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A Discrete New Generalized Two Parameter Lindley Distribution: Properties, Estimation and Applications

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Abstract The main objective of this paper is to introduce a flexible two-parameter distribution which has not been considered in the literature before. The proposed distribution is referred to as a discrete new generalized two parameter Lindley distribution. Discrete Lindley and Geometric distributions are sub-models of the proposed distribution. Its probability mass function exhibits different shapes including decreasing, unimodal and decreasing-increasing-decreasing. Our proposed distribution has only two-parameters and its hazard rate function can accommodate increasing, constant, decreasing and bathtub shapes. Moreover, this distribution can describe equi and over dispersed data. Several distributional properties are obtained and several reliability characteristics are derived such as cumulative distribution function, hazard rate function, second hazard rate function, mean residual life function, reversed hazard rate function, accumulated hazard rate function and also its order statistics. In addition, the study of the shapes of the hazard rate function is provided analytically and also by plots. Estimation of the parameters is done using the maximum likelihood method. A simulation study is conducted to assess the performance of the maximum likelihood estimators. Finally, the flexibility of the model is illustrated using three real data sets.

Keywords Mixture discrete distribution; Bathtub shaped; Index of dispersion; Moments; Order statistics; Maximum likelihood estimation.

AMS 2010 subject classifications 60E05, 62N05, 62F10

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1. Introduction

In statistical literature, several distributions have been proposed as discrete analogues of continuous ones. These discretized distributions have useful applications in reliability analysis, for example, where time can be interpreted as a discrete random variable in many cases such as: the number of times a piece of equipment is operated; the life time of a certain device being measured by the number of cycles it completes etc. Various discretization methods have been considered. For more details, readers are encouraged to see Chakraborty (2015). One of the discretization methods is based on the reliability function. If the underlying continuous failure time Y has the reliability function $R_Y(y)$, the probability mass function (PMF) of the discrete random variable X associated with that continuous distribution can be written as

$$P_X(x) = R_Y(x) - R_Y(x+1), \quad x = 0, 1, 2, \ldots. \tag{1}$$

This method was proposed by Nakagawa and Osaki (1975) to discretize a continuous distribution, the discrete Weibull distribution, and obtain some reliability properties. After that, many researchers used this method to propose new discretized distributions such as Gómez-Déniz and Calderín-Ojeda (2011) who introduced the discrete Lindley (dL) distribution and recently, Opone *et al.* (2021) who introduced discrete Marshall-Olkin Weibull

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distribution. This method gained great attention mainly because it preserves same functional form of the reliability function (RF) that is $R_Y(x) = R_X(x)$. Consequently, many reliability characteristics and properties should remain unchanged. This method is also useful in practice because if the RF of the continuous random variable has a compact form, then, the resulting PMF will be in a compact form. Possibly this method is the easiest method of construction (Chakraborty (2015)). As a result, the method of discretizing using the reliability function will be used in constructing the discretized distribution used in this study.

Although, many discretized distributions have been proposed in the literature, few of them possess bathtub (BT) hazard rate (HR) functions. Some examples of the distributions having this property follow. Nooghabi *et al.* (2011) introduced the 3-parameter discrete modified Weibull distribution. Nekoukhou and Bidram (2015) proposed a 3-parameter exponentiated discrete Weibull distribution. Jayakumar and Babu (2019) obtained a 5-parameter discrete Additive Weibull Geometric distribution and El-Morshedy *et al.* (2019) proposed a two-parameter exponentiated discrete Lindley (EDLi) distribution. Recently, Opone *et al.* (2021) proposed the 3-parameter discrete Marshall-Olkin Weibull distribution.

Most of the above mentioned papers do not study the shape of the HR function analytically depending mainly on plots to illustrate the shape of the HR; plots can be quite misleading in many cases as will be shown in this paper in Section 2 through an analytical study of the shapes of the HR function of EDLi distribution given by El-Morshedy *et al.* (2019).

Lindley (1958) proposed the one parameter Lindley distribution and Ghitany *et al.* (2008) showed that it has more flexibility than the exponential distribution in modeling some lifetime data sets. It has only increasing HR function. Therefore, several generalizations were conducted to accommodate various shapes such as increasing, decreasing, constant, BT and upside-down bathtub shaped hazard rates. The new generalized two parameter Lindley (NG2PL) distribution was proposed by Ekhosuehi *et al.* (2018) as one of the generalization of the one-parameter Lindley (1958). The unknown parameters were estimated by maximum likelihood (ML) method and it was illustrated through a graph that the HR can take various shapes, increasing, decreasing, constant and BT. The RF of the NG2PL distribution is a two-component mixture of Exponential (θ) and Gamma $(\alpha,\ \theta)$ distributions with mixing proportion $p = \frac{\theta}{\theta+1}$. The RF of NG2PL distribution as given in Ekhosuehi *et al.* (2018) is

$$R_Y(y;\ \alpha,\theta) = \frac{\theta\,\Gamma(\alpha)\ e^{-\theta y} + \Gamma(\alpha,\ \theta y)}{(\theta+1)\ \Gamma(\alpha)}, \qquad y > 0 \tag{2}$$

where $\alpha,\theta > 0$, $\Gamma(\alpha)$ and $\Gamma(\alpha,\ \theta y)$ are respectively the complete gamma function and upper incomplete gamma function.

The importance of this distribution comes from several reasons. It appears that the NG2PL distribution is the only two-parameter distribution from the Lindley family of continuous distributions whose HR can take increasing, constant, decreasing, and BT shapes. Also, there are two important sub-models from the NG2PL distribution; namely the exponential distribution when $\alpha = 1$ and the Lindley distribution when $\alpha = 2$. Moreover, the dNG2PL distribution can be represented as a mixture distribution.

The main objective of this paper is introducing a new discrete distribution by discretizing the NG2PL distribution proposed by Ekhosuehi *et al.* (2018). The new discretized distribution will be referred to as discrete new generalized two parameter Lindley (dNG2PL) distribution. In addition to being a two-parameter model, the new model has some advantages. It can be represented as a mixture distribution of two discrete distributions. The importance of mixture distributions comes from their ability to model the case of heterogeneous populations and hence they have many applications in medical sciences, biology, engineering, finance and economics. In addition, dNG2PL distribution can fit over and equi dispersed data. Moreover, it possesses four shapes of HR namely, increasing, decreasing, constant and BT.

The rest of this paper is organized as follows. Section 2 presents an analytical study of the shape of the HR function of EDLi distribution. Sections 3 and 4 introduce discrete new generalized two parameter Lindley distribution and its distributional properties, respectively. In Section 5, the reliability characteristics of the proposed distribution are obtained. This includes a theoretical study of its hazard rate function and its shapes. In Section 6, maximum likelihood estimation of the parameters of the proposed distribution is developed. Finally, in Sections 7 and 8, respectively a simulation study and three real data sets are given to illustrate the results.

2. Hazard Rate Shapes for Continuous and Discrete Exponentiated Lindley Distributions

The main objective of this Section is to study theoretically the HR shapes of the EDLi distribution which was proposed by El-Morshedy *et al.* (2019) who illustrated the shapes of its HR merely by plots. This distribution is the discrete analogue of the continuous exponentiated Lindley (EL) distribution distributed by Nadarajah *et al.* (2012). First the HR shapes of the continuous EL distribution will be studied. Then a proposition given by Noughabi *et al.* (2013), which determines the shapes of the HR of the discrete model in terms of the continuous one, will be used to determine the shape of the HR of EDLi distribution.

2.1. The Hazard Rate of the Continuous Exponentiated Lindley distribution

The shape of the HR function of the continuous EL distribution proposed by Nadarajah *et al.* (2012) will be studied using the approach of Glaser (1980). The main results of Glaser (1980) are given in Appendix B. The probability density function of the continuous EL distribution is given by

$$f_1\,(y;\ \alpha,\ \lambda) = \frac{\alpha\,\lambda^2}{1+\lambda}\,(1+y)\ \ [V(y)]^{\alpha-1}\,e^{-\lambda\,y},\ y>0,\,\alpha>0,\,\lambda>0, \tag{3}$$

where

$$V(y) = 1-\frac{1+\lambda+\lambda\,y}{1+\lambda}\,e^{-\lambda\,y},$$

let

$$\eta_1(y) = -\frac{\partial\ \ln\ f_1\,(y;\ \alpha,\lambda)}{\partial\,y}.$$

It can be shown that

$$\eta_1(y)\ = \lambda-(1+y)^{-1}-\frac{c\,(\alpha-1)\,e^{-\lambda\,y}\,(1+y)}{V(y)}, \tag{4}$$

where $c=\frac{\lambda^2}{1+\lambda}$.
It can also be noted that

$$\lim_{y\to 0}\ \eta_{\,1}(y) = \begin{cases} \infty & if\,\alpha<1, \\ \lambda-1 & if\,\alpha=1, \\ -\infty & if\,\alpha>1. \end{cases}$$

$$\lim_{y\to\,\infty}\ \eta_{\,1}(y) = \lambda.$$

Hence, if $\alpha<1, \eta_1$ (y) cannot be increasing or upside down bathtub and for $\alpha\geq 1,\ \eta_{\,1}$(y) cannot be decreasing or BT.

The following lemma provides the shapes of the HR function.

Lemma 1

Let $h_1(y;\ \alpha,\lambda)$ denotes the HR function of the continuous EL distribution.

i) If $\alpha\geq 1$, then $h_1(y;\ \alpha,\lambda)$ is increasing.

ii) If $\alpha<1$, then $h_1(y;\ \alpha,\lambda)$ is either decreasing or bathtub.

Proof

i) **Part a: For $\alpha = 1$.**

Referring to (4), $(1+y)^{\,-1}$ is a decreasing function in y indicating that $\eta_{\,1}(y)$ is an increasing function in y, hence $h_1(y;\ \alpha,\lambda)$is an increasing function in y.

Part b: For $\alpha > 1$.
Both $(1+y)^{-1}$ and $\frac{c(\alpha-1)e^{-\lambda y}(1+y)}{V(y)}$ are decreasing functions in y indicating that $\eta_1(y)$ and consequently $h_1(y;\ \alpha,\lambda)$ are increasing functions in y.
Hence, part (i) of Lemma (1) is established.

ii) **For $\alpha < 1$,** $-(1+y)^{-1}$ is an increasing function in y and $\frac{-c(\alpha-1)e^{-\lambda y}(1+y)}{V(y)} \geq 0$ is a decreasing function in y.

$\because \lim_{y\to 0} \eta_1(y) = \infty$ for $\alpha < 1$

Then initially $\eta_1(y)$ is a decreasing function in y. We have two cases:

a) **Case 1:** If the rate of decrease of the function $\frac{-c(\alpha-1)e^{-\lambda y}(1+y)}{V(y)}$ is always greater than the rate of increase of the function $-(1+y)^{-1}$ then $\eta_1(y)$ and hence $h_1(y;\ \alpha,\lambda)$ will always be decreasing functions in y for $\alpha < 1$.

b) **Case 2:** If the rate of decrease of the function $\frac{-c(\alpha-1)e^{-\lambda y}(1+y)}{V(y)}$ is greater than the rate of increase of the function $-(1+y)^{-1}$ for $y < y_0$ and the situation is reversed for $y > y_0$ then $\eta_1(y)$ will be decreasing for $y < y_0$ and increasing for $y > y_0$. Thus, $\eta_1(y)$ will attain its minimum at $y = y_0$. Hence, $\eta_1(y)$ has a BT shape. Based on Glaser's (1980) Lemma given in Appendix B, we evaluate $\lim_{y\to 0} f_1(y;\ \alpha,\lambda)$ using (3), it can be shown that $\lim_{y\to 0} V(y) = 0$

$$\because \alpha < 1, \text{ then} \lim_{y\to 0} f_1(y;\ \alpha,\lambda) = \infty.$$

Hence for case 2, $h_1(y;\ \alpha,\lambda)$ has a BT shape and part (ii) of Lemma (1) is established.

2.2. The Hazard Rate of the Exponentiated Discrete Lindley distribution

Let $h_1(x;\ \alpha,\lambda)$ denote the HR function of the EDLi distribution. Using the proposition of Noughabi *et al.* (2013) given in the Appendix B, it clear that the HR function of EDLi distribution has the following shapes:
i) If $\alpha \geq 1$, $h_1(x;\ \alpha,\lambda)$ is increasing in x.
ii) If $\alpha < 1$, $h_1(x;\ \alpha,\lambda)$ is either decreasing, bathtub or increasing.
Hence the only possible shapes of the HR function of EDLi distribution are increasing, decreasing and BT. The other shapes of the HR function stated by El-Morshedy *et al.* (2019) namely decreasing-increasing-decreasing, increasing-decreasing-increasing, J- shaped and unimodal are incorrect.

3. Discrete New Generalized Two-Parameter Lindley Distribution

In this Section, we will obtain the dNG2PL distribution by discretizing the NG2PL distribution proposed by Ekhosuehi *et al.* (2018).
Using the discretization method based on the RF given in (2), the PMF of the dNG2PL distribution, is given by

$$P_X(x;\ \alpha,\ \theta) = \frac{1}{(\theta+1)\,\Gamma(\alpha)}\{[\theta\,\Gamma(\alpha)\ e^{-\theta x} + \Gamma(\alpha,\ \theta x)] - [\theta\,\Gamma(\alpha)\ e^{-\theta(x+1)} + \Gamma(\alpha,\ \theta(x+1))]\}, \quad (5)$$

where $\Gamma(a,\ z)$is the upper incomplete gamma function.
After some simplifications, $P_X(x;\ \alpha,\ \theta)$ will take the form

$$P_X(x;\ \alpha,\ \theta) = \frac{1}{(\theta+1)\Gamma(\alpha)}\left\{\theta\,\Gamma(\alpha)\ (1-e^{-\theta})\,e^{-\theta x} + \Gamma(\alpha,\theta x,\theta(x+1))\right\},\ \ x = 0,1,2,\dots;\ \ (\alpha,\theta) > 0, \quad (6)$$

where $\Gamma(\alpha,\theta x,\theta(x+1)) = \int_{\theta x}^{\theta(x+1)} u^{\alpha-1}\,e^{-u}\,du\ = \int_{x}^{x+1} \theta^{\alpha}\,u^{\alpha-1}\,e^{-\theta u}\,du.$

The RF is given by

$$R_X(x;\ \alpha,\theta) = \frac{1}{(\theta+1)\,\Gamma(\alpha)} \left\{\theta\,\Gamma(\alpha)\ e^{-\theta\,x} + \Gamma(\alpha,\,\theta x)\right\}, \quad x = 0,1,2,\dots\,; \quad (\alpha,\,\theta)\ > 0. \tag{7}$$

It is known that, using the reliability function method, the RF of NG2PL distribution and dNG2PL distribution are the same.

The cumulative distribution function of dNG2PL distribution is as follows

$$\begin{aligned} F_X(x;\ \alpha,\theta) &= 1 - P(X > x) = 1 - R_X(x+1;\ \alpha,\theta) \\ &= 1 - \frac{1}{(\theta+1)\,\Gamma(\alpha)} \left\{\theta\,\Gamma(\alpha)\ e^{-\theta\,(x+1)} + \Gamma(\alpha,\,\theta\,(x+1))\right\}, \quad x = 0,1,2,\dots;\ \ (\alpha,\theta\,)\ > 0. \end{aligned} \tag{8}$$

The graphical presentation of the PMF of dNG2PL distribution for some fixed values of the parameters is shown in Figure 1. It is observed that the PMF plots have decreasing, unimodal and decreasing-increasing-decreasing shapes.

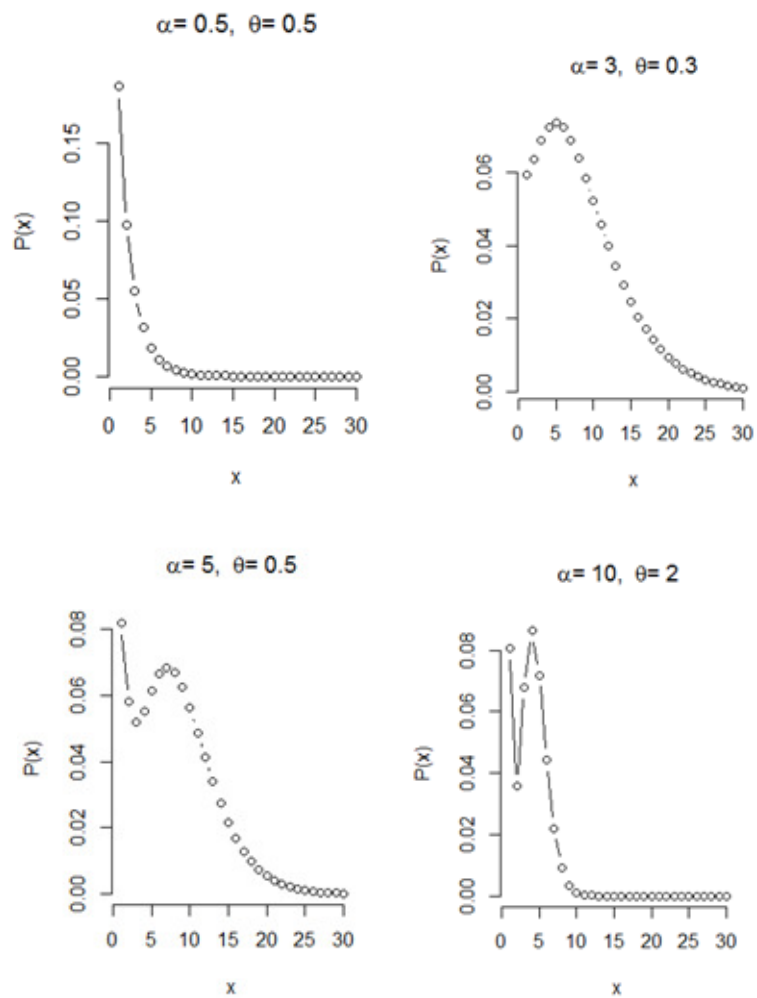

Figure 1. **The PMF plots of dNG2PL distribution with different combinations for α and θ**

Particular Cases

- For $\alpha = 1$, the dNG2PL distribution in (5) reduces to the geometric distribution.

- For $\alpha = 2$, the dNG2PL distribution in (5) reduces to the one parameter discrete Lindley distribution.

4. Distributional Properties

In this Section, the mixture representation of the PMF of dNG2PL distribution, the moments and the order statistics are derived.

4.1. Mixture Representation of the Probability Mass Function

The NG2PL distribution can be represented as mixture distribution as shown by Ekhosuehi *et al.* (2018). We shall prove in this subsection that the PMF of dNG2PL distribution can also be written as a mixture distribution. This representation will be used in simulating observations from the dNG2PL distribution as explained in Section 7.

The RF of the NG2PL distribution as a mixture representation is given as follows

$$R_Y\,(y;\ \alpha,\theta) = \ p\int_y^{\infty}\theta\, e^{-\theta u}\,du\, + (1-p)\frac{\theta^{\alpha}}{\Gamma(\alpha)}\ \int_y^{\infty}u^{\alpha-1}\ e^{-\theta\, u}\ du \tag{9}$$

where $y > 0$ and $p = \frac{\theta}{\theta+1}$.

Result
The PMF of dNG2PL distribution can be represented as a two component mixture of Geometric$(e^{-\theta})$ distribution and discrete Gamma(α,θ) distribution. Therefore, the PMF of dNG2PL distribution as mixture distribution is

$$P_X(x\ ;\ \alpha,\ \theta) = p\ (e^{-\theta\, x} - e^{-\theta\,(x+1)}\,) +\ (1-p)\ \int_x^{x+1}\frac{\theta^{\alpha}}{\Gamma(\alpha)}\,u^{\alpha-1}\,e^{-\theta\, u}du, \quad x = 0,1,2,...\ ;\quad (\alpha,\theta) > 0, \tag{10}$$

where $(e^{-\theta\, x} - e^{-\theta\,(x+1)}\,)$ is the PMF of the Geometric $(e^{-\theta})$distribution which is a discrete analogue of exponential (θ) distribution (See, Roy (1993)) and $\frac{\theta^{\,k}}{\Gamma(k)}\ \int_x^{x+1}\ z^{\,k-1}\ e^{-\theta\ k}\ dz$ denotes the PMF of the discrete gamma distributions as proposed by Chakraborty and Chakravarty (2012).
Proof:
The RF of dNG2PL distribution as a mixture distribution is as follows

$$R_Y(x;\ \alpha,\theta) =\ p\int_x^{\infty}\theta\, e^{-\theta\, u}du\ +\ (1-p)\ \int_x^{\infty}\frac{\theta^{\alpha}}{\Gamma(\alpha)}\,u^{\alpha-1}\,e^{-\theta\, u}du.$$

Therefore, the PMF of the mixture of dNG2PL distribution can be derived as follows

$$
\begin{aligned}
P_X(x;\ \alpha,\ \theta) &= R_Y(x;\ \alpha,\theta) - R_Y(x+1;\ \alpha,\theta),\\
&= p\int_x^{\infty} \theta e^{-\theta u} du + (1-p)\int_x^{\infty} \frac{\theta^{\alpha}}{\Gamma(\alpha)} u^{\alpha-1} e^{-\theta\, u} du - \left[p\int_{x+1}^{\infty} \theta\, e^{-\theta\, u} du\ +\ (1-p)\int_{x+1}^{\infty} \frac{\theta^{\alpha}}{\Gamma(\alpha)}\, u^{\alpha-1} e^{-\theta\, u} du\right],\\
&= p\left[\int_x^{\infty} \theta\, e^{-\theta\, u} du - \int_{x+1}^{\infty} \theta\, e^{-\theta\, u}\, du\right] + (1-p)\left[\int_x^{\infty} \frac{\theta^{\alpha}}{\Gamma(\alpha)}\, u^{\alpha-1}\, e^{-\theta\, u} du - \int_{x+1}^{\infty} \frac{\theta^{\alpha}}{\Gamma(\alpha)}\, u^{\alpha-1}\, e^{-\theta\, u} du\right],\\
&= p\int_x^{x+1} \theta\, e^{-\theta\, u} du + (1-p)\int_x^{x+1} \frac{\theta^{\alpha}}{\Gamma(\alpha)}\, u^{\alpha-1}\, e^{-\theta\, u} du,\\
&= p\,(e^{-\theta\, x} - e^{-\theta(x+1)})\ + (1-p)\int_x^{x+1} \frac{\theta^{\alpha}}{\Gamma(\alpha)} u^{\alpha-1}\, e^{-\theta\, u} du,
\end{aligned}
$$

where $x = 0, 1, 2, ...;\ (\alpha,\theta) > 0,$
Hence, the result given by (10) follows.

4.2. Moments

Moments can be used to study the characteristics of a distribution. So, in this subsection, we will derive the moments of the dNG2PL distribution.

4.2.1. The Non-Central Moments of Discrete New Generalized two-Parameter Lindley Distribution Suppose that X is a discrete non-negative integer valued random variable with RF given in (7). Chakraborti *et al.* (2017) proposed the *r*-th moment of a nonnegative random variable X and derived formulas in terms of the RF. The *r*-th moment of X for dNG2PL distribution is given by

$$
\mu_r' \ =\ E\,(X^r)\ =\ \sum_{x=0}^{\infty}((x+1)^r - x^r)\, R(x+1;\ \alpha,\theta), \qquad r \geq 1,
$$

$$
\mu_r' \ =\ \frac{1}{(\theta+1)\,\Gamma(\alpha)} \sum_{x=0}^{\infty} \left((x+1)^r - x^r\right) \left\{\theta\,\Gamma(\alpha)\ e^{-\theta\,(x+1)}\ + \Gamma(\alpha,\, \theta\,(x+1))\right\}. \tag{11}
$$

As a special case, the mean and variance of dNG2PL distribution are obtained as follows

$$
\mu_1{}' = \frac{1}{(\theta+1)\,\Gamma(\alpha)} \sum_{x=0}^{\infty} \left\{\theta\,\Gamma(\alpha)\ e^{-\theta\,(x+1)}\ + \Gamma(\alpha,\, \theta\,(x+1))\right\}. \tag{12}
$$

$$
\begin{aligned}
\mathrm{Var}\,(X)\ &=\ \mu_2'\ -\ (\mu_1')^2\\
&= \frac{1}{(\theta+1)\,\Gamma(\alpha)} \sum_{x=0}^{\infty} (2x+1) \left\{\theta\,\Gamma(\alpha)\ e^{-\theta\,(x+1)}\ + \Gamma(\alpha,\, \theta\,(x+1))\right\}\\
&\quad - \left[\frac{1}{(\theta+1)\,\Gamma(\alpha)} \sum_{x=0}^{\infty} \left\{\theta\,\Gamma(\alpha)\ e^{-\theta\,(x+1)}\ + \Gamma(\alpha,\, \theta\,(x+1))\right\}\right]^2.
\end{aligned} \tag{13}
$$

The mean and variance of the dNG2PL distribution for different values of θ and α are calculated numerically and displayed in Table 1. From this table, we observe that for fixed values of α as the values of θ increase both the values of the mean and the variance decrease. Also, the values of the mean and variance of the dNG2PL distribution increase as the parameter α increases for a fixed value of θ.

4.2.2. Index of Dispersion Index of dispersion (ID) is one of the properties of a distribution. ID is defined as the ratio between the variance to the mean. ID indicates whether the distribution is suitable for over or under or equi-dispersed data. If ID $> 1(< 1)$ the distribution is over dispersed (under dispersed) and if ID=1 the distribution is equi- dispersed. The ID of the dNG2PL distribution is calculated numerically for different values of θ and α. Table 1 provides these results. From Table 1, we can observe that depending on the values of parameters, the variance can be equal or greater than the mean. Therefore, dNG2PL distribution is appropriate for modeling equi or over dispersed data.

Table 1: Values of the mean, (variance) and ID for different combinations of α and θ

.

α / θ	**0.5**	**0.7**	**1**	**3**	**6**
0.5	0.942 (2.576) 2.735	0.632 (1.355) 2.144	0.390 (0.657) 1.685	0.044 (0.046) 1.045	0.002 (0.002) 1.000
0.9	1.401 (3.499) 2.498	0.900 (1.774) 1.971	0.530 (0.841) 1.587	0.056 (0.060) 1.071	0.002 (0.002) 1.000
1.5	2.212 (5.704) 2.579	1.402 (2.820) 2.011	0.789 (1.221) 1.548	0.069 (0.071) 1.029	0.003 (0.003) 1.000
2	2.885 (7.801) 2.704	1.784 (3.802) 2.131	1.073 (1.867) 1.740	0.104 (0.116) 1.115	0.004 (0.004) 1.000
4	5.478 (19.540) 3.567	3.463 (10.078) 2.910	2.040 (4.735) 2.321	0.249 (0.301) 1.209	0.024 (0.026) 1.083
8	10.943 (66.387) 6.067	6.864 (34.227) 4.986	4.044 (16.037) 3.966	0.563 (1.075) 1.909	0.026 (0.138) 1.095

4.2.3. Skewness and Kurtosis In this subsection, we will compute the values of skewness and kurtosis for different combinations of the parameters α and θ. The skewness and the kurtosis are given respectively by
Skewness $= \frac{\mu_3' - 3\,\mu_2'\,\mu_1' \ + 2(\mu_1')^3}{(Var(X))^{3/2}}$, and
Kurtosis$= \frac{\mu_4' - 4\,\mu_1'\ \mu_3'\ + 6\,(\mu_1'\,)^2\,\mu_2'\ - 3\,(\mu_1')^4}{(Var(X)\,)^4}$.

Table 2 presents these results. From Table 2, the skewness coefficient is always positive which means the dNG2PL distribution is skewed to the right. In addition, the kurtosis is greater than 3 for most values which, means this distribution has heavier tails than a normal distribution in these cases.

Table 2: Values of the skewness and (kurtosis) for different combinations of α and θ

.

α / θ	0.5	1	3	6
0.5	2.681 (9.904)	2.782 (10.011)	4.976 (25.662)	21.202 (447.366)
1.5	1.756 (4.423)	1.924 (5.018)	4.083 (17.884)	18.816 (351.932)
4	0.907 (0.905)	1.208 (1.478)	2.432 (6.129)	6.567 (43.695)

4.2.4. The Moment Generating Function Suppose X is a non-negative integer valued random variable, then for any $t \in \mathbb{R}$, we have

$$M_X(t) = E(e^{t\,x}) = 1 + \sum_{x=0}^{\infty}(e^{t(x+1)} - e^{t\,x})\,R(x+1;\ \alpha,\theta)$$

[see Song and Wang (2019)].

Therefore, the moment generating function of dNG2PL distribution is given by

$$M_X(t) = 1 + \frac{1}{(\theta+1)\,\Gamma(\alpha)}\sum_{x=0}^{\infty}(e^{t(x+1)} - e^{tx})\left\{\theta\,\Gamma(\alpha)\ e^{-\theta\,(x+1)} + \Gamma(\alpha,\,\theta\,(x+1))\right\}. \tag{14}$$

4.2.5. Probability Generating Function Based on a result given by Song and Wang (2019), the probability generating function is given by

$$\begin{aligned} E(t^X) &= 1 + \sum_{x=0}^{\infty}(t^{x+1} - t^x)\,R(x+1;\ \alpha,\theta), \\ &= 1 + \frac{1}{(\theta+1)\,\Gamma(\alpha)}\sum_{x=0}^{\infty}(t^{x+1} - t^x)\left\{\theta\,\Gamma(\alpha)\ e^{-\theta\,(x+1)} + \Gamma(\alpha,\,\theta\,(x+1))\right\}, \quad |t| \le 1. \end{aligned} \tag{15}$$

4.3. Order Statistics

Order statistics is a fundamental tool in inference and non-parametric statistics. In this subsection, we establish some results for the dNG2PL distribution related to order statistics. Let $X_1, X_2, ..., X_n$ be a random sample from dNG2PL distribution and $X_{(1)}, X_{(2)}, ..., X_{(n)}$ represents the corresponding order statistics. Then the cumulative distribution function of *r*-th order statistic is given by

$$F_{X_{(r)}}(x_{(r)}) = \sum_{r=i}^{n}\binom{n}{r}(F_X(x_{(r)};\ \alpha,\theta))^r(1 - F_X(x_{(r)};\ \alpha,\theta))^{n-r}$$

$$\begin{aligned} F_{X_{(r)}}(x_{(r)}) = &\left(\frac{1}{(\theta+1)\,\Gamma(\alpha)}\right)^n \sum_{r=i}^{n}\binom{n}{r}\left[(\theta+1)\,\Gamma(\alpha) - \ \theta\,\Gamma(\alpha)\ e^{-\theta\,(x_{(r)}+1)} - \Gamma(\alpha,\,\theta\,(x_{(r)}+1))\right]^r \\ &\times\ \left[\theta\,\Gamma(\alpha)\ e^{-\theta\,(x_{(r)}+1)} + \Gamma(\alpha,\,\theta\,(x_{(r)}+1))\right]^{n-r}. \end{aligned} \tag{16}$$

Using the binomial expansion theorem for $(F_X(x_r;\ \alpha,\theta))^r$, we obtain

$$(F_X(x_r;\ \alpha,\theta))^r=\sum_{j=0}^{r}\left(\begin{array}{c} r \\ j \end{array}\right)(-1)^j\left\{\frac{1}{(\theta+1)\,\Gamma(\alpha)}\left[\theta\,\Gamma(\alpha)\ e^{-\theta\,(x_{(r)}+1)}+\Gamma(\alpha,\,\theta\,(x_{(r)}+1))\right]\right\}^j. \tag{17}$$

Therefore,

$$F_{X_{(r)}}(x_{(r)})=\sum_{r=i}^{n}\sum_{j=0}^{r}\left(\begin{array}{c} n \\ r \end{array}\right)\left(\begin{array}{c} r \\ j \end{array}\right)(-1)^j\left\{\frac{1}{(\theta+1)\,\Gamma(\alpha)}\left[\theta\,\Gamma(\alpha)\ e^{-\theta\,(x_{(r)}+1)}+\Gamma(\alpha,\,\theta\,(x_{(r)}+1))\right]\right\}^{n-r+j}. \tag{18}$$

The PMF of the r-th order statistic is given by

$$f_{X_{(r)}}(x_{(r)})\ =\ \left(\begin{array}{c} n \\ r \end{array}\right)(F_X(x_{(r)};\ \alpha,\theta))^{r-1}\,(1-F_X(x_{(r)};\ \alpha,\theta))^{n-r}\,P_X\,(x_{(r)};\ \alpha,\theta)$$

$$\begin{aligned} f_{X_{(r)}}(x_{(r)})=&\left(\begin{array}{c} n \\ r \end{array}\right)\left(\frac{1}{(\theta+1)\,\Gamma(\alpha)}\right)^n\left[(\theta+1)\,\Gamma(\alpha)-\ \theta\,\Gamma(\alpha)\ e^{-\theta\,(x_{(r)}+1)}-\Gamma(\alpha,\,\theta\,(x_{(r)}+1))\right]^{r-1}\\ &\times\left[\theta\,\Gamma(\alpha)\ e^{-\theta\,(x_{(r)}+1)}+\Gamma(\alpha,\,\theta\,(x_{(r)}+1))\right]^{n-r}\\ &\times\left\{\theta\,\Gamma(\alpha)\ (1-e^{-\theta})\ e^{-\theta\,x_{(r)}}+\ \Gamma(\alpha,\,\theta x_{(r)},\theta(x_{(r)}+1))\right\}. \end{aligned} \tag{19}$$

Important special cases of the order statistics are the minimum and maximum values of a sample. The PMF of $X_{(1)}\ =\ \min\,(X_1,\ X_2,....,X_n)$ is

$$\begin{aligned} f_{X_{(1)}}(x_{(1)})=&\frac{n}{[\,(\theta+1)\,\Gamma(\alpha)]^n}\ \left[\theta\,\Gamma(\alpha)\ e^{-\theta\,(x_{(1)}+1)}+\Gamma(\alpha,\,\theta\,(x_{(1)}+1))\right]^{n-1}\\ &\times\left\{\theta\,\Gamma(\alpha)\ (1-e^{-\theta})\ e^{-\theta\,x_{(1)}}+\Gamma(\alpha,\,\theta x,\theta(x_{(1)}+1))\right\}. \end{aligned} \tag{20}$$

The PMF of $X_{(n)}\ =\ \max\,(X_1,\ X_2,....,X_n)$ is

$$\begin{aligned} f_{X_{(n)}}(x_{(n)})=&\frac{n}{[\,(\theta+1)\,\Gamma(\alpha)]^n}\left[(\theta+1)\,\Gamma(\alpha)-\ \theta\,\Gamma(\alpha)\ e^{-\theta\,(x_{(n)}+1)}-\Gamma(\alpha,\,\theta\,(x_{(n)}+1))\right]^{n-1}\\ &\times\left\{\theta\,\Gamma(\alpha)\ (1-e^{-\theta})\ e^{-\theta\,x_{(n)}}+\ \Gamma(\alpha,\,\theta x_{(n)},\theta(x_{(n)}+1))\right\}. \end{aligned} \tag{21}$$

5. Reliability Characteristics

In this Section, we will study theoretically the shapes of the HR of dNG2PL distribution and some reliability characteristics.

5.1. The Hazard Rate Function

In this subsection, we will derive the HR of dNG2PL distribution. Also, we will show that the HR of dNG2PL distribution has increasing, decreasing, constant and BT shapes.
The HR of dNG2PL distribution is given by

$$\begin{aligned} h_X(x;\ \alpha,\theta)=&\ \frac{P_X(x;\ \alpha,\theta)}{R_Y(x;\ \alpha,\theta)}=1-\frac{R_Y(x+1;\ \alpha,\theta)}{R_Y(x;\ \alpha,\theta)},\\ =&\ 1-\ \frac{\theta\,\Gamma(\alpha)\ e^{-\theta\,(x+1)}+\Gamma(\alpha,\,\theta(x+1))}{\theta\,\Gamma(\alpha)\ e^{-\theta\,x}+\Gamma(\alpha,\,\theta x)}. \end{aligned} \tag{22}$$

In an analogous manner to Section 2, the shapes of the HR of the continuous NG2PL distribution will be studied analytically using the approach of Glaser (1980) and then the propostion given by Noughabi *et al.* (2013) will be used to determine the shapes of the HR of the dNG2PL distribution.

Theorem:
The HR of NG2PL distribution given by Ekhosueski *et al.* (2018) has the following shapes
a) Decreasing if $0 < \alpha < 1$.
b) Constant if $\alpha = 1$.
c) Increasing if $1 < \alpha \leq 2$,
d) Bathtub if $\alpha > 2$.
Proof:
The probability density function of NG2PL distribution is given by

$$f_Y(y;\ \alpha, \theta) = \frac{\theta^2}{(\theta+1)}\,(1 + \frac{\theta^{\alpha-2}\,y^{\alpha-1}}{\Gamma(\alpha)})\ e^{-\theta\,y},\quad y > 0\,;\quad (\alpha, \theta) > 0. \tag{23}$$

Let $\eta(y) = -\frac{\partial\ \ln f_Y(y;\ \alpha,\theta)}{\partial\, y}$ as defined by Glaser (1980).
Therefore, $\eta(y)$ and $\eta'(y)$ for the NG2PL distribution are derived respectively as follows

$$\eta(y) = \theta - \frac{(\alpha-1)\ \theta^{\alpha-2}\,y^{\alpha-2}}{\Gamma(\alpha)\ +\ \theta^{\alpha-2}\,y^{\alpha-1}}, \tag{24}$$

$$\eta'(y) = \frac{\Gamma(\alpha)\,[-(\alpha-1)\ (\alpha-2)\ \Gamma(\alpha)\ \theta^{\alpha-2}\,y^{\alpha-3}\ -\ (\alpha-1)\ (\alpha-2)\,\theta^{2\,\alpha-4}\,y^{2\,\alpha-4} + (\alpha-1)^{2}\,\theta^{2\,\alpha-4}\,y^{2\alpha-4}\,]}{[\Gamma(\alpha)\ +\ \theta^{\alpha-2}\,y^{\alpha-1}]^{2}}. \tag{25}$$

The denominator in (25) is always positive. So, we will examine the numerator. The 2^{nd} and 3^{rd} terms of the numerator of $\eta'(y)$ are given by

$$-\,(\alpha-1)\ (\alpha-2)\,\theta^{2\,\alpha-4}\,y^{2\,\alpha-4} + (\alpha-1)^{2}\,\theta^{2\,\alpha-4}\,y^{2\alpha-4}\ =\ (\alpha-1)\ \theta^{2\,\alpha-4}\,y^{2\,\alpha-4}.$$

Hence, the numerator can be simplified as follows

$$\frac{\Gamma(\alpha)\ \theta^{\alpha-2}}{y^2}\,[-(\alpha-1)\ (\alpha-2)\ \Gamma(\alpha)\ y^{\alpha-1}\ + (\alpha-1)\ \theta^{\alpha-2}\,y^{\alpha-2}\,]$$

a) If $0 < \alpha < 1$, $\eta'(y)\ < 0\ \Rightarrow\ h_Y(y;\ \alpha, \theta)$ is decreasing.
b) If $\alpha = 1$ then $\eta'(y) = 0$ and $h_Y(y;\ \alpha, \theta)$ is constant.
c) If $1 < \alpha \leq 2$ then the numerator of $\eta'(y)$ will be positive, hence, $\eta'(y) > 0\ \Rightarrow\ h_Y(y;\ \alpha, \theta)$ is increasing.
d) If $\alpha > 2$ then we have that $\eta'(y)$ changes sign from negative to positive at $y = y_0$
where $y_0 = [\frac{(\alpha-2)\,\Gamma(\alpha)}{\theta^{\alpha-2}}]^{\frac{1}{\alpha-1}}$. Hence $\eta(y)$ has a BT shape.
Based on the lemma of Glaser (1980) if
$\delta = \lim_{y\to 0} g(y)\,\eta(y)\ > 1,\ \ g(y) = \frac{1}{h_Y(y;\ \alpha,\theta)}$, then $h_Y(y;\ \alpha, \theta)$ is BT.
For the NG2PL distribution

$$\lim_{y\to 0}\eta(y)\ =\ \lim_{y\to 0}(\theta\ - \frac{(\alpha-1)\,\theta^{\alpha-2}\ y^{\alpha-2}}{\Gamma(\alpha) +\ \theta^{\alpha-2}\,y^{\alpha-1}})\ =\ \theta$$

and

$$\lim_{y\to 0} g(y) = \lim_{y\to 0}\frac{\theta\,\Gamma(\alpha)\ +\ \Gamma(\alpha)}{\theta^2\,\Gamma(\alpha)} = \frac{\theta+1}{\theta^2}.$$

Therefore, $\delta = \lim_{y\to 0} \eta(y)\,g(y)\ = 1 + \frac{1}{\theta}\ > 1$. As a result, the HR of NG2PL distribution is BT for $\alpha > 2$.

According to the proposition of Noughabi *et al.* (2013), the HR of dNG2PL distribution has the following shapes
a) Decreasing for $0 < \alpha < 1$,
b) Constant for $\alpha = 1$,
c) Increasing for $1 < \alpha \leq 2$,
d) BT for $\alpha > 2$ except if the change point of $h_Y(y;\ \alpha, \theta)$ say y^* lies between 0 and 1 and $h_X(0) \leq h_X(1)$ then $h_X(x;\ \alpha, \theta)$ is increasing.

To find y^* we have to solve $h'_Y(y) = 0$. The HR of NG2PL distribution which is given by Ekhosuehi *et al.* (2018) is as follows

$$h_Y(y;\ \alpha, \theta) = \frac{\theta^2\, e^{-\theta\, y}\, (\Gamma(\alpha)\, + \theta^{\alpha-2}\ y^{\alpha-1})}{\theta\, \Gamma(\alpha)\ e^{-\theta\, y}\, +\, \Gamma(\alpha,\, \theta\, y)}, \quad y > 0;\ \ (\,\alpha,\, \theta) > 0, \tag{26}$$

The first derivative $h'_Y(y)$ is obtained as follows

$$\begin{aligned} h'_Y(y) =\ & \theta^{\alpha+1}\, (\alpha - 1)\, \Gamma(\alpha)\, y^{\alpha-2}\, e^{-2\,\theta y} \\ & +\ \Gamma(\alpha, \theta y)\, [-\theta^3\, \Gamma(\alpha)\, e^{-\theta\, y}\, -\, \theta^{\alpha+1}\, y^{\alpha-1}\, e^{-\theta\, y}\, +\, \theta^{\alpha}\, (\alpha - 1)\, y^{\alpha-2}\, e^{-\theta\, y}] \\ & +\ [\theta^{\alpha+2}\ \Gamma(\alpha)\, y^{\alpha-1}\, e^{-2\,\theta\, y}\, + \theta^{2\,\alpha}\ y^{2\,\alpha-2}\, e^{-2\,\theta y}]. \end{aligned} \tag{27}$$

It is clear that there does not exist a closed form expression for the change point y^*, hence it has to be obtained numerically.
The corresponding change point of dNG2PL distribution will be denoted by x^*. Table 3 displays the root of $h'_Y(y)$, y^* and its corresponding change point, x^* for different combinations of $(\alpha,\ \theta)$. Specifically $(\alpha,\ \theta) = (8,\ 0.3\ ),\ (4, 0.8\ ), (8,\ 4), (5,\ 0.1\ ),\ (4,\ 0.4),\ (6,\ \ 0.5),\ (7,\ 0.4)$, and $(7,\ 0.3)$.

Table 3: Root of $h'_Y(y)$, y^*, the solution of $h_X(x_i) - h_X(x_{i+1})$ and the change point x^* for dNG2PL distribution

$(\alpha,\ \theta)$	y^*	$h_X(x_i) - h_X(x_{i+1})$	x^*
$\alpha = 8, \theta = 0.3$	8.751	0.001	$x_{i+1} = 9$
$\alpha = 4, \theta = 0.8$	1.635	0.052	$x_{i+1} = 2$
$\alpha = 8, \theta = 4$	1.041	0.240	$x_{i+1} = 2$
$\alpha = 5, \theta = 0.1$	8.982	0.0001	$x_{i+1} = 9$
$\alpha = 4, \theta = 0.4$	2.338	0.005	$x_{i+1} = 3$
$\alpha = 6, \theta = 0.5$	4.065	0.001	$x_{i+1} = 5$
$\alpha = 7, \theta = 0.4$	5.870	0.004	$x_{i+1} = 6$
$\alpha = 7, \theta = 0.3$	7.365	0.001	$x_{i+1} = 8$

Figure 2 illustrates graphically the various shapes of the HR of the dNG2PL distribution.

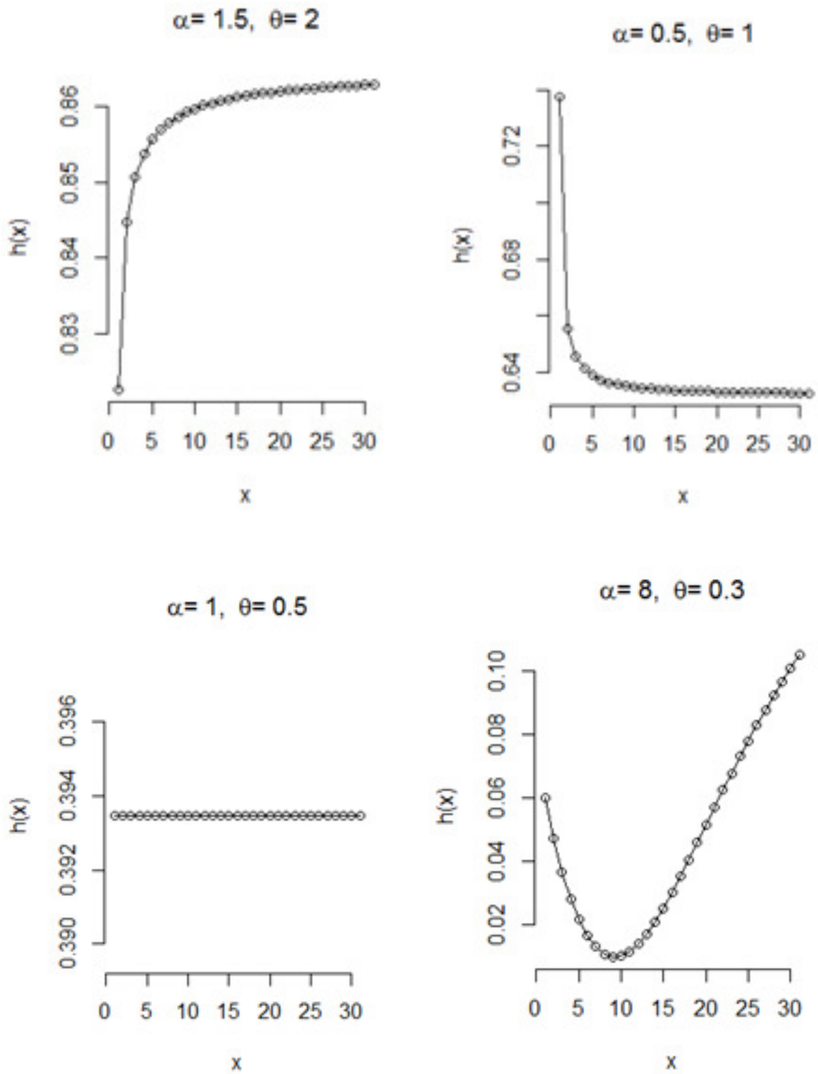

Figure 2. **The HR plots of dNG2PL distribution with different combinations for α and θ**

5.2. The Second Hazard Rate Function

The second failure rate is

$$\begin{aligned} SFR(x) &= \log\left(\frac{R_X(x;\ \alpha,\theta)}{R_X(x+1;\ \alpha,\theta)}\right) \\ &= \log\left\{\frac{\theta\,\Gamma(\alpha)\ e^{-\theta\,x}\ +\Gamma(\alpha,\,\theta\,x)}{\theta\,\Gamma(\alpha)\ e^{-\theta\,(x+1)}\ +\Gamma(\alpha,\,\theta\,(x+1))}\right\},\quad x=0,1,2,...;\ \ (\alpha,\,\theta\,)\ >0. \end{aligned} \tag{28}$$

5.3. The Reversed Hazard Rate Function

Using (6) and (8), the reversed HR function is

$$r\,(x;\ \alpha,\theta)\ =\frac{P_X(x;\ \alpha,\theta)}{F_X(x;\ \alpha,\theta)}\ =\ \frac{\theta\,\Gamma(\alpha)\ (1-e^{-\theta})\ e^{-\theta\,x}\ +\ \Gamma(\alpha,\,\theta x,\theta(x+1))}{(\theta+1)\,\Gamma(\alpha)-\ \ \theta\,\Gamma(\alpha)\ e^{-\theta\,(x+1)}\ +\Gamma(\alpha,\,\theta\,(x+1))} \tag{29}$$
$$x=0,1,2,...;\ \ (\alpha,\,\theta\,)\ >0.$$

5.4. Mean Residual Life Function

The mean residual life function as in Kemp (2004) is given by

$$\begin{aligned} L_{\,x}\ &=\ \sum_{j\geq\,x}\prod_{i=x}^{j}(1-h(X=i))\ , \\ &=\sum_{j\geq\,x}\prod_{i=x}^{j}(\frac{\theta\,\Gamma(\alpha)\ e^{-\theta\,(i+1)}\ +\Gamma(\alpha,\,\theta(i+1))}{\theta\,\Gamma(\alpha)\ e^{-\theta\,i}\ +\Gamma(\alpha,\,\theta i)}),\quad x=0,1,2,...;\ (\alpha,\,\theta\,)\ >0. \end{aligned} \tag{30}$$

A result in Kemp (2004) shows the link between increasing HR/ decreasing HR with decreasing / increasing mean residual life function. According to Kemp's result, the dNG2PL distribution has decreasing mean residual life function when $1<\alpha\leq 2$ and has increasing mean residual life function when $0<\alpha<1$.

5.5. Accumulated Hazard Rate

The accumulated HR function is equal to the area under the step function plot of the hazard function. The accumulated HR function is given as

$$H_X\,=\,\sum_{j=0}^{x}h\,(X=j),$$

$$\begin{aligned} H_X\,&=\,\sum_{j=0}^{x}h\,(X=j), \\ &=\,\sum_{j=0}^{x}\left[1-\ \frac{\theta\,\Gamma(\alpha)\ e^{-\theta\,(j+1)}\ +\Gamma(\alpha,\,\theta(j+1))}{\theta\,\Gamma(\alpha)\ e^{-\theta\,j}\ +\Gamma(\alpha,\,\theta j)}\right]. \end{aligned} \tag{31}$$

[see, Kemp (2004)].

6. Maximum Likelihood Estimation

In this Section, we discuss maximum likelihood estimation of the two parameters of the dNG2PL distribution. Also, the asymptotic confidence intervals of the two parameters will be derived.
Let $X_1, X_2, ..., X_n$ be a random sample of size n from dNG2PL distribution. Then the likelihood function is

$$L=\left[\frac{1}{(\theta+1)\,\Gamma(\alpha)}\right]^n\prod_{i=1}^{n}\left\{\theta\,\Gamma(\alpha)\ (1-e^{-\theta})\ e^{-\theta\,x_i}\ +\ \Gamma(\alpha,\,\theta x_i,\theta(x_i+1))\right\}, \tag{32}$$

$$\ln L=\ -n\ \ln(\theta+1)-n\ \ln(\Gamma(\alpha))\ +\ \sum_{i=1}^{n}\ln\,(O_1), \tag{33}$$

where

$$O_1 = \theta\,\Gamma(\alpha)\;(1-e^{-\theta})\;e^{-\theta\,x} \;+\; \Gamma(\alpha,\,\theta x,\theta(x+1)).$$

$$\frac{\boldsymbol{\partial\, ln\, L}}{\boldsymbol{\partial\,\alpha}} = -n\;\psi(\alpha)\;+\;\sum_{i=1}^{n}\frac{D_{1,\alpha}}{O_1}. \tag{34}$$

where
$\Psi(\alpha) = \frac{\partial}{\partial\,\alpha}\ln\left(\Gamma(\alpha)\right)$ is the digamma function (See, Abramowitz and Stegun (1972)),

$$D_{1,\,\alpha} = \frac{\partial\,O_1}{\partial\alpha} = \theta\,\Gamma(\alpha)\;\psi(\alpha)\,(1-e^{-\theta})\,e^{-\theta x}\;+\;(\ln(\theta)\;+\;\ln(u))\;\Gamma(\alpha,\;\theta x,\;\theta(x+1)),$$

$$\frac{\partial\,\Gamma(\alpha)}{\partial\,\alpha} \;=\; \Gamma(\alpha)\;\psi(\alpha).$$

$$\frac{\boldsymbol{\partial\, ln\, L}}{\boldsymbol{\partial\,\theta}} = \frac{-n}{\theta+1} + \sum_{i=1}^{n}\frac{1}{O_1}\left[\Gamma(\alpha)\,D_{1,\,\theta} + D_{2,\,\theta}\right], \tag{35}$$

where

$$D_{1,\,\theta} = \frac{\partial}{\partial\,\theta}\theta\,(1-e^{-\theta})\,e^{-\theta x} = (1-e^{-\theta})\,e^{-\theta x} + \theta\,e^{-\theta\,(x+1)}\; - \theta\,x\,(1-e^{-\theta})\,e^{-\theta x},$$
$$D_{2,\,\theta} = \;\frac{\partial}{\partial\,\theta}\Gamma(\alpha,\theta x,\theta(x+1)) = \frac{1}{\theta}[\alpha\,\Gamma(\alpha,\theta x,\theta(x+1)) -\;\Gamma(\alpha+1,\theta x,\theta(x+1))\,],$$

Hence, we have to solve the two ln-likelihood equations $\frac{\partial\,\ln\,L}{\partial\,\alpha}\; = 0$ and $\frac{\partial\,\ln\,L}{\partial\,\theta}\; = 0$.

It is clear that the solution of these two non-linear equations can not be done analytically. Numerical solution of these equations provides the ML estimates of α and θ respectively.
The second order partial derivatives of the likelihood function which are useful in the computation of the variance-covariance matrix for the estimators and also in constructing confidence intervals for the unknown parameters α and θ, are presented as follows.

$$\begin{aligned}\frac{\boldsymbol{\partial^2\, ln\, L}}{\boldsymbol{\partial\,\alpha^2}} = -n\,\Psi'(\alpha)\;+\sum_{i=1}^{n}\frac{1}{{O_1}^2}\{O_1\,[\theta\;(1-e^{-\theta})\,e^{-\theta x}\;\big(\Gamma(\alpha)\,\Psi'(\alpha) + (\Gamma(\alpha))^2\Psi(\alpha)\,\big)\\ +\,(\ln{(\theta)}+\ln{(u)})^2\;\Gamma(\alpha,\theta x,\theta(x+1))] -\; {D^2}_{1,\,\alpha}\}.\end{aligned} \tag{36}$$

$$\begin{aligned}\frac{\boldsymbol{\partial^2\, ln\, L}}{\boldsymbol{\partial\,\theta^2}} = \frac{n}{(\theta+1)^2} + \sum_{i=1}^{n}\frac{1}{{O_1}^2}\{O_1[\Gamma(\alpha)\,e^{-\theta\,(x+1)}(2-2\theta x-\theta)\\ +\,\Gamma(\alpha)\,(1-e^{-\theta})\,(-2\,x\,e^{-\theta\,x} + \theta\,x^2\;e^{-\theta\,x})\\ +\,\frac{1}{\theta}\,((\alpha-\frac{1}{\theta})\,D_{2,\,\theta} - D_{3,\,\theta})\,] - (\Gamma(\alpha)\,D_{1,\,\theta}\; + D_{2,\,\theta})\;D_{4,\,\theta}\}.\end{aligned} \tag{37}$$

$$\frac{\boldsymbol{\partial^2\, ln\, L}}{\boldsymbol{\partial\theta\,\partial\alpha}} = \sum_{i=1}^{n}\frac{1}{{O_1}^2}\{O_1\,[\Gamma(\alpha)\psi(\alpha)\,D_{1,\,\theta} + (\ln(\theta)+\ln(u))\;D_{2,\,\theta} + \frac{1}{\theta}\Gamma(\alpha,\theta x,\theta(x+1))\,] - D_{1,\,\alpha}\,D_{4,\,\theta}\}. \tag{38}$$

where $\Psi'(\alpha)\; = \frac{\partial\;\Psi(\alpha)}{\partial\;\alpha}$ is the polyGamma[1,α] function,

$$D_{3,\theta} = \frac{\partial}{\partial \theta}\Gamma(\alpha+1,\theta x,\theta(x+1)) = \frac{1}{\theta}[(\alpha+1)\,\Gamma(\alpha+1,\theta x,\theta(x+1)) - \Gamma(\alpha+2,\theta x,\theta(x+1))\,],$$

$$D_{4,\theta} = \frac{\partial O_1}{\partial \theta} = \Gamma(\alpha)\,D_{1,\theta} + D_{2,\theta}.$$

The asymptotic variance-covariance matrix of the ML estimators of α and θ is given by

$$I^{-1} = \begin{bmatrix} E(-\frac{\partial^2 \ln L}{\partial \alpha^2}) & E(-\frac{\partial^2 \ln L}{\partial \alpha\,\partial \theta}) \\ E(-\frac{\partial^2 \ln L}{\partial \theta\,\partial \alpha}) & E(-\frac{\partial^2 \ln L}{\partial \theta^2}) \end{bmatrix}^{-1}$$

Based on the above matrix, one can estimate the entries approximately by
$E(-\frac{\partial^2 \ln L}{\partial \alpha^2}) = -\left.\frac{\partial^2 \ln L}{\partial \alpha^2}\right|_{\alpha=\widehat{\alpha},\,\theta=\widehat{\theta}}$ and so on.

For large samples, the vector $\widehat{\Phi} = (\widehat{\alpha},\ \widehat{\theta})$ is consistent and asymptotically normal. That is $\lim_{n\to\infty} \widehat{\Phi} = \Phi$ and also $\widehat{\Phi} \sim N(0, I^{-1})$.

Hence, the $100(1-\delta)\%$ asymptotic confidence intervals for the parameters α and θ are given respectively by $\widehat{\alpha} \pm z_{\delta/2}\sqrt{V(\widehat{\alpha})}$ and $\widehat{\theta} \pm z_{\delta/2}\sqrt{V(\widehat{\theta})}$.

where $V(\widehat{\alpha})$ and $V(\widehat{\theta})$ are the diagonal elements of I^{-1} and $z_{\delta/2}$ is the $(1-\frac{\delta}{2})$ th quantile of the standard normal distribution.

7. Simulation study

Here, a simulation study is conducted to assess the performance of the ML estimators of the model parameters of dNG2PL distribution. We consider this for various sample sizes $n = 20,\ 50,\ 100,\ 150$ and three different combinations of the parameters α and θ which cover decreasing-increasing-decreasing shape of PMF for $(\alpha,\theta) = (5,\ 0.5)$, unimodal shape of PMF for (3, 0.3) and decreasing shape of PMF for (0.9, 0.1).

The algorithm for the simulation study is given as follows
Step 1: Generate N=1000 samples of size *n* and specify the initial values of the two parameters α, θ.
Step 2: Generate $u_i \sim U(0,1)$, $i = 1, 2, \ldots, n$.
Step 3: If $u_i \leq p = \frac{\theta}{\theta+1}$ generate $X_i \sim \text{Geometric}\left(e^{-\theta}\right)$ distribution, otherwise generate $X_i \sim$ discrete gamma $(\alpha,\ \theta)$ distribution.
Step 4: Compute the ML estimates for the two unknown parameters.
Step 5: Repeat steps from 2 to 4, N times.
Step 6: Compute the average value (AV) of the estimators, the estimated average bias (AB), estimated mean square error (MSE), and confidence interval (C.I) for each parameter for each sample size.
For the parameter θ, the average value (AV) of the estimates for a given value of the parameter and fixed sample size n is calculated as $AV\left(\widehat{\theta}\right) = \frac{1}{N}\sum_{i=1}^{N}\widehat{\theta}_i$,
AB = $\frac{1}{N}\sum_{i=1}^{N}\left(\widehat{\theta}_i - \theta\right)$, $MSE\left(\widehat{\theta}\right) = \frac{1}{N}\sum_{i=1}^{N}\left(\widehat{\theta}_i - \theta\right)^2$, and
the 95 % asymptotic C.I = ($\widehat{\theta}_i \pm 1.96\sqrt{V(\widehat{\theta})}$). Similarly, the corresponding measures for the parameter α may be obtained.

Since the ML estimators of the parameters cannot be obtained in closed forms, the Newton Raphson method is used to compute the parameters estimates. We use maxLik function in R programme to compute the ML estimates. The AVs along with their ABs, standard errors (SE), MSEs and the width (W) of the C.Is of the two parameters (α, θ) are obtained and presented in Table 4 for $(\alpha,\ \theta) = (5,\ \ 0.5)$, Table 5 for $(\alpha,\ \theta) = (3,\ 0.3)$ and Table 6 for $(\alpha,\ \theta) = (0.9,\ 0.1)$.

From Tables (4-6), we observe that as the sample size increases the AB, SE and MSE of the two estimators decrease. This result is expected as increasing the sample size means observing more data and hence more information is provided. The numerical results reveal that two properties of good estimators are satisfied which are asymptotic unbiasedness and consistency. For the C.Is, as the sample size increases the width of the intervals becomes narrower. Also, the average estimates of the parameters get closer to the initial values.

Table 4: Simulation results for α= 5, θ=0.5

n	**Parameter**	**AV**	**AB**	**SE**	**MSE**	**W**
20	α	5.303	0.303	1.727	6.887	6.770
20	θ	0.527	0.027	0.158	0.055	0.621
50	α	5.110	0.110	1.081	2.498	4.237
50	θ	0.511	0.011	0.097	0.020	0.381
100	α	5.055	0.055	0.755	1.145	2.960
100	θ	0.506	0.006	0.067	0.009	0.264
150	α	5.026	0.026	0.610	0.712	2.393
150	θ	0.503	0.003	0.054	0.006	0.214

Table 5: Simulation results for α= 3, θ=0.3

n	**Parameter**	**AV**	**AB**	**SE**	**MSE**	**W**
20	α	3.309	0.309	1.292	4.056	5.065
20	θ	0.325	0.025	0.117	0.030	0.457
50	α	3.096	0.096	0.866	1.755	3.395
50	θ	0.307	0.007	0.076	0.013	0.300
100	α	3.055	0.055	0.661	0.980	2.592
100	θ	0.304	0.004	0.057	0.007	0.224
150	α	3.018	0.018	0.553	0.669	2.168
150	θ	0.301	0.001	0.048	0.005	0.186

Table 6: Simulation results for α= 0.9, θ=0.1

n	**Parameter**	**AV**	**AB**	**SE**	**MSE**	**W**
20	α	1.206	0.306	0.468	1.140	1.836
20	θ	0.136	0.036	0.054	0.012	0.213
50	α	0.985	0.085	0.223	0.139	0.876
50	θ	0.114	0.014	0.028	0.002	0.110
100	α	0.954	0.054	0.147	0.047	0.578
100	θ	0.110	0.010	0.019	0.001	0.074
150	α	0.948	0.048	0.118	0.030	0.463
150	θ	0.109	0.009	0.015	0.001	0.060

8. Real Data

In this Section, we illustrate the flexibility of our model (dNG2PL) distribution using three real datasets. The first two of them are discrete real data sets whereas the last one is count data set.

Discrete data:

For each data set, the dNG2PL distribution is compared with some competing distributions based on some criteria. These criteria are the ln-likelihood (ln L), Kolmogrov-Smirnov (K-S) statistic and its corresponding p-value, Akaike Information Criterion (AIC) and Akaike Information Criterion with correction (AICc). They are calculated for all the compared distributions in order to verify which distribution fits better to the data. The distribution with the smallest values of the K-S, AIC and AICc and also largest ln L value and highest p-value is considered the best for a given data. Here, $AIC = \ -2\ ln\ L\ +\ 2k,\ \ AICc = \ -2\ln\ L\ +\ \frac{2kn}{n-k-1}$ where L is the likelihood function evaluated at the maximum likelihood estimates, k is the number of parameters and n is the sample size. The competing distributions whose PMFs are given in Appendix A are dL, EDLi, Discrete power Lindley (dPL), discrete Weibull (dWE), discrete Burr (dB) and discrete weighted exponential (dWEX) distributions. The K-S statistic was calculated using ks.test function in R.

First data set:
This data is data set 424 of Hand *et al.* (1993). The data consists of the time to death (in weeks) of AG positive leukemia patients. The data are 1, 1, 4, 5, 16, 22, 26, 39, 56, 65, 65, 100, 108, 121, 134, 143, and 156.

Table 7: Parameter estimates and goodness of fit for various models fitted for the time to death (in weeks) of AG positive leukemia patients

	Estimate	**SE**	**ln L**	**K-S**	**p-val**	**AIC**	**AICc**
dNG2PL	$\widehat{\theta}$=0.013	0.005	-87.205	**0.148**	**0.851**	**178.409**	**179.267**
	$\widehat{\alpha}$=0.823	0.253					
dL	$\widehat{\theta}$=0.031	0.005	-91.858	0.215	0.412	185.717	185.983
EDLi	$\widehat{a}$=0.981	0.006	-87.234	0.149	0.843	178.468	179.326
	$\widehat{b}$=0.436	0.131					
dPL	$\widehat{\theta}$=0.146	0.070	-87.763	0.149	0.847	179.526	180.383
	$\widehat{\alpha}$=0.643	0.109					
dWE	$\widehat{q}$=0.978	0.019	-87.352	0.15	0.841	178.704	179.562
	$\widehat{\beta}$ =0.925	0.191					
dB	$\widehat{\theta}$=0.995	0.001	-96.167	0.312	0.074	196.333	197.19
	$\widehat{\alpha}$=63.563	1.155					
dWEX	$\widehat{\lambda}$=0.016	0.004	-87.365	0.152	0.825	178.73	179.587
	$\widehat{\alpha}$=219.567	4.194					

Table 7 presents the ML estimates with their SEs as well as the ln L, K-S statistic and its p-value, AIC and AICc. Although all compared distributions fit the data, the proposed model has the lowest value of the K-S statistics and its corresponding p-value has largest value. Also, our proposed model has largest ln L and smallest AIC and AICc values. Consequently, our proposed model can be considered the best model against the competing models based on the given criteria.

Second data set:
The second data comprises the 2003 final examination marks of 48 slow space students in mathematics in the Indian Institute of Technology at Kanpur. This data set was previously analyzed by Bakouch *et al.* (2014). The data are 4, 5, 6, 6, 7, 7, 8, 11, 12, 12, 13, 14, 14, 15, 15, 15, 15, 15, 18, 18, 18, 19, 19, 19, 20, 21, 21, 23, 23, 23, 25, 27, 28, 29, 31, 34, 34, 37, 39, 40, 44, 50, 50, 58, 60, 65, 70, and 86.

Table 8: Parameter estimates and goodness of fit for various models fitted for the final examinations marks

	Estimate	**SE**	**ln L**	**K-S**	**p-val**	**AIC**	**AICc**
dNG2PL	$\widehat{\theta}$=0.083	0.018	-198.072	**0.089**	**0.843**	400.144	400.411
	$\widehat{\alpha}$=2.277	0.470					
dL	$\widehat{\theta}$=0.073	0.007	-198.262	0.105	0.668	**398.524**	**398.611**
EDLi	$\widehat{a}$=0.919	0.011	-197.429	0.094	0.785	398.858	399.125
	$\widehat{b}$=1.349	0.308					
dPL	$\widehat{\theta}$=0.052	0.020	-197.788	0.098	0.743	399.575	399.842
	$\widehat{\alpha}$=1.101	0.106					
dWE	$\widehat{q}$=0.994	0.003	-198.608	0.109	0.614	401.217	401.483
	$\widehat{\beta}$ =1.546	0.166					
dB	$\widehat{\theta}$=0.952	0.019	-247.484	0.385	0	498.968	499.234
	$\widehat{\alpha}$=6.743	2.666					
dWEX	$\widehat{\lambda}$=0.076	0.015	-197.599	0.092	0.81	399.199	399.465
	$\widehat{\alpha}$=0.006	0.377					

The ML estimates of the parameters, SEs, K-S statistics and the corresponding p-values, AIC and AICc are given in Table 8. From Table 8, it is observed that all models fit the data set except the dB distribution. However, the dNG2PL distribution has the smallest K-S statistic and largest p-value whereas the dL distribution has the smallest values of AIC and AICc critera.

Count Data:
This data consists of the counts of cysts of kidneys using steroids. This data was analyzed by El-Morshedy *et al.* (2019). The dNG2PL distribution is compared with dL, dPL, EDLi, discrete Gamma (dG), dB, dWE and dWEX distributions. The count data is displayed in Table 9 , we compute the ML estimates of the parameters of the considered distributions, ln L, chi-square (χ^2) statistic, corresponding p-value, degrees of freedom (df), AIC and AICc criteria. It is observed from Table 9, based on the p-value, that all the distributions fit the data except the dL and dWEX distributions. The dNG2PL distribution has the smallest χ^2value and the largest p-value. Based on AIC and AICc criteria, the EDLi distribution has the smallest values followed by the dNG2PL distribution.

Table 9: Parameter estimates and goodness of fit for various models fitted for the counts of cysts of kidneys using steroids

X	**Freq**	**dNG2PL**	**dL**	**dPL**	**EDLi**	**dG**	**dB**	**dWE**	**dWEX**
0	65	64.93	40.25	63.67	64.97	64.59	64.74	63.64	45.75
1	14	15.18	29.83	17.32	14.39	15.78	19.18	17.45	27.07
2	10	9.03	18.36	9.32	9.01	9.08	8.48	9.30	15.66
3	6	5.95	10.35	5.71	6.14	5.87	4.63	5.68	9.06
4	4	4.11	5.53	3.75	4.33	4.00	2.86	3.73	5.24
5	2	2.91	2.86	2.58	3.10	2.81	1.92	2.56	3.03
6	2	2.09	1.44	1.83	2.24	2.02	1.36	1.82	1.76
7	2	1.52	0.71	1.33	1.62	1.47	1.01	1.32	1.02
8	1	1.11	0.35	0.98	1.18	1.08	0.78	0.98	0.59
9	1	0.82	0.17	0.74	0.85	0.80	0.61	0.74	0.34
10	1	0.60	0.08	0.57	0.62	0.60	0.49	0.57	0.20
11	2	1.75	0.07	2.20	1.55	0.45	0.41	2.21	0.11
Total	**110**	**110**	**110**	**110**	**110**	**110**	**110**	**110**	**110**

	Estimate	SE	ln L	x2	df	p-val	AIC	AICc
dNG2PL	$\widehat{\theta}$=0.275	0.058	-167.131	**0.501**	3	**0.919**	**338.263**	**338.375**
	$\widehat{\alpha}$=0.328	0.075						
dL	$\widehat{\theta}$=0.829	0.059	-189.11	43.479	4	0.000	380.220	380.257
dPL	$\widehat{\theta}$=1.314	0.126	-167.947	0.984	3	0.805	339.893	339.893
	$\widehat{\alpha}$=0.563	0.062						
EDLi	$\widehat{a}$=0.672	0.048	-166.95	0.507	3	0.917	**337.900**	**338.012**
	$\widehat{b}$=0.264	0.056						
dG	$\widehat{k}$=0.401	0.084	-167.28	0.558	3	0.906	338.561	338.673
	$\widehat{\theta}$=0.231	0.057						
dB	$\widehat{\theta}$=0.278	0.045	-171.139	2.469	3	0.481	346.278	346.390
	$\widehat{\alpha}$=1.053	0.167						
dWE	$\widehat{q}$=0.421	0.047	-167.989	1.037	3	0.792	339.977	340.089
	$\widehat{\beta}$=0.629	0.073						
dWEX	$\widehat{\lambda}$=0.547	0.052	-179.195	23.634	3	0.000	362.391	362.502
	$\widehat{\alpha}$=106.397	1.101						

9. Conclusion

In this research, a discrete two-parameter analogue of Ekhosuehi et al.'s continuous new generalized two-parameter Lindley distribution is proposed. According to theory, the shapes of the hazard rate of the dNG2PL distribution are increasing, decreasing, constant, and BT. In addition; the mean residual life function of the dNG2PL distribution is decreasing or increasing. Several characteristics of the dNG2PL distribution are investigated, such as the mixture representation of the PMF of dNG2PL distribution, the moments of dNG2PL distribution especially the mean and variance, measures of skewness and kurtosis, moment generating function, probability generating function and the order statistics. The unknown parameters are estimated using the maximum likelihood method. To evaluate the estimators' efficiency, a simulation study is carried out. The efficiency of the estimators is investigated using the average bias, the standard error and the mean square error. We found that, asymptotic unbiasedness and consistency properties are satisfied. Furthermore, two discrete real data sets and one count data set are examined in order to demonstrate the model's utility. Based on the studied criteria, the results show that the dNG2PL distribution gives a better fit than numerous current distributions.

Appendix (A): Compared Distributions

Compared distributions are

(a) dL distribution (Gómez-Déniz and Calderín-Ojeda (2011)) which is a special case of dNG2PL distribution when $\alpha = 2$.

(b) EDLi distribution (El-Morshedy *et al.* (2019)) which has the following PMF,

$$P_X(x) = \frac{1}{(1-\log a)^b}\left[\Lambda(x+1;a,b) - \Lambda(x;a,b)\right], \quad x = 0,1,2,...; \quad 0 < a < 1, \ b > 0.$$

where a is a scale parameter, b is a shape parameter and $\Lambda(x;a,b) = \{1 - a^x + [(1+x)\ a^x - 1]\, log\,(a)\,\}^b$.

(c) dPL distribution (Oliveira *et al.* (2018)) with the following PMF,

$$P_X(x) = \left(1 + \frac{\beta\ x^{\alpha}}{\beta+1}\right)\gamma^{x^{\alpha}} - \left(1 + \frac{\beta\ (x+1)^{\alpha}}{\beta+1}\right)\gamma^{(x+1)^{\alpha}}, \ x = 0,1,2,\ldots; \quad \alpha,\beta > 0, \ \gamma = e^{-\beta}.$$

(d) dB distribution (Krishna and Pundir (2009)) which has the following PMF,

$$P_X(x) = \theta^{\log(1+x^{\alpha})} - \theta^{\log(1+(1+x)^{\alpha})}, \quad x = 0, 1, 2, \ldots; \quad \alpha > 0, \quad 0 < \theta < 1.$$

(e) dWE distribution (Nakagawa and Osaki (1975)) which has the following PMF,

$$P_X(x) = q^{x^{\beta}} - q^{(x+1)^{\beta}}, \quad x = 0, 1, 2, ...; \ \beta > 0 \ \text{ and } \ 0 < q < 1.$$

(f) dWEX distribution (Khongthip *et al.* (2018))

which has the following PMF,

$$P_X(x) = \frac{1 - e^{(\alpha+1)\lambda} + (\alpha+1)\left(e^{\lambda} - 1\right) e^{(x+1)\alpha\lambda}}{\alpha \, e^{(\alpha+1)(x+1)\lambda}}, \quad x = 0, 1, 2, ...; (\alpha, \lambda) > 0.$$

(g) dG distribution (Chakraborty and Chakravarty (2012)) which is given as

$$P_X(x) = \frac{1}{\Gamma(k)} \Gamma(k, \frac{x}{\theta}, \frac{x+1}{\theta}), \quad x = 0, 1, 2, ...; \quad (k, \theta) > 0,$$

where $\Gamma(k, \frac{x}{\theta}, \frac{x+1}{\theta}) = \int_x^{x+1} \frac{1}{\theta^k} u^{k-1} e^{-u/\theta} \, du.$

Appendix B

Glaser (1980) established the following theorem

Theorem 1 (Glaser (1980)): Let Y be a non-negative continuous random variable with twice differentiable probability density function $f(y)$ and HR function $h_Y(y)$. Let $\eta(y) = -\frac{\partial \ln f_Y(y)}{\partial y}$.

a) If $\eta(y)$ is increasing, then $h_Y(y)$ is increasing.

b) If $\eta(y)$ is decreasing, then $h_Y(y)$ is decreasing.

c) If $\eta(y)$ is BT and if there exists a value t_0 such that $h'_Y(t_0) = 0$ then $h_Y(y)$ is BT, otherwise $h_Y(y)$ is increasing .

d) If $\eta(y)$is upside-down bathtub and if there exists a value t_0 such that $h'_Y(t_0) = 0$ then $h_Y(y)$ is upside-down bathtub, otherwise $h_Y(y)$ is decreasing.

Since determining t_0 in cases c and d may be difficult, Glaser (1980) provided the following lemma which simplifies the problem.

Lemma (Glaser (1980))

Let $\varepsilon = \lim_{y\to 0} f_Y(y)$ and $\delta = \lim_{y\to 0} g(y)\,\eta(y)$
where $g(y) = \frac{1}{h_Y(y)}$, then

1. Suppose that $\eta(y)$ is BT, then
   a) If either $\varepsilon = 0$ or $\delta < 1$, $h_Y(y)$ is increasing.
   b) If either $\varepsilon = \infty$ or $\delta > 1$, $h_Y(y)$ is BT.
2. Suppose that $\eta(y)$ is upside down bathtub, then
   a) If either $\varepsilon = 0$ or $0 < \delta < 1$, then $h_Y(y)$ is upside down bathtub.
   b) If either $\varepsilon = \infty$ or $\delta > 1$, then $h_Y(y)$ is decreasing.

Proposition (Noughabi *et al.* (2013)): Let $R_Y(.)$ be the reliability function of a continuous lifetime model, and $y_i,\ i = 0, 1, 2, \ldots$ be real values and let $h_Y\,(.)$ be the HR of the continuous model and $h_X(.)$ be the HR of its discrete analogue.

a) If $h_Y\,(.)$ is increasing (decreasing) on (a, b), then $h_X\,(x_i)\,,\ \ i = m,\ m+1,\ \ldots, n$ is an increasing (decreasing) sequence, where $a \leq x_m \leq x_n \leq b$.

b) Let $h_Y\,(.)$ be BT (upside-down bathtub) shaped with the change point, in $(x_i,\ \ x_{i+1})$, then $h_X\,(.)$ is BT (UBT) shaped. Moreover, if $h_X(x_i) - h_X(x_{i+1}) > 0$ then the change point is x_{i+1}. Otherwise, its change point is x_i.

This proposition shows that when the hazard rate of the continuous model is BT-shaped, the discrete model has an increasing or BT shaped hazard rate function. In other words, when $h_Y\,(.)$ exhibits a BT (upside-down bathtub) shaped, the discrete skeleton $h_X\,(.)$ is also BT-shaped (UBT-shaped), but when the change point of $h_Y\,(.)$ is between x_0 and x_1 and $h_X\,(x_0)\ \leq (\geq)\, h_X\,(x_1)$, the sequence $h_X\,(x_i)\,,\ i = 0, 1, 2, \ldots$ is increasing (decreasing).

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