.003 31	0.010 34	0.006 20	0.012 80	0.009 51	0.020 54	0.020 39	−0.243 73
	n	d_1	d_2	d_3	d_4	d_5	d_6	c_7	d_8	d_9	d_{10}
	2	−11.111 10	11.111 10								
	3	−5.609 46	−0.503 48	6.112 94							
	4	−3.800 80	−0.307 06	−0.352 92	4.460 78						
	5	−2.916 13	−0.199 74	−0.244 37	−0.301 37	3.661 61					
	6	−2.396 79	−0.146 15	−0.171 33	−0.218 12	−0.249 93	3.182 32				
	7	−2.061 40	−0.112 63	−0.119 86	−0.159 53	−0.198 56	−0.238 34	2.890 32			
	8	−1.832 33	−0.082 19	−0.094 77	−0.129 15	−0.139 72	−0.177 80	−0.226 16	2.682 12		
	9	−1.669 10	−0.061 57	−0.074 84	−0.098 91	−0.106 46	−0.140 24	−0.172 94	−0.202 30	2.526 35	
	10	−1.548 97	−0.047 63	−0.058 09	−0.073 55	−0.085 52	−0.109 94	−0.129 73	−0.171 06	−0.201 78	2.426 27

b	n	c_1	c_2	c_3	c_4	c_5	c_6	c_7	c_8	c_9	c_{10}
10	**2**	2.099 15	−1.099 15								
	3	1.554 47	0.045 34	−0.599 81							
	4	1.374 30	0.028 16	0.032 69	−0.435 15						
	5	1.285 69	0.019 34	0.022 20	0.027 33	−0.354 55					
	6	1.233 78	0.013 82	0.015 62	0.019 71	0.026 02	−0.308 94				
	7	1.200 11	0.010 40	0.011 64	0.015 23	0.020 00	0.017 72	−0.275 09			
	8	1.177 05	0.007 79	0.009 74	0.011 33	0.016 35	0.012 00	0.017 18	−0.251 45		
	9	1.159 95	0.006 96	0.006 20	0.009 89	0.013 50	0.008 40	0.013 27	0.021 51	−0.239 69	
	10	1.147 60	0.005 02	0.005 13	0.008 24	0.010 44	0.007 12	0.008 10	0.019 85	0.018 11	−0.229 62
	n	d_1	d_2	d_3	d_4	d_5	d_6	c_7	d_8	d_9	d_{10}
	2	−12.091 90	12.091 90								
	3	−6.101 74	−0.494 62	6.596 36							
	4	−4.119 23	−0.305 58	−0.363 43	4.788 24						
	5	−3.145 76	−0.208 61	−0.248 12	−0.292 54	3.895 04					
	6	−2.578 81	−0.148 35	−0.176 30	−0.209 30	−0.261 61	3.374 37				
	7	−2.207 60	−0.110 64	−0.132 41	−0.159 89	−0.195 29	−0.227 12	3.032 94			
	8	−1.951 40	−0.081 71	−0.111 32	−0.116 61	−0.154 75	−0.163 55	−0.214 88	2.794 21		
	9	−1.764 00	−0.072 61	−0.072 56	−0.100 86	−0.123 48	−0.124 15	−0.172 08	−0.196 53	2.626 26	
	10	−1.629 14	−0.051 42	−0.060 86	−0.082 76	−0.090 03	−0.110 19	−0.115 69	−0.178 44	−0.188 73	2.507 26

Table 4. $\frac{Var(\hat{\mu})}{\sigma^2}$, $\frac{Var(\hat{\sigma})}{\sigma^2}$ and $Cov(\hat{\mu}, \hat{\sigma})$ when $a = 1$.

b	n								
	2	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
1	0.444 37	0.361 11	0.341 84	0.336 05	0.334 25	0.333 64	0.333 43	0.333 37	0.333 37
	0.777 60	0.444 45	0.367 46	0.344 34	0.337 03	0.334 59	0.333 75	0.333 51	0.333 50
	−0.555 44	−0.388 90	−0.350 38	−0.338 80	−0.335 18	−0.333 96	−0.333 53	−0.333 42	−0.333 42
2	0.187 41	0.145 84	0.133 92	0.129 18	0.127 01	0.125 98	0.125 49	0.125 24	0.125 11
	0.687 01	0.312 54	0.205 31	0.162 55	0.143 13	0.133 90	0.129 42	0.127 20	0.126 07
	−0.312 29	−0.187 51	−0.151 76	−0.137 53	−0.131 03	−0.127 96	−0.126 47	−0.125 72	−0.125 34
3	0.106 63	0.081 68	0.074 01	0.070 63	0.068 92	0.067 91	0.067 44	0.067 14	0.066 94
	0.706 28	0.306 77	0.184 23	0.130 16	0.102 66	0.087 29	0.078 95	0.074 02	0.071 02
	−0.226 54	−0.126 71	−0.096 06	−0.082 53	−0.075 67	−0.071 61	−0.069 76	−0.068 54	−0.067 75
4	0.069 47	0.052 78	0.047 53	0.045 10	0.043 77	0.043 01	0.042 54	0.042 24	0.042 04
	0.736 33	0.319 34	0.187 95	0.127 16	0.094 28	0.075 10	0.063 27	0.055 80	0.050 93
	−0.180 64	−0.097 20	−0.070 94	−0.058 78	−0.052 18	−0.048 36	−0.046 00	−0.044 50	−0.043 52
5	0.049 00	0.037 07	0.033 26	0.031 44	0.030 44	0.029 83	0.029 42	0.029 17	0.028 98
	0.763 48	0.334 65	0.197 20	0.131 99	0.095 71	0.073 64	0.059 34	0.049 97	0.043 50
	−0.151 10	−0.079 59	−0.056 69	−0.045 81	−0.039 77	−0.036 10	−0.033 69	−0.032 15	−0.031 05
6	0.036 49	0.027 54	0.024 65	0.023 26	0.022 47	0.021 97	0.021 65	0.021 42	0.021 27
	0.786 96	0.348 93	0.207 18	0.139 10	0.100 28	0.076 11	0.060 17	0.049 24	0.041 57
	−0.130 35	−0.067 71	−0.047 48	−0.037 75	−0.032 20	−0.028 74	−0.026 47	−0.024 90	−0.023 81
7	0.028 22	0.021 27	0.019 00	0.017 91	0.017 27	0.016 87	0.016 60	0.016 41	0.016 28
	0.806 29	0.361 57	0.216 48	0.146 16	0.105 70	0.080 06	0.062 80	0.050 73	0.042 12
	−0.114 68	−0.059 09	−0.040 94	−0.032 16	−0.027 10	−0.023 90	−0.021 74	−0.020 21	−0.019 16
8	0.022 50	0.016 96	0.015 12	0.014 23	0.013 73	0.013 40	0.013 03	0.012 96	0.012 89
	0.821 60	0.372 64	0.224 96	0.152 85	0.111 27	0.084 52	0.054 72	0.049 25	0.043 79
	−0.102 41	−0.052 55	−0.036 10	−0.028 09	−0.023 49	−0.020 51	−0.017 20	−0.016 59	−0.015 98
9	0.018 37	0.013 81	0.012 33	0.011 59	0.011 16	0.010 88	0.010 70	0.010 56	0.010 46
	0.837 04	0.381 83	0.232 67	0.159 19	0.116 45	0.088 74	0.069 84	0.056 25	0.046 18
	−0.092 81	−0.047 24	−0.032 37	−0.025 00	−0.020 73	−0.017 96	−0.016 08	−0.014 72	−0.013 70
10	0.015 28	0.011 50	0.010 25	0.009 63	0.009 27	0.009 04	0.008 88	0.008 76	0.008 67
	0.848 04	0.390 45	0.239 43	0.164 89	0.121 25	0.092 94	0.073 45	0.059 30	0.048 72
	−0.084 70	−0.043 10	−0.029 37	−0.022 59	−0.018 59	−0.016 02	−0.014 27	−0.012 98	−0.012 01

3.2. *BLUP of a Future Record*

Suppose we have the following sequence or record values $X_{U(1)}, X_{U(2)}, ..., X_{U(m)}$ and we want to predict $X_{U(n)}$, where $1 \leq m < n$. Since our distribution function belongs to a location - scale family, the BLUP of $X_{U(n)}$ can be obtained from the following linear model (see, Goldberger [11])

$$\hat{X}_{U(n)} = (\hat{\mu} + \alpha_n \ \hat{\sigma}) + w^T \Sigma^{-1} (X_U - \hat{\mu} \ \mathbf{1} - \hat{\sigma} \ \alpha), \tag{16}$$

where w^T is the vector of covariances between the nth record statistic and the previous m records.

4. Simulation Study and a Real Data Example

Example 4.1. To check the efficiency of all previous work, a simulation study is conducted. For Kumaraswamy distribution with $a = 1$ and $b = 10$, a random sample of size 100 was generated and then the upper records among them were observed. The picked upper records are $0.0662548, 0.105291, 0.278982, 0.285683$ and 0.350102. Now we use (16) to predict the $5th$ record from the previous four records.

$$X_U = \begin{bmatrix} 0.0662548 \\ 0.105291 \\ 0.278982 \\ 0.285683 \end{bmatrix}, \alpha = \begin{bmatrix} 0.0909 \\ 0.1736 \\ 0.2487 \\ 0.3170 \end{bmatrix}, \Sigma = \begin{bmatrix} 0.00639 & 0.00626 & 0.00569 & 0.00517 \\ 0.00626 & 0.01143 & 0.01039 & 0.00945 \\ 0.00569 & 0.01039 & 0.01423 & 0.01294 \\ 0.00517 & 0.00945 & 0.01294 & 0.01575 \end{bmatrix}$$

$$w = \begin{bmatrix} 0.00470 \\ 0.00859 \\ 0.01176 \\ 0.01431 \end{bmatrix}, \text{coeficients of } \hat{\mu} = \begin{bmatrix} 1.37430 \\ 0.02816 \\ 0.03269 \\ -0.43515 \end{bmatrix}, \text{and of } \hat{\sigma} = \begin{bmatrix} -4.11923 \\ -0.30558 \\ -0.36343 \\ 4.78824 \end{bmatrix}, \alpha_5 = 0.3791.$$

By calculating, $\hat{\mu} = -0.0211761, \hat{\sigma} = 0.96145$ and $\hat{X}_{U(5)} = 0.345213$. When comparing the predicted $5th$ record with the actual one (0.350102), we will find that the two values are close.

Example 4.2. In Table 5, a real-life data for a specific month (September) from 1990 to 2014 for the capacity of the Shasta reservoir in California, USA (in terms of acre-foot). The data has been transformed by using the transformation $x = (z - z_{min})/(z_{max} - z_{min})$ and the resulted numbers has the interval [0,1]. z_{min} will equal to zero while z_{max} which is the maximum capacity of the reservoir will equal to 4552000 acre-feet.

Table 5. September capacity for Shasta reservoir from 1990 to 2014

$Year$	$Capacity$	$Tansformed\,data$	$Year$	$Capacity$	$Tansformed\,data$
1990	1637368	0.359 70	2003	3159376	0.694 06
1991	1339851	0.294 34	2004	2182851	0.479 54
1992	1683200	0.369 77	2005	3034837	0.666 70
1993	3101762	0.681 41	2006	3205145	0.704 12
1994	2101642	0.461 70	2007	1879144	0.412 82
1995	3136430	0.689 02	2008	1384481	0.304 15
1996	3088810	0.678 56	2009	1773947	0.389 71
1997	2308339	0.507 10	2010	3318779	0.729 08
1998	3441073	0.755 95	2011	3341094	0.733 98
1999	3327499	0.731 00	2012	2591560	0.569 32
2000	2985131	0.655 78	2013	1950985	0.428 60
2001	2199643	0.483 23	2014	1157084	0.254 19
2002	2558201	0.561 99			

Now, to fit the data after the transformation to Kumaraswamy distribution we will use the methods of Maximum Likelihood and Moments to estimate the parameters. After that we will apply Cramér-von Mises, Kolmogorov-Smirnov and Anderson-Darling goodness of fit tests to choose the parameters.

Table 6. Fitting transformed data to Kumaraswamy distribution

$Distribution/TestingMethod$	$Kumaraswamy[a,b]$	
Moments	$\hat{a} = 3.52629$ $\hat{b} = 5$	
	$P - value$	$Statistics$
Cramér-von Mises	0.551631	0.107001
Kolmogorov-Smirnov	0.479015	0.161973
Anderson-Darling	0.584629	0.667214
Maximum Likelihood	$\hat{a} = 3.74572$ $\hat{b} = 6$	
	$P - value$	$Statistics$
Cramér-von Mises	0.440213	0.134571
Kolmogorov-Smirnov	0.293705	0.189228
Anderson-Darling	0.479055	0.799829

Clearly when $\hat{a} = 3.52629$ and $\hat{b} = 5$, the values of all goodness of fit tests are better. After that, the records among the values are $0.35970, 0.36977, 0.68141, 0.68902$ and 0.75595. When using the same steps in Example 4.1 to predict the $5th$ upper record from the first four records, we will get $\hat{X}_{U(5)} = 0.74018$ which has a small difference from the original value (0.75595).

5. Conclusion

The work that has been done in this paper focuses on predicting a future upper record based on Kumaraswamy distributed data with a very small part of error in the prediction by using the method of BLUE for estimating the parameters and the method of BLUP for the prediction process. The results are useful when people are interesting into knowing the next biggest number for some natural phenomena. Section 2 introduces new formulas for the jth moment of the nth upper record and the product moment of two not necessarily sequential upper records along with two various versions for the expression introduced to make the programming process (on a mathematical software such as MATHEMATICA) of the expression much easier. And of course, using these mathematical softwares gives credibility for the work that has been done.In section 3, the method of finding the BLUE for the parameters along with the rule for predicting the PLUP was founded along with some tables that were built for specific most repeated values of the parameters to make it easier for the practitioners to do their work. The simulated data in Example 4.1 was generated to clarify the steps that is needed to be done to find the predictive value of the future upper record value and to show that it will bring a good result for the predicted record. While in Example 4.2, a real data was tested for their distribution and whether if follows Kumaraswamy distribution or not by using different testing methods and found that it does. Then the same steps used in Example 4.1 applied here and we found an even better results.

Acknowledgement

This publication was supported by Deanship of Scientific Research, Prince Sattam Bin Abdul-Aziz University, Alkharj, Saudi Arabia. Our special thanks for their encouragement and supporting of this research.
The author would like to thank the anonymous referees along with the associate editor for constructive suggestions and comments that improved the representation substantially.

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