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A Primal-Dual Interior-Point Algorithm Based on a Kernel Function with a New Barrier Term

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Abstract In this paper, we propose a path-following interior-point method (IPM) for solving linear optimization (LO) problems based on a new kernel function (KF). The latter differs from other KFs in having an exponential-hyperbolic barrier term that belongs to the hyperbolic type, recently developed by I. Touil and W. Chikouche [22, 23]. The complexity analysis for large-update primal-dual IPMs based on this KF yields an $\mathcal{O}\left(\sqrt{n}\log^2 n \log \frac{n}{\epsilon}\right)$ iteration bound which improves the classical iteration bound. For small-update methods, the proposed algorithm enjoys the favorable iteration bound, namely, $\mathcal{O}\left(\sqrt{n}\log \frac{n}{\epsilon}\right)$. We back up these results with some preliminary numerical tests which show that our algorithm outperformed other algorithms with better theoretical convergence complexity.
To our knowledge, this is the first feasible primal-dual interior-point algorithm based on an exponential-hyperbolic KF.

Keywords Linear programming, Interior-point methods, Kernel function, Complexity analysis, Large and small-update methods.

AMS 2010 subject classifications 90C05, 90C51, 90C31

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1. Introduction

Linear programming, also called LO, is a simple way to perform optimization. LO can be applied to different practical fields such as economics, engineering and operations research. The most efficient methods to solve LO problems are the IPMs. These methods begun to gain popularity since the landmark paper of Karmarker [13] in 1984. After that, Peng et al. [19] introduced the concept of kernel based IPMs. They used a direction determined by a so-called self-regular barrier function. Later, Bai et al. [4] introduced the class of eligible KFs. Since then, KFs became the object of attention of so many researchers as they not only serve to define a measure of the distance between the iterate and the central path, but also play a crucial role in improving the computational complexity of an interior-point algorithm. This led to a rich literature diversified mainly by the type of the barrier term in the KF. See [9, 24, 6, 10, 12, 16, 11, 26] for more informations on interior point algorithms based on KFs. It's worth noting that the latest type is the hyperbolic one which was recently introduced by Touil and Chikouche [22, 23].

Another way to determine search directions was proposed by Darvay [7]. He presented a new method using the technique of algebraically equivalent transformation. See [17, 8, 15] for some recent works based on this technique.

In this paper, our main contribution is a primal-dual IPM based on the following new KF.

$$\psi(t) = \frac{t^2-1}{2} + \sinh^2(1)\left(e^{\coth(t)-\coth(1)} - 1\right), \ \forall t > 0. \tag{1}$$

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We structure our paper as follows. In Section 2, we briefly recall the basics of IPMs for LO. Then, we present some properties of the new KF in Section 3. The estimate of the step size and the decrease behavior of the new barrier function are discussed in Section 4, where we derive the iteration bound of our algorithm for large- and small-update methods. In Section 5, we present numerical tests on five different examples to illustrate the effectiveness of the proposed algorithm and compare the results with other available KFs. In the final section we conclude with some remarks.

Let us finish this introduction with some notations used in the whole paper : $\|\cdot\|$ denotes the Euclidean norm of a vector. $\mathbb{R}^n_+$ and $\mathbb{R}^n_{++}$ denote the nonnegative and the positive orthants respectively. For given vectors $x, s \in \mathbb{R}^n$, $X = \mathrm{diag}(x)$ denotes the $n \times n$ diagonal matrix whose diagonal entries are the components of x, and the vector xs indicate the component-wise product of x and s. Finally, if $f(x), g(x) \geq 0$ are two real valued functions of a real nonnegative variable, the notation $f(x) = \mathcal{O}(g(x))$ means that $f(x) \leq Cg(x)$ for some positive constant C and $f(x) = \Theta(g(x))$ means that $C_1 g(x) \leq f(x) \leq C_2 g(x)$ for two positive constants C_1 and C_2.

2. Preliminaries

In this section, we briefly describe the basics of IPMs for LO. We start by considering the standard LO problem

$$(P)\begin{cases} \min c^T x \\ Ax = b, \\ x \geq 0, \end{cases}$$

and it's dual problem

$$(D)\begin{cases} \max b^T y \\ A^T y + s = c, \\ s \geq 0, \end{cases}$$

where $A \in \mathbb{R}^{m \times n}$ with rank$(A) = m \leq n$, $c \in \mathbb{R}^n$ and $b \in \mathbb{R}^m$ are given. We assume that both (P) and (D) satisfy the interior-point condition (IPC); that is, there exists (x^0, y^0, s^0) such that

$$Ax^0 = b, \;\; x^0 > 0, \;\; A^T y^0 + s^0 = c, \;\; s^0 > 0.$$

It is well known that, under the above assumption, the optimal solution of the primal-dual pair can be obtained by solving the following system of equations

$$\begin{cases} Ax = b, x \geq 0, \\ A^T y + s = c, s \geq 0, \\ xs = 0. \end{cases} \tag{2}$$

The idea underlying primal-dual IPMs is to replace the complementarity condition $xs = 0$ in (2) by the nonlinear equation $xs = \mu e$, with parameter $\mu > 0$ which leads to the following system

$$\begin{cases} Ax = b, x \geq 0, \\ A^T y + s = c, s \geq 0, \\ xs = \mu e, \quad \mu > 0. \end{cases} \tag{3}$$

Since the IPC is satisfied and the rank of matrix A is full, the perturbed system (3) has a unique solution for each $\mu > 0$. The set of unique solutions $\{(x_\mu, y_\mu, s_\mu) : \mu > 0\}$ forms a well-behaved curve, called central path which plays an important role in convergence analysis of the IPMs. If $\mu \to 0$, then the limit of the central path exists and yields optimal solutions for (P) and (D).

Applying the damped Newton method to the perturbed system (3) produces the following system for the

search direction $(\Delta x, \Delta y, \Delta s)$
$$\begin{cases} A\Delta x = 0, \\ A^T\Delta y + \Delta s = 0, \\ s\Delta x + x\Delta s = \mu e - xs. \end{cases} \tag{4}$$
By taking a step size α along the search direction, the new iterate is constructed according to

$$\begin{aligned} x_+ &:= x + \alpha\Delta x \\ y_+ &:= y + \alpha\Delta y \\ s_+ &:= s + \alpha\Delta s, \end{aligned}$$

for some $0 < \alpha \leq 1$ satisfying $(x_+, s_+) > 0$.
Let the scaled vector v and the scaled search directions d_x and d_s be defined as follows

$$v = \sqrt{\frac{xs}{\mu}}, \quad d_x = \frac{v\Delta x}{x}, \quad d_s = \frac{v\Delta s}{s}. \tag{5}$$

System (4) is then rewritten in the following form

$$\begin{cases} \overline{A}d_x = 0, \\ \overline{A}^T\Delta y + d_s = 0, \\ d_x + d_s = v^{-1} - v, \end{cases} \tag{6}$$

where $\overline{A} = \frac{1}{\mu}AV^{-1}X, \;\; V = \mathrm{diag}(v), \;\; X = \mathrm{diag}(x)$.
Observe that the right-hand side in the last equation of (6) is equal to minus gradient of the proximity function

$$\Psi(v) = \sum_{i=1}^{n} \psi_c(v_i),$$

where
$$\psi_c(t) = \frac{t^2 - 1}{2} - \log t,$$
is the so-called KF of the barrier function Ψ. The main idea of kernel-based IPMs is to replace ψ_c by any strictly convex function $\psi :]0, +\infty[\to [0, +\infty[$ which is minimal at $t = 1$ with $\psi(1) = 0$. Thus, for a different function ψ, one gets a different search direction. In this work, we replace ψ_c by the new KF previously defined in (1).

Coming back to system (6), we can convert it to

$$\begin{cases} \overline{A}d_x = 0, \\ \overline{A}^T\Delta y + d_s = 0, \\ d_x + d_s = -\nabla\Psi(v). \end{cases} \tag{7}$$

Since A has full row rank, system (7) has a unique solution. Furthermore, the vectors d_x and d_s are orthogonal and thus
$$d_x = d_s = 0 \Leftrightarrow \nabla\Psi(v) = 0 \Leftrightarrow v = e \Leftrightarrow \Psi(v) = 0 \Leftrightarrow x = x_\mu \text{ and } s = s_\mu.$$
We also define a proximity measure to the central path as follows:

$$\sigma(v) = \frac{1}{2}\left\|d_x + d_s\right\| = \frac{1}{2}\left\|\nabla\Psi(v)\right\|. \tag{8}$$

The generic primal-dual IPM is summarized in the following algorithm.

Algorithm : Generic Interior-Point Algorithm for LO

Input
a threshold parameter $\tau \geq 1$;
an accuracy parameter $\epsilon > 0$;
a fixed barrier update parameter $\theta \in]0, 1[$; (x^0, y^0, s^0) satisfy the IPC and $\mu^0 = 1$ such that $\Phi\left(x^0, s^0; \mu^0\right) := \Psi(v^0) \leq \tau$.
begin
$x :\ = x^0; s :\ = s^0; \mu :\ = \mu^0$;
while $n\mu \geq \epsilon$ **do**
  begin (outer iteration)
  $\mu :\ = (1 - \theta)\mu$;
  while $\Phi(x, s; \mu) := \Psi(v) > \tau$ **do**
    begin (inner iteration)
    Solve system (7) and use (5) to obtain $(\Delta x, \Delta y, \Delta s)$;
    Choose a suitable step size α;
    $x := x + \alpha\Delta x; y := y + \alpha\Delta y; s := s + \alpha\Delta s; v := \sqrt{\frac{xs}{\mu}}$;
  end while (inner iteration)
end while (outer iteration)

3. The new kernel function and its properties

In this section, we provide some easily obtained properties of the new KF which are used in the complexity analysis. Recall that our KF ψ is defined as follows:

$$\psi(t) = \frac{t^2 - 1}{2} + \sinh^2(1)\left(e^{\coth(t) - \coth(1)} - 1\right), \ \forall t > 0.$$

For conveniency, we give the first three derivatives of ψ

$$\psi'(t) = t - \frac{\sinh^2(1)}{\sinh^2(t)} e^{\coth(t) - \coth(1)}, \tag{9}$$

$$\psi''(t) = 1 + \sinh^2(1) e^{\coth(t) - \coth(1)} \left(2\frac{\coth(t)}{\sinh^2(t)} + \frac{1}{\sinh^4(t)}\right), \tag{10}$$

and

$$\psi'''(t) = -\sinh^2(1) e^{\coth(t) - \coth(1)} \left(4\frac{\coth^2(t)}{\sinh^2(t)} + \frac{6\coth(t) + 2}{\sinh^4(t)} + \frac{1}{\sinh^6(t)}\right). \tag{11}$$

From (10), we see that $\psi''(t) \geq 1,\ \forall t > 0$, thus we have the following lemma.

Lemma 3.1 (Lemma 2.1 in [3])
Let $\psi(t)$ be defined as in (1). Then

$$\frac{1}{2}(t-1)^2 \leq \psi(t) \leq \frac{1}{2}(\psi'(t))^2, \quad \forall t > 0.$$

Corollary 3.2 (Corollary 2.2 in [3])
Let $\sigma(v)$ be defined as in (8). Then, for any $v > 0$, we have

$$\sigma(v) \geq \sqrt{\frac{\Psi(v)}{2}}.$$

Remark 3.3
Through the paper we assume that $\tau \geq 2$. Using Corollary 3.2 and the assumption that $\Psi(v) \geq \tau$, we have

$$\sigma(v) \geq 1.$$

The following lemma provide an important feature of the hyperbolic cotangent function that enable us to prove the e-convexity of the new KF.

Lemma 3.4 (Lemma 3.2 in [23])
Let ψ be the function defined in (1). Then, we have

$$2t\coth(t) - 1 > 0, \ \forall t > 0. \tag{12}$$

Let $\varrho : [0, +\infty[\longrightarrow [1, +\infty[$ be the inverse function of $\psi(t)$ for $t \geq 1$, and $\rho : [0, +\infty[\longrightarrow]0, 1]$ the inverse function of $-\dfrac{1}{2}\psi'(t)$ for $0 < t \leq 1$. Then we have the following lemma.

Lemma 3.5
One has

- $\sqrt{1+2s} \leq \varrho(s) \leq 1 + \sqrt{2s}, \forall s \in [0, +\infty).$
- $\coth(t) \leq \log\left(e^{\coth(1)}(2z+1)\right)$, For all $(z, t) \in [0, +\infty[\times]0, 1]$ such that $z = -\frac{1}{2}\psi'(t)$.

Proof. The first item can be easily obtained using (10), Lemma 3.1 and the fact that

$$\psi(t) = \int_1^t \int_1^x \psi''(y)dydx.$$

As for the second item, let $z \geq 0$ and $t \in]0, 1]$ such that $z = -\frac{1}{2}\psi'(t)$, then $\rho(z) = t$.
Using (9), we have

$$\begin{aligned} 2z &= -\psi'(t) \\ &= -t + \frac{\sinh^2(1)}{\sinh^2(t)} e^{\coth(t) - \coth(1)}. \end{aligned}$$

Since $\sinh$ is a monotonically increasing function, we obtain

$$e^{\coth(t) - \coth(1)} \leq (2z+1),$$

which implies that

$$\coth(t) \leq \log(e^{\coth(1)}(2z+1)).$$

This proves the lemma. □

The next lemma reveals some key properties of the new KF.

Lemma 3.6
Let ψ be as defined in (1). Then,

(i) $t\psi''(t) - \psi'(t) > 0, \ \forall t > 0.$

(ii) $t\psi''(t) + \psi'(t) > 0, \ \forall t > 0$, i.e. $\psi(t)$ is exponentially convex on $]0, +\infty[$.

(iii) ψ'' is monotonically decreasing on $]0, +\infty[$.

Proof. For the first and second item, using (9) and (10), we have

$$t\psi''(t) - \psi'(t) = \frac{\sinh^2(1)}{\sinh^2(t)} e^{\coth(t) - \coth(1)} \left(2t\coth(t) + 1 + \frac{t}{\sinh^2(t)}\right) > 0,$$

and

$$t\psi''(t) + \psi'(t) = 2t + \frac{\sinh^2(1)}{\sinh^2(t)} e^{\coth(t)-\coth(1)} \left(2t\coth(t) - 1 + \frac{t}{\sinh^2(t)}\right) > 0,$$

by taking into acount (12) of Lemma 3.4.
For the third item, using (11), we have $\psi'''(t) < 0$ for all $t > 0$, then ψ'' decreases monotonically. This completes the proof. □

4. Analysis of the algorithm

4.1. Growth behavior of the barrier function

We proceed by studying the effect of updating the barrier parameter μ on the value of the function $\Psi(v)$.

Theorem 4.1 (Theorem 3.2 in [4])
For any positive vector v and $\beta \geq 1$, we have

$$\Psi(\beta v) \leq n\psi\left(\beta\varrho\left(\frac{\Psi(v)}{n}\right)\right).$$

Lemma 4.2
Let $0 \leq \theta < 1$ and $v_+ = \frac{v}{\sqrt{1-\theta}}$. If $\Psi(v) \leq \tau$, then we have

$$\Psi(v_+) \leq \frac{\psi''(1)}{2} \frac{\left(\theta\sqrt{n} + \sqrt{2\tau}\right)^2}{1-\theta}.$$

Proof. The result is obtained using the same arguments as in Lemma 6.3 in [4]. □

As a direct consequence, we have the following corollary.

Corollary 4.3
Let θ be such that $0 < \theta < 1$. If $\Psi(v) \leq \tau$, then

$$\Psi(v_+) \leq \frac{3\coth(1) + 1}{(1-\theta)} \left(\theta\sqrt{n} + \sqrt{2\tau}\right)^2 := \Psi_0.$$

Ψ_0 is an upper bound for $\Psi(v_+)$ during the process of the algorithm.

4.2. Decrease of the proximity during a (damped) Newton step

The purpose of this subsection is to compute a default step size α such that (x_+, y_+, s_+) defined in the algorithm are feasible and the proximity function decreases sufficiently. First, we consider the decrease in Ψ as a function of α noted f and defined by

$$f(\alpha) = \Psi(v_+) - \Psi(v),$$

and we assume that the step size α satisfies

$$v + \alpha d_x > 0 \text{ and } v + \alpha d_s > 0.$$

Due to the $e-$convexity of ψ, we have

$$\Psi(v_+) \leq \frac{1}{2}\left(\Psi(v + \alpha d_x) + \Psi(v + \alpha d_s)\right).$$

For simplicity, we put $\sigma := \sigma(v)$. Following the same procedure as in [4], we have the following theorem.

Theorem 4.4
Let us set $\bar{\alpha} = \frac{1}{\psi''(\rho(2\sigma))}$, as the default step size. Then

$$f(\bar{\alpha}) \leq -\frac{\sigma^2}{\psi''(\rho(2\sigma))}. \tag{13}$$

We can obtain the upper bound for the decreasing value of the proximity function in an inner iteration by the next theorem.

Theorem 4.5
If $\bar{\alpha}$ is the default step size and $\sigma \geq 1$, we have

$$f(\bar{\alpha}) \leq -\frac{\sqrt{\Psi(v)}}{80\log^2\left(e^{\coth(1)}(\sqrt{\Psi(v)}+1)\right)}. \tag{14}$$

Proof. From (10) we have

$$\begin{aligned}\psi''(t) =& 1 + (t-\psi'(t))\left(2\coth(t) + \frac{1}{\sinh^2(t)}\right)\\ &\leq 1 + 3(1-\psi'(t))\coth^2(t).\end{aligned}$$

Putting $t = \rho(2\sigma)$, we get

$$\psi''(\rho(2\sigma)) \leq 1 + 3(1+4\sigma)\coth^2(\rho(2\sigma)).$$

Thus, Lemma 3.5 implies that

$$\begin{aligned}\psi''(\rho(2\sigma)) &\leq 1 + 3(1+4\sigma)\log^2\left(e^{\coth(1)}(4\sigma+1)\right)\\ &\leq 4(1+4\sigma)\log^2\left(e^{\coth(1)}(4\sigma+1)\right)\\ &\leq 4(\sigma+4\sigma)\log^2\left(e^{\coth(1)}(4\sigma+1)\right)\\ &= 20\sigma\log^2\left(e^{\coth(1)}(4\sigma+1)\right),\end{aligned}$$

where the last inequality is obtained using Remark 3.3. Hence, from (13) it follows that

$$f(\bar{\alpha}) \leq -\frac{\sigma}{20\log^2\left(e^{\coth(1)}(4\sigma+1)\right)}.$$

We can easily verify that the function $g(t) := -\frac{t}{20\log^2\left(e^{\coth(1)}(4t+1)\right)}$ is monotonically decreasing for $t \geq 1$. Using Corollary 3.2, we get $\sigma \geq \frac{\sqrt{\Psi(v)}}{4}$, which gives

$$f(\bar{\alpha}) \leq -\frac{\sqrt{\Psi(v)}}{80\log^2\left(e^{\coth(1)}(\sqrt{\Psi(v)}+1)\right)},$$

which completes the proof. □

4.3. Iteration complexity

Now, we compute how many inner iterations are required to return to the situation where $\Psi(v) \leq \tau$ after $\mu-$update. Let us define the value of $\Psi(v)$ after $\mu-$update as Ψ_0, and the subsequent values in the same outer iteration as $\Psi_i, i = 1, \ldots, K$, where K stands for the total number of inner iterations in the outer iteration. The decrease on each inner iteration is given by (14) , that is,

$$\Psi_{i+1} \leq \Psi_i - \kappa {\Psi_i}^{1-\gamma}, \;\; i = 0, 1 \ldots, K-1,$$

with

$$\kappa = \frac{1}{80 \log^2 \left(e^{\coth(1)} (\sqrt{\Psi_0} + 1) \right)},$$

and

$$\gamma = \frac{1}{2}.$$

Lemma 4.6 (Proposition 2.2 in [18])
Let $t_0, t_1, ..., t_k$ be a sequence of positive numbers such that

$$t_{k+1} \leq t_k - \beta t_k^{1-\gamma}, k = 0, 1, ..., K-1,$$

where $\beta > 0$ and $0 < \gamma \leq 1$. Then

$$K \leq \left[\frac{t_0^\gamma}{\beta \gamma} \right].$$

As a consequence, by taking $t_k = \Psi_k$, $\beta = \frac{1}{80 \log^2 \left(e^{\coth(1)} (\sqrt{\Psi_0}+1) \right)}$ and $\gamma = \frac{1}{2}$, we get the following lemma.

Lemma 4.7
One has

$$K \leq \mathcal{O} \left(\log^2 (\Psi_0) \Psi_0^{\frac{1}{2}} \right).$$

We arrive at the final result of this section which summarizes the complexity bound.

Theorem 4.8
Let Ψ_0 be an upper bound for $\Psi(v_+)$ and $\tau \geq 2$. Then, the total number of iterations to obtain an approximation solution with $n\mu \leq \epsilon$ is bounded by

$$\mathcal{O} \left(\log^2 (\Psi_0) \Psi_0^{\frac{1}{2}} \frac{\log \frac{n}{\epsilon}}{\theta} \right).$$

Proof. Recall that an upper bound for the total number of iterations is obtained by multiplying the upper bound K by the number of barrier parameter updates, which is bounded above by $\frac{1}{\theta}(\log \frac{n}{\epsilon})$ (see Lemma II.17 in [21]). Thus, we obtain the result due to the above lemma. □

For small-update methods with $\tau = \mathcal{O}(1)$ and $\theta = \Theta\left(\frac{1}{\sqrt{n}}\right)$, Corollary 4.3 implies that $\Psi_0 = \mathcal{O}(1)$. Hence, the complexity of the primal-dual interior point algorithm for linear programming problem based the new KF is $\mathcal{O}\left(\sqrt{n} \log \frac{n}{\epsilon}\right)$ iterations complexity.

As for large-update methods i.e., $\tau = \mathcal{O}(n)$ and $\theta = \Theta(1)$, Corollary 4.3 implies that $\Psi_0 = \mathcal{O}(n)$. Thus, we obtain $\mathcal{O}\left(\sqrt{n} \log^2 n \log \frac{n}{\epsilon}\right)$ iterations complexity.

5. Numerical tests

In this section, we carried out thorough numerical experiments to show the computational performance of the proposed algorithm comparing it with other algorithms based on the KFs provided in Table 1. Our experiments are implemented in MATLAB R2012b using a Supermicro dual-2.80 GHz Intel Core i5 server with 4.00 Go RAM. We have taken $\epsilon = 10^{-8}$, $\tau = n$, and $\theta \in \{0.1, 0.3, 0.5, 0.7, 0.9, 0.99\}$.

Table 1. Considered kernel functions.

Kernel functions	Complexity	Reference
$\psi_c(t) = \frac{t^2-1}{2} - \log t$	$\mathcal{O}\left(n \log \frac{n}{\epsilon}\right)$	[21]
$\psi_1(t) = \frac{t^2-1}{2} + \frac{e^{p(\frac{1}{t}-1)}-1}{p},\ p=2$	$\mathcal{O}\left(\sqrt{n} \log^2 n \log \frac{n}{\epsilon}\right)$	[1]
$\psi_2(t) = \frac{t^2-1}{2} - \int_1^t e^{p(\frac{1}{x}-1)}dx,\ \ p = \log(1+n)$	$\mathcal{O}\left(\sqrt{n} \log n \log \frac{n}{\epsilon}\right)$	[5]
$\psi_3(t) = \frac{t^2-1}{2} - \log t + \frac{1}{8}\tan^2(\pi\frac{1-t}{4t+2})$	$\mathcal{O}\left(n^{\frac{2}{3}} \log \frac{n}{\epsilon}\right)$	[20]
$\psi_4(t) = \frac{1+2\coth(1)}{2\sinh^2(1)}(t^2-1) + \coth^2(t) - \coth^2(1) - \log t$	$\mathcal{O}\left(n^{\frac{2}{3}} \log \frac{n}{\epsilon}\right)$	[22]
$\psi_{\text{new}}(t) = \frac{t^2-1}{2} + \sinh^2(1)\left(e^{\coth(t)-\coth(1)} - 1\right)$	$\mathcal{O}\left(\sqrt{n} \log^2 n \log \frac{n}{\epsilon}\right)$	New

To analyze the computational performance fairly, we choose a practical step size α as in [14] i.e., $\alpha = \min(\alpha_x, \alpha_s)$, with

$$\alpha_x = \min_{i=1,\ldots,n} \begin{cases} -\dfrac{x_i}{\Delta x_i} & \text{if } \Delta x_i < 0, \\ 1 & \text{elsewhere,} \end{cases}$$

and

$$\alpha_s = \min_{i=1,\ldots,n} \begin{cases} -\dfrac{s_i}{\Delta s_i} & \text{if } \Delta s_i < 0, \\ 1 & \text{elsewhere.} \end{cases}$$

This choice of α garantees the strict positivity of the new point. Moreover, we increase the step size by a fixed factor $0 < \beta < 1$ (in our case we choose $\beta = 0.9$). We conducted comparative numerical tests between the KFs provided in Table 1 on the following test problems.

Example 5.1 ([2])

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & -3 \end{pmatrix}, \quad b = (1, 0.5)^t, \quad c = (1, 2, 3, 4)^t,$$

where the initial feasible solutions are defined as follows

$$x^0 = (0.5, 0.27, 0.14, 0.09)^t, \quad y^0 = (0, 0)^t \text{ and } s^0 = (1, 2, 3, 4)^t.$$

The optimal solution is given by

$$x^* = (0.87500, 0, 0, 0.12500)^t,\ y^* = (1.75000, -0.75000)^t, \text{ and } s^* = (0.00000, 1.00000, 1.25000, 0.00000)^t.$$

Example 5.2 ([2])

$$A = \begin{pmatrix} 2 & 1 & 1 & 0 & 0 \\ 1 & 2 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{pmatrix}, \quad b = (8, 7, 3)^t, \quad c = (-4, -5, 0, 0, 0)^t,$$

where the initial feasible solutions are defined as follows

$$x^0 = (2.85, 1.9, 0.4, 0.35, 1.1)^t, \quad y^0 = (-1.2, -1.8, -0.5)^t \text{ and } s^0 = (0.2, 0.3, 1.2, 1.8, 0.5)^t.$$

Example 5.3 ([2])

$$A = \begin{pmatrix} 2 & 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & -1 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}, \quad b = (0,0,1)^t, \quad c = (3,-1,1,0,0,0)^t,$$

where the initial feasible solutions are defined as follows
$x^0 = (0.06757, 0.13258, 0.13302, 0.26774, 0.13302, 0.2664)^t$, $y^0 = (-2,-2,-3)^t$
and $s^0 = (10,4,6,1,5,1)^t$.

Example 5.4 ([2])

$$A = \begin{pmatrix} 0 & 1 & 2 & -1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 2 & 3 & 4 & -1 & 0 & 1 & 0 & 0 \\ -1 & 0 & -2 & 1 & 2 & 0 & 0 & 1 & 0 \\ 1 & 2 & 0 & -1 & -2 & 0 & 0 & 0 & 1 \\ 1 & 3 & 4 & 2 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad b = (1,2,3,2,1)^t, \quad c = (1,0,-2,1,1,0,0,00)^t,$$

where the initial feasible solutions are defined as follows
$x^0 = (0.1819, 0.0699, 0.063, 0.1105, 0.2012, 0.6732, 1.1885, 2.835, 2.1912)^t$,
$y^0 = (-1.3843, -0.8751, -0.4241, -0.4463, -3.0424)^t$
and $s^0 = (4.9398, 13.1544, 14.7156, 9.1788, 4.5072, 1.3843, 0.8751, 0.4241, 0.4463)^t$.

The summary of results is given in the following table.

Table 2. Number of inner iterations for fixed size examples.

Ex	θ	ψ_c	ψ_1	ψ_2	ψ_3	ψ_4	ψ_{new}
Example 1	0.1	**188**	**188**	**188**	**188**	200	**188**
	0.3	**56**	**56**	**56**	**56**	70	**56**
	0.5	**29**	**29**	**29**	**29**	44	**29**
	0.7	**17**	**17**	**17**	**17**	23	**17**
	0.9	**11**	13	**11**	**11**	14	**11**
Example 2	0.1	**191**	**191**	**191**	**191**	215	**191**
	0.3	**57**	**57**	**57**	**57**	75	**57**
	0.5	**29**	**29**	**29**	**29**	39	**29**
	0.7	**17**	**17**	**17**	**17**	21	**17**
	0.9	11	**9**	10	11	**9**	**9**
Example 3	0.1	**192**	**192**	**192**	**192**	204	**192**
	0.3	**57**	60	**57**	**57**	66	**57**
	0.5	**30**	33	**30**	**30**	33	**30**
	0.7	**17**	20	20	**17**	20	18
	0.9	33	21	**19**	39	25	**19**
Example 4	0.1	**196**	**196**	**196**	**196**	212	**196**
	0.3	**58**	**58**	**58**	**58**	78	**58**
	0.5	31	**30**	**30**	31	42	**30**
	0.7	28	24	24	**22**	24	24
	0.9	24	**16**	17	23	20	23

From Table 2, it becomes clear that smaller values of the parameter θ influence the iteration count negatively. Thus, for the following variable size problem, we only choose $\theta \in \{0.9, 0.99\}$ for seven different sizes $n = 2m$ where $m \in \{5, 25, 50, 100, 200, 400, 1000\}$.

Example 5.5 ([25])
The matrix A is defined as

$$A(i,j) = \begin{cases} 1 & \text{if } i = j \text{ or } j = i + m, \\ 0 & \text{otherwise,} \end{cases}$$

$c = -e$ and $b(i) = 2$, $i = 1, ..., m$.
We start by an initial point (x^0, y^0, s^0) such that
$x^0(i) = 1.5$, $x^0(i+m) = 0.5$, $y^0(i) = -2$, for $i = 1, ..., m$, and $s^0 = e$.

Table 3. Number of inner iterations for Example 5

θ	m	ψ_c	ψ_1	ψ_2	ψ_3	ψ_4	ψ_{new}
$\theta = 0.9$	5	11	**9**	10	11	**9**	**9**
	25	12	**10**	**10**	12	**10**	**10**
	50	12	**10**	**10**	12	**10**	**10**
	100	13	**11**	**11**	13	**11**	**11**
	200	13	**11**	**11**	13	**11**	**11**
	400	13	**11**	**11**	13	**11**	**11**
	1000	15	**12**	**12**	15	**12**	**12**
$\theta = 0.99$	5	11	**10**	11	11	**10**	**10**
	25	11	**10**	**10**	11	**10**	**10**
	50	13	**12**	**12**	13	**12**	**12**
	100	13	**12**	**12**	13	**12**	**12**
	200	13	**12**	**12**	13	**12**	**12**
	400	13	**12**	**12**	13	**12**	**12**
	1000	13	**12**	**12**	13	**12**	**12**

Recall that the numerical results were obtained by performing our algorithm with the KFs defined in Table 1 on five test problems. For each example, we used **bold** font to highlight the best, i.e., the smallest, iteration number.

Although, most of the considered KFs in Table 1 have better theoretical convergence complexity, numerical results show that by using our new KF, with exponential-hyperbolic barrier term, the best iteration complexity was achieved in 91% of the realized experiments.

6. Conclusions and remarks

This paper proposes a new KF for solving linear programming problems that differs from the existing KFs, since its barrier term is of exponential-hyperbolic type. Using some mild properties, a simple analysis for the primal-dual IPMs based on the proximity function induced by the new KF shows that our algorithm has $\mathcal{O}\left(\sqrt{n}\log^2 n \log \frac{n}{\epsilon}\right)$ and $\mathcal{O}\left(\sqrt{n}\log \frac{n}{\epsilon}\right)$ iteration complexities for large- and small-update methods respectively. Finally, the numerical results show that the new proposed KF is well promising and outperforms some existing KFs in the literature.

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