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Equilibrium stacking in finite uniform approximation of 3-person games played with staircase-function strategies

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Abstract A method of finite uniform approximation of 3-person games played with staircase-function strategies is presented. A continuous staircase 3-person game is approximated to a staircase trimatrix game by sampling the player's pure strategy value set. The set is sampled uniformly so that the resulting staircase trimatrix game is cubic. An equilibrium of the staircase trimatrix game is obtained by stacking the equilibria of the subinterval trimatrix games, each defined on an interval where the pure strategy value is constant. The stack is an approximate solution to the initial staircase game. The (weak) consistency, equivalent to the approximate solution acceptability, is studied by how much the players' payoff and equilibrium strategy change as the sampling density minimally increases. The consistency includes the payoff, equilibrium strategy support cardinality, equilibrium strategy sampling density, and support probability consistency. The most important parts are the payoff consistency and equilibrium strategy support cardinality (weak) consistency, which are checked in the quickest and easiest way. However, it is practically reasonable to consider a relaxed payoff consistency, by which the player's payoff change in an appropriate approximation may grow at most by ε as the sampling density minimally increases. The weak consistency itself is a relaxation to the consistency, where the minimal decrement of the sampling density is ignored. An example is presented to show how the approximation is fulfilled for a case of when every subinterval trimatrix game has pure strategy equilibria.

Keywords Game Theory, Payoff Functional, Staircase-Function Strategy, Trimatrix Game, Approximate Equilibrium Consistency, Equilibrium Stacking

AMS 2010 subject classifications 91A06, 91A10, 91A50, 18F20, 65D99, 41A99

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1. Introduction

In rationalizing the distribution of limited resources, 3-person noncooperative games are applied as often as a game model representing two sides becomes impossible or inconsistent [15, 17, 23, 25]. Although trimatrix games are the simplest 3-person games, equilibrium points in such games are not always determinable. An infinite or continuous 3-person game may not even have an equilibrium. Nevertheless, the best choice is to approximate an infinite or continuous game to a finite one, which always has an equilibrium [17, 18, 9, 24].

A possible action of the player called a pure strategy can be as a simple (point) action, as well as a process consisting of an order of simple actions. In the simplest case, the player's pure strategy is a short action (operation, move, maneuver, etc.) whose duration is negligible and thus is represented as just a (time) point. In a far more complicated case, the player's pure strategy is a function of time [21, 20, 16, 10], so the player's action is a complex process [6, 8, 26, 19].

Trimatrix games played with point strategies have been studied much deeper compared to 3-person games played with function-strategies. This is frankly explained by that a trimatrix game is finite, whereas a 3-person

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game played with function-strategies must be infinite or continuous (unless every player has a finite set of one's function-strategies). A pure strategy situation consisting of three function-strategies each defined on a time interval (or, generally speaking, on a finite measure set) is mapped into a real value by a player's functional [22, 10, 16]. The real value is the player's payoff [21, 10, 16, 20]. If each of the players is allowed to use just a finite set of one's function-strategies, the 3-person game can be rendered down to a trimatrix game, whichever the complexity of the functionals is [17, 11, 12].

2. Motivation to finite approximation of 3-person games

To render a 3-person game with strategies as (time) functions down to a trimatrix game, there are two fundamental conditions: discretization of time and the set of possible values of every player's function-strategy. The first fundamental condition can be presumed to be naturally fulfilled due to any process is interpreted static on a sufficiently short time span [1, 7, 25, 26]. Then a time interval, on which the pure strategy is defined, is naturally broken into a set of subintervals, on which the strategy is (approximately) constant. Moreover, the system to be game-modeled may be managed so that the form of the strategies players will use is forcedly defined. An example of this is the problem of rationalizing industrial wastewater treatment modeled by a dyadic 3-person game, in which the system is managed by a government [15]. In such a game-model set-up, points of discontinuities (breakpoints) being the same for all the players (subjects of environmental pollution) are defined by the government.

With the time discretization, the second fundamental condition allows to have finite sets of the players' pure strategies. While the players may use strategies of whichever form they want, the number of their factual actions has a natural limit in any non-everlasting game [2, 7, 13, 14, 25, 26]. Thus, the set of function-strategies used in a 3-person game is finite anyway, and any (non-everlasting) 3-person game is played as if it is a trimatrix game.

To approximate a continuous game means obtaining its simplified version whose exact solution would be an approximate but still acceptable solution to the initial game [3, 12, 16]. The simplified game is usually finite. The continuous game approximation is based on sampling the sets of players' pure strategies [11, 12, 16]. An approximate solution (e. g., an equilibrium situation) is considered acceptable if it changes minimally by changing the sampling step minimally. The solution change is a complex notion involving the solution payoff, the solution strategy support cardinality, the support density, and the closeness of the solution strategies.

Obviously, this method cannot be applied straightforwardly to approximate a 3-person game with staircase-function strategies. However, on every time subinterval the players' strategies are constant, so the game on this subinterval can be directly approximated by the method. It remains only to properly "glue" together the subinterval approximations.

3. Objective and tasks to be fulfilled

Issued from the impossibility of solving 3-person games played with staircase-functions strategies, the objective is to develop a method of finite approximation of such games. For achieving the objective, the following tasks are to be fulfilled:

1. To formalize a 3-person game, in which the players' strategies are functions of time.
2. To formalize a 3-person game, in which the players' strategies are staircase functions. In such a game, the set of the player's pure strategies is a continuum of staircase functions of (discrete) time.
3. To state conditions of sampling the set of possible values of the player's pure strategy so that the game be defined on a product of staircase-function finite spaces.
4. To state conditions of the appropriate finite approximation.
5. To discuss applicability and significance of the method for the game theory and to make an appropriate conclusion on it.

The paper proceeds as follows. Section 4 introduces a 3-person game with strategies as functions. A 3-person game with staircase-function strategies is formalized in Section 5. Section 6 describes how the pure strategy value axis is sampled. The question of whether an approximate solution can be accepted or not is answered in Section 7.

A visual exemplification is presented in Section 8 showing a carefully reasoned analysis of the approximation. The study is discussed and concluded in the last two sections.

4. A 3-person game with strategies as functions

In a 3-person game, in which the player's pure strategy is a function of time, denote a strategy of the first, second, and third players by $x(t)$, $y(t)$, and $z(t)$, respectively. Let these function-strategies be defined almost everywhere on interval $[t_1;\ t_2]$ by $t_2 > t_1$. These function-strategies are presumed to be bounded, i. e.

$$a_{\min} \leqslant x(t) \leqslant a_{\max} \text{ by } a_{\min} < a_{\max}, \tag{1}$$

$$b_{\min} \leqslant y(t) \leqslant b_{\max} \text{ by } b_{\min} < b_{\max}, \tag{2}$$

and

$$c_{\min} \leqslant z(t) \leqslant c_{\max} \text{ by } c_{\min} < c_{\max}. \tag{3}$$

Besides, the square of the function-strategy is presumed to be Lebesgue-integrable. Thus, pure strategies of the player belong to a rectangular functional space of functions of time:

$$X = \{x(t),\ t \in [t_1;\ t_2],\ t_1 < t_2 :\ a_{\min} \leqslant x(t) \leqslant a_{\max} \text{ by } a_{\min} < a_{\max}\} \subset \mathbb{L}_2[t_1;\ t_2], \tag{4}$$

$$Y = \{y(t),\ t \in [t_1;\ t_2],\ t_1 < t_2 :\ b_{\min} \leqslant y(t) \leqslant b_{\max} \text{ by } b_{\min} < b_{\max}\} \subset \mathbb{L}_2[t_1;\ t_2], \tag{5}$$

and

$$Z = \{z(t),\ t \in [t_1;\ t_2],\ t_1 < t_2 :\ c_{\min} \leqslant z(t) \leqslant c_{\max} \text{ by } c_{\min} < c_{\max}\} \subset \mathbb{L}_2[t_1;\ t_2] \tag{6}$$

are the sets of the players' pure strategies, respectively.

The player's payoff in situation

$$\{x(t),\ y(t),\ z(t)\} \tag{7}$$

is presumed to be an integral functional [10, 16]. Thus, the first, second, and third players' payoffs are

$$F\big(x(t),\ y(t),\ z(t)\big) = \int\limits_{[t_1;\ t_2]} f\big(x(t),\ y(t),\ z(t),\ t\big) d\mu(t), \tag{8}$$

$$G\big(x(t),\ y(t),\ z(t)\big) = \int\limits_{[t_1;\ t_2]} g\big(x(t),\ y(t),\ z(t),\ t\big) d\mu(t), \tag{9}$$

$$H\big(x(t),\ y(t),\ z(t)\big) = \int\limits_{[t_1;\ t_2]} h\big(x(t),\ y(t),\ z(t),\ t\big) d\mu(t), \tag{10}$$

respectively, where

$$f\big(x(t),\ y(t),\ z(t),\ t\big), \tag{11}$$

$$g\big(x(t),\ y(t),\ z(t),\ t\big), \tag{12}$$

$$h\big(x(t),\ y(t),\ z(t),\ t\big) \tag{13}$$

are functions of $x(t)$, $y(t)$, $z(t)$, explicitly including time t. Therefore, the continuous 3-person game

$$\Big\langle \{X,\ Y,\ Z\},\ \Big\{F\big(x(t),\ y(t),\ z(t)\big),\ G\big(x(t),\ y(t),\ z(t)\big),\ H\big(x(t),\ y(t),\ z(t)\big)\Big\}\Big\rangle \tag{14}$$

with function-strategies is defined on product

$$X \times Y \times Z \subset \mathbb{L}_2[t_1;\ t_2] \times \mathbb{L}_2[t_1;\ t_2] \times \mathbb{L}_2[t_1;\ t_2] \tag{15}$$

of rectangular functional spaces (4) — (6) of players' pure strategies. In practical reality, 3-person game (14) with strategies as functions is presumed to be played discretely through time interval $[t_1;\ t_2]$. Then a function-strategy becomes staircase. The number of subintervals at which the player's pure strategy is constant must be the same for every player. This is defined by the (physical, economical, biological, social, etc.) laws of a system modeled by the game.

5. A 3-person game with staircase-function strategies

So, denote by N the number of subintervals at which the player's pure strategy is constant, where $N \in \mathbb{N} \backslash \{1\}$. Then the player's pure strategy is a staircase function having only at most N different values. Let $\left\{\tau^{(i)}\right\}_{i=1}^{N-1}$ be the time points at which the staircase-function strategy changes its value, where

$$t_1 = \tau^{(0)} < \tau^{(1)} < \tau^{(2)} < \ldots < \tau^{(N-1)} < \tau^{(N)} = t_2. \tag{16}$$

The time interval breaking by (16) is the same for every player. This is naturally defined by the laws of the system. Then

$$\left\{x\left(\tau^{(i)}\right)\right\}_{i=0}^{N},\ \left\{y\left(\tau^{(i)}\right)\right\}_{i=0}^{N},\ \left\{z\left(\tau^{(i)}\right)\right\}_{i=0}^{N} \tag{17}$$

are the values of the players' strategies in a play-off of game (14). Obviously, points $\left\{\tau^{(i)}\right\}_{i=0}^{N}$ are not necessarily to be equidistant.

The staircase-function strategies are right-continuous [4]:

$$\lim_{\substack{\varepsilon>0 \\ \varepsilon\to 0}} x\left(\tau^{(i)}+\varepsilon\right) = x\left(\tau^{(i)}\right), \tag{18}$$

$$\lim_{\substack{\varepsilon>0 \\ \varepsilon\to 0}} y\left(\tau^{(i)}+\varepsilon\right) = y\left(\tau^{(i)}\right), \tag{19}$$

$$\lim_{\substack{\varepsilon>0 \\ \varepsilon\to 0}} z\left(\tau^{(i)}+\varepsilon\right) = z\left(\tau^{(i)}\right), \tag{20}$$

for $i = \overline{1,\ N-1}$, whereas

$$\lim_{\substack{\varepsilon>0 \\ \varepsilon\to 0}} x\left(\tau^{(i)}-\varepsilon\right) \neq x\left(\tau^{(i)}\right), \tag{21}$$

$$\lim_{\substack{\varepsilon>0\\ \varepsilon\to 0}} y\left(\tau^{(i)}-\varepsilon\right) \neq y\left(\tau^{(i)}\right), \tag{22}$$

$$\lim_{\substack{\varepsilon>0\\ \varepsilon\to 0}} z\left(\tau^{(i)}-\varepsilon\right) \neq z\left(\tau^{(i)}\right), \tag{23}$$

for $i=\overline{1,\ N-1}$. It is easy to see that a strategy value on subinterval $\left[\tau^{(N-1)};\ \tau^{(N)}\right]$ should not change, i. e.

$$x\left(\tau^{(N-1)}\right) = x\left(\tau^{(N)}\right),$$

$$y\left(\tau^{(N-1)}\right) = y\left(\tau^{(N)}\right),$$

$$z\left(\tau^{(N-1)}\right) = z\left(\tau^{(N)}\right).$$

So,

$$\lim_{\substack{\varepsilon>0\\ \varepsilon\to 0}} x\left(\tau^{(N)}-\varepsilon\right) = x\left(\tau^{(N)}\right), \tag{24}$$

$$\lim_{\substack{\varepsilon>0\\ \varepsilon\to 0}} y\left(\tau^{(N)}-\varepsilon\right) = y\left(\tau^{(N)}\right), \tag{25}$$

$$\lim_{\substack{\varepsilon>0\\ \varepsilon\to 0}} z\left(\tau^{(N)}-\varepsilon\right) = z\left(\tau^{(N)}\right). \tag{26}$$

Constant values (17) by (16) mean that game (14) can be thought of as it is a succession of N continuous 3-person games

$$\left\langle \left\{[a_{\min};\ a_{\max}],\ [b_{\min};\ b_{\max}],\ [c_{\min};\ c_{\max}]\right\},\ \left\{F\left(\alpha_i,\ \beta_i,\ \gamma_i\right),\ G\left(\alpha_i,\ \beta_i,\ \gamma_i\right),\ H\left(\alpha_i,\ \beta_i,\ \gamma_i\right)\right\}\right\rangle \tag{27}$$

defined on parallelepiped

$$[a_{\min};\ a_{\max}] \times [b_{\min};\ b_{\max}] \times [c_{\min};\ c_{\max}] \tag{28}$$

by

$$\begin{gathered}\alpha_i = x(t) \in [a_{\min};\ a_{\max}] \ \text{ and } \ \beta_i = y(t) \in [b_{\min};\ b_{\max}] \ \text{ and } \ \gamma_i = z(t) \in [c_{\min};\ c_{\max}] \\ \forall\, t \in \left[\tau^{(i-1)};\ \tau^{(i)}\right) \ \text{ for } \ i=\overline{1,\ N-1} \ \text{ and } \ \forall\, t \in \left[\tau^{(N-1)};\ \tau^{(N)}\right],\end{gathered} \tag{29}$$

where the factual players' payoffs in situation $\{\alpha_i,\ \beta_i,\ \gamma_i\}$ are

$$F\left(\alpha_i,\ \beta_i,\ \gamma_i\right) = \int\limits_{\left[\tau^{(i-1)};\ \tau^{(i)}\right)} f\left(\alpha_i,\ \beta_i,\ \gamma_i,\ t\right) d\mu(t) \ \ \forall\, i=\overline{1,\ N-1} \tag{30}$$

by

$$F\left(\alpha_N,\ \beta_N,\ \gamma_N\right) = \int\limits_{\left[\tau^{(N-1)};\ \tau^{(N)}\right]} f\left(\alpha_N,\ \beta_N,\ \gamma_N,\ t\right) d\mu(t), \tag{31}$$

$$G(\alpha_i,\ \beta_i,\ \gamma_i) = \int\limits_{\left[\tau^{(i-1)};\ \tau^{(i)}\right)} g(\alpha_i,\ \beta_i,\ \gamma_i,\ t)\, d\mu(t)\ \ \forall\, i = \overline{1,\ N-1} \tag{32}$$

by

$$G(\alpha_N,\ \beta_N,\ \gamma_N) = \int\limits_{\left[\tau^{(N-1)};\ \tau^{(N)}\right]} g(\alpha_N,\ \beta_N,\ \gamma_N,\ t)\, d\mu(t), \tag{33}$$

and

$$H(\alpha_i,\ \beta_i,\ \gamma_i) = \int\limits_{\left[\tau^{(i-1)};\ \tau^{(i)}\right)} h(\alpha_i,\ \beta_i,\ \gamma_i,\ t)\, d\mu(t)\ \ \forall\, i = \overline{1,\ N-1} \tag{34}$$

by

$$H(\alpha_N,\ \beta_N,\ \gamma_N) = \int\limits_{\left[\tau^{(N-1)};\ \tau^{(N)}\right]} h(\alpha_N,\ \beta_N,\ \gamma_N,\ t)\, d\mu(t). \tag{35}$$

Thus, game (14) with staircase-function strategies can be called staircase. A pure-strategy situation in the staircase game (14) is a succession of N situations $\left\{\{\alpha_i,\ \beta_i,\ \gamma_i\}\right\}_{i=1}^{N}$ in games (27), where each situation corresponds to its subinterval. The succession allows considering players' payoffs in situation (7) in a simpler form.

Theorem 1
In a pure-strategy situation of the staircase game (14), represented as a succession of N games (27), functionals (8) — (10) are re-written as subinterval-wise sums

$$F\big(x(t),\ y(t),\ z(t)\big) = \sum_{i=1}^{N} F(\alpha_i,\ \beta_i,\ \gamma_i) =$$
$$= \sum_{i=1}^{N-1} \int\limits_{\left[\tau^{(i-1)};\ \tau^{(i)}\right)} f(\alpha_i,\ \beta_i,\ \gamma_i,\ t)\, d\mu(t) + \int\limits_{\left[\tau^{(N-1)};\ \tau^{(N)}\right]} f(\alpha_N,\ \beta_N,\ \gamma_N,\ t)\, d\mu(t), \tag{36}$$

$$G\big(x(t),\ y(t),\ z(t)\big) = \sum_{i=1}^{N} G(\alpha_i,\ \beta_i,\ \gamma_i) =$$
$$= \sum_{i=1}^{N-1} \int\limits_{\left[\tau^{(i-1)};\ \tau^{(i)}\right)} g(\alpha_i,\ \beta_i,\ \gamma_i,\ t)\, d\mu(t) + \int\limits_{\left[\tau^{(N-1)};\ \tau^{(N)}\right]} g(\alpha_N,\ \beta_N,\ \gamma_N,\ t)\, d\mu(t), \tag{37}$$

$$H\big(x(t),\ y(t),\ z(t)\big) = \sum_{i=1}^{N} H(\alpha_i,\ \beta_i,\ \gamma_i) =$$
$$= \sum_{i=1}^{N-1} \int\limits_{\left[\tau^{(i-1)};\ \tau^{(i)}\right)} h(\alpha_i,\ \beta_i,\ \gamma_i,\ t)\, d\mu(t) + \int\limits_{\left[\tau^{(N-1)};\ \tau^{(N)}\right]} h(\alpha_N,\ \beta_N,\ \gamma_N,\ t)\, d\mu(t). \tag{38}$$

Proof
Situation $\{\alpha_i,\ \beta_i,\ \gamma_i\}$ is tied to half-subinterval $\left[\tau^{(i-1)};\ \tau^{(i)}\right)$ by $i=\overline{1,\ N-1}$ and to subinterval $\left[\tau^{(N-1)};\ \tau^{(N)}\right]$ by $i=N$. Each of functions (11)—(13) in this situation is some function of time t. Denote a function corresponding to (11) by $\psi_i\left(t\right)$. For situation $\{\alpha_i,\ \beta_i,\ \gamma_i\}$ function

$$\psi_i\left(t\right)=0\ \ \forall\, t\notin\left[\tau^{(i-1)};\ \tau^{(i)}\right), \tag{39}$$

and for situation $\{\alpha_N,\ \beta_N,\ \gamma_N\}$ function

$$\psi_N\left(t\right)=0\ \ \forall\, t\notin\left[\tau^{(N-1)};\ \tau^{(N)}\right]. \tag{40}$$

Therefore,

$$f\left(x\left(t\right),\ y\left(t\right),\ z\left(t\right),\ t\right)=\sum_{i=1}^{N}\psi_i\left(t\right) \tag{41}$$

in a pure-strategy situation (7) of the staircase game (14), by using (39) and (40). Consequently,

$$\begin{gathered}
F\big(x\left(t\right),\ y\left(t\right),\ z\left(t\right)\big)=\int\limits_{[t_1;\ t_2]}f\big(x\left(t\right),\ y\left(t\right),\ z\left(t\right),\ t\big)d\mu\left(t\right)=\\
=\sum_{i=1}^{N-1}\int\limits_{\left[\tau^{(i-1)};\ \tau^{(i)}\right)}\psi_i\left(t\right)d\mu\left(t\right)+\int\limits_{\left[\tau^{(N-1)};\ \tau^{(N)}\right]}\psi_N\left(t\right)d\mu\left(t\right)=\\
=\sum_{i=1}^{N-1}\int\limits_{\left[\tau^{(i-1)};\ \tau^{(i)}\right)}f\left(\alpha_i,\ \beta_i,\ \gamma_i,\ t\right)d\mu\left(t\right)+\int\limits_{\left[\tau^{(N-1)};\ \tau^{(N)}\right]}f\left(\alpha_N,\ \beta_N,\ \gamma_N,\ t\right)d\mu\left(t\right)=\\
=\sum_{i=1}^{N}F\left(\alpha_i,\ \beta_i,\ \gamma_i\right)
\end{gathered} \tag{42}$$

in a pure-strategy situation (7) of the staircase game (14). Obviously, subinterval-wise sums (37) and (38) are proved similarly to (39)—(42). □

Although Theorem 1 does not provide a method of solving the 3-person staircase game, it provides a fundamental decomposition of the game. By this decomposition each subinterval game (27) can be solved separately. The solutions of subinterval games are then stacked (stitched) together.

6. Sampling along the pure strategy value axis

In a classical 3-person game (27), the players have their sets of pure strategies $[a_{\min};\ a_{\max}]$, $[b_{\min};\ b_{\max}]$, and $[c_{\min};\ c_{\max}]$. Let these sets be sampled uniformly with a step determined by an integer S, $S\in\mathbb{N}$. So,

$$A\left(S\right)=\left\{a^{(s)}\right\}_{s=1}^{S+1}=\left\{a_{\min}+\frac{s-1}{S}\cdot\left(a_{\max}-a_{\min}\right)\right\}_{s=1}^{S+1}\subset[a_{\min};\ a_{\max}] \tag{43}$$

and

$$B\left(S\right)=\left\{b^{(s)}\right\}_{s=1}^{S+1}=\left\{b_{\min}+\frac{s-1}{S}\cdot\left(b_{\max}-b_{\min}\right)\right\}_{s=1}^{S+1}\subset[b_{\min};\ b_{\max}] \tag{44}$$

and

$$C(S)=\left\{c^{(s)}\right\}_{s=1}^{S+1}=\left\{c_{\min}+\frac{s-1}{S}\cdot\left(c_{\max}-c_{\min}\right)\right\}_{s=1}^{S+1}\subset\left[c_{\min};\;c_{\max}\right] \tag{45}$$

are the sampled pure strategy sets of the first, second, and third players, respectively. The roughest sampling is by $S=1$, when

$$A(1)=\left\{a^{(1)},\;a^{(2)}\right\}=\left\{a_{\min},\;a_{\max}\right\} \tag{46}$$

and

$$B(1)=\left\{b^{(1)},\;b^{(2)}\right\}=\left\{b_{\min},\;b_{\max}\right\} \tag{47}$$

and

$$C(1)=\left\{c^{(1)},\;c^{(2)}\right\}=\left\{c_{\min},\;c_{\max}\right\}. \tag{48}$$

With the sampling by (43)—(45), the succession of N continuous 3-person games (27) by (16)—(26) and (29)—(35) becomes a succession of N trimatrix $(S+1)\times(S+1)\times(S+1)$ games

$$\left\langle\left\{\left\{a^{(m)}\right\}_{m=1}^{S+1},\;\left\{b^{(j)}\right\}_{j=1}^{S+1},\;\left\{c^{(q)}\right\}_{q=1}^{S+1}\right\},\;\left\{\mathbf{F}_i(S),\;\mathbf{G}_i(S),\;\mathbf{H}_i(S)\right\}\right\rangle \tag{49}$$

with first player's 3-dimensional payoff matrices

$$\mathbf{F}_i(S)=\left[\varphi_{imjq}(S)\right]_{(S+1)\times(S+1)\times(S+1)} \tag{50}$$

whose elements are

$$\varphi_{imjq}(S)=\int\limits_{\left[\tau^{(i-1)};\;\tau^{(i)}\right)} f\left(a^{(m)},\;b^{(j)},\;c^{(q)},\;t\right)d\mu(t)\;\;\text{for}\;\;i=\overline{1,\;N-1} \tag{51}$$

and

$$\varphi_{Nmjq}(S)=\int\limits_{\left[\tau^{(N-1)};\;\tau^{(N)}\right]} f\left(a^{(m)},\;b^{(j)},\;c^{(q)},\;t\right)d\mu(t), \tag{52}$$

the second player's 3-dimensional payoff matrices

$$\mathbf{G}_i(S)=\left[\rho_{imjq}(S)\right]_{(S+1)\times(S+1)\times(S+1)} \tag{53}$$

whose elements are

$$\rho_{imjq}(S)=\int\limits_{\left[\tau^{(i-1)};\;\tau^{(i)}\right)} g\left(a^{(m)},\;b^{(j)},\;c^{(q)},\;t\right)d\mu(t)\;\;\text{for}\;\;i=\overline{1,\;N-1} \tag{54}$$

and

$$\rho_{Nmjq}(S)=\int\limits_{\left[\tau^{(N-1)};\;\tau^{(N)}\right]} g\left(a^{(m)},\;b^{(j)},\;c^{(q)},\;t\right)d\mu(t), \tag{55}$$

and the third player's 3-dimensional payoff matrices

$$\mathbf{H}_i(S) = [\theta_{imjq}(S)]_{(S+1)\times(S+1)\times(S+1)} \tag{56}$$

whose elements are

$$\theta_{imjq}(S) = \int\limits_{\left[\tau^{(i-1)};\, \tau^{(i)}\right)} h\left(a^{(m)},\, b^{(j)},\, c^{(q)},\, t\right) d\mu(t) \;\; \text{for} \;\; i = \overline{1,\, N-1} \tag{57}$$

and

$$\theta_{Nmjq}(S) = \int\limits_{\left[\tau^{(N-1)};\, \tau^{(N)}\right]} h\left(a^{(m)},\, b^{(j)},\, c^{(q)},\, t\right) d\mu(t). \tag{58}$$

Note that with the sampling by (46) — (48), there is a succession of 3-person dyadic games [15, 18].

So, if integer S for game (14) by condition (29) is somehow selected, the continuous staircase game is approximated and represented as a succession of N trimatrix $(S+1)\times(S+1)\times(S+1)$ games. It is well-known that a finite 3-person game always has an equilibrium either in pure or mixed strategies. Thus the game is rendered to a trimatrix game in order to obtain a staircase equilibrium. However, there is a much easier way to solve a finite staircase 3-person game.

Theorem 2
Game (14) on product (15) of rectangular functional spaces (4) — (6) made a finite staircase game by condition (29) and sampling (43) — (45) is always solved as a stack of successive equilibria of N trimatrix games (49) by (50) — (58). The player's payoff in the stacked equilibrium is the sum of the respective subinterval equilibrium payoffs.

Proof
An equilibrium situation in the trimatrix game always exists, either in pure or mixed strategies. Denote by

$$\mathbf{U}_i(S) = \left[u_i^{(m)}(S)\right]_{1\times(S+1)}$$

and

$$\mathbf{Z}_i(S) = \left[z_i^{(j)}(S)\right]_{1\times(S+1)}$$

and

$$\mathbf{W}_i(S) = \left[w_i^{(q)}(S)\right]_{1\times(S+1)}$$

the mixed strategies of the first, second, and third players, respectively, in trimatrix game (49). The respective sets of mixed strategies of the players are

$$\mathcal{U} = \left\{\mathbf{U}_i(S) \in \mathbb{R}^{S+1} : u_i^{(m)}(S) \geqslant 0,\; \sum_{m=1}^{S+1} u_i^{(m)}(S) = 1\right\} \tag{59}$$

and

$$\mathcal{Z} = \left\{\mathbf{Z}_i(S) \in \mathbb{R}^{S+1} : z_i^{(j)}(S) \geqslant 0,\; \sum_{j=1}^{S+1} z_i^{(j)}(S) = 1\right\} \tag{60}$$

and

$$\mathcal{W} = \left\{\mathbf{W}_i(S) \in \mathbb{R}^{S+1} : w_i^{(q)}(S) \geqslant 0,\; \sum_{q=1}^{S+1} w_i^{(q)}(S) = 1\right\}, \tag{61}$$

so

$$\mathbf{U}_i(S) \in \mathcal{U},$$
$$\mathbf{Z}_i(S) \in \mathcal{Z},$$
$$\mathbf{W}_i(S) \in \mathcal{W},$$

and

$$\{\mathbf{U}_i(S),\ \mathbf{Z}_i(S),\ \mathbf{W}_i(S)\} \tag{62}$$

is a situation in game (49). Let

$$\{\mathbf{U}_i^*(S),\ \mathbf{Z}_i^*(S),\ \mathbf{W}_i^*(S)\} = \left\{ \left[u_i^{(m)*}(S)\right]_{1\times(S+1)},\ \left[z_i^{(j)*}(S)\right]_{1\times(S+1)},\ \left[w_i^{(q)*}(S)\right]_{1\times(S+1)} \right\} \tag{63}$$

by

$$\mathbf{U}_i^*(S) \in \mathcal{U},$$
$$\mathbf{Z}_i^*(S) \in \mathcal{Z},$$
$$\mathbf{W}_i^*(S) \in \mathcal{W}$$

be an equilibrium situation in game (49). In situation (63) the players receive their payoffs

$$\{p_i^*(S),\ r_i^*(S),\ v_i^*(S)\}. \tag{64}$$

Then

$$\left\{\{\mathbf{U}_i^*(S),\ \mathbf{Z}_i^*(S),\ \mathbf{W}_i^*(S)\}\right\}_{i=1}^N =$$
$$= \left\{ \left\{ \left[u_i^{(m)*}(S)\right]_{1\times(S+1)},\ \left[z_i^{(j)*}(S)\right]_{1\times(S+1)},\ \left[w_i^{(q)*}(S)\right]_{1\times(S+1)} \right\} \right\}_{i=1}^N \tag{65}$$

is a set of equilibrium situations of N games (49) by (50) — (58). So, the stack of equilibria

$$\{\mathbf{U}_i^*(S)\}_{i=1}^N = \left\{ \left[u_i^{(m)*}(S)\right]_{1\times(S+1)} \right\}_{i=1}^N \tag{66}$$

is a stacked ("staircase") strategy of the first player in the staircase game (14). The equilibria

$$\{\mathbf{Z}_i^*(S)\}_{i=1}^N = \left\{ \left[z_i^{(j)*}(S)\right]_{1\times(S+1)} \right\}_{i=1}^N \tag{67}$$

are stacked likewise into a "staircase" strategy of the second player, and

$$\{\mathbf{W}_i^*(S)\}_{i=1}^N = \left\{ \left[w_i^{(q)*}(S)\right]_{1\times(S+1)} \right\}_{i=1}^N \tag{68}$$

is a "staircase" strategy of the third player in this game. Then for the set of equilibria (65), inequalities

$$\sum_{m=1}^{S+1}\sum_{j=1}^{S+1}\sum_{q=1}^{S+1} \varphi_{imjq}(S)\, u_i^{(m)}(S)\, z_i^{(j)*}(S)\, w_i^{(q)*}(S) =$$
$$= \sum_{m=1}^{S+1}\sum_{j=1}^{S+1}\sum_{q=1}^{S+1} u_i^{(m)}(S)\, z_i^{(j)*}(S)\, w_i^{(q)*}(S) \int\limits_{\left[\tau^{(i-1)};\ \tau^{(i)}\right)} f\left(a^{(m)},\ b^{(j)},\ c^{(q)},\ t\right) d\mu(t) \leqslant$$

$$\leqslant \sum_{m=1}^{S+1} \sum_{j=1}^{S+1} \sum_{q=1}^{S+1} u_i^{(m)*}(S)\, z_i^{(j)*}(S)\, w_i^{(q)*}(S) \int\limits_{\left[\tau^{(i-1)};\, \tau^{(i)}\right)} f\left(a^{(m)},\ b^{(j)},\ c^{(q)},\ t\right) d\mu(t) =$$

$$= \sum_{m=1}^{S+1} \sum_{j=1}^{S+1} \sum_{q=1}^{S+1} \varphi_{imjq}(S)\, u_i^{(m)*}(S)\, z_i^{(j)*}(S)\, w_i^{(q)*}(S) = p_i^*(S) \quad \forall\, \mathbf{U}_i(S) \in \mathcal{U} \ \text{ for } \ i = \overline{1,\ N-1}, \tag{69}$$

$$\sum_{m=1}^{S+1} \sum_{j=1}^{S+1} \sum_{q=1}^{S+1} \varphi_{Nmjq}(S)\, u_N^{(m)}(S)\, z_N^{(j)*}(S)\, w_N^{(q)*}(S) =$$

$$= \sum_{m=1}^{S+1} \sum_{j=1}^{S+1} \sum_{q=1}^{S+1} u_N^{(m)}(S)\, z_N^{(j)*}(S)\, w_N^{(q)*}(S) \int\limits_{\left[\tau^{(N-1)};\, \tau^{(N)}\right]} f\left(a^{(m)},\ b^{(j)},\ c^{(q)},\ t\right) d\mu(t) \leqslant$$

$$\leqslant \sum_{m=1}^{S+1} \sum_{j=1}^{S+1} \sum_{q=1}^{S+1} u_N^{(m)*}(S)\, z_N^{(j)*}(S)\, w_N^{(q)*}(S) \int\limits_{\left[\tau^{(N-1)};\, \tau^{(N)}\right]} f\left(a^{(m)},\ b^{(j)},\ c^{(q)},\ t\right) d\mu(t) =$$

$$= \sum_{m=1}^{S+1} \sum_{j=1}^{S+1} \sum_{q=1}^{S+1} \varphi_{Nmjq}(S)\, u_N^{(m)*}(S)\, z_N^{(j)*}(S)\, w_N^{(q)*}(S) = p_N^*(S) \quad \forall\, \mathbf{U}_N(S) \in \mathcal{U} \tag{70}$$

and inequalities

$$\sum_{m=1}^{S+1} \sum_{j=1}^{S+1} \sum_{q=1}^{S+1} \rho_{imjq}(S)\, u_i^{(m)*}(S)\, z_i^{(j)}(S)\, w_i^{(q)*}(S) =$$

$$= \sum_{m=1}^{S+1} \sum_{j=1}^{S+1} \sum_{q=1}^{S+1} u_i^{(m)*}(S)\, z_i^{(j)}(S)\, w_i^{(q)*}(S) \int\limits_{\left[\tau^{(i-1)};\, \tau^{(i)}\right)} g\left(a^{(m)},\ b^{(j)},\ c^{(q)},\ t\right) d\mu(t) \leqslant$$

$$\leqslant \sum_{m=1}^{S+1} \sum_{j=1}^{S+1} \sum_{q=1}^{S+1} u_i^{(m)*}(S)\, z_i^{(j)*}(S)\, w_i^{(q)*}(S) \int\limits_{\left[\tau^{(i-1)};\, \tau^{(i)}\right)} g\left(a^{(m)},\ b^{(j)},\ c^{(q)},\ t\right) d\mu(t) =$$

$$= \sum_{m=1}^{S+1} \sum_{j=1}^{S+1} \sum_{q=1}^{S+1} \rho_{imjq}(S)\, u_i^{(m)*}(S)\, z_i^{(j)*}(S)\, w_i^{(q)*}(S) = r_i^*(S) \quad \forall\, \mathbf{Z}_i(S) \in \mathcal{Z} \ \text{ for } \ i = \overline{1,\ N-1}, \tag{71}$$

$$\sum_{m=1}^{S+1} \sum_{j=1}^{S+1} \sum_{q=1}^{S+1} \rho_{Nmjq}(S)\, u_N^{(m)*}(S)\, z_N^{(j)}(S)\, w_N^{(q)*}(S) =$$

$$= \sum_{m=1}^{S+1} \sum_{j=1}^{S+1} \sum_{q=1}^{S+1} u_N^{(m)*}(S)\, z_N^{(j)}(S)\, w_N^{(q)*}(S) \int\limits_{\left[\tau^{(N-1)};\, \tau^{(N)}\right]} g\left(a^{(m)},\ b^{(j)},\ c^{(q)},\ t\right) d\mu(t) \leqslant$$

$$\leqslant \sum_{m=1}^{S+1} \sum_{j=1}^{S+1} \sum_{q=1}^{S+1} u_N^{(m)*}(S)\, z_N^{(j)*}(S)\, w_N^{(q)*}(S) \int\limits_{\left[\tau^{(N-1)};\, \tau^{(N)}\right]} g\left(a^{(m)},\ b^{(j)},\ c^{(q)},\ t\right) d\mu(t) =$$

$$= \sum_{m=1}^{S+1} \sum_{j=1}^{S+1} \sum_{q=1}^{S+1} \rho_{Nmjq}(S)\, u_N^{(m)*}(S)\, z_N^{(j)*}(S)\, w_N^{(q)*}(S) = r_N^*(S) \quad \forall\, \mathbf{Z}_N(S) \in \mathcal{Z} \tag{72}$$

and inequalities

$$\sum_{m=1}^{S+1}\sum_{j=1}^{S+1}\sum_{q=1}^{S+1}\theta_{imjq}(S)\,u_i^{(m)*}(S)\,z_i^{(j)*}(S)\,w_i^{(q)}(S)=$$
$$=\sum_{m=1}^{S+1}\sum_{j=1}^{S+1}\sum_{q=1}^{S+1}u_i^{(m)*}(S)\,z_i^{(j)*}(S)\,w_i^{(q)}(S)\int\limits_{\left[\tau^{(i-1)};\ \tau^{(i)}\right)}h\left(a^{(m)},\ b^{(j)},\ c^{(q)},\ t\right)d\mu(t)\leqslant$$
$$\leqslant\sum_{m=1}^{S+1}\sum_{j=1}^{S+1}\sum_{q=1}^{S+1}u_i^{(m)*}(S)\,z_i^{(j)*}(S)\,w_i^{(q)*}(S)\int\limits_{\left[\tau^{(i-1)};\ \tau^{(i)}\right)}h\left(a^{(m)},\ b^{(j)},\ c^{(q)},\ t\right)d\mu(t)=$$
$$=\sum_{m=1}^{S+1}\sum_{j=1}^{S+1}\sum_{q=1}^{S+1}\theta_{imjq}(S)\,u_i^{(m)*}(S)\,z_i^{(j)*}(S)\,w_i^{(q)*}(S)=v_i^*(S)\quad\forall\,\mathbf{W}_i(S)\in\mathcal{W}\ \text{ for }\ i=\overline{1,\ N-1},\tag{73}$$

$$\sum_{m=1}^{S+1}\sum_{j=1}^{S+1}\sum_{q=1}^{S+1}\theta_{Nmjq}(S)\,u_N^{(m)*}(S)\,z_N^{(j)*}(S)\,w_N^{(q)}(S)=$$
$$=\sum_{m=1}^{S+1}\sum_{j=1}^{S+1}\sum_{q=1}^{S+1}u_N^{(m)*}(S)\,z_N^{(j)*}(S)\,w_N^{(q)}(S)\int\limits_{\left[\tau^{(N-1)};\ \tau^{(N)}\right]}h\left(a^{(m)},\ b^{(j)},\ c^{(q)},\ t\right)d\mu(t)\leqslant$$
$$\leqslant\sum_{m=1}^{S+1}\sum_{j=1}^{S+1}\sum_{q=1}^{S+1}u_N^{(m)*}(S)\,z_N^{(j)*}(S)\,w_N^{(q)*}(S)\int\limits_{\left[\tau^{(N-1)};\ \tau^{(N)}\right]}h\left(a^{(m)},\ b^{(j)},\ c^{(q)},\ t\right)d\mu(t)=$$
$$=\sum_{m=1}^{S+1}\sum_{j=1}^{S+1}\sum_{q=1}^{S+1}\theta_{Nmjq}(S)\,u_N^{(m)*}(S)\,z_N^{(j)*}(S)\,w_N^{(q)*}(S)=v_N^*(S)\quad\text{and}\ \ \forall\,\mathbf{W}_N(S)\in\mathcal{W}\tag{74}$$

hold. So, inequalities

$$\sum_{i=1}^{N-1}\sum_{m=1}^{S+1}\sum_{j=1}^{S+1}\sum_{q=1}^{S+1}\varphi_{imjq}(S)\,u_i^{(m)}(S)\,z_i^{(j)*}(S)\,w_i^{(q)*}(S)+$$
$$+\sum_{m=1}^{S+1}\sum_{j=1}^{S+1}\sum_{q=1}^{S+1}\varphi_{Nmjq}(S)\,u_N^{(m)}(S)\,z_N^{(j)*}(S)\,w_N^{(q)*}(S)=$$
$$=\sum_{i=1}^{N-1}\left(\sum_{m=1}^{S+1}\sum_{j=1}^{S+1}\sum_{q=1}^{S+1}u_i^{(m)}(S)\,z_i^{(j)*}(S)\,w_i^{(q)*}(S)\int\limits_{\left[\tau^{(i-1)};\ \tau^{(i)}\right)}f\left(a^{(m)},\ b^{(j)},\ c^{(q)},\ t\right)d\mu(t)\right)+$$
$$+\sum_{m=1}^{S+1}\sum_{j=1}^{S+1}\sum_{q=1}^{S+1}u_N^{(m)}(S)\,z_N^{(j)*}(S)\,w_i^{(q)*}(S)\int\limits_{\left[\tau^{(N-1)};\ \tau^{(N)}\right]}f\left(a^{(m)},\ b^{(j)},\ c^{(q)},\ t\right)d\mu(t)\leqslant$$
$$\leqslant\sum_{i=1}^{N-1}\left(\sum_{m=1}^{S+1}\sum_{j=1}^{S+1}\sum_{q=1}^{S+1}u_i^{(m)*}(S)\,z_i^{(j)*}(S)\,w_i^{(q)*}(S)\int\limits_{\left[\tau^{(i-1)};\ \tau^{(i)}\right)}f\left(a^{(m)},\ b^{(j)},\ c^{(q)},\ t\right)d\mu(t)\right)+$$

$$+\sum_{m=1}^{S+1}\sum_{j=1}^{S+1}\sum_{q=1}^{S+1} u_N^{(m)*}(S)\, z_N^{(j)*}(S)\, w_i^{(q)*}(S) \int\limits_{\left[\tau^{(N-1)};\ \tau^{(N)}\right]} f\left(a^{(m)},\ b^{(j)},\ c^{(q)},\ t\right) d\mu(t) =$$

$$= \sum_{i=1}^{N-1}\sum_{m=1}^{S+1}\sum_{j=1}^{S+1}\sum_{q=1}^{S+1} \varphi_{imjq}(S)\, u_i^{(m)*}(S)\, z_i^{(j)*}(S)\, w_i^{(q)*}(S) +$$

$$+\sum_{m=1}^{S+1}\sum_{j=1}^{S+1}\sum_{q=1}^{S+1} \varphi_{Nmjq}(S)\, u_N^{(m)*}(S)\, z_N^{(j)*}(S)\, w_N^{(q)*}(S) = \sum_{i=1}^{N} p_i^*(S) = p^*(S) \tag{75}$$

and

$$\sum_{i=1}^{N-1}\sum_{m=1}^{S+1}\sum_{j=1}^{S+1}\sum_{q=1}^{S+1} \rho_{imjq}(S)\, u_i^{(m)*}(S)\, z_i^{(j)}(S)\, w_i^{(q)*}(S) +$$

$$+\sum_{m=1}^{S+1}\sum_{j=1}^{S+1}\sum_{q=1}^{S+1} \rho_{Nmjq}(S)\, u_N^{(m)*}(S)\, z_N^{(j)}(S)\, w_N^{(q)*}(S) =$$

$$= \sum_{i=1}^{N-1}\left(\sum_{m=1}^{S+1}\sum_{j=1}^{S+1}\sum_{q=1}^{S+1} u_i^{(m)*}(S)\, z_i^{(j)}(S)\, w_i^{(q)*}(S) \int\limits_{\left[\tau^{(i-1)};\ \tau^{(i)}\right)} g\left(a^{(m)},\ b^{(j)},\ c^{(q)},\ t\right) d\mu(t)\right) +$$

$$+\sum_{m=1}^{S+1}\sum_{j=1}^{S+1}\sum_{q=1}^{S+1} u_N^{(m)*}(S)\, z_N^{(j)}(S)\, w_i^{(q)*}(S) \int\limits_{\left[\tau^{(N-1)};\ \tau^{(N)}\right]} g\left(a^{(m)},\ b^{(j)},\ c^{(q)},\ t\right) d\mu(t) \leqslant$$

$$\leqslant \sum_{i=1}^{N-1}\left(\sum_{m=1}^{S+1}\sum_{j=1}^{S+1}\sum_{q=1}^{S+1} u_i^{(m)*}(S)\, z_i^{(j)*}(S)\, w_i^{(q)*}(S) \int\limits_{\left[\tau^{(i-1)};\ \tau^{(i)}\right)} g\left(a^{(m)},\ b^{(j)},\ c^{(q)},\ t\right) d\mu(t)\right) +$$

$$+\sum_{m=1}^{S+1}\sum_{j=1}^{S+1}\sum_{q=1}^{S+1} u_N^{(m)*}(S)\, z_N^{(j)*}(S)\, w_i^{(q)*}(S) \int\limits_{\left[\tau^{(N-1)};\ \tau^{(N)}\right]} g\left(a^{(m)},\ b^{(j)},\ c^{(q)},\ t\right) d\mu(t) =$$

$$= \sum_{i=1}^{N-1}\sum_{m=1}^{S+1}\sum_{j=1}^{S+1}\sum_{q=1}^{S+1} \rho_{imjq}(S)\, u_i^{(m)*}(S)\, z_i^{(j)*}(S)\, w_i^{(q)*}(S) +$$

$$+\sum_{m=1}^{S+1}\sum_{j=1}^{S+1}\sum_{q=1}^{S+1} \rho_{Nmjq}(S)\, u_N^{(m)*}(S)\, z_N^{(j)*}(S)\, w_N^{(q)*}(S) = \sum_{i=1}^{N} r_i^*(S) = r^*(S) \tag{76}$$

and

$$\sum_{i=1}^{N-1}\sum_{m=1}^{S+1}\sum_{j=1}^{S+1}\sum_{q=1}^{S+1} \theta_{imjq}(S)\, u_i^{(m)*}(S)\, z_i^{(j)*}(S)\, w_i^{(q)}(S) +$$

$$+\sum_{m=1}^{S+1}\sum_{j=1}^{S+1}\sum_{q=1}^{S+1} \theta_{Nmjq}(S)\, u_N^{(m)*}(S)\, z_N^{(j)*}(S)\, w_N^{(q)}(S) =$$

$$= \sum_{i=1}^{N-1}\left(\sum_{m=1}^{S+1}\sum_{j=1}^{S+1}\sum_{q=1}^{S+1} u_i^{(m)*}(S)\, z_i^{(j)*}(S)\, w_i^{(q)}(S) \int\limits_{\left[\tau^{(i-1)};\ \tau^{(i)}\right)} h\left(a^{(m)},\ b^{(j)},\ c^{(q)},\ t\right) d\mu(t)\right) +$$

$$+\sum_{m=1}^{S+1}\sum_{j=1}^{S+1}\sum_{q=1}^{S+1} u_N^{(m)*}(S)\, z_N^{(j)*}(S)\, w_i^{(q)}(S) \int\limits_{\left[\tau^{(N-1)};\; \tau^{(N)}\right]} h\left(a^{(m)},\, b^{(j)},\, c^{(q)},\, t\right) d\mu(t) \leqslant$$

$$\leqslant \sum_{i=1}^{N-1}\left(\sum_{m=1}^{S+1}\sum_{j=1}^{S+1}\sum_{q=1}^{S+1} u_i^{(m)*}(S)\, z_i^{(j)*}(S)\, w_i^{(q)*}(S) \int\limits_{\left[\tau^{(i-1)};\; \tau^{(i)}\right)} h\left(a^{(m)},\, b^{(j)},\, c^{(q)},\, t\right) d\mu(t)\right) +$$

$$+\sum_{m=1}^{S+1}\sum_{j=1}^{S+1}\sum_{q=1}^{S+1} u_N^{(m)*}(S)\, z_N^{(j)*}(S)\, w_i^{(q)*}(S) \int\limits_{\left[\tau^{(N-1)};\; \tau^{(N)}\right]} h\left(a^{(m)},\, b^{(j)},\, c^{(q)},\, t\right) d\mu(t) =$$

$$= \sum_{i=1}^{N-1}\sum_{m=1}^{S+1}\sum_{j=1}^{S+1}\sum_{q=1}^{S+1} \theta_{imjq}(S)\, u_i^{(m)*}(S)\, z_i^{(j)*}(S)\, w_i^{(q)*}(S) +$$

$$+\sum_{m=1}^{S+1}\sum_{j=1}^{S+1}\sum_{q=1}^{S+1} \theta_{Nmjq}(S)\, u_N^{(m)*}(S)\, z_N^{(j)*}(S)\, w_N^{(q)*}(S) = \sum_{i=1}^{N} v_i^*(S) = v^*(S) \tag{77}$$

hold as well. Therefore, inequalities (75) — (77) along with using Theorem 1 allow to conclude that the stack of successive equilibria (65) is an equilibrium situation in game (14) by (29) sampled by (43) — (45). The players' payoffs

$$\{p^*(S),\; r^*(S),\; v^*(S)\} \tag{78}$$

in this situation are calculated as the sum of the respective subinterval equilibrium payoffs as it is done in the right side of inequalities (75) — (77). □

As the solutions of the subinterval trimatrix games are independent, these games are solved in parallel, without caring of the succession. Once N equilibrium situations in the subinterval trimatrix games are found, they are successively stacked (stitched) and the stack, according to Theorem 2, is an equilibrium in the staircase game (14) sampled by (43) — (45).

In general case, the sampling along the pure strategy value axis can be non-uniform, and each player can have its own number of the sampled points (it is $S+1$ in the above-considered proposition, whereas S is the number of pure strategy value intervals). Nevertheless, this is quite specific case, so it is not considered now.

7. Approximate equilibrium solution consistency

Obviously, integer S for approximating game (14) by condition (29) cannot be selected arbitrarily. The conditions of the appropriate finite approximation can be stated by using the known method of obtaining the approximate solution of a continuous game [16, 12]. There are four groups of the conditions, whereas the requirement of the smooth sampling of the payoff functionals is inapplicable here [16].

An easy-to-find condition of the finite approximation appropriateness is the equilibrium payoff change:

$$|p_i^*(S) - p_i^*(S+1)| \leqslant |p_i^*(S-1) - p_i^*(S)| \quad \text{for} \quad i = \overline{1,\; N} \tag{79}$$

and

$$|r_i^*(S) - r_i^*(S+1)| \leqslant |r_i^*(S-1) - r_i^*(S)| \quad \text{for} \quad i = \overline{1,\; N} \tag{80}$$

and

$$|v_i^*(S) - v_i^*(S+1)| \leqslant |v_i^*(S-1) - v_i^*(S)| \quad \text{for} \quad i = \overline{1,\; N}. \tag{81}$$

Each of conditions (79) — (81) means that, as the sampling density minimally increases, the player's equilibrium payoff change in an appropriate approximation should not grow. This is the basic requirement to an approximate equilibrium.

Definition 1
The stack of successive equilibrium situations (65) is called payoff-S-consistent if inequalities (79) — (81) hold. Stack (66) of equilibrium strategies is called first-player-payoff-S-consistent if inequalities (79) hold. Stack (67) of equilibrium strategies is called second-player-payoff-S-consistent if inequalities (80) hold. Stack (68) of equilibrium strategies is called third-player-payoff-S-consistent if inequalities (81) hold.

It is clear that if each of stacks (66) — (68) is payoff-S-consistent for the respective player, then the stack of successive equilibrium situations (65) is payoff-S-consistent. Nevertheless, even the basic requirement may be unfeasible. Indeed, among those $3N$ inequalities just a few ones may be violated so that the difference between the left and right absolute values is very small (insignificant). Or, they nearly all may be violated, but the differences are insignificantly (negligibly) small. Then, it is useful and practically reasonable to consider the payoff consistency adding a relaxation.

Definition 2
An approximate equilibrium (65) from stacks (66) — (68) is called ε-payoff-S-consistent if inequalities

$$|p_i^*(S) - p_i^*(S+1)| - \varepsilon \leqslant |p_i^*(S-1) - p_i^*(S)| \text{ by some } \varepsilon > 0 \text{ for } i = \overline{1,\ N} \tag{82}$$

and

$$|r_i^*(S) - r_i^*(S+1)| - \varepsilon \leqslant |r_i^*(S-1) - r_i^*(S)| \text{ by some } \varepsilon > 0 \text{ for } i = \overline{1,\ N} \tag{83}$$

and

$$|v_i^*(S) - v_i^*(S+1)| - \varepsilon \leqslant |v_i^*(S-1) - v_i^*(S)| \text{ by some } \varepsilon > 0 \text{ for } i = \overline{1,\ N} \tag{84}$$

hold. Stack (66) of equilibrium strategies is called first-player-ε-payoff-S-consistent if inequalities (82) hold. Stack (67) of equilibrium strategies is called second-player-ε-payoff-S-consistent if inequalities (83) hold. Stack (68) of equilibrium strategies is called third-player-ε-payoff-S-consistent if inequalities (84) hold.

The next to payoff consistency condition is the change of the equilibrium strategy support cardinality. Denote the supports of the equilibrium strategies of the players by

$$\operatorname{supp} \mathbf{U}_i^*(S) = \{m_u\}_{u=1}^{U_i(S)} \subset \{m\}_{m=1}^{S+1} \tag{85}$$

by the respective support probabilities

$$\left\{u_i^{(m_u)*}(S)\right\}_{u=1}^{U_i(S)} \subset \left\{u_i^{(m)*}(S)\right\}_{m=1}^{S+1} \tag{86}$$

and

$$\operatorname{supp} \mathbf{Z}_i^*(S) = \{j_z\}_{z=1}^{Z_i(S)} \subset \{j\}_{j=1}^{S+1} \tag{87}$$

by the respective support probabilities

$$\left\{z_i^{(j_z)*}(S)\right\}_{z=1}^{Z_i(S)} \subset \left\{z_i^{(j)*}(S)\right\}_{j=1}^{S+1} \tag{88}$$

and

$$\operatorname{supp} \mathbf{W}_i^*(S) = \{q_w\}_{w=1}^{W_i(S)} \subset \{q\}_{q=1}^{S+1} \tag{89}$$

by the respective support probabilities

$$\left\{w_i^{(q_w)*}\left(S\right)\right\}_{w=1}^{W_i(S)} \subset \left\{w_i^{(q)*}\left(S\right)\right\}_{q=1}^{S+1}. \tag{90}$$

Then inequalities

$$U_i\left(S+1\right) \geqslant U_i\left(S\right) \quad \text{for} \quad i=\overline{1,\ N} \tag{91}$$

and

$$Z_i\left(S+1\right) \geqslant Z_i\left(S\right) \quad \text{for} \quad i=\overline{1,\ N} \tag{92}$$

and

$$W_i\left(S+1\right) \geqslant W_i\left(S\right) \quad \text{for} \quad i=\overline{1,\ N} \tag{93}$$

require that, by minimally increasing the sampling density, the cardinalities of the supports not decrease.

Definition 3
An approximate equilibrium (65) from stacks (66) — (68) is called weakly support-cardinality-S-consistent if inequalities (91) — (93) hold. Support (85) is called weakly first-player-support-cardinality-S-consistent if inequalities (91) hold. Support (87) is called weakly second-player-support-cardinality-S-consistent if inequalities (92) hold. Support (89) is called weakly third-player-support-cardinality-S-consistent if inequalities (93) hold.

Obviously, requirements (91) — (93) can be supplemented (strengthened) by considering a minimal decrement of the sampling density. Then inequalities

$$U_i\left(S\right) \geqslant U_i\left(S-1\right) \quad \text{for} \quad i=\overline{1,\ N} \tag{94}$$

and

$$Z_i\left(S\right) \geqslant Z_i\left(S-1\right) \quad \text{for} \quad i=\overline{1,\ N} \tag{95}$$

and

$$W_i\left(S\right) \geqslant W_i\left(S-1\right) \quad \text{for} \quad i=\overline{1,\ N} \tag{96}$$

are required.

Definition 4
An approximate equilibrium (65) from stacks (66) — (68) is called support-cardinality-S-consistent if inequalities (91) — (96) hold. Support (85) is called first-player-support-cardinality-S-consistent if inequalities (91) and (94) hold. Support (87) is called second-player-support-cardinality-S-consistent if inequalities (92) and (95) hold. Support (89) is called third-player-support-cardinality-S-consistent if inequalities (93) and (96) hold.

Just as Definition 2 may be thought of as it is a "relaxation" of Definition 1 for the payoff consistency, Definition 3 is a "relaxed" (weak) version of the support cardinality consistency by Definition 4. Obviously, if each of stacks (66) — (68) is (weakly) support-cardinality-S-consistent for the respective player, then the stack of successive equilibrium situations (65) is (weakly) support-cardinality-S-consistent.

The third group of the conditions is the support index distance, which determines the support density defined by the sampling density by integer S. As the sampling density minimally increases, the maximal gap between the support indices should not increase. Let $m_u\left(S\right)$ and $j_z\left(S\right)$ and $q_w\left(S\right)$ be the respective support indices corresponding to integer S on a subinterval by (29). Then inequalities

$$\max_{u=\overline{1,\ U_i(S+1)-1}} \left[m_{u+1}\left(S+1\right) - m_u\left(S+1\right)\right] \leqslant \max_{u=\overline{1,\ U_i(S)-1}} \left[m_{u+1}\left(S\right) - m_u\left(S\right)\right] \quad \text{for} \quad i=\overline{1,\ N} \tag{97}$$

and

$$\max_{z=\overline{1,\ Z_i(S+1)-1}} \left[j_{z+1}\left(S+1\right)-j_z\left(S+1\right)\right] \leqslant \max_{z=\overline{1,\ Z_i(S)-1}} \left[j_{z+1}\left(S\right)-j_z\left(S\right)\right] \quad \text{for} \quad i=\overline{1,\ N} \tag{98}$$

and

$$\max_{w=\overline{1,\ W_i(S+1)-1}} \left[q_{w+1}\left(S+1\right)-q_w\left(S+1\right)\right] \leqslant \max_{w=\overline{1,\ W_i(S)-1}} \left[q_{w+1}\left(S\right)-q_w\left(S\right)\right] \quad \text{for} \quad i=\overline{1,\ N} \tag{99}$$

are required.

Definition 5
An approximate equilibrium (65) from stacks (66) — (68) is called weakly sampling-density-S-consistent if inequalities (97) — (99) hold. Stack (66) is called weakly first-player-sampling-density-S-consistent if inequalities (97) hold. Stack (67) is called weakly second-player-sampling-density-S-consistent if inequalities (98) hold. Stack (68) is called weakly third-player-sampling-density-S-consistent if inequalities (99) hold.

Just like in the case of the support cardinality consistency, requirements (97) — (99) can be supplemented (strengthened) by considering a minimal decrement of the sampling density. Then inequalities

$$\max_{u=\overline{1,\ U_i(S)-1}} \left[m_{u+1}\left(S\right)-m_u\left(S\right)\right] \leqslant \max_{u=\overline{1,\ U_i(S-1)-1}} \left[m_{u+1}\left(S-1\right)-m_u\left(S-1\right)\right] \quad \text{for} \quad i=\overline{1,\ N} \tag{100}$$

and

$$\max_{z=\overline{1,\ Z_i(S)-1}} \left[j_{z+1}\left(S\right)-j_z\left(S\right)\right] \leqslant \max_{z=\overline{1,\ Z_i(S-1)-1}} \left[j_{z+1}\left(S-1\right)-j_z\left(S-1\right)\right] \quad \text{for} \quad i=\overline{1,\ N} \tag{101}$$

and

$$\max_{w=\overline{1,\ W_i(S)-1}} \left[q_{w+1}\left(S\right)-q_w\left(S\right)\right] \leqslant \max_{w=\overline{1,\ W_i(S-1)-1}} \left[q_{w+1}\left(S-1\right)-q_w\left(S-1\right)\right] \quad \text{for} \quad i=\overline{1,\ N} \tag{102}$$

are required.

Definition 6
An approximate equilibrium (65) from stacks (66) — (68) is called sampling-density-S-consistent if inequalities (97) — (102) hold. Stack (66) is called first-player-sampling-density-S-consistent if inequalities (97) and (100) hold. Stack (67) is called second-player-sampling-density-S-consistent if inequalities (98) and (101) hold. Stack (68) is called third-player-sampling-density-S-consistent if inequalities (99) and (102) hold.

Clearly, Definition 5 is a "relaxed" (weak) version of the sampling density consistency by Definition 6. Besides, these definitions are applicable to a stack whose equilibria are not of pure strategies. If each of stacks (66) — (68) is (weakly) sampling-density-S-consistent for the respective player, then the stack of successive equilibrium situations (65) is (weakly) sampling-density-S-consistent. It is worth noting that the support index distance condition is always violated if a mixed equilibrium strategy is of just two nonzero probabilities. For instance, if

$$\operatorname{supp} \mathbf{U}_i^*\left(S\right)=\left\{m_u\right\}_{u=1}^{2} \subset \left\{m\right\}_{m=1}^{S+1}$$

by support probabilities

$$\left\{u_i^{(m_u)*}\left(S\right)\right\}_{u=1}^{2}=\left\{u_i^{(1)*}\left(S\right),\ u_i^{(S+1)*}\left(S\right)\right\}$$

and

$$\operatorname{supp} \mathbf{U}_i^*\left(S+1\right)=\left\{m_u\right\}_{u=1}^{2} \subset \left\{m\right\}_{m=1}^{S+2}$$

by support probabilities

$$\left\{u_i^{(m_u)*}\left(S+1\right)\right\}_{u=1}^{2}=\left\{u_i^{(1)*}\left(S+1\right),\ u_i^{(S+2)*}\left(S+1\right)\right\}$$

for some i, then the i-th inequality in (97) is violated because

$$\max_{u=\overline{1,\ U_i(S+1)-1}} [m_{u+1}(S+1) - m_u(S+1)] = (S+2) - 1 = S+1$$

and

$$\max_{u=\overline{1,\ U_i(S)-1}} [m_{u+1}(S) - m_u(S)] = (S+1) - 1 = S.$$

At the same time, a support still can be support-cardinality-S-consistent in this case as

$$U_i(S+1) = 2 \geqslant 2 = U_i(S).$$

So, the support index distance condition is far more fragile than the support cardinality condition.

The fourth group of the conditions is the closeness of the equilibrium strategies (as polyline functions) by the sampling density minimally increment. Denote by $\mu_1(i;\ m,\ S)$ a polyline whose vertices are probabilities $\left\{u_i^{(m)*}(S)\right\}_{m=1}^{S+1}$, by $\mu_2(i;\ j,\ S)$ a polyline whose vertices are probabilities $\left\{z_i^{(j)*}(S)\right\}_{j=1}^{S+1}$, and by $\mu_3(i;\ q,\ S)$ a polyline whose vertices are probabilities $\left\{w_i^{(q)*}(S)\right\}_{q=1}^{S+1}$. Then, by minimally increasing the sampling density, the "neighboring" polylines should not be farther from each other, i. e.

$$\max_{[0;\ 1]} |\mu_1(i;\ m,\ S) - \mu_1(i;\ m,\ S+1)| \leqslant \max_{[0;\ 1]} |\mu_1(i;\ m,\ S-1) - \mu_1(i;\ m,\ S)| \quad \text{for} \quad i = \overline{1,\ N} \tag{103}$$

and

$$\max_{[0;\ 1]} |\mu_2(i;\ j,\ S) - \mu_2(i;\ j,\ S+1)| \leqslant \max_{[0;\ 1]} |\mu_2(i;\ j,\ S-1) - \mu_2(i;\ j,\ S)| \quad \text{for} \quad i = \overline{1,\ N} \tag{104}$$

and

$$\max_{[0;\ 1]} |\mu_3(i;\ q,\ S) - \mu_3(i;\ q,\ S+1)| \leqslant \max_{[0;\ 1]} |\mu_3(i;\ q,\ S-1) - \mu_3(i;\ q,\ S)| \quad \text{for} \quad i = \overline{1,\ N} \tag{105}$$

along with

$$\|\mu_1(i;\ m,\ S) - \mu_1(i;\ m,\ S+1)\| \leqslant \|\mu_1(i;\ m,\ S-1) - \mu_1(i;\ m,\ S)\| \quad \text{in} \quad \mathbb{L}_2[0;\ 1] \quad \text{for} \quad i = \overline{1,\ N} \tag{106}$$

and

$$\|\mu_2(i;\ j,\ S) - \mu_2(i;\ j,\ S+1)\| \leqslant \|\mu_2(i;\ j,\ S-1) - \mu_2(i;\ j,\ S)\| \quad \text{in} \quad \mathbb{L}_2[0;\ 1] \quad \text{for} \quad i = \overline{1,\ N} \tag{107}$$

and

$$\|\mu_3(i;\ q,\ S) - \mu_3(i;\ q,\ S+1)\| \leqslant \|\mu_3(i;\ q,\ S-1) - \mu_3(i;\ q,\ S)\| \quad \text{in} \quad \mathbb{L}_2[0;\ 1] \quad \text{for} \quad i = \overline{1,\ N}. \tag{108}$$

Definition 7
An approximate equilibrium (65) from stacks (66) — (68) is called probability-S-consistent if inequalities (103) — (108) hold. Stack (66) is called first-player-probability-S-consistent if inequalities (103) and (106) hold. Stack (67) is called second-player-probability-S-consistent if inequalities (104) and (107) hold. Stack (68) is called third-player-probability-S-consistent if inequalities (105) and (108) hold.

If inequalities (79) — (81) and (91) — (108) hold for some i, then trimatrix game (49), assigned to the subinterval between $\tau^{(i-1)}$ and $\tau^{(i)}$, has a consistent approximate solution to the corresponding continuous game (27) by (29) of the equilibrium type. If inequalities (79) — (81), (91) — (93), (97) — (99), and (103) — (108) hold for some i, the consistency is weak. On this basis, the (weak) consistency of an approximate equilibrium solution to a staircase game (14) is formulated.

Definition 8
The stack of successive equilibrium situations (65) is called a weakly S-consistent approximate equilibrium solution of the continuous staircase 3-person game (14) if inequalities (79) — (81), (91) — (93), (97) — (99), and (103) — (108) hold.

Obviously, a weakly S-consistent approximate equilibrium solution is strengthened if inequalities (94) — (96) and (100) — (102) strengthening the support cardinality and sampling density consistency hold as well. Although it is a rare case in practice, but it is nonetheless worth to define it.

Definition 9
The stack of successive equilibrium situations (65) is called an S-consistent approximate equilibrium solution of the continuous staircase 3-person game (14) if inequalities (79) — (81) and (91) — (108) hold.

Clearly, an S-consistent approximate equilibrium solution is weakly consistent. Theoretically, the approximate solution consistency proposes a better approximation than the weak consistency. However, it is a quite rare case when an equilibrium (even in a classical continuous 3-person game, with ordinary point strategies) appears to be even weakly consistent. Therefore, it is more reasonable to consider not the (weak) consistency "stack" by (Definition 8) Definition 9 but its components by Definitions 1 — 7. Even if an approximate equilibrium is not weakly consistent, it may be, e. g., payoff-consistent or ε-payoff-S-consistent (by a tolerated ε). Such ε might be thought of as a payoff consistency concession [11, 12, 14, 15, 20, 25]. This can be sufficient to accept it as an appropriate approximate solution to the staircase game (14).

To ascertain whether the stack of successive equilibrium situations (65) is weakly consistent or not, the three bunches of N trimatrix games (49) should be solved, where the sampling density is defined by integers $S-1$, S, $S+1$. Nevertheless, the consistency meant by some sampling density integer S does not guarantee that every player will select such sampling density. Moreover, a trimatrix game may have more than just one equilibrium situation. So, every player first decides on one's payoff consistency, where inequalities (82) — (84) are checked by some ε (its selection is not a trivial task also). If a player's equilibria stack is ε-payoff-S-consistent, the respective stack may be accepted as an approximate equilibrium solution to the staircase game (14). An approximate solution may be accepted in a stronger sense, if it is ε-payoff-S-consistent for any $S \geqslant S_*$ starting from some $S_* \in \mathbb{N}$ by a tolerated ε. This is a fundamental step in the finite approximation.

8. A visual exemplification

Consider a case in which $t \in [0.12\pi;\ 0.2\pi]$, the sets of pure strategies of the players are

$$X = \{x(t),\ t \in [0.12\pi;\ 0.2\pi]:\ 5 \leqslant x(t) \leqslant 6\} \subset \mathbb{L}_2[0.12\pi;\ 0.2\pi] \tag{109}$$

and

$$Y = \{y(t),\ t \in [0.12\pi;\ 0.2\pi]:\ 2 \leqslant y(t) \leqslant 4\} \subset \mathbb{L}_2[0.12\pi;\ 0.2\pi] \tag{110}$$

and

$$Z = \{z(t),\ t \in [0.12\pi;\ 0.2\pi]:\ 1 \leqslant z(t) \leqslant 2.6\} \subset \mathbb{L}_2[0.12\pi;\ 0.2\pi], \tag{111}$$

respectively. Each of the players is allowed to change its pure strategy value at time points

$$\left\{\tau^{(i)}\right\}_{i=1}^{7} = \{0.12\pi + 0.01\pi i\}_{i=1}^{7} \subset [0.12\pi;\ 0.2\pi]. \tag{112}$$

So, the player possesses 8-subinterval staircase function-strategies defined on interval $[0.12\pi;\ 0.2\pi]$. The players' payoff functionals are

$$F\big(x(t),\ y(t),\ z(t)\big) = \int\limits_{[0.12\pi;\ 0.2\pi]} z(t) \cdot \sin\left(0.45t \cdot x(t)\, y(t) - \frac{\pi}{9}\right) d\mu(t), \tag{113}$$

$$G\big(x\,(t)\,,\ y\,(t)\,,\ z\,(t)\big) = \int\limits_{[0.12\pi;\ 0.2\pi]} y\,(t) \cdot \cos\left(0.2t \cdot x\,(t)\,z\,(t) + \frac{\pi}{6}\right) d\mu\,(t), \tag{114}$$

$$H\big(x\,(t)\,,\ y\,(t)\,,\ z\,(t)\big) = \int\limits_{[0.12\pi;\ 0.2\pi]} x\,(t) \cdot \sin\left(0.75t \cdot y\,(t)\,z\,(t) - \frac{\pi}{5}\right) d\mu\,(t). \tag{115}$$

Hence, the 3-person staircase game is represented as a succession of eight continuous 3-person games each defined on a subinterval of set

$$\left\{ \left\{ [0.11\pi + 0.01\pi i;\ 0.12\pi + 0.01\pi i) \right\}_{i=1}^{7},\ [0.19\pi;\ 0.2\pi] \right\}. \tag{116}$$

For these eight games, with the sampling by (43) — (45), the pure strategy sets of the players' are

$$A\,(S) = \left\{ a^{(s)} \right\}_{s=1}^{S+1} = \left\{ 5 + \frac{s-1}{S} \right\}_{s=1}^{S+1} \subset [5;\ 6] \tag{117}$$

and

$$B\,(S) = \left\{ b^{(s)} \right\}_{s=1}^{S+1} = \left\{ 2 + \frac{2s-2}{S} \right\}_{s=1}^{S+1} \subset [2;\ 4] \tag{118}$$

and

$$C\,(S) = \left\{ c^{(s)} \right\}_{s=1}^{S+1} = \left\{ 1 + \frac{1.6s-1.6}{S} \right\}_{s=1}^{S+1} \subset [1;\ 2.6]\,, \tag{119}$$

respectively. Sets (117) — (119) for each of the subintervals in (116) imply the succession of eight trimatrix games

$$\left\langle \left\{ \left\{ a^{(m)} \right\}_{m=1}^{S+1},\ \left\{ b^{(j)} \right\}_{j=1}^{S+1},\ \left\{ c^{(q)} \right\}_{q=1}^{S+1} \right\},\ \left\{ \mathbf{F}_i\,(S)\,,\ \mathbf{G}_i\,(S)\,,\ \mathbf{H}_i\,(S) \right\} \right\rangle =$$
$$= \left\langle \left\{ \left\{ 5 + \frac{m-1}{S} \right\}_{m=1}^{S+1},\ \left\{ 2 + \frac{2j-2}{S} \right\}_{j=1}^{S+1},\ \left\{ 1 + \frac{1.6q-1.6}{S} \right\}_{q=1}^{S+1} \right\},\ \left\{ \mathbf{F}_i\,(S)\,,\ \mathbf{G}_i\,(S)\,,\ \mathbf{H}_i\,(S) \right\} \right\rangle$$
$$\text{for}\ \ i = \overline{1,\ 8}. \tag{120}$$

In games (120), there are eight first player's payoff matrices (50) whose elements are

$$\varphi_{imjq}\,(S) = \int\limits_{[0.11\pi+0.01\pi i;\ 0.12\pi+0.01\pi i)} c^{(q)} \cdot \sin\left(0.45a^{(m)}b^{(j)}t - \frac{\pi}{9}\right) d\mu\,(t)\ \ \text{for}\ \ i = \overline{1,\ 7} \tag{121}$$

and

$$\varphi_{8mjq}\,(S) = \int\limits_{[0.19\pi;\ 0.2\pi]} c^{(q)} \cdot \sin\left(0.45a^{(m)}b^{(j)}t - \frac{\pi}{9}\right) d\mu\,(t), \tag{122}$$

there are eight second player's payoff matrices (53) whose elements are

$$\rho_{imjq}\,(S) = \int\limits_{[0.11\pi+0.01\pi i;\ 0.12\pi+0.01\pi i)} b^{(j)} \cdot \cos\left(0.2a^{(m)}c^{(q)}t + \frac{\pi}{6}\right) d\mu\,(t)\ \ \text{for}\ \ i = \overline{1,\ 7} \tag{123}$$

and

$$\rho_{8mjq}(S) = \int\limits_{[0.19\pi;\ 0.2\pi]} b^{(j)} \cdot \cos\left(0.2a^{(m)}c^{(q)}t + \frac{\pi}{6}\right) d\mu(t), \tag{124}$$

and there are eight third player's payoff matrices (56) whose elements are

$$\theta_{imjq}(S) = \int\limits_{[0.11\pi+0.01\pi i;\ 0.12\pi+0.01\pi i)} a^{(m)} \cdot \sin\left(0.75b^{(j)}c^{(q)}t - \frac{\pi}{5}\right) d\mu(t) \ \text{ for } \ i = \overline{1,\ 7} \tag{125}$$

and

$$\theta_{8mjq}(S) = \int\limits_{[0.19\pi;\ 0.2\pi]} a^{(m)} \cdot \sin\left(0.75b^{(j)}c^{(q)}t - \frac{\pi}{5}\right) d\mu(t). \tag{126}$$

The sectional views of the first player's payoffs $F(\alpha_i,\ 2,\ \gamma_i)$ and $F(\alpha_i,\ 4,\ \gamma_i)$ on each subinterval of set (116) are shown in Figure 1 for $\beta_i = 2$ and $\beta_i = 4$. It is clear that hypersurfaces $F(\alpha_i,\ \beta_i,\ \gamma_i)$ on open parallelepiped

$$(5;\ 6) \times (2;\ 4) \times (1;\ 2.6) \tag{127}$$

do not have many extremums. The sectional views of the second player's payoffs $G(5,\ \beta_i,\ \gamma_i)$ and $G(6,\ \beta_i,\ \gamma_i)$ shown on each subinterval of set (116) in Figure 2 for $\alpha_i = 5$ and $\alpha_i = 6$ are roughly similar. The maximum at $\beta_i = 4$ and $\gamma_i = 1$ by any α_i is quite noticeable. As the time progresses, the maximum and its vicinity do not change. The sectional views of the third player's payoffs $H(5,\ \beta_i,\ \gamma_i)$ and $H(6,\ \beta_i,\ \gamma_i)$ on each subinterval of set (116) are shown in Figure 3 for $\alpha_i = 5$ and $\alpha_i = 6$. Hypersurfaces $H(\alpha_i,\ \beta_i,\ \gamma_i)$ on parallelepiped (127) "appear" to be humpy-like but still they are roughly similar.

At $S = 2$ the finite 3-person staircase game has a pure strategy equilibrium. It is obtained as a stack of successive pure strategy equilibria of eight trimatrix games (120). Except for subinterval $[0.16\pi;\ 0.17\pi)$, every $3 \times 3 \times 3$ trimatrix game on subintervals

$$[0.12\pi;\ 0.13\pi),$$
$$[0.13\pi;\ 0.14\pi),$$
$$[0.14\pi;\ 0.15\pi),$$
$$[0.15\pi;\ 0.16\pi),$$
$$[0.17\pi;\ 0.18\pi),$$
$$[0.18\pi;\ 0.19\pi),$$
$$[0.19\pi;\ 0.2\pi]$$

has two pure strategy equilibria. The $3 \times 3 \times 3$ game on subinterval $[0.16\pi;\ 0.17\pi)$ has just one. This exception exists only for the sampling with $S = 2$. In other words, every subinterval trimatrix game has two pure strategy equilibria by sampling with $S \geqslant 3$. This fact has been ascertained for the sampling by up to $S = 56$.

According to Theorem 2, the finite 3-person staircase game has $2^8 = 256$ pure strategy equilibria (each of which is a stack of the respective subinterval game equilibria). To select a single stacked equilibrium, a single equilibrium should be selected on each subinterval. For this, a selection criterion is defined, by which the sum of the players' payoffs is maximal. Figure 4 shows 8-subinterval-stacks of every player's equilibria for $S = \overline{2,\ 36}$ (the circled line corresponds to an equilibrium by which the sum of the players' payoffs is maximal), where the first, second, and third players' stacks are shown in the left, middle, and right subplot, respectively. It is seen that these equilibria have a tendency to converge (see the circled lines in Figure 5 with 8-subinterval-stacks of the player's equilibria for $S = 30$).

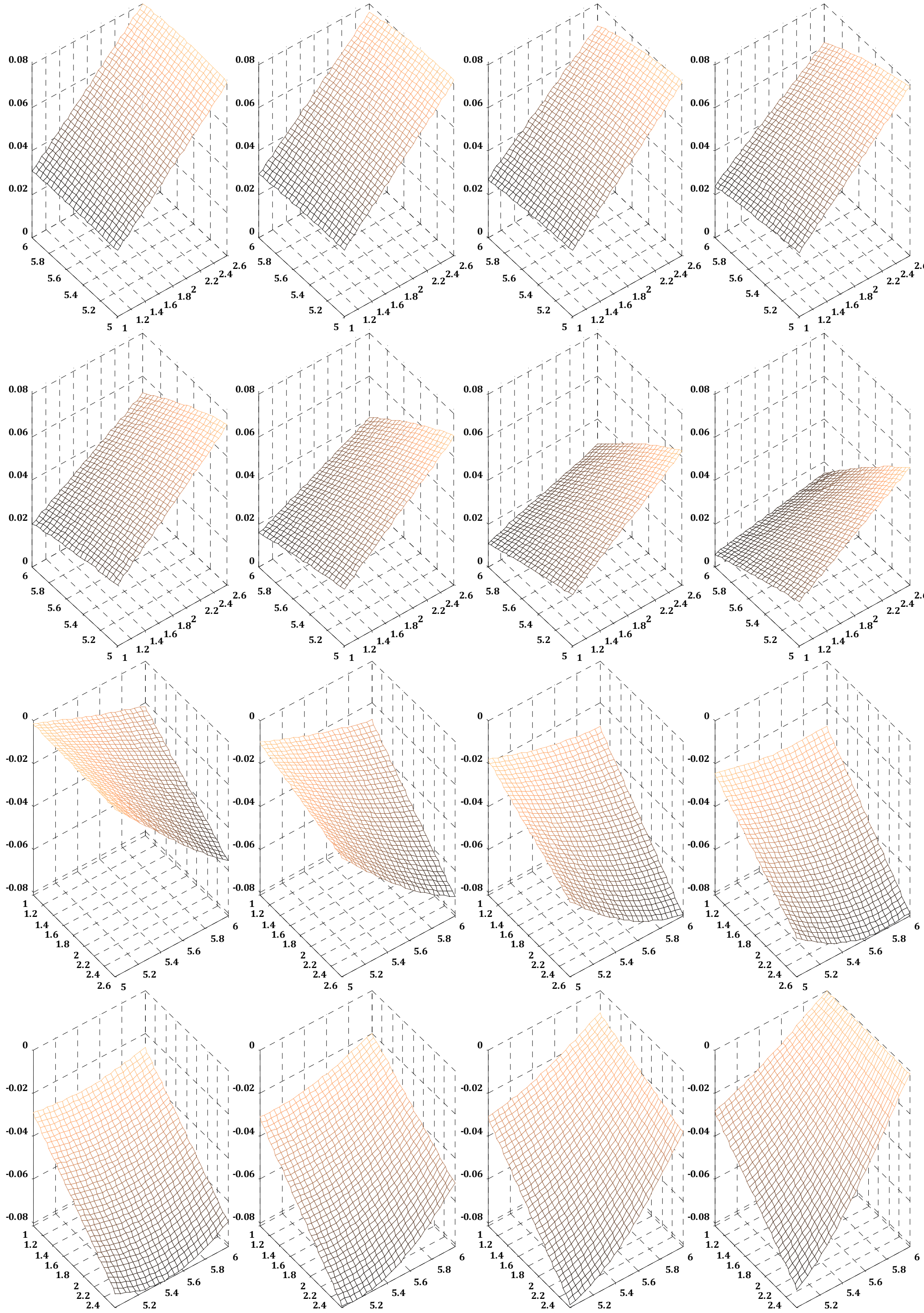

Figure 1. The sectional views of the first player's payoffs on each subinterval of set (116) by $\beta_i = 2$ (the first and second rows) and $\beta_i = 4$ (the third and fourth rows)

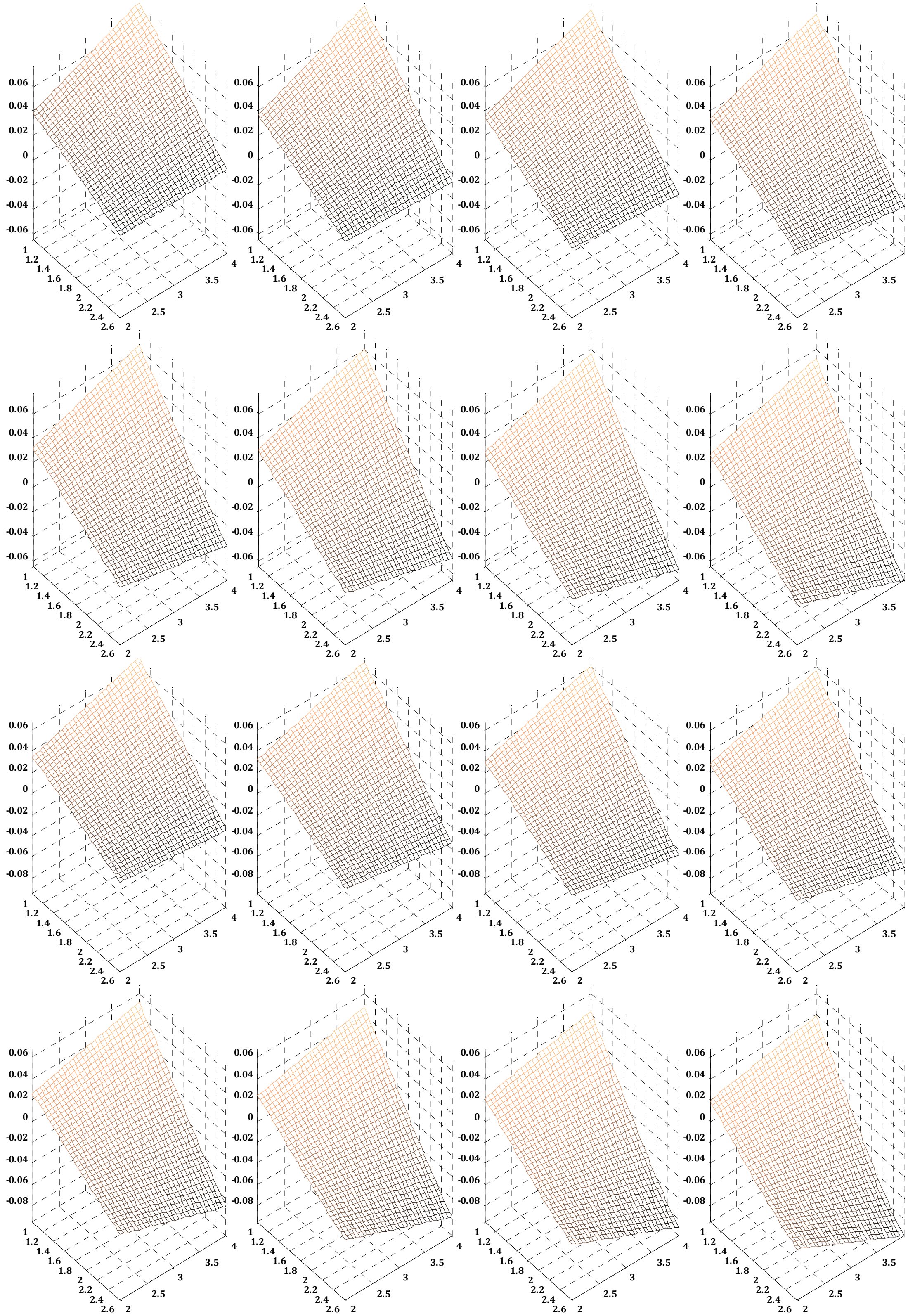

Figure 2. The sectional views of the second player's payoffs on each subinterval of set (116) by $\alpha_i = 5$ (the first and second rows) and $\alpha_i = 6$ (the third and fourth rows)

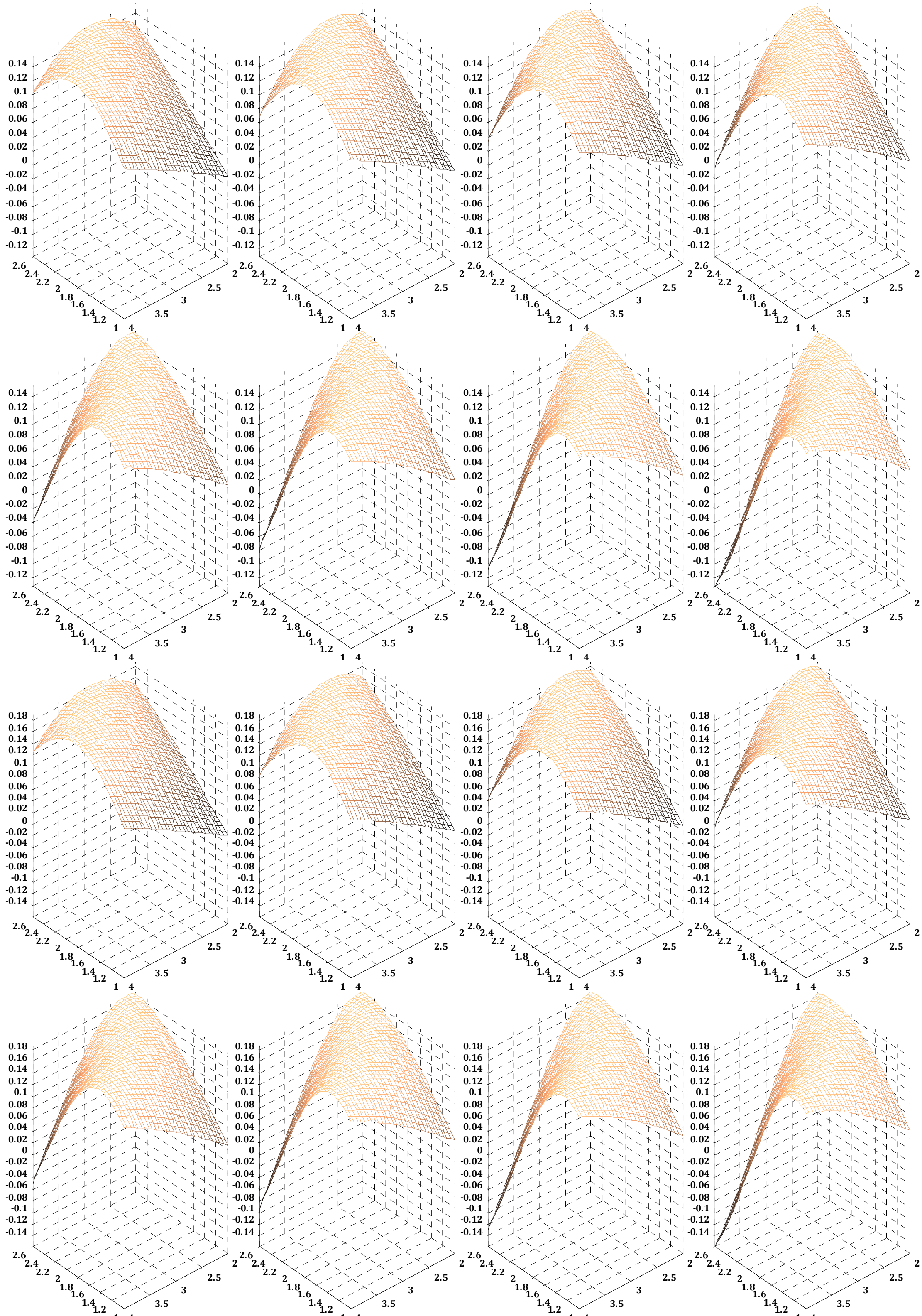

Figure 3. The sectional views of the third player's payoffs on each subinterval of set (116) by $\alpha_i = 5$ (the first and second rows) and $\alpha_i = 6$ (the third and fourth rows)

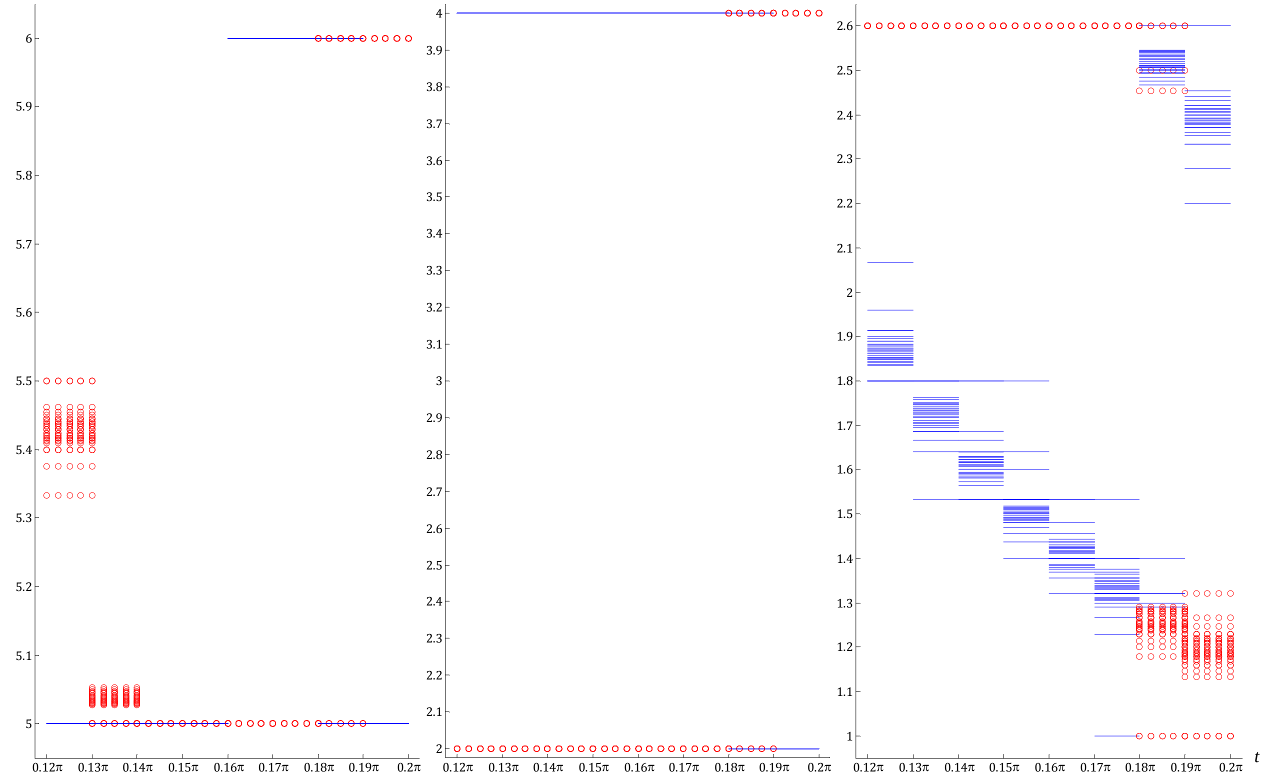

Figure 4. The pile of 8-subinterval-stacks of the player's equilibria for $S = \overline{2,\ 36}$

A trivial fact here is that every approximate equilibrium from the stacks is support-cardinality-S-consistent (by Definition 4). The more interesting one is that the stack in Figure 5 (see only the circled lines) is payoff-30-consistent. Besides, there is a payoff-23-consistent stack (although it hardly can be seen distinctly in Figure 4).

Approximate equilibrium solutions are ε-payoff-S-consistent for

$$\varepsilon = 0.01 \cdot p_i^* (S)$$

in (82),

$$\varepsilon = 0.01 \cdot r_i^* (S)$$

in (83), and

$$\varepsilon = 0.01 \cdot v_i^* (S)$$

in (84) at $i = \overline{1,\ 8}$ by

$$S \in \{6,\ 7,\ 13,\ 14,\ 23,\ 25,\ 30,\ 32,\ 34,\ 35,\ 37,\ 41,\ 42,\ 46,\ 47,\ 48,\ 49,\ 50,\ 51,\ 53,\ 55\}.$$

At the same time, approximate equilibrium solutions by

$$S \in \{3,\ 4,\ 5,\ 9,\ 11,\ 12,\ 15\}$$

are not ε-payoff-S-consistent for

$$\varepsilon = 0.25 \cdot p_i^* (S)$$

in (82),

$$\varepsilon = 0.25 \cdot r_i^* (S)$$

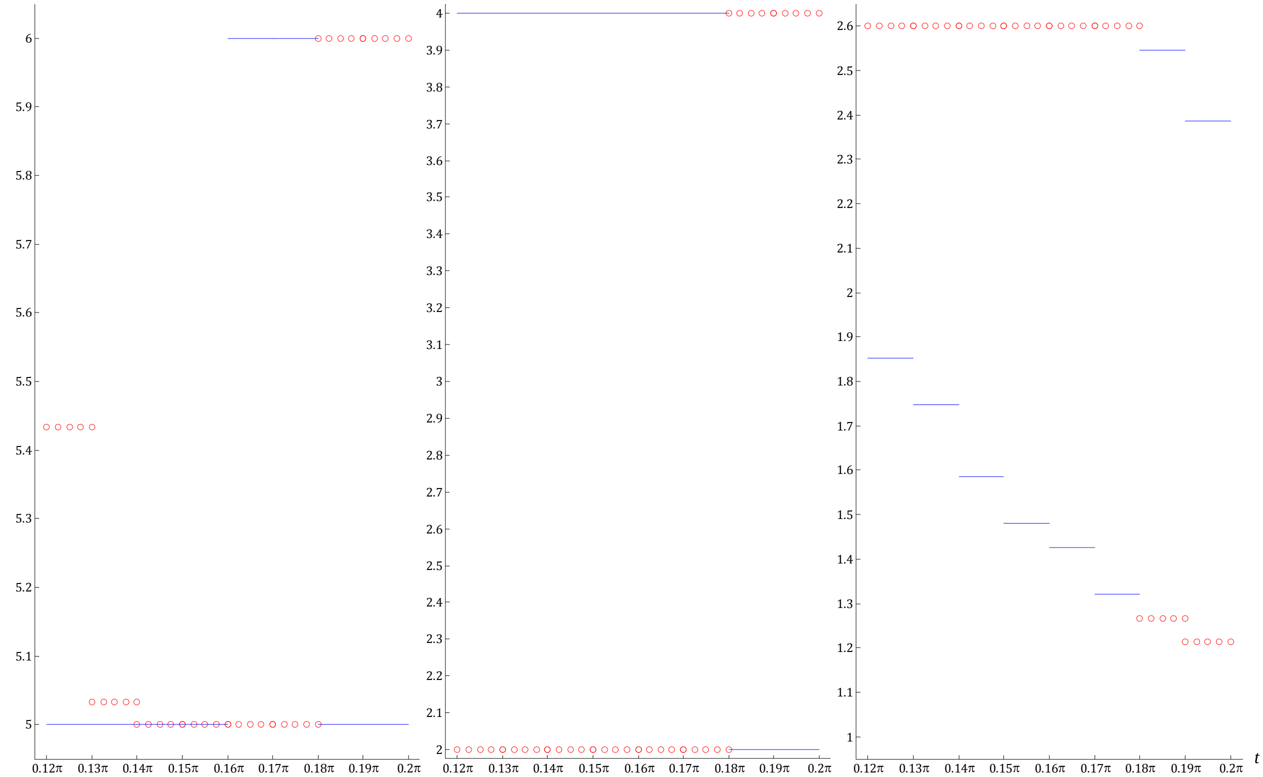

Figure 5. The pile of 8-subinterval-stacks of the player's equilibria for $S = 30$

in (83), and

$$\varepsilon = 0.25 \cdot v_i^* (S)$$

in (84) at $i = \overline{1,\ 8}$ nor they are ε-payoff-S-consistent for

$$\varepsilon = 0.5 \cdot p_i^* (S)$$

in (82),

$$\varepsilon = 0.5 \cdot r_i^* (S)$$

in (83), and

$$\varepsilon = 0.5 \cdot v_i^* (S)$$

in (84) at $i = \overline{1,\ 8}$ by

$$S \in \{3,\ 4,\ 5,\ 9,\ 11,\ 15\}.$$

Some convergence nevertheless develops as the sampling density increases. This is confirmed by Figure 6 showing how the players' payoffs (78) in the 3-person staircase game vary as number S increases. Indeed, the payoff gradually ceases oscillation (fluctuation) as $S \geqslant 17$ (Figure 7).

The payoff fluctuations are not equivalent, though. The second player's payoff badly fluctuates with respect to the first and third players, and this is clearly seen even in Figure 6. Besides, the second player's payoff is negative. However, the second player has no choice but to select the circled-line-stack in Figure 5, which is the same at $S = 20$. Thus, the best decision for the second player is to use stack $\{\mathbf{Z}_i^* (20)\}_{i=1}^{8}$ that implies using $y^* (t) = 2$ by $t \in [0.12\pi;\ 0.18\pi)$ and $y^* (t) = 4$ by $t \in [0.18\pi;\ 0.2\pi]$. The respective decisions from the first and third players are to use stacks $\{\mathbf{U}_i^* (20)\}_{i=1}^{8}$ and $\{\mathbf{W}_i^* (20)\}_{i=1}^{8}$ which are similar to those circled-line-stacks in Figure 5 for these players. This is the acceptable approximate equilibrium in the 3-person staircase game.

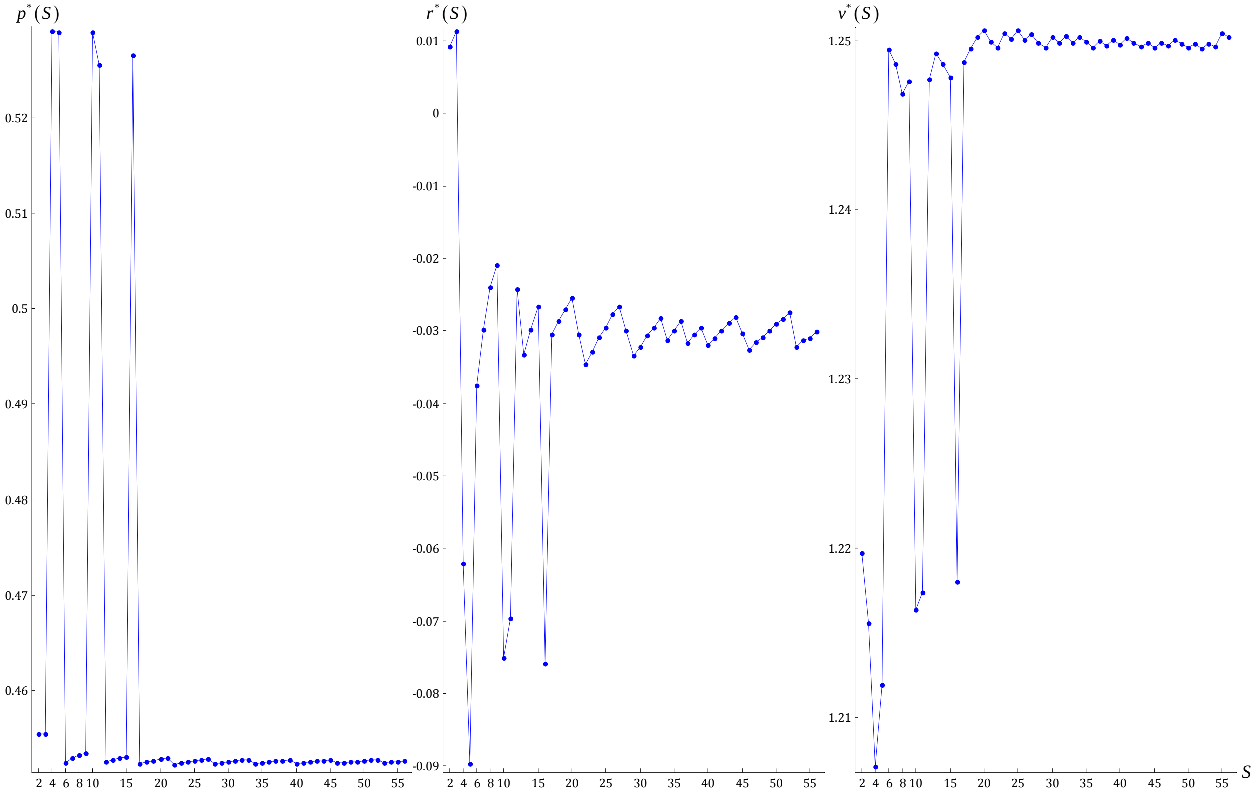

Figure 6. The player's payoff (from equilibria by which the sum of the players' payoffs is maximal) versus $S = \overline{2,\ 56}$ in the 3-person staircase game (approximated by the sampling)

Even the considered example (being relatively trivial as the acceptable approximate equilibrium exists in pure strategies) clearly shows that many additional problems may arise when fulfilling the appropriate finite approximation of a 3-person staircase game. The multiplicity of subinterval equilibria may induce instability of the players' behavior [5, 7, 19, 13, 14], if there is no criterion of the single equilibrium selection. The behavior instability worsens as the number of subinterval equilibria increases or just the number of subintervals increases. All the more so when an equilibrium is in mixed strategies (no equilibrium in pure strategies). The behavior instability is a serious problem in noncooperative games having multiple equilibria differing in the player's payoffs [13, 14]. It is particularly solved by equilibria refinement with using domination efficiency along with maximin (maximinimin for trimatrix games) and the superoptimality rule [14]. So, the above-applied criterion of maximizing the sum of the players' payoffs (the collective utility maximization) is not single.

Solving the sampled staircase game straightforwardly, without considering each subinterval 3-person game separately, is intractable. For instance, by sampling the exemplified game, where each of the players uses 8-subinterval staircase function-strategies, with $S = 20$, the resulting $21^8 \times 21^8 \times 21^8$ matrix game cannot be solved in a reasonable time span. Instead of this, eight trimatrix $21 \times 21 \times 21$ games are solved. It has taken about 40 milliseconds (on an Intel Core i7 processor) not including the time spent on calculating entries (121)—(126) of matrices $\mathbf{F}_i(S)$, $\mathbf{G}_i(S)$, $\mathbf{H}_i(S)$. The entire process has taken up to a minute.

9. Discussion

Obviously, solving subinterval trimatrix games separately and then stacking their solutions is a far more efficient way to obtain an approximate solution of the sampled staircase game. Stacking the subinterval games' pure-strategy

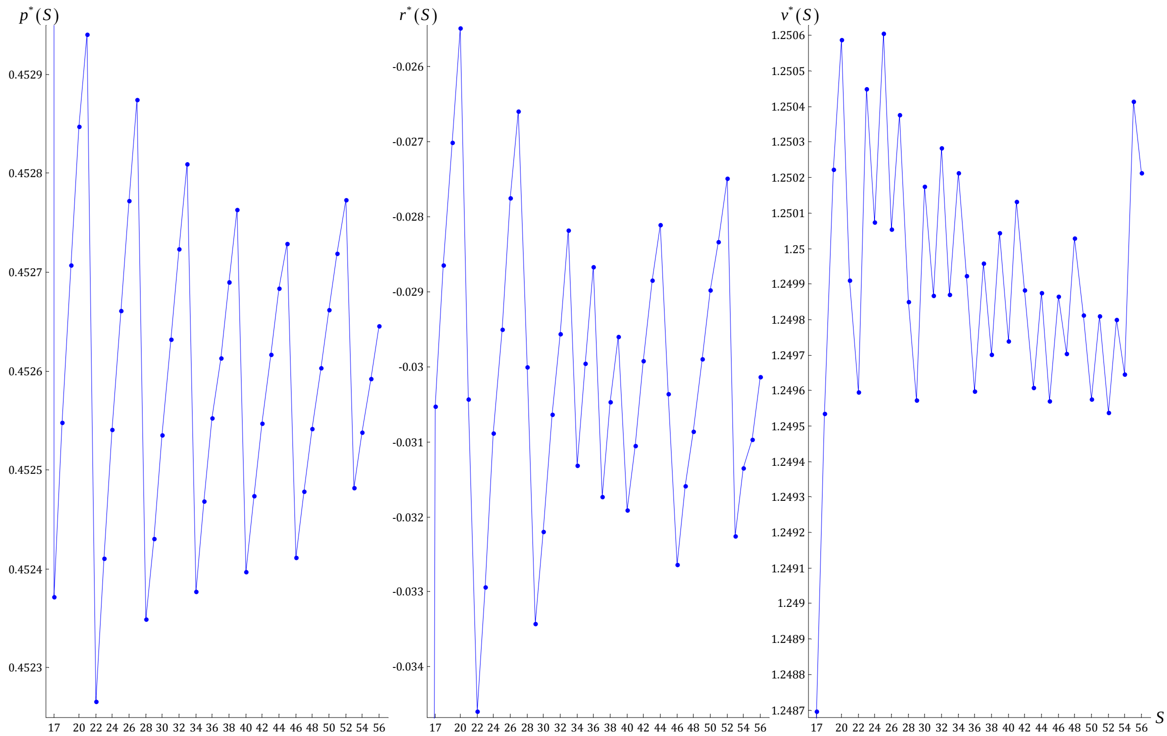

Figure 7. The player's payoff versus $S = \overline{17,\ 56}$ (a zoom-in from Figure 6)

equilibria is fulfilled trivially. When pure-strategy equilibria and mixed-strategy equilibria are stacked, the stacking is fulfilled as well implying that the resulting pure-mixed-strategy solution (equilibrium) of the sampled staircase game is realized successively, subinterval by subinterval, spending the same amount of time to implement both pure strategy and mixed strategy solutions (equilibria). Along with that, a staircase equilibrium in mixed strategies mixed with pure strategies is not a simple mathematical construction.

Although the case when an approximate equilibrium solution is (weakly) S-consistent is theoretically possible, the most important conception is the ε-payoff consistency by Definition 2. However, it is not proved that limits

$$\lim_{S\to\infty} p_i^*(S) \quad \forall\, i = \overline{1,\ N} \tag{128}$$

and

$$\lim_{S\to\infty} r_i^*(S) \quad \forall\, i = \overline{1,\ N} \tag{129}$$

and

$$\lim_{S\to\infty} v_i^*(S) \quad \forall\, i = \overline{1,\ N} \tag{130}$$

exist and they are equal to the respective equilibrium payoffs of the subinterval continuous games. Second, if limits (128) — (130) exist, it is not proved that this is followed by that an approximate equilibrium solution is ε-payoff-S-consistent for any $S \geqslant S_*$ (for some $S_* \in \mathbb{N}$). Even the ε-payoff consistency of a pure-strategy staircase equilibrium is fragile enough (just like in the above-considered example). It is clear that the fragility of a consistency type (payoff, support cardinality, support index distance, closeness of the equilibrium strategies) loosens as the number of subintervals increases.

Time variable t explicitly included into (8) — (10) means that the system changes (and the players modify their actions) as time goes by. This is quite a natural convention, but if time t is not explicitly included into the function under the integrals in (8) — (10), then the payoff value depends only on the length of the subinterval. If the length does not change, every subinterval has the same trimatrix game. The triviality of the equal-length-subinterval solution is explained by a standstill of the players' strategies.

10. Conclusion

A 3-person game defined on a product of staircase-function continuous spaces is approximated to a staircase trimatrix game by sampling the player's pure strategy value set. The set is sampled uniformly so the resulting staircase trimatrix game is cubic. Owing to Theorem 2, an equilibrium of the staircase trimatrix game is obtained by stacking the equilibria of the subinterval trimatrix games, each defined on an interval where the pure strategy value is constant.

The stack of the subinterval trimatrix game equilibria is an approximate equilibrium solution to the initial staircase game. The (weak) consistency of the approximate solution is studied by how much the players' payoff and equilibrium strategy change as the sampling density minimally increases. Thus, the consistency, equivalent to the approximate solution acceptability, includes the payoff (Definitions 1 and 2), equilibrium strategy support cardinality (Definitions 3 and 4), equilibrium strategy sampling density (Definitions 5 and 6), and support probability consistency (Definition 7). The most important parts are the payoff consistency and equilibrium strategy support cardinality (weak) consistency, which are checked in the quickest and easiest way. In addition, it is practically reasonable to consider a relaxed payoff consistency. The relaxed payoff consistency by (82) — (84) means that, as the sampling density minimally increases, the player's payoff change in an appropriate approximation may grow at most by ε. The weak consistency (Definition 8) itself is a relaxation to the consistency by Definition 9, where the minimal decrement of the sampling density is ignored.

Therefore, the suggested method of finite approximation of 3-person staircase games consists in the uniform sampling, solving subinterval trimatrix games, and stacking their equilibria if they are consistent. The finite approximation is regarded appropriate if at least the respective approximate (stacked) equilibrium is ε-payoff consistent (Definition 2) for every player. The presented method is a significant contribution to the (noncooperative) 3-person game theory. It allows approximately solving a continuous 3-person game with staircase-function strategies in a far simpler manner by "breaking" this game into a succession of subinterval games. The subinterval games are solved incomparably faster, so the method is practically applicable. Once the (weak) consistency by Definition 8 or the ε-payoff consistency by Definition 2 is confirmed, the approximate equilibrium solution can be accepted [3, 13, 14, 7, 11, 12].

A serious problem is the multiplicity of subinterval equilibria. This problem may not be unambiguously settled. Its gravity deepens as the number of subintervals increases due to then the number of stacks (by the collective utility maximization criterion) immensely grows. A future research of staircase game finite approximation should concern this question.

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