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Analysis of Household Income, Expenditure and Consumption Survey Research Data for North Sinai Governorate in Egypt Using Length Biased Truncated Lomax Distribution

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Abstract The length biased truncated Lomax distribution is introduced in this study as a weighted form of the truncated Lomax distribution. The length biased truncated Lomax distribution's essential distributional features are investigated. In the case of complete and type-II censored data, the maximum likelihood method is provided for estimating population parameter. The model parameter asymptotic confidence interval is calculated. To demonstrate the pattern of the estimate, a sample generation algorithm is supplied, as well as a Monte Carlo simulation analysis. We can see from the simulation research that as the censoring level is increased, the mean squared error of parameter estimates decrease's for all given values. With increasing sample size, the mean squared error and average length of parameter estimates decrease. The estimates get increasingly accurate as the sample size grows higher, suggesting that its asymptotically unbiased. Furthermore, in all cases, the mean squared error diminishes as the sample size grows, indicating that the estimates of parameter are consistent. Modelling to medical data and the percentage of household spending on education out of total household expenditure from the household income, expenditure and consumption survey (HIECS) data are used to show the importance of the new model. The Kumaraswamy, beta, truncated power Lomax, truncated Weibull, and one parameter-beta distributions perform poorly in comparison to the suggested distribution.

Keywords Truncated Lomax distribution, Maximum likelihood, Length biased distributions, Monte Carlo simulation

AMS 2010 subject classifications 62G30, 62M20

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1. Introduction

The weighted distribution (WD) approach covers model formulation and data assessment into a specific methodology. WDs arise often in research involving reliability, survival analysis, family data analysis, physics, bio medicine, econometrics, ecology, renewal processes, branching processes, and ecology, among other disciplines as described in [13, 22, 23, 24, 29], and references therein. In reality, different distributions occur when observations from a sample are gathered with differing probability. Assume T is a non-negative random variable (RV) with the probability density function (pdf) $f_w(t)$, the pdf of a weighted RV T' is as follows:

$$f_w\ (t) = \frac{w(t)g(t)}{E[w(t)]}\ , \qquad t > 0, \tag{1}$$

where $w(t)$ is a non-negative weight function and $E(w(t)) > 0$. Distinct selections of $w(t)$produce different WDs, that is

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- For $w(t) = t^s, s > 0$, the pdf in (1) is Known as the WD of order s.
- For $s = 1$ *or* $s = 2$ *or* $s = 3$, the pdf (1) is called as the length-biased (size-biased) (LB), the area-biased and volume biased distributions, respectively.

In lifetime analysis, the Lomax (Lo) distribution is a crucial model. It has been broadly applied in a number of disciplines, covering income and wealth analyses, as well as biological sciences, business failure modelling, model firm size and queuing problems, reliability modelling and life testing (see, [7, 11, 14, 15, 18]) respectively. The cumulative distribution function (cdf) and the pdf of the Lo distribution with shape parameter α and scale parameter λ are defined by:

$$F_{Lo}(t;\alpha,\lambda) = 1 - \lambda^{\alpha}(\lambda+t)^{-\alpha}, \qquad t > 0,$$

and

$$f_{Lo}(t;\alpha,\lambda) = \alpha\lambda^{\alpha}\,(\lambda+t)^{-(\alpha+1)}, \qquad t > 0.$$

Numerous new modifications and expansions of the Lo distribution have subsequently been developed, such as; beta Lo [21], gamma-Lo [12], Weibull Lo [27], power Lo [25], generalized Lo [5], beta transmuted Lo [19], alpha power Lo [10], truncated Lo inverse Lo [3], and distributions among others. Some recent studies about Lo distribution can be found in [8, 9]. Furthermore, versions of WD related to Lo distribution have been handled by some authors. The reader can refer to LB weighted Lo distribution [4] and LB weighted exponentiated Lo distribution [2]. A truncated distribution is a conditional distribution whose scope is restricted to a smaller region than the parent distribution's. The truncated distributions have been widely used, mostly in reliability and life-testing investigations. Our enthusiasm here with the newly truncated Lo (TLo) distribution reported in [16] with one shape parameter. The TLo distribution's pdf is stated as

$$f_{TLo}(t;\alpha) = \frac{\alpha\,(1+t)^{-(\alpha+1)}}{1-2^{-\alpha}}; \qquad 0 < t < 1,\ \alpha > 0. \tag{2}$$

The cdf corresponding to (2) is as follows

$$F_{TLo}(t;\alpha) = \frac{1-(1+t)^{-\alpha}}{1-2^{-\alpha}} \qquad 0 < t < 1,\ \alpha > 0. \tag{3}$$

In light of the importance of the TLo distribution as well as the concept of the WD, we offer a weighted version of the TLo distribution termed the length biased TLo (LBTLo) distribution. The LBTLo distribution (i) can be thought as an alternative model to the Kumaraswamy, beta with one and two parameters, truncated power Lo, and truncated Weibull distributions (as will be seen in Section 5), (ii) can be carried higher hazard rate, and (iii) has a broader range of uses in some areas as medical and demographic specially; the percentage of household spending on education out of total household expenditure from HIECS data. This data is a measure of HIECS, among other household surveys undertaken by statistical organizations in many countries throughout the world, HIECS is very important. This survey provides a large amount of data for measuring household and individual living standards, as well as creating databases for measuring poverty, designing social assistance programmes, and providing necessary weights for compiling consumer price indices, which are considered to be an important indicator for assessing inflation.

The motivation of creating the LBTLo model can be summarized in the following points:

- The new model is more flexible, and the applications suggested that it be used.
- The new model is very simple and has one parameter.
- The pdf of the new model can be uni-modal and right skewed, and the hrf can be increasing.
- to make the kurtosis more flexible as compared to the baseline base line Lomax model.

- for producing skewness for symmetrical and asymmetrical distributions.
- for constructing heavy-tailed densities that are not longer-tailed for modeling various real-life data
- for providing consistently better fits than other generated Lomax extensions.

We investigate the estimation of the population parameter using the maximum likelihood (ML) technique in the situation of complete and type II censoring (T2C) data. The length biased version's application to real data is evaluated. The following is how the rest of the article is structured. The weighted version of the TLo distribution is described in Section 2. Section 3 delves into the structural characteristics of the LBTLo distribution. Section 4 discusses the point and confidence interval (CI) estimators of population parameter using the ML approach. In the same Section 4, a simulation study is presented. The use of real data to study the implementation of the LBTLo distribution is explained in Section 5. Finally, some observations are provided.

2. The Length Biased Truncated Lomax distribution

The LBTLo distribution is created by taking the weight function $w(t) = t$ and inserting (2) into (1), as shown in the following formulation.

A non-negative continuous RV, T, is said to follow the LBTLo distribution with positive shape parameter if its pdf is of the form:

$$f_{LBTLo}\,(t;\ \alpha) = \frac{\alpha(1-\alpha)}{\left[2^{-\alpha}\,(1+\alpha)-1\right]}\ t(1+t)^{-(\alpha+1)}, \qquad 0<t<1, \quad \alpha>0, \tag{4}$$

or it can be written as

$$f_{LBTLo}\,(t;\ \alpha) = \alpha(1-\alpha)\Lambda(\alpha)t(1+t)^{-(\alpha+1)}, \qquad 0<t<1, \quad \alpha>0, \tag{5}$$

where, $\Lambda(\alpha) = \frac{1}{\left[2^{-\alpha}\,(1+\alpha)-1\right]}$. The associated cdf of the LBTLo distribution can determined as below

$$F_{LBTLo}\,(t;\ \alpha) = \alpha(1-\alpha)\Lambda(\alpha)\int_0^t t(1+t)^{-(\alpha+1)}\,\mathrm{d}\,t,$$

using the partial integration by letting $u = t, \quad dv = (1+t)^{-(\alpha+1)}\,dt$. Then the final form of the cdf is

$$F_{LBTLo}\,(t;\ \alpha) = \Lambda(\alpha)\left[(1+t)^{-\alpha}\ (1+\alpha t)-1\right]. \tag{6}$$

Quantiles are fundamental for estimation and simulation of distribution parameters. The quantile function of T can be obtained by inverting (6) as

$$u\ = \Lambda(\alpha)\left[(1+t)^{-\alpha}\ (1+\alpha t)-1\right] = 0, \quad 1<u<0,$$

where U is uniform (0, 1). Also we can generate random number of generation by solve the previous equation numerically.

The LBTLo's survival function (sf) and hazard rate function (hrf) are determined by,

$$\overline{F}_{LBTLo}\,(t;\alpha) = 1-\Lambda(\alpha)\left[(1+t)^{-\alpha}\ (1+\alpha t)-1\right],$$

and

$$h_{LBTLo}\,(t;\ \alpha) = \frac{\alpha(1-\alpha)\Lambda(\alpha)t(1+t)^{-(\alpha+1)}}{1-\Lambda(\alpha)\left[(1+t)^{-\alpha}\ (1+\alpha t)-1\right]}.$$

The pdfs and hrfs for the LBTLo distribution are shown in Figure 1 for various values of α.

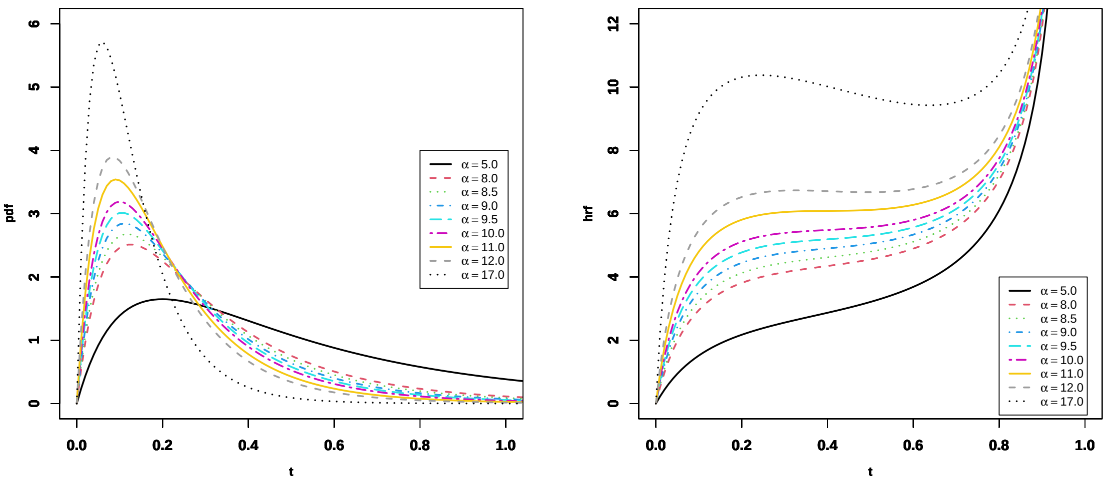

Figure 1. The pdf and hrf plots for LBTLo distribution.

The pdf can be uni-modal and right skewed, and the hrf can be increasing, as seen in Figure 1.

3. The LBTLo Distribution's Statistical Properties

Some structural characteristics of the LBTLo distribution are explored in this section.

3.1. Shapes of probability density function

The behaviour of pdf of the LBTLo distribution $f_{LBTLo}(t)$ at $t = 0$ and $t = 1$ are respectively given

$$at \qquad t \to 0 \qquad \lim_{t\to 0} f_{LBTLo}(t) = 0,$$

$$at \qquad t \to 1 \qquad \lim_{t\to 1} f_{LBTLo}(t) = 0.$$

3.2. Mode of LBTLo distribution

The mode of the LBTLo distribution is obtained by solving the equation $f'_{LBTLo}(t) = 0$

$$f'_{LBTLo}(t) = \alpha(1-\alpha)\Lambda(\alpha)\left[(1+t)^{-(\alpha+1)} - (\alpha+1)t(1+t)^{-(\alpha+2)}\right],$$

$$f'_{LBTLo}(t) = 0 \qquad \Rightarrow \qquad t_0 = \frac{1}{\alpha},$$

for

$$f''_{LBTLo}(t) = \alpha(1-\alpha)\Lambda(\alpha)\left[(\alpha+1)(\alpha+2)t_0(1+t_0)^{-(\alpha+1)} - 2(\alpha+1)(1+t_0)^{-(\alpha+2)}\right] < 0,$$

hence $f_{LBTLo}(t)$ has local maximum at t_0.

3.3. Moments & Incomplete Moments

In statistical analysis, particularly in practical works, the moments of a random variable are required. The r^{th} moment for $r = 1, 2, ...$ of the LBTLo distribution is readily determined by using pdf (4), as follows

$$\mu'_r = \alpha(1-\alpha)\Lambda(\alpha) \int_0^1 t^{r+1} \, (1+t)^{-(\alpha+1)} \, \mathrm{d}\, t.$$

For the last term of the preceding equation, use the subsequent binomial expansion

$$(1+t)^{-(\alpha+1)} = \sum_{i=0}^{\infty} (-1)^i \binom{\alpha+i}{i} t^i,$$

provides the form of r^{th} moment as,

$$\mu'_r = \alpha(1-\alpha)\Lambda(\alpha) \sum_{i=0}^{\infty} (-1)^i \binom{\alpha+i}{i} \, \frac{1}{r+i+2}. \tag{7}$$

The r^{th} central moment of T from the LBTLo distribution is as follows:

$$\mu_r = E(T - \mu'_r)^r = \sum_{i=0}^{r} (-1)^i \binom{r}{i} \left(\mu'_r\right)^i \mu'_{r-i}\,.$$

The coefficient of skewness (CS) and coefficient of kurtosis (CK) are defined by

$$CS = \frac{\mu_3}{\mu_2^{\frac{3}{2}}} \;\; and \; CK = \frac{\mu_4}{\mu_2^2}\,.$$

Let T denote a RV with pdf (5) . Using and the specification of T's moment generating function, we find

$$M_t(x) = \sum_{r=0}^{\infty} \frac{t^r}{r!} \mu'_r = \sum_{i,r=0}^{\infty} \frac{t^r}{r!} (-1)^i \binom{\alpha+i}{i} \frac{\alpha(1-\alpha)\Lambda(\alpha)}{r+i+2}.$$

Numerical comparison of the LBTLo and one-parameter Lo distributions including; variance (V), coefficient of variation (CV), CS, and CK for specific parameter choices are given in Tables 1 and 2 .

We can see from Tables 1 and 2 that when α increases, the $\mu'_1, \mu'_2, \mu'_3, \mu'_4$and V decrease, while the CS, CK, and CV increase. Both models are skewed to right with leptokurtic curves. Also we can note that the values of $\mu'_1, \mu'_2, \mu'_3, \mu'_4$and V for the Lo model are less than the LBTLo model. While the values of CS, CK, and CV for Lo model are more than the LBTLo model.

Furthermore, the rth incomplete moment, say $\varphi_r\,(t)$ is given by:

$$\varphi_r\,(t) = \int_0^t t^r \, f_{LBTLo}\,(t;\,\alpha)\,\mathrm{d}\,t = \alpha(1-\alpha)\Lambda(\alpha) \sum_{i=0}^{\infty} (-1)^i \binom{\alpha+i}{i} \int_0^t t^{r+i+1}\,\mathrm{d}\,t,$$

then,

$$\varphi_r(t) = \sum_{i=0}^{\infty} (-1)^i \binom{\alpha+i}{i} t^{r+i+2} \frac{\alpha(1-\alpha)\Lambda(\alpha)}{r+i+2}.$$

Table 1. Results of $\mu_1', \mu_2', \mu_3', \mu_4'$,V, CS, CK and CV for the LBTLo model

$\alpha\downarrow$	$\mu_1'\uparrow$	$\mu_2'\uparrow$	$\mu_3'\uparrow$	$\mu_4'\uparrow$	$V\uparrow$	$CS\downarrow$	$CK\downarrow$	$CV\downarrow$
5	0.41	0.231	0.154	0.113	0.062	0.508	2.297	0.609
5.7	0.381	0.204	0.132	0.095	0.059	0.639	2.522	0.638
6.5	0.349	0.176	0.109	0.077	0.054	0.788	2.843	0.668
8	0.296	0.132	0.075	0.05	0.045	1.051	3.61	0.715
8.5	0.28	0.12	0.066	0.043	0.041	1.133	3.904	0.727
9	0.265	0.109	0.058	0.037	0.038	1.211	4.214	0.738
9.5	0.252	0.099	0.051	0.032	0.035	1.285	4.533	0.748
10	0.239	0.09	0.045	0.027	0.033	1.353	4.859	0.756
11	0.216	0.074	0.035	0.02	0.028	1.476	5.512	0.768
12	0.197	0.062	0.027	0.015	0.023	1.576	6.135	0.775

Table 2. Results of $\mu_1', \mu_2', \mu_3', \mu_4'$,V, CS, CK and CV for the Lo model

$\alpha\downarrow$	$\mu_1'\uparrow$	$\mu_2'\uparrow$	$\mu_3'\uparrow$	$\mu_4'\uparrow$	$V\uparrow$	$CS\downarrow$	$CK\downarrow$	$CV\downarrow$
5	0.203	0.083	0.047	0.031	0.042	1.49	4.907	1.01
5.7	0.185	0.071	0.038	0.024	0.036	1.639	5.636	1.027
6.5	0.167	0.058	0.029	0.018	0.03	1.799	6.535	1.045
8	0.138	0.041	0.018	0.01	0.022	2.054	8.275	1.07
8.5	0.13	0.036	0.016	0.0086	0.019	2.124	8.834	1.075
9	0.123	0.033	0.013	0.0071	0.017	2.186	9.368	1.079
9.5	0.116	0.029	0.011	0.0059	0.016	2.239	9.868	1.082
10	0.11	0.026	0.0099	0.0049	0.014	2.284	10.329	1.083
11	0.099	0.021	0.0074	0.0035	0.012	2.352	11.112	1.084
12	0.091	0.018	0.0056	0.0024	0.0096	2.394	11.701	1.082

The Lorenz and Bonferroni curves, which are widely employed in income and wealth distributions, are an important use of the first incomplete moment. They are obtained in the following order:

$$Lz(t) = \frac{\sum_{i=0}^{\infty}(-1)^i \binom{\alpha+i}{i} \frac{t^{i+3}}{i+3}}{\sum_{i=0}^{\infty}(-1)^i \binom{\alpha+i}{i} \frac{1}{i+3}},$$

$$Bo(t) = \frac{Lz(t)}{F_{LBTLo}(t)} = \frac{\sum_{i=0}^{\infty}(-1)^i \binom{\alpha+i}{i} \frac{t^{i+3}}{i+3}}{\Lambda(\alpha)[(1+t)^{-\alpha}(1+\alpha t)-1]\sum_{i=0}^{\infty}(-1)^i \binom{\alpha+i}{i} \frac{1}{i+3}}$$

Some numerical values of Lz and Bo curves are displayed in Table 3.

Table 3. Some numerical values of Lz and Bo curves

$\alpha\downarrow$	t=0.8		t=1.5		t=2.0		t=2.5	
	Lz	Bo	Lz	Bo	Lz	Bo	Lz	Bo
5	1.26	1.393	0.733	0.652	0.633	0.539	0.585	0.488
5.7	1.216	1.316	0.78	0.717	0.702	0.626	0.666	0.587
6.5	1.173	1.245	0.829	0.782	0.772	0.716	0.748	0.689
8	1.111	1.149	0.9	0.876	0.873	0.844	0.863	0.833
8.5	1.094	1.125	0.918	0.899	0.897	0.875	0.89	0.867
9	1.08	1.105	0.934	0.919	0.918	0.901	0.913	0.895
9.5	1.068	1.088	0.946	0.935	0.935	0.922	0.931	0.918
10	1.057	1.073	0.957	0.948	0.949	0.939	0.946	0.936
11	1.04	1.05	0.973	0.968	0.969	0.963	0.968	0.962
12	1.028	1.034	0.983	0.981	0.981	0.978	0.981	0.978

from Table 3 at t=0.8 and the value α increased then the value of Lz and Bo are decreased. But when t=1.5,2 and 2.5 and the value α increased then the value of Lz and Bo are increasing.
For eny value od α when the value of t is increasing then the value of Lz and Bo are decreasing.

3.4. Information Measures

The Rényi entropy (RE), presented in [26], is defined by

$$I_R(\varepsilon) = (1-\varepsilon)^{-1}\ \log\left\{\int_0^1 \left[f_{LBTLo}\left(t;\alpha\right)\right]^{\varepsilon} \mathrm{d}\,t\right\}, \varepsilon \neq 1, \qquad \varepsilon > 0.$$

Then, the RE of LBTLo distribution is given by

$$I_R(\varepsilon) = (1-\varepsilon)^{-1}\ \ \log\left\{\left[\alpha\left(1-\alpha\right)\Lambda\left(\alpha\right)\right]^{\varepsilon}\int_0^1 t^{\varepsilon}\left(1+t\right)^{-\varepsilon(\alpha+1)}\mathrm{d}\,t\right\}.$$

Now, we consider the generalized binomial theory in the last, then the RE of LBTLo distribution is

$$I_R(\varepsilon) = (1-\varepsilon)^{-1}\ \log\left\{\sum_{i=0}^{\infty}(-1)^i\binom{\varepsilon(\alpha+1)+i-1}{i}\frac{[\alpha(1-\alpha)\Lambda(\alpha)]^{\varepsilon}}{\varepsilon+i+1}\right\}.$$

The Havrda and Charvat entropy (HaCE) measure (see [17]) is defined by:

$$HC_R\left(\varepsilon\right) = \frac{1}{2^{1-\varepsilon}-1}\left(\left\{\int_0^{\infty}\left[f_{LBTLo}\left(t;\alpha\right)\right]^{\varepsilon}\mathrm{d}\,t\right\}^{\frac{1}{\varepsilon}} - 1\right), \qquad \varepsilon \neq 1, \qquad \varepsilon > 0.$$

Hence, the HaCE of LBTLo distribution is given by:

$$HC_R(\varepsilon) = \frac{1}{2^{1-\varepsilon}-1}\left(\left\{\sum_{i=0}^{\infty}(-1)^i\binom{\varepsilon(\alpha+1)+i-1}{i}\frac{[\alpha(1-\alpha)\Lambda(\alpha)]^{\varepsilon}}{\varepsilon+i+1}\right\}^{\frac{1}{\varepsilon}} - 1\right).$$

The Arimoto entropy (ArE) measure (see [6]) of the LBTLo distribution is defined by:

$$A_R\left(\varepsilon\right) = \frac{\varepsilon}{1-\varepsilon}\left(\left\{\int_0^{\infty}\left[f_{LBTLo}\left(t;\alpha\right)\right]^{\varepsilon}\mathrm{d}\,t\right\}^{\frac{1}{\varepsilon}} - 1\right), \qquad \varepsilon \neq 1, \qquad \varepsilon > 0.$$

Hence the ArE of LBTLo distribution is given by:

$$A_R(\varepsilon) = \frac{\varepsilon}{1-\varepsilon}\left(\left\{\sum_{i=0}^{\infty}(-1)^i\binom{\varepsilon(\alpha+1)+i-1}{i}\frac{[\alpha(1-\alpha)\Lambda(\alpha)]^{\varepsilon}}{\varepsilon+i+1}\right\}^{\frac{1}{\varepsilon}}-1\right).$$

The Tsallis entropy (TsE) measure (see [28]) is defined by:

$$T_R(\varepsilon) = \frac{1}{\varepsilon-1}\left\{1-\int_0^{\infty}\left[f_{LBTLo}(t;\alpha)\right]^{\varepsilon}\mathrm{d}\,t\right\}, \quad \varepsilon \neq 1, \quad \varepsilon > 0.$$

Hence the TsE of LBTLo distribution is obtained as follows:

$$T_R(\varepsilon) = \frac{1}{\varepsilon-1}\left\{1-\sum_{i=0}^{\infty}(-1)^i\binom{\varepsilon(\alpha+1)+i-1}{i}\frac{[\alpha(1-\alpha)\Lambda(\alpha)]^{\varepsilon}}{\varepsilon+i+1}\right\}.$$

Some of the numerical values of RE, HaCE, ArE and TsE for some selected parameter values are given in Tables 4, 5, 6 and 7 .

Table 4. Numerical values of RE, HaCE, ArE and TsE for the LBTLo model at $\varepsilon = 0.8$

$\alpha\downarrow$	$RE\downarrow$	$HaCE\downarrow$	$ArE\downarrow$	$TsE\downarrow$
5	-0.035	-0.107	-0.079	-0.079
5.7	-0.05	-0.153	-0.113	-0.113
6.5	-0.071	-0.217	-0.161	-0.161
8	-0.12	-0.363	-0.268	-0.27
8.5	-0.139	-0.416	-0.307	-0.309
9	-0.157	-0.47	-0.346	-0.35
9.5	-0.177	-0.525	-0.387	-0.39
10	-0.196	-0.58	-0.427	-0.431
11	-0.235	-0.689	-0.506	-0.512
12	-0.273	-0.794	-0.582	-0.591

Table 5. Numerical values of RE, HaCE, ArE and TsE for the LBTLo model at $\varepsilon = 1.5$

$\alpha\downarrow$	$RE\downarrow$	$HaCE\downarrow$	$ArE\downarrow$	$TsE\downarrow$
5	-0.059	-0.24	-0.139	-0.141
5.7	-0.084	-0.345	-0.199	-0.202
6.5	-0.117	-0.49	-0.281	-0.287
8	-0.186	-0.817	-0.461	-0.478
8.5	-0.21	-0.935	-0.526	-0.548
9	-0.234	-1.057	-0.591	-0.619
9.5	-0.258	-1.181	-0.657	-0.692
10	-0.281	-1.306	-0.723	-0.765
11	-0.326	-1.557	-0.854	-0.912
12	-0.369	-1.804	-0.981	-1.057

Table 6. Numerical values of RE, HaCE, ArE and TsE for the LBTLo model at $\varepsilon = 2.0$

$\alpha \downarrow$	$RE \downarrow$	$HaCE \downarrow$	$ArE \downarrow$	$TsE \downarrow$
5	-0.073	-0.368	-0.176	-0.184
5.7	-0.103	-0.533	-0.251	-0.267
6.5	-0.141	-0.765	-0.352	-0.382
8	-0.217	-1.299	-0.569	-0.649
8.5	-0.243	-1.499	-0.646	-0.75
9	-0.268	-1.709	-0.723	-0.854
9.5	-0.293	-1.925	-0.802	-0.963
10	-0.317	-2.148	-0.88	-1.074
11	-0.363	-2.609	-1.036	-1.304
12	-0.405	-3.083	-1.188	-1.541

Table 7. Numerical values of RE, HaCE, ArE and TsE for the LBTLo model at $\varepsilon = 2.5$

$\alpha \downarrow$	$RE \downarrow$	$HaCE \downarrow$	$ArE \downarrow$	$TsE \downarrow$
5	-0.086	-0.532	-0.266	-0.229
5.7	-0.118	-0.781	-0.378	-0.337
6.5	-0.16	-1.137	-0.529	-0.49
8	-0.24	-1.997	-0.856	-0.861
8.5	-0.266	-2.335	-0.974	-1.006
9	-0.292	-2.696	-1.094	-1.162
9.5	-0.317	-3.08	-1.216	-1.327
10	-0.341	-3.483	-1.339	-1.501
11	-0.387	-4.347	-1.587	-1.873
12	-0.43	-5.279	-1.834	-2.275

From Tables 4, 5, 6 and 7 we can note when α increases, the RE, HaCE, ArE and TsE decrease. Also the RE, HaCE, ArE and TsE are decreasing with increases value of ε.

4. Parameter Estimators in Right-Censored Samples

In medical survival analysis and industrial life testing, it is critical to decrease the cost and/or duration of a life-testing study, thus one may vote to end the test early, resulting in the so-called censored sample approach. One of the most prevalent types of censoring is T2C. The life testing experiment for the T2C is ended after a set number of failures, say r.

The subsections below offer ML estimators of the LBTLo model parameter based on T2C as well as an approximated CI is created. Furthermore, some numerical illustration are performed.

4.1. ML Estimators viz T2C

Let $T_{1:n} < T_{2:n} < ... < T_{n:n}$ be a T2C of size r resulting from a life test on n items whose lifetimes are described by the LBTLo model with parameter α.The log likelihood function, indicated by $\ln\, l$, of r failures and (n-r) censored values is given by

$$\ln\ l = \ln\ c + r\ \ln\ \alpha + r\ \ln\left(1-\alpha\right) + r\ \ln\left[2^{-\alpha}\left(1+\alpha\right)-1\right] + \sum_{i=1}^{r} \ln\left(t_i\right) - \left(\alpha+1\right)\sum_{i=1}^{r} \ln\ \left[1+t_i\right] + \left(n-r\right)\ln\left(A_r\right),$$

where, $c = \frac{n!}{(r-1!)(n-r)!}$, $A_r = 1 - \frac{(1+t_r)^{-\alpha}(1+\alpha t_r)-1}{[2^{-\alpha}(1+\alpha)-1]}$, and we use t_i instead of $t_{i:n}$ for clarity. The first derivative of the log-likelihood function with respect to α is obtained as follows

$$\frac{d\ \ln\ l}{d\ \alpha} = \frac{r}{\alpha} - \frac{r}{1-\alpha} + \frac{r2^{-\alpha}[1-(1+\alpha)\ \ \ln\ 2]}{2^{-\alpha}(1+\alpha)-1} - \sum_{i=1}^{r} \ln[1+t_i\ ] + (n-r)\frac{(d\ A_r/d\ \alpha)}{A_r},$$

where

$$\frac{d\ A_r}{d\ \alpha} = \frac{(B_r)^{-\alpha}[(K_r)(\ln\ (B_r)) - t_r]\, D_r - 2^{-\alpha}[(\ln\ 2)\ (1+\alpha)-1]\left[(B_r)^{-\alpha}(K_r)-1\right]}{D_r^2},$$

$$B_r = 1 + t_r,\ K_r = 1 + \alpha t_r \quad and\ D_r = 2^{-\alpha}(1+\alpha) - 1.$$

The ML estimator of α is obtained by setting $\frac{d\ \ln\ l}{d\ \alpha}$ equal to zero and solving numerically. In the case of a complete sample, we obtain the ML of the model parameter for r = n.

For the LBTLo distribution, an approximate CI of α is obtained. The asymptotic distribution of ML estimates (MLEs) of elements of parameter α is known to be given by $(\widehat{\alpha} - \alpha) \to N\left(0, I^{-1}(\alpha)\right)$, where I^{-1} is the unknown parameter's variance covariance matrix, so the approximate 100(1- v) percent two-sided CI for α is given by $\widehat{\alpha} \pm Z_{\frac{v}{2}}\sqrt{var\,(\widehat{\alpha})}$ where, $Z_{\frac{v}{2}}$ is the upper percentile of the standard normal distribution.

4.2. Numerical Results

This subsection includes a simulation study to evaluate the estimators' behaviour in the cases of complete and T2C. Mathematica 9 is used to calculate mean squared errors (MSEs), lower bound (LB) of CI, upper bound (UB) of CI, and average length (AL) of 90 percent and 95 percent of estimated value of α. The following algorithm is created in the following manner:

- From the LBTLo distribution, 5000 random samples of size n = 30, 50, 100, 150, 200, and 300 are created.
- The true parameter values are α = 6.5, α= 5.7, and α= 5.
- Three levels of censorship are chosen: r = 50%, 70% (T2C), and 100% (complete sample).
- For selected values of α , the MLEs, MSEs, LB, UB, and AL are calculated.
- Based on complete and T2C, numerical outcomes are provided in Tables 8, 9, and 10.

Table 8. Accuracy measures of LBTLo estimates for $\alpha = 6.5$ under T2C

n	t_r	MLE	MSE	LB(90)	UB(90)	AL(90)	LB(95)	UB(95)	AL(95)
30	50%	5.8549	0.4162	4.8596	6.8502	1.9907	4.6690	7.0408	2.3719
	70%	5.9371	0.3169	4.9623	6.9118	1.9495	4.7757	7.0985	2.3228
	100%	5.9696	0.2813	4.9967	6.9425	1.9458	4.8104	7.1288	2.3184
50	50%	6.1894	0.0965	5.3624	7.0164	1.6540	5.2040	7.1747	1.9707
	70%	6.2157	0.0808	5.4108	7.0207	1.6100	5.2566	7.1749	1.9183
	100%	6.2437	0.0657	5.4413	7.0460	1.6047	5.2877	7.1996	1.9119
100	50%	6.7535	0.0643	6.0216	7.4854	1.4639	5.8814	7.6256	1.7442
	70%	6.7320	0.0538	6.0198	7.4442	1.4244	5.8835	7.5806	1.6971
	100%	6.7051	0.0421	5.9970	7.4132	1.4162	5.8614	7.5488	1.6874
150	50%	6.2829	0.0471	5.6974	6.8684	1.1710	5.5853	6.9806	1.3953
	70%	6.3026	0.0390	5.7317	6.8736	1.1420	5.6223	6.9829	1.3606
	100%	6.3151	0.0342	5.7462	6.8840	1.1378	5.6373	6.9930	1.3557
200	50%	6.2713	0.0423	5.7646	6.7779	1.0133	5.6676	6.8749	1.2073
	70%	6.3184	0.0330	5.8236	6.8131	0.9895	5.7289	6.9078	1.1790
	100%	6.3337	0.0277	5.8407	6.8267	0.9861	5.7462	6.9211	1.1749
300	50%	6.6299	0.0169	6.1706	7.0893	0.9187	6.0826	7.1773	1.0947
	70%	6.5614	0.0038	6.1143	7.0085	0.8943	6.0286	7.0941	1.0655
	100%	6.5123	0.0002	6.0681	6.9565	0.8884	5.9831	7.0415	1.0585

Table 9. Accuracy measures of LBTLo estimates for $\alpha = 5.7$ under T2C

n	t_r	MLE	MSE	LB(90)	UB(90)	AL(90)	LB(95)	UB(95)	AL(95)
30	50%	5.64844	0.410931	4.6494	6.64747	1.99807	4.45809	6.83878	2.38069
	70%	5.70521	0.36008	4.73382	6.67659	1.94277	4.54781	6.8626	2.31479
	100%	5.73868	0.348962	4.77075	6.7066	1.93584	4.58541	6.89195	2.30654
50	50%	5.84359	0.398011	5.02361	6.66358	1.63997	4.86659	6.8206	1.954
	70%	5.8414	0.368383	5.04476	6.63804	1.59328	4.89221	6.79059	1.89838
	100%	5.84219	0.343094	5.04942	6.63495	1.58553	4.89761	6.78676	1.88914
100	50%	5.61085	0.216208	4.90621	6.31549	1.40928	4.77127	6.45042	1.67914
	70%	5.60169	0.176085	4.91761	6.28576	1.36815	4.78662	6.41676	1.63014
	100%	5.57796	0.16765	4.89783	6.25809	1.36025	4.7676	6.38832	1.62073
150	50%	5.7783	0.11234	5.20081	6.35579	1.15498	5.09023	6.46637	1.37614
	70%	5.79492	0.094904	5.23359	6.35624	1.12266	5.1261	6.46373	1.33763
	100%	5.80009	0.090784	5.24125	6.35893	1.11768	5.13424	6.46594	1.33171
200	50%	5.64835	0.084193	5.14973	6.14697	0.997237	5.05425	6.24245	1.1882
	70%	5.65168	0.08329	5.16757	6.13579	0.968219	5.07487	6.22849	1.15362
	100%	5.65029	0.079846	5.1686	6.13198	0.963384	5.07636	6.22422	1.14786
300	50%	5.68577	0.073459	5.23955	6.13199	0.892448	5.1541	6.21744	1.06334
	70%	5.72608	0.071268	5.29216	6.16	0.86784	5.20907	6.24309	1.03402
	100%	5.73502	0.069075	5.30301	6.16702	0.864004	5.22029	6.24974	1.02945

Table 10. Accuracy measures of LBTLo estimates for $\alpha = 5.0$ under T2C

n	t_r	MLE	MSE	LB(90)	UB(90)	AL(90)	LB(95)	UB(95)	AL(95)
	50%	5.4080	0.4117	3.7057	6.5103	2.8046	3.4371	6.7788	3.3417
30	70%	5.3561	0.1031	3.7056	6.4065	2.7009	3.4471	6.6651	3.2180
	100%	4.8278	0.0321	3.5908	6.2649	2.6741	3.3347	6.5209	3.1862
	50%	5.1581	0.2250	4.1736	6.1426	1.9690	3.9851	6.3311	2.3460
50	70%	4.8487	0.1026	3.9967	5.9008	1.9040	3.8144	6.0831	2.2686
	100%	4.8518	0.0223	4.0059	5.8977	1.8918	3.9247	6.0788	2.2541
	50%	4.8770	0.1044	3.8776	5.4764	1.5988	3.7245	5.6294	1.9050
100	70%	4.8642	0.0556	3.9903	5.5380	1.5477	3.8422	5.6862	1.8440
	100%	4.8885	0.0167	4.0383	5.5787	1.5404	3.8909	5.7262	1.8354
	50%	4.8828	0.0137	4.1892	5.5763	1.3871	4.0564	5.7091	1.6527
150	70%	4.8936	0.0093	4.2320	5.5752	1.3432	4.1034	5.7038	1.6004
	100%	4.9290	0.0017	4.2901	5.6280	1.3379	4.1620	5.7560	1.5941
	50%	4.9143	0.0073	4.3479	5.4807	1.1328	4.2394	5.5892	1.3498
200	70%	5.0311	0.0040	4.4814	5.5809	1.0995	4.3761	5.6861	1.3100
	100%	5.0479	0.0012	4.5007	5.5951	1.0944	4.3959	5.6999	1.3040
	50%	4.9995	0.0010	4.4084	5.3905	0.9820	4.3144	5.4845	1.1701
300	70%	5.0113	0.0028	4.5849	5.5377	0.9527	4.4937	5.6289	1.1352
	100%	5.0290	0.0008	4.5553	5.5027	0.9474	4.46464	5.5934	1.1289

From the above tables, we observe the following

- The MSEs and ALs of parameter estimate decrease with increasing value of sample size.
- The MSEs and ALs of parameter estimate decrease with increasing the censoring level r.
- The lengths of the CI become narrower as n increases.
- The estimate values of parameter α become more accurate, implying that its asymptotically unbiased with increasing sample size.
- The confidence interval's overall performance is quite good for the parameter values investigated.
- The mean squared error decreases as the sample size increases in all situations, demonstrating that the various estimates of α are consistent.
- The AL of CIs increases as the confidence levels increase from 90% to 95%.

5. Data Analysis

In this section, two good data sets are studied to show how the LBTLo distribution outperforms other models. Comparing the new model to some models; namely, Kumaraswamy (Kum), truncated power Lomax (TPLo), beta (B), truncated Weibull (TWe) and the one-parameter beta (BI). We obtain the MLEs, and standard errors (SEs) of the model parameters. To compare the distribution models, we consider criteria like; Akaike information criterion (AIC), the consistent AIC (CAIC), Bayesian IC (BIC), Hannan-Quinn IC (HQIC), Kolmogorov–Smirnov (KS) test and p-value (PV) test. The wider distribution, on the other hand, refers to lower AIC, CAIC, BIC, HQIC, KS, and the greatest value of PV.

First, we look at statistics on the number of months it takes for renal dialysis patients to get infected, as stated in [20]. The "times of infection" data set is: {2.5, 2.5, 3.5, 3.5, 3.5, 4.5, 5.5, 6.5, 6.5, 7.5, 7.5, 7.5, 7.5, 8.5, 9.5, 10.5, 11.5, 12.5, 12.5, 13.5, 14.5, 14.5, 21.5, 21.5, 22.5, 22.5, 25.5, 27.5}. We now perform a normalization procedure by dividing this data by thirty, resulting in values between 0 and 1. The data set has been changed to: {0.08333, 0.08333, 0.116667, 0.850000, 0.116667, 0.116667, 0.15000, 0.18333, 0.216667, 0.916667, 0.216667, 0.25000, 0.25000, 0.25000, 0.25000, 0.28333, 0.316667, 0.35000, 0.38333, 0.416667, 0.416667, 0.750000, 0.450000, 0.483333, 0.483333, 0.716667, 0.716667, 0.750000}

The second data represent the percentage of household spending on education out of total household expenditure from the Household Income, Expenditure and Consumption Survey 2015 research data for North Sinai Governorate from Central Agency for Public Mobilization and Statistics (CAPMAS), which is 91 observations for households who actually spend on education. The HIECS is of great importance among other household surveys conducted by statistical agencies in various countries around the world. This survey provides a large amount of data to rely in measuring the living standards of households and individuals, as well as establishing databases that serve in measuring poverty, designing social assistance programs, and providing necessary weights to compile consumer price indices, considered to be an important indicator to assess inflation.

0.122132, 0.122765, 0.097939, 0.039479, 0.001468, 0.005376, 0.055259, 0.041587, 0.038622, 0.012409, 0.108181, 0.045738, 0.0175, 0.036216, 0.019058, 0.044541, 0.030088, 0.012575, 0.066238, 0.063552, 0.034373, 0.081124, 0.071778, 0.014717, 0.027474, 0.070955, 0.008435, 0.028732, 0.040941, 0.033612, 0.070127, 0.121961, 0.05718, 0.04093, 0.000294, 0.080076, 0.087248, 0.013935, 0.054817, 0.038729, 0.046207, 0.053197, 0.02905, 0.102578, 0.079925, 0.036808, 0.035223, 0.130772, 0.096788, 0.196098, 0.073585, 0.078273, 0.02836,0. 015997, 0.008347, 0.012098, 0.047497, 0.0992, 0.122806, 0.071555, 0.250766, 0.093927, 0.029984, 0.083865, 0.054542, 0.04806, 0.033336, 0.033878, 0.068899, 0.024866, 0.025393, 0.254662, 0.319948, 0.125383, 0.060451, 0.147953, 0.076783, 0.024211, 0.054605, 0.021549, 0.041629, 0.139228, 0.134643, 0.044348, 0.030344, 0.036264, 0.063436, 0.063795, 0.048082, 0.066341, 0.039385

Figure 2 illustrates the boxplots for the proposed data.The total time test (TTT) plot (see [1]) is an essential graphical technique to check if the data can be applied to a given distribution or not, this is the TTT plot's empirically determined version is given by plotting $T\left(\frac{r}{n}\right) = \left\{\sum_{i=1}^{n}\left[y_{i:n} + (n-r)\, y_{i:n}\right]\right\} / \left(\sum_{i=1}^{n} y_{i:n}\right)$, against $\frac{r}{n}$, where $r = 1, ..., n$ *and* $y_{i:n}$, $(i = 1, ..., n)$ are the order statistics of the sample. The hrf is constant if the TTT plot is graphically displayed as a straight diagonal, but increasing (or decreasing) if the TTT plot is concave (or convex), (see [1]). If the TTT plot is initially convex and then concave, the hrf is U-shaped (bathtub); otherwise, the hrf is unimodal. Figure 3 shows the TTT plots for both data sets, which show that the empirical hrf of the first and second data sets are increasing. The plots of the profile log likelihood for the both data sets are reported in Figure 4.

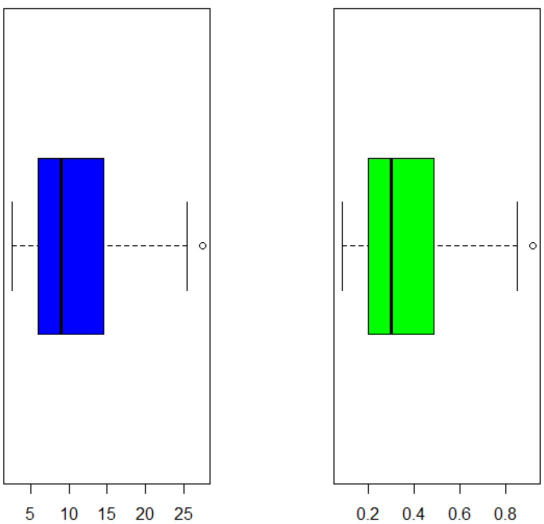

Figure 2. Box plots with color blue for the first data and the color green for the second data.

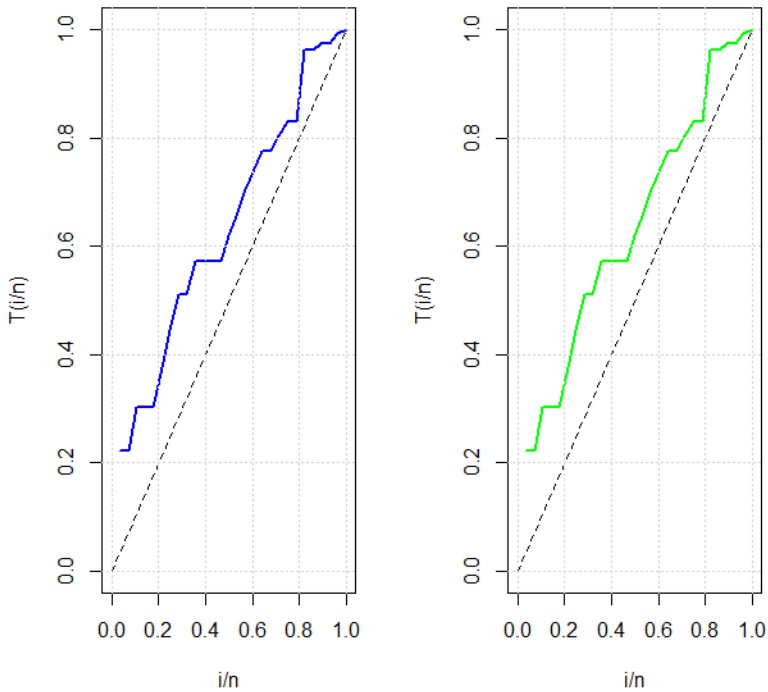

Figure 3. TTT plots with color blue for the first data and the color green for the second data.

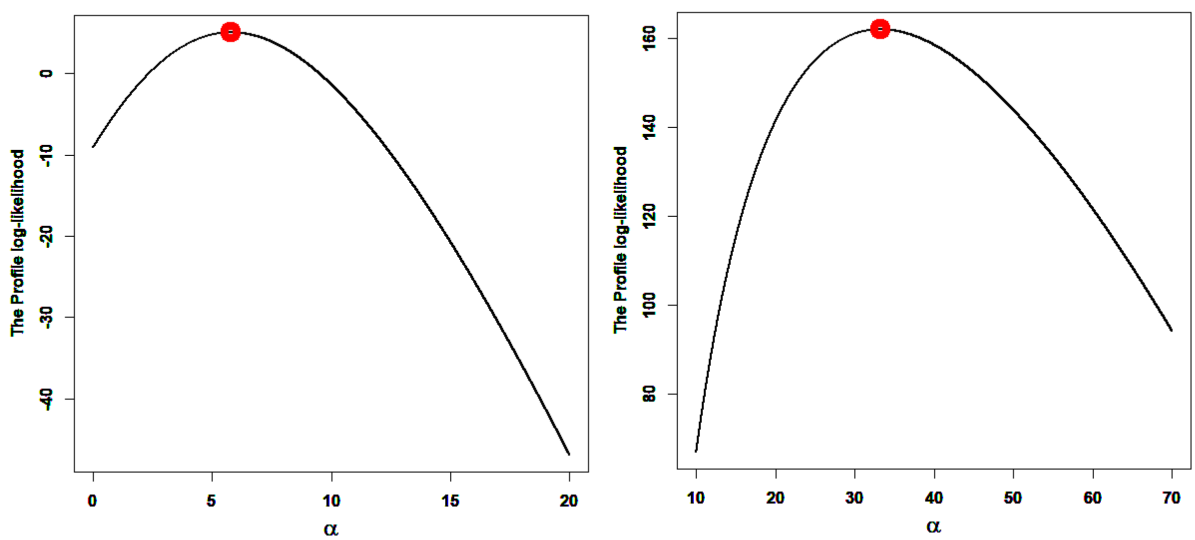

Figure 4. The profile log likelihood for the both data sets.

The MLEs of the six competing models, as well as their SEs and AIC, CAIC, BIC, HQIC, PV, and KS values for both data sets, are shown in Tables 11 and 12.

Table 11. MLEs, SEs and measures of fitting for the first data set

Distributions	MLE and SE		AIC	CAIC	BIC	HQIC	PV	KS
	α	β						
LBTLo	5.78 (1.11)		-8.16	-8.006	-8.713	-7.753	0.9559	0.0967
Kum	1.27 (1.11)	2.08 (0.57)	-3.325	-2.845	-4.431	-2.51	0.6629	0.1377
TPLo	4.03 (1.61)	1.63 (0.28)	-5.425	-4.972	-6.557	-4.637	0.8597	0.114
B	1.36 (0.33)	2.11 (0.55)	-3.555	-3.075	-4.661	-2.741	0.6321	0.1412
TWe	3.33 (1.19)	1.52 (0.27)	-4.883	-4.403	-5.989	-4.068	0.8022	0.1216
BI	2.90 (0.44)		-2.747	-2.593	-3.3	-2.34	0.1553	0.2136

Table 12. The MLEs, SEs and measures of fitting for the second data set

Distributions	MLE and SE		AIC	CAIC	BIC	HQIC	PV	KS
	α	β						
LBTLo	33.18 (2.42)		- 322.117	- 322.072	- 322.158	- 321.104	0.865	0.063
Kum	26.35 (7.05)	1.24 (0.10)	- 320.202	- 320.066	- 320.284	- 318.176	0.636	0.078
TPLo	32.46 (7.82)	1.3 (0.10)	- 321.918	- 321.782	-322	- 319.892	0.653	0.077
B	1.44 (0.19)	20.86 (3.30)	- 321.202	- 321.066	- 321.284	- 319.176	0.720	0.73
TWe	29.49 (7.42)	1.27 (0.10)	- 321.237	- 321.101	- 321.319	- 319.211	0.630	0.078
BI	28.72 (2.17)		- 316.738	- 316.689	- 316.775	- 315.721	0.3871	0.095

We find that the LBTLo distribution with one-parameter provides a better fit than the others five models. It has the smallest values of AIC, CAIC, BIC, HQIC, KS and the greatest value of PV among those considered here.

Moreover, the plots of empirical cdf and empirical pdf displayed in Figures 5 and 7 respectively. Furthermore, the PP plots of all competitive models for both data sets are displayed in Figures 6 and 8.

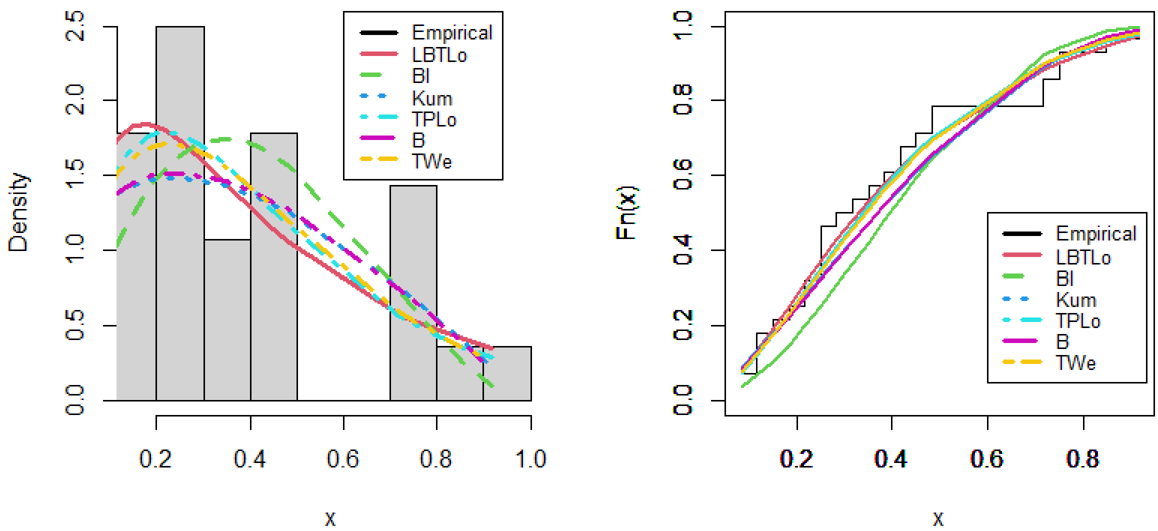

Figure 5. Estimated pdf and cdf of competitive models for the first dataset

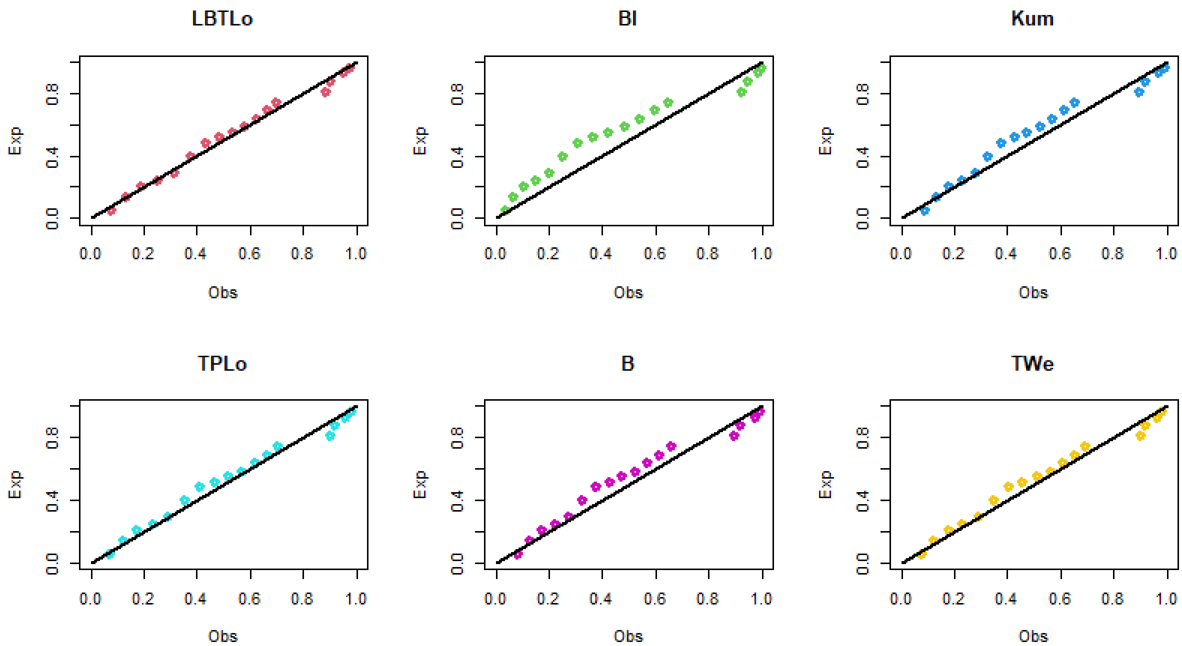

Figure 6. PP plots of the fitted models for the first data set

Also from Figures 6 and 8 we can see that the LBTLo distribution provides a better fit than the other five competitive models for both data sets.

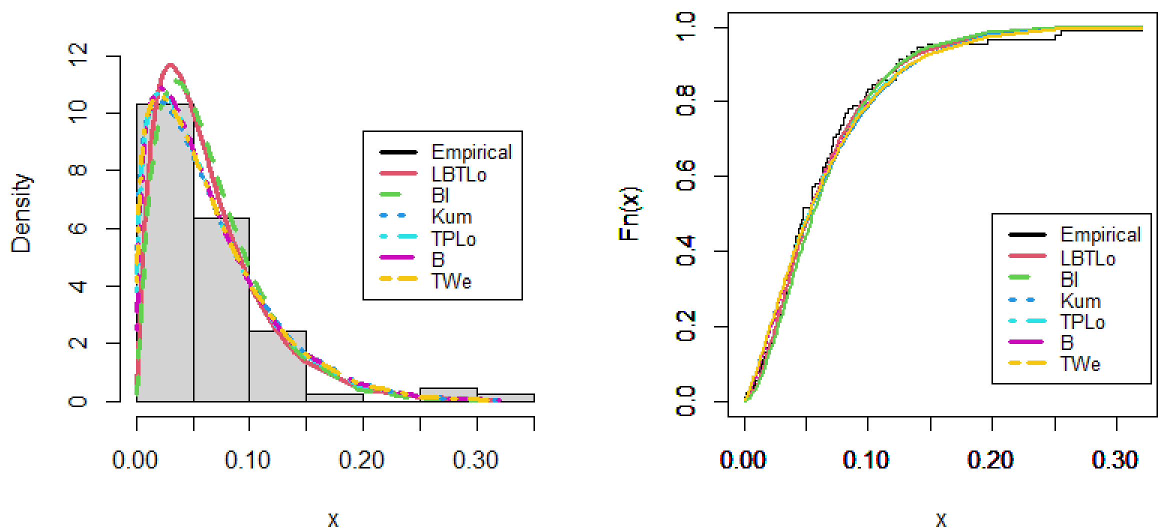

Figure 7. Estimated pdf and cdf of competitive models for the second data set

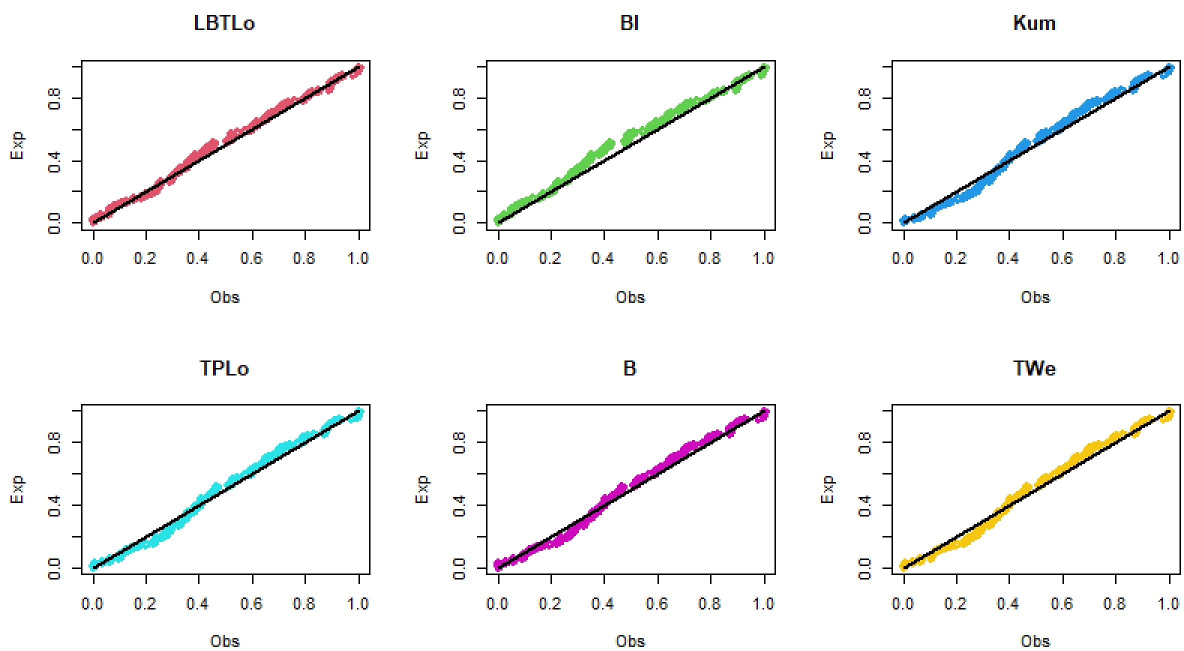

Figure 8. PP plots of the fitted models for the second data set.

6. Summary and Conclusion

The length biased truncated Lomax distribution with one parameter is a novel weighted distribution proposed in this work. The ordinary moments, central moments, mean, skewness, kurtosis, moment generating function, incomplete moment, Lorenz and Bonferroni curves, quantile function, probability weighted moments, and order statistics are all investigated as structural features of the new distribution.

Four different types of entropy are computed. Some numerical values of moments and the four different types of entropy for various values for the parameter α are calculated. We use censoring sampling scheme to estimate the population parameter using maximum likelihood approach. We run a Monte Carlo simulation to see how the maximum likelihood estimates behave in finite samples. From the simulation study we note that the MSEs and average length of parameter estimate decrease with increasing sample size. The behaviour of parameter estimate improves with increasing the censoring level r. As sample size gets larger, the CI becomes smaller, and the length of the confidence interval increases as confidence levels get bigger. The significance and potentiality of the proposed model are empirically demonstrated using two real data sets.

Appendix: List of Abbreviations

Abbreviation	Definition
AIC	Akaike information criterion
AL	Average length
ArE	Arimoto entropy
B	Beta
BI	Beta with one parameter
BIC	Bayesian information criterion
CAIC	Correct Akaike information criterion
cdf	Cumulative distribution function
CI	Confidence interval
CK	Coefficient of kurtosis
CS	Coefficient of skewness
CV	Coefficient of variation
HaCE	Havrda and Charvat entropy
HIECS	Household expenditure from the household income, expenditure and consumption survey
HQIC	Hannan-Quinn information criterion
hrf	Hazard rate function
KS	Kolmogorov–Smirnov
Kum	Kumaraswamy
LB	Length-biased
LB	lower bound
LBTLo	Length biased truncated Lomax
Lo	Lomax
ML	Maximum likelihood
MLEs	ML estimates
MSEs	Mean squared errors
pdf	Probability density function
PV	P-value
RE	R´enyi entropy
RV	Random variable
sf	Survival function
SE	Standard error
T2C	Type II censoring
TLo	Truncated Lomax
TPLo	Truncated power Lomax
TsE	Tsallis entropy
TTT	Total time test
TWe	Truncated Weibull
UB	Upper bound
V	variance
WD	Weighted distribution

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