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# Statistical Analysis Based on Adaptive Progressive Hybrid Censored Data From Lomax Distribution

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**Abstract** In this article, we consider statistical inferences about the unknown parameters of the Lomax distribution based on the Adaptive Type-II Progressive Hybrid censoring scheme, this scheme can save both the total test time and the cost induced by the failure of the units and increases the efficiency of statistical analysis. The estimation of the parameters is derived using the maximum likelihood ($MLE$) and the Bayesian procedures. The Bayesian estimators are obtained based on the symmetric and asymmetric loss functions. There are no explicit forms for the Bayesian estimators, therefore, we propose Lindley's approximation method to compute the Bayesian estimators. A comparison between these estimators is provided by using extensive simulation. A real-life data example is provided to illustrate our proposed estimators.

**Keywords** Lomax distribution, maximum likelihood, Lindley's approximation, adaptive progressive censoring

**AMS 2010 subject classifications** 62E10, 62N01, 62N02, 62G30

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## 1. Introduction

In survival analysis, a sample of size n is subjected to a life test for the purpose of observing their failure times. The data recorded is then used to model a time-to-failure distribution. This strategy may be impractical, costly, and time consuming. The experimenter may also need to terminate the study before recording the failure times for all the subjects under consideration due to time constraints and facility restrictions. In addition, some functioning test subjects may need to be eliminated so they can be used in another test or to gather degradation related information about failure time, which is generally the case when testing subjects is costly such as when clinical equipment is needed. Moreover, in some cases, the failure is deliberate and expected, however it does not happen due to operator flaws, equipment malfunction, test irregularity, etc. Samples that result from such situations are called censored samples.

A censoring scheme, that can reach a balance between the total time spent for the experiment; the number of subjects used in the experiment; and the efficiency of statistical inference primarily based on the outcomes of the experiment is preferable.

The most common censoring schemes are Type-I and Type-II censoring schemes. In Type-I, the test is ceased at a pre-specified time T, while in Type-II censoring, the experiment is terminated when a specific number, $m$, of failures have been observed. Additional reduction in the experimental time and the cost were required, necessitating the use of a hybrid censoring scheme, which can be deemed as a mixture of Type-I and Type-II censoring schemes. In this scheme the experiment terminates at a pre-specified time. However, all these censoring schemes, do not allow intermediate elimination of active units throughout the experiment other than at the final termination point.

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As a result of this inflexibility, several progressive censoring schemes have been proposed in the literature to model removals of subjects during lifetime experiments. Various authors have discussed inference under progressive censoring using different lifetime distributions. Among others we list, Balakrishnan and Asgharzadeh (2005), Panahi et al. (2021), Helu and Samawi (2017) and recently, Helu et al. (2020). For a comprehensive review of progressive censoring, we refer readers to Balakrishnan and Cramer (2014). In Progressive Type-II censoring, $n$ independent items are placed at the same time on a life testing experiment and only $m(< n)$ failures are completely observed. The censoring occurs progressively in $m$ stages as follows, when the first failure is observed, a random sample of size $R_1$ is immediately drawn and removed from the test. Then after the failure of the second item, another sample of size $R_2$ is randomly selected and removed from the remaining survival units. Continuing this process until the $mth$ failure, $X_{m:m:n}$, is observed and all remaining $R_m = n - R - \cdots - R_{m-1} - m$ surviving units are removed from the experiment, with $X_{1:m:n} \leq X_{2:m:n} \leq \cdots \leq X_{m:m:n}$ being the ordered failure times resulting from the progressively Type-II censored experiment. For notation simplicity, we will write $X_i$ for $X_{i:m:n}$. One of the disadvantages of the Type-II progressive censoring scheme is that the duration of the experiment may be very long if subjects are highly reliable. To resolve such situation Kundu and Joarder (2006) introduced the Type-II Progressive hybrid (P-II hybrid) censoring scheme. The P-II hybrid scheme is a combination of the hybrid and the progressive censoring schemes. It is more flexible according to Kundu and Joarder (2006), Panahi (2017), and Wang (2018). In the P-II hybrid censoring scheme, the experiment terminates at a pre-specified time $T^* = min(T, X_m)$, where $T > 0$ and the integer $m$ are pre-assigned. In this scheme, the total time required to terminate the experiment does not exceed T. However, the main disadvantage of the P-II hybrid-censoring scheme is that the number of observed failures is random. Accordingly, it can turn out to be a very small number (even zero), thus the traditional statistical inference methods may not be valid or they may have low efficiency of the estimator(s) of the model parameter(s). In order to deal with this disadvantage, Ng et al. (2009) proposed the Adaptive Type-II Progressive Hybrid censoring scheme ($Adaptive$-IIPH) which is an adjustment of the P-II hybrid censoring scheme. This new scheme, not only saves the total test time and the cost induced by the failure of the units, but also increases the efficiency of the statistical analysis as well as ensuring that $m$ items are obtained. In recent years, the $Adaptive$-IIPH censoring scheme has been studied by a vast number of authors, among others is Cui et al. (2019) who discussed the problem of estimating the Weibull distribution parameters in a constant-stress accelerated life test. Ye et al. (2014) estimated the parameters of the extreme value distribution using the maximum likelihood technique ($MLE$). Zheng and Shi (2013) estimated the parameters of the exponential distribution using EM algorithm based on $Adaptive$-IIPH. Nassar et al. (2018) derived the maximum likelihood and the Bayes estimators for the unknown parameters of the Weibull distribution. Chen and Gui (2020) discussed the problem of estimating the parameters of the bathtub-shaped failure rate function. Panahi et al. (2021) derived the maximum likelihood and Bayes estimates for the Bur Type III distribution. Kohansal and Shoaee (2019) studied the statistical inferences for a multicomponent stress-strength reliability model. Yan and Wang (2020) derived the Bayes estimates and the credible intervals of the alpha power generalized exponential distribution based on $Adaptive$-IIPH.

In this study we assume that the lifetimes have a Lomax distribution. This distribution has been introduced by Lomax (1954) as a model for business failure data. It belongs to the class of decreasing failure rate distributions (Chahkandi and Ganjali, 2009). Moreover, it provides a very good alternative to common lifetimes distribution such as exponential, Weibull, or gamma distributions when the experimenter assumes that the population may be a heavy-tailed distribution (see Bryson, 1974). The origin and other aspects of this distribution, as well as areas of application, can be found in Arnold (1983).

Inferential issues for the Lomax distribution based on censored data have been addressed by Childs et al. (2001) who considered the order statistics from the nonidentical right truncated Lomax distribution. Howlader and Hossain (2002) considered Bayesian estimation of the survival function of the Lomax distribution. Elfattah et al. (2007) derived the Bayesian and the non-Baysian estimators for the same sample size from the Lomax distribution based on the progressively Type-I censoring. El-Sherpieny et al. (2020) used maximum product spacing to estimate the parameters of the power Lomax distribution.

The contents of this article are organized as follows: In Section 2, we introduce the notation and describe the Adaptive Type-II Progressive censoring scheme. In Section 3, when the underlying lifetime distribution is Lomax, we derive the maximum likelihood estimator. In Section 4, Bayesian estimates are computed under four different loss functions using Lindley's method. In Section 5, the efficiency of the estimators based on the proposed censoring scheme is compared by means of extensive Monte Carlo simulations. A real-life data set is analyzed in Section 6 to illustrate the proposed methods of estimation. Finally, the results and conclusion of the current study are discussed in Section 7.

## 2. Adaptive Type-II Progressive Hybrid censoring scheme

Assume there are $n$ units in a life-testing experiment, and the effective sample size $m(< n)$ is determined in advance, as well as the censoring scheme $(R_1, R_2..., R_m)$, however, the values of some of the $R_i$ may change as the experiment progresses. Suppose the experimenter provides an ideal total test time $T$, however, we are allowed to extend the experiment beyond $T$. If the $mth$ failure occurs before time $T$ (i.e. $X_m < T$), the experiment is carried out in the same way as Type-II Progressive censoring and stops at time $X_m$ with the pre-fixed censoring scheme $(R_1, R_2, ..., R_m)$. Otherwise, if the experimental time has passed $T$, but the number of observed failures has not yet reached $m$, we would leave as many surviving units as possible, hoping to see more failures in a short period of time, allowing us to complete the experiment in the most effective way possible (see David and Nagaraja (2003)), i.e. if $X_j < T < X_{j+1}, j = 0, 1, \ldots, m-1$, we do not withdraw any items from the experiment by setting $R_{j+1} = R_{j+2} = \cdots = R_{m-1} = 0$ and $R_m = n - m - \sum_{i=1}^{j} R_i$. This setting can be seen as a design that guarantees $m$ observed failure times while keeping the total test time not too far away from the ideal test time $T$ (see Figure 1). Note that if $T = 0$, then we have a conventional Type-II censoring scheme, while, if $T \to \infty$, then $Adaptive$-IIPH reduces to a Progressive Type-II censoring scheme. If the failure times of the $n$ subjects originally on the test are from a continuous distribution with cdf $F(x)$ and pdf $f(x)$, then the likelihood function as given by Ng et al. (2009) is:

$$f(x_1, \ldots, x_m) = d_j \prod_{i=1}^{m} f(x_i) \prod_{i=1}^{j} [1 - F(x_i)]^{Ri} [1 - F(x_m)]^{n-m-\sum_{i=1}^{j} R_i}, \tag{1}$$

where $0 < x_1, x_2, \ldots, x_m < \infty$, and $d_j = \prod_{i=1}^{m} \left( n - m - \sum_{k=1}^{\min\left\{ \begin{array}{c} i-1, \\ j \end{array} \right\}} R_k \right)$

## 3. Maximum Likelihood Estimation

### 3.1. Model description

The lifetime of a particular component has an exponential distribution with a failure rate $\lambda$; let $\lambda$ follows a gamma distribution with a shape parameter $\theta$ and a scale parameter $\beta$, then the failure time $X$ of a component selected at random from such a mixed population has a Lomax distribution with probability density function given by:

$$\begin{aligned} f(x;\theta,\beta) &= \int_0^\infty \lambda e^{-\lambda x} \frac{\lambda^{\theta-1} e^{\lambda/\beta}}{\beta^\theta \Gamma(\theta)} d\lambda \\ &= \frac{\beta\theta}{(1+\beta x)^{\theta+1}}, \qquad x > 0, \beta > 0,\ \theta > 0, \end{aligned} \tag{2}$$

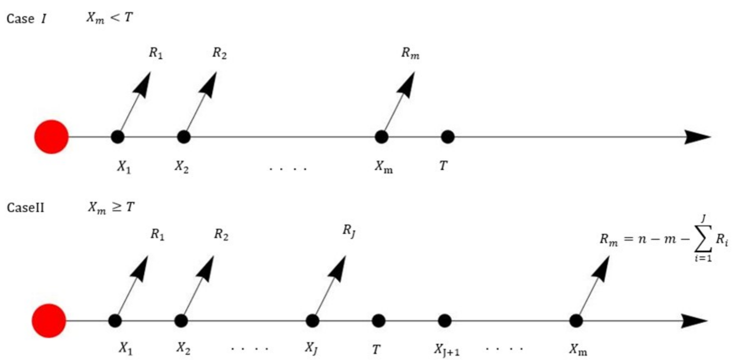


Figure 1. Schematic representation of Adaptive Type-II Progressive censoring. Case 1: Experiment terminates before time T, Case 2: Experiment terminates after time T.

with a cumulative distribution function (cdf) defined as:

$$F(x; \theta, \beta) = 1 - \frac{1}{(1 + \beta x)^{\theta}}, \quad x > 0, \beta > 0, \theta > 0. \tag{3}$$

### 3.2. *Maximum likelihood estimators*

Suppose that $n$ independent subjects are placed on a test and that the lifetime distribution of each unit is given by (2). The ordered $m$ failures are observed under the $Adaptive$-IIPH censoring scheme $\mathbf{R} = (R_1, ..., R_m)$. Thus in accordance with (2), (3) and (1), the log-likelihood function of $\theta$ and $\beta$ based on $Adaptive$-IIPH censored data becomes

$$\begin{aligned} \ln L(\underline{x}|\theta, \beta) \quad \propto \quad & m \ln(\theta\beta) - (\theta + 1) \sum_{i=1}^{m} \ln(1 + \beta x_i) - \theta \sum_{i=1}^{j} R_i \ln(1 + \beta x_i) \\ & -\theta \left( n - m - \sum_{i=1}^{j} R_i \right) \ln(1 + \beta x_m). \end{aligned} \tag{4}$$

The $MLEs$ of the parameters $\theta$ and $\beta$ can be obtained by deriving (4) with respect to $\theta$ and $\beta$ and equating the normal equations to 0 as follows:

$$\begin{aligned}\frac{\partial \ln L(\underline{x}|\theta,\beta)}{\partial\theta} &= \frac{m}{\theta} - \sum_{i=1}^{m}\ln(1+\beta x_i) - \sum_{i=1}^{j} R_i \ln(1+\beta x_i) + \left(n - m - \sum_{i=1}^{j} R_i\right)\ln(1+\beta x_m) \\ &= 0, \end{aligned} \tag{5}$$

$$\begin{aligned}\frac{\partial \ln L(\underline{x}|\theta,\beta)}{\partial\beta} &= \frac{m}{\beta} - (\theta+1)\sum_{i=1}^{m}\frac{x_i}{1+\beta x_i} - \theta\sum_{i=1}^{j}\frac{R_i\, x_i}{1+\beta x_i} + \frac{\theta\left(n-m-\sum_{i=1}^{j}R_i\right)x_m}{1+\beta x_m} \\ &\quad - \sum_{i=1}^{m}\frac{R_i(\theta x_i)^{-\beta}e^{-(\theta x_i)^{-\beta}}\ln(\theta x_i)}{1-e^{-(\theta x_i)^{-\beta}}} = 0. \end{aligned} \tag{6}$$

Notice that there is no explicit solution to Eqs.(5) & (6). Therefore, we implement Newton-Raphson method by using SAS/IML language to obtain $MLEs$ of $\theta$ and $\beta$. The maximum likelihood estimators of $\theta$ and $\beta$are exist and unique (see Wingo (1993); Helu et al. (2015)).

## 4. Bayesian Estimation

In this section, we derive the Bayes estimate of the parameters of the Lomax distribution under various loss functions. A commonly used loss function is the squared error loss function ($SQR$)

$$L(\widehat{\theta},\theta) = (\widehat{\theta}-\theta)^2, \tag{7}$$

The Bayes estimate under (7) is the posterior mean. Given by $\widehat{\theta}_{SQR} = E_\pi\theta$. The $SQR$ is widely employed in the Bayesian inference due to its computational simplicity. It is a symmetric loss function that gives equal weight to overestimation as well as underestimation. However, this is not a good criteria from a practical point of view. For example, Feynman (1987) stated that in the disaster of the space shuttle, Challenger, the management might had (Or may have) overestimated the average life or reliability of solid fuel rocket booster. In estimating reliability and failure rate functions, an overestimation causes more damage than underestimation. To resolve such a situation, asymmetrical loss functions are more appropriate. Varian (1975) introduced the Linear- Exponential loss function ($Linex$) in response to the criticism of the $SQR$. The $Linex$ loss function is defined as follows:

$$L(\widehat{\theta},\theta) = \exp(\lambda(\widehat{\theta}-\theta)) - \lambda(\widehat{\theta}-\theta) - 1, \quad \lambda \neq 0 \tag{8}$$

The magnitude of $\lambda$ reflects the degree of symmetry while the sign of $\lambda$ reflects the direction of symmetry. Zellner (1986) obtained the Bayesian estimator under ($Linex$) loss function by minimizing the posterior expected loss as follows:

$$\widehat{\theta}_{LIN} = -\frac{1}{\lambda}\ln E_\pi(e^{-\lambda\theta}) \tag{9}$$

provided that $E_\pi(e^{-\lambda\theta})$ exists and finite.

The $Linex$ loss function is suitable for situations where overestimation may lead to serious consequences, and it is known for its flexibility and popularity to estimate the location parameter. On the other hand, Basu and Ebrahimi (1991) and Parsian and Farsipour (1993) found that the $Linex$ loss function is unsuitable for estimating the scale parameter and other quantities. Hence, Basu and Ebrahimi (1991) defined a modified $Linex$ loss function instead. Later on, Calabria and Pulcini (1996) proposed a suitable alternative to the modified $Linex$ loss function that is the General entropy ($GE$) loss function.

The General entropy loss function is defined as

$$L(\widehat{\theta},\theta)=\left(\frac{\widehat{\theta}}{\theta}\right)^{\lambda}-\lambda\log\left(\frac{\widehat{\theta}}{\theta}\right)-1,\ \ \lambda\neq 0. \tag{10}$$

Where $\lambda$ reflects the magnitude and degree of symmetry. Calabria and Pulcini (1996) obtained the Bayesian estimator under (10) as follows:

$$\widehat{\theta}_{GE}=\left(E_{\pi}(\theta^{-\lambda})\right)^{\frac{-1}{\lambda}}, \tag{11}$$

provided that $E_{\pi}(\theta^{-\lambda})$ exists and finite, moreover, $\widehat{\theta}_{GE}=\widehat{\theta}_{SQR}$ when $\lambda=-1$.

Norstrom (1996) introduced the Precautionary loss function as follows:

$$L(\widehat{\theta},\theta)=\frac{(\theta-\widehat{\theta})^2}{\widehat{\theta}}, \tag{12}$$

and claimed that it is useful to derive conservative estimators since it approaches infinity near the origin and prevents underestimation. This loss function is suitable for situations where underestimation may lead to serious consequences.

The Bayesian estimator under (12) is given by

$$\widehat{\theta}_{PRE}=\sqrt{E_{\pi}(\theta^2)}, \tag{13}$$

provided that $E_{\pi}(\theta^2)$ exists and finite, notice that $\widehat{\theta}_{PRE}=\widehat{\theta}_{GE}$ when $\lambda=-2$.

A common feature of lifetime distributions with a shape parameter is that the Bayes estimators can not be expressed in closed forms. We suggest using Lindley's approximation to derive the Bayes estimators of the unknown parameters of the Lomax distribution based on progressively Type-II censored sample.

### *4.1. Lindley's approximation method*

It is assumed that $\theta$ and $\beta$ have independent gamma priors;

$$\theta\sim\pi_1(\theta)=\frac{\theta^{a-1}e^{-b\theta}b^a}{\Gamma(a)};\qquad\qquad \beta\sim\pi_2(\beta)=\frac{\beta^{c-1}e^{-c\beta}d^c}{\Gamma(d)} \tag{14}$$

where $a,b,c$ and $d$ are assumed to be known and non-negative. The joint prior for $\theta$ and $\beta$ is $\pi(\theta,\beta)\propto\theta^{a-1}e^{-b\theta}\beta^{c-1}e^{-d\beta}$. Use the same set-ups as in Section 3. If $\underline{x}=(x_1,x_2,...,x_m)$ is $Adaptive$-IIPH censored sample from Lomax$(\theta,\beta)$, then the joint posterior pdf of $\theta$ and $\beta$ is given by

$$\begin{aligned}\pi(\theta,\beta|\underline{x}) &= \frac{L(\underline{x}|\theta,\beta)\pi(\theta,\beta)}{\int_0^{\infty}\int_0^{\infty}L(\underline{x}|\theta,\beta)\pi(\theta,\beta)d\theta d\beta}\\ &= \frac{\theta^{a-1}\beta^{c-1}e^{-b\theta}e^{-d\beta}\prod_{i=1}^{m}\frac{\theta\beta}{(1+\beta x_i)^{\theta+1}}\prod_{i=1}^{j}\left[(1+\beta x_i)^{-\theta}\right]^{R_i}(1+\beta x_m)^{n-m-\sum_{i=1}^{j}R_i}}{\int_0^{\infty}\int_0^{\infty}\theta^{a-1}\beta^{c-1}e^{-b\theta}e^{-d\beta}\prod_{i=1}^{m}\frac{\theta\beta}{(1+\beta x_i)^{\theta+1}}\prod_{i=1}^{j}\left[(1+\beta x_i)^{-\theta}\right]^{R_i}(1+\beta x_m)^{n-m-\sum_{i=1}^{j}R_i}d\theta d\beta}\end{aligned} \tag{15}$$

Therefore, the Bayes estimators of any function of $\theta$ and $\beta$ say $u(\theta,\beta)$ is the posterior expected value. Let $u(\theta,\beta)$ be a function of $\theta$ and $\beta$, then the expected value of $u(\theta,\beta)$ is given by:

$$\widehat{u}=E_{\pi}(u(\theta,\beta)|\underline{x})=\frac{\int_0^{\infty}\int_0^{\infty}u(\theta,\beta)e^{\rho(\theta,\beta)+l(\underline{x}|\theta,\beta)}d\theta d\beta}{\int_0^{\infty}\int_0^{\infty}e^{\rho(\theta,\beta)+l(\underline{x}|\theta,\beta)}d\theta d\beta} \tag{16}$$

where, $\rho(\theta,\beta) = \ln \pi(\theta,\beta)$ and $l(\underline{x}|\theta,\beta) = \ln L(\underline{x}|\theta,\beta)$.

It can be noticed that $\widehat{u}$ is in the form of a ratio of two integrals which can not be simplified to a closed form. Hence Lindley's approximation method is applied to obtain the Bayes estimators of $\theta$ and $\beta$, see Lindley (1980). Then (16) is reduced to the following numerical expression.

$$\begin{aligned}\widehat{u} \;=\;& u(\widehat{\theta},\widehat{\beta}) + 0.5[(\widehat{u}_{\theta\theta} + 2\widehat{u}_\theta\widehat{\rho}_\theta)\widehat{\sigma}_{\theta\theta} + (\widehat{u}_{\beta\theta} + 2\widehat{u}_\beta\widehat{\rho}_\theta)\widehat{\sigma}_{\beta\theta} + (\widehat{u}_{\theta\beta} + 2\widehat{u}_\theta\widehat{\rho}_\beta)\widehat{\sigma}_{\theta\beta} + \\ & (\widehat{u}_{\beta\beta} + 2\widehat{u}_\beta\widehat{\rho}_\beta)\widehat{\sigma}_{\beta\beta}] + 0.5[(\widehat{u}_\theta\widehat{\sigma}_{\theta\theta} + \widehat{u}_\beta\widehat{\sigma}_{\theta\beta})(\widehat{l}_{\theta\theta\theta}\widehat{\sigma}_{\theta\theta} + \widehat{l}_{\theta\beta\theta}\widehat{\sigma}_{\theta\beta} + \widehat{l}_{\beta\theta\theta}\widehat{\sigma}_{\beta\theta} + \widehat{l}_{\beta\beta\theta}\widehat{\sigma}_{\beta\beta}) + \\ & (\widehat{u}_\theta\widehat{\sigma}_{\beta\theta} + \widehat{u}_\beta\widehat{\sigma}_{\beta\beta})(\widehat{l}_{\beta\theta\theta}\widehat{\sigma}_{\theta\theta} + \widehat{l}_{\theta\beta\beta}\widehat{\sigma}_{\theta\beta} + \widehat{l}_{\beta\theta\beta}\widehat{\sigma}_{\beta\theta} + \widehat{l}_{\beta\beta\beta}\widehat{\sigma}_{\beta\beta})].\end{aligned}$$

where, $\widehat{\theta}$ and $\widehat{\beta}$ are the $MLE$'s of $\theta$ and $\beta$ respectively, $\widehat{u}_{\theta\theta} = \frac{\partial^2 u(\theta,\beta)}{\partial\theta\partial\theta}|_{(\widehat{\theta},\widehat{\beta})}$, $\widehat{\rho}_\theta = \frac{(a-1)}{\widehat{\theta}} - b$, $\widehat{\rho}_\beta = \frac{(c-1)}{\widehat{\beta}} - d$. Other expressions can be defined similarly (see the Appendix).

### *4.2. Bayes estimates under symmetric loss function*

- Approximate Bayes estimate of $\theta$ under squared loss function.

If $u(\theta,\beta) = \theta, \quad u_\theta = 1, \quad u_\beta = u_{\beta\beta} = u_{\theta\theta} = u_{\theta\beta} = u_{\beta\theta} = 0$, then,

$$\widehat{\theta}_{SQR} = \widehat{\theta} + \widehat{\rho}_\theta\widehat{\sigma}_{\theta\theta} + \widehat{\rho}_\beta\widehat{\sigma}_{\theta\beta} + \widehat{\sigma}^2_{\beta\theta}\widehat{l}_{\theta\beta\beta} + \frac{\left[\widehat{\sigma}^2_{\theta\theta}\widehat{l}_{\theta\theta\theta} + 3\widehat{\sigma}_{\theta\theta}\widehat{\sigma}_{\beta\theta}\widehat{l}_{\theta\beta\theta} + \widehat{\sigma}_{\beta\beta}\widehat{\sigma}_{\theta\theta}\widehat{l}_{\beta\beta\theta} + \widehat{\sigma}_{\beta\theta}\widehat{\sigma}_{\beta\beta}\widehat{l}_{\beta\beta\beta}\right]}{2}.$$

- Approximate Bayes estimate of $\beta$ under squared loss function.

If $u(\theta,\beta) = \beta, \quad u_\beta = 1, \quad u_{\beta\beta} = u_{\theta\beta} = u_{\beta\theta} = u_\theta = u_{\theta\theta} = 0$, then

$$\widehat{\beta}_{SQR} = \widehat{\beta} + \widehat{\rho}_\theta\widehat{\sigma}_{\beta\theta} + \widehat{\rho}_\beta\widehat{\sigma}_{\beta\beta} + \widehat{\sigma}^2_{\theta\beta}\widehat{l}_{\theta\beta\theta} + \frac{\left[\widehat{\sigma}^2_{\beta\beta}\widehat{l}_{\beta\beta\beta} + 3\widehat{\sigma}_{\beta\beta}\widehat{\sigma}_{\theta\beta}\widehat{l}_{\theta\beta\beta} + \widehat{\sigma}_{\beta\beta}\widehat{\sigma}_{\theta\theta}\widehat{l}_{\beta\theta\theta} + \widehat{\sigma}_{\theta\beta}\widehat{\sigma}_{\theta\theta}\widehat{l}_{\theta\theta\theta}\right]}{2}.$$

### *4.3. Bayes estimates under asymmetric loss functions*

**(1) The Bayes estimates under $Linex$ loss function**

- Approximate Bayes estimate of $\theta$ under $Linex$ loss function

If, $u(\theta,\beta) = e^{-\lambda\theta}, u_\theta = -\lambda e^{-\lambda\theta}, u_{\theta\theta} = \lambda^2 e^{-\lambda\theta}, u_\beta = u_{\beta\beta} = u_{\theta\beta} = u_{\beta\theta} = 0.$

Then,

$$\begin{aligned}E_\pi(e^{-\lambda\theta}|\underline{x}) \;=\;& e^{-\lambda\widehat{\theta}} + \frac{\widehat{u}_{\theta\theta}\widehat{\sigma}_{\theta\theta}}{2} + \widehat{u}_\theta\left[\left(\widehat{\rho}_\theta + \widehat{\sigma}_{\beta\theta}\widehat{l}_{\theta\beta\theta}\right)\widehat{\sigma}_{\theta\theta} + \left(\widehat{\rho}_\beta + \widehat{\sigma}_{\beta\theta}\widehat{l}_{\theta\beta\beta}\right)\widehat{\sigma}_{\theta\beta}\right] \\ & + \frac{\widehat{u}_\theta\left[\widehat{\sigma}^2_{\theta\theta}\widehat{l}_{\theta\theta\theta} + \widehat{\sigma}_{\theta\theta}\widehat{\sigma}_{\beta\beta}\widehat{l}_{\beta\beta\theta} + \widehat{\sigma}_{\beta\theta}\widehat{\sigma}_{\theta\theta}\widehat{l}_{\beta\theta\theta} + \widehat{\sigma}_{\beta\theta}\widehat{\sigma}_{\beta\beta}\widehat{l}_{\beta\beta\beta}\right]}{2},\end{aligned}$$

hence, the Bayes estimate of $\theta$ is obtained by

$$\widehat{\theta}_{LIN} = -\frac{1}{\lambda}\ln E_\pi(e^{-\lambda\theta}|\underline{x}).$$

- Approximate Bayes estimate of $\beta$ under $Linex$ loss function

If $u(\theta,\beta) = e^{-\lambda\beta}, u_\beta = -\lambda e^{-\lambda\beta}, u_{\beta\beta} = \lambda^2 e^{-\lambda\beta}, u_\theta = u_{\theta\theta} = u_{\theta\beta} = u_{\beta\theta} = 0.$

Then,

$$E_\pi(e^{-\lambda\beta}|\underline{x}) = e^{-\lambda\widehat{\beta}} + \frac{\widehat{u}_\beta\widehat{\sigma}_{\beta\beta}\left(\widehat{\sigma}_{\beta\beta}\widehat{l}_{\beta\beta\beta} + 3\widehat{\sigma}_{\theta\beta}\widehat{l}_{\beta\beta\theta} + \widehat{\sigma}_{\theta\theta}\widehat{l}_{\beta\theta\theta}\right)}{2} + \frac{(\hat{u}_{\beta\beta} + 2\widehat{u}_\beta\widehat{\rho}_\beta)\,\widehat{\sigma}_{\beta\beta} + \widehat{u}_\beta\widehat{\sigma}_{\theta\beta}\widehat{\sigma}_{\theta\theta}\widehat{l}_{\theta\theta\theta}}{2} + \widehat{u}_\beta\left(\widehat{\rho}_\theta + \widehat{\sigma}_{\theta\beta}\widehat{l}_{\theta\beta\theta}\right)\widehat{\sigma}_{\theta\beta},$$

hence, the Bayes estimate of $\beta$ is obtained by

$$\widehat{\beta}_{LIN} = -\frac{1}{\lambda}\ln E_\pi(e^{-\lambda\beta}|\underline{x}).$$

**(2) The Bayes estimates under General entropy loss function**

- Approximate Bayes estimate of $\theta$ under General entropy

$$\text{If}\quad u(\theta,\beta) = \theta^{-\lambda}, u_\theta = -\tfrac{\lambda}{\theta^{\lambda+1}}, u_{\theta\theta} = \tfrac{\lambda(\lambda+1)}{\theta^{\lambda+2}}, u_\beta = u_{\beta\beta} = u_{\theta\beta} = u_{\beta\theta} = 0.$$

$$E_\pi(\theta^{-\lambda}|\underline{x}) = \widehat{\theta}^{-\lambda} + \frac{(\widehat{u}_{\theta\theta} + 2\widehat{u}_\theta\widehat{\rho}_\theta)\widehat{\sigma}_{\theta\theta}}{2} + \widehat{u}_\theta\left(\widehat{\rho}_\beta\widehat{\sigma}_{\theta\beta} + \widehat{\sigma}_{\theta\theta}\widehat{\sigma}_{\beta\theta}\widehat{l}_{\theta\beta\theta} + \widehat{\sigma}^2_{\beta\theta}\widehat{l}_{\theta\beta\beta}\right) + \frac{\widehat{u}_\theta\left[\widehat{\sigma}^2_{\theta\theta}\widehat{l}_{\theta\theta\theta} + \widehat{\sigma}_{\theta\theta}\left(\widehat{\sigma}_{\beta\beta}\widehat{l}_{\beta\beta\theta} + \widehat{\sigma}_{\beta\theta}\widehat{l}_{\beta\theta\theta}\right) + \widehat{\sigma}_{\beta\theta}\widehat{\sigma}_{\beta\beta}\widehat{l}_{\beta\beta\beta}\right]}{2},$$

so, the Bayes estimate of $\theta$ is $\widehat{\theta}_{GE} = [E_\pi(\theta^{-\lambda}|\underline{x})]^{\frac{-1}{\lambda}}$.

- Approximate Bayes estimator of $\beta$ under the General entropy.

$$\text{If}\quad u(\theta,\beta) = \beta^{-\lambda}, u_\beta = -\tfrac{\lambda}{\beta^{\lambda+1}}, u_{\beta\beta} = \tfrac{\lambda(\lambda+1)}{\beta^{\lambda+2}}, u_\theta = u_{\theta\theta} = u_{\theta\beta} = u_{\beta\theta} = 0.$$

Then,

$$E_\pi(\beta^{-\lambda}|\underline{x}) = \widehat{\beta}^{-\lambda} + \widehat{u}_\beta\left(\widehat{\rho}_\theta\widehat{\sigma}_{\beta\theta} + \widehat{\rho}_\beta\widehat{\sigma}_{\beta\beta}\right) + \frac{\hat{u}_{\beta\beta}\widehat{\sigma}_{\beta\beta} + \widehat{u}_\beta\widehat{\sigma}_{\theta\beta}\widehat{\sigma}_{\theta\theta}\widehat{l}_{\theta\theta\theta}}{2} + \frac{\widehat{u}_\beta\left[\left(\widehat{\sigma}_{\beta\beta}\widehat{l}_{\beta\beta\beta} + \widehat{\sigma}_{\theta\theta}\widehat{l}_{\beta\theta\theta}\right)\widehat{\sigma}_{\beta\beta} + \left(2\widehat{\sigma}_{\theta\beta}\widehat{l}_{\theta\beta\theta} + 3\widehat{\sigma}_{\beta\beta}\widehat{l}_{\beta\beta\theta}\right)\widehat{\sigma}_{\theta\beta}\right]}{2},$$

hence, $\widehat{\beta}_{GE} = [E_\pi(\beta^{-\lambda}|\underline{x})]^{\frac{-1}{\lambda}}$.

**(3) The Bayes estimates under Precautionary loss function.**

- Approximate Bayes estimate of $\theta$ under the Precautionary loss function.

$$\text{If}\ \ u(\theta,\beta) = \theta^2,\ \ u_\theta = 2\theta,\ \ u_{\theta\theta} = 2,\ \ u_\beta = u_{\beta\beta} = u_{\theta\beta} = u_{\beta\theta} = 0.$$

Then

$$E_\pi(\theta^2|\underline{x}) = \widehat{\theta}^2 + \widehat{u}_\theta\widehat{\rho}_\beta\widehat{\sigma}_{\theta\beta} + \frac{\left[(\widehat{u}_{\theta\theta} + 2\widehat{u}_\theta\widehat{\rho}_\theta)\widehat{\sigma}_{\theta\theta} + \widehat{u}_\theta\left(\widehat{\sigma}^2_{\theta\theta}\widehat{l}_{\theta\theta\theta} + \widehat{\sigma}_{\beta\theta}\widehat{l}_{\theta\beta\theta}\right)\right]}{2} + \frac{\widehat{u}_\theta\left[\widehat{\sigma}_{\theta\theta}\left(\widehat{\sigma}_{\beta\beta}\widehat{l}_{\beta\beta\theta} + \widehat{\sigma}_{\beta\theta}\widehat{l}_{\beta\theta\theta}\right) + \widehat{\sigma}_{\beta\theta}\left(2\widehat{\sigma}_{\beta\theta}\widehat{l}_{\theta\beta\beta} + \widehat{\sigma}_{\beta\beta}\widehat{l}_{\beta\beta\beta}\right)\right]}{2},$$

therefore, $\widehat{\theta}_{PRE} = \sqrt{E_\pi(\theta^2|\underline{x})}$.

- Approximate Bayes estimate of $\beta$ under the Precautionary loss function.

$$\text{If} \quad u(\theta,\beta)=\beta^2,\ \ u_\beta=2\beta,\ \ u_{\beta\beta}=2\ \ u_\theta=u_{\theta\theta}=u_{\theta\beta}=u_{\beta\theta}=0.$$

$$\text{Then, } E_\pi(\beta^2|\underline{x}) = \widehat{\beta}^2+\widehat{u}_\beta\widehat{\rho}_\theta\widehat{\sigma}_{\beta\theta}+\frac{\left[(\hat{u}_{\beta\beta}+2\widehat{u}_\beta\widehat{\rho}_\beta)\widehat{\sigma}_{\beta\beta}+\widehat{u}_\beta\widehat{\sigma}_{\theta\beta}\widehat{\sigma}_{\theta\theta}\widehat{l}\widehat{\sigma}^2_{\beta\beta}\widehat{l}_{\beta\beta\beta}\right]}{2} + \frac{\widehat{u}_\beta\left[2\widehat{\sigma}^2_{\theta\beta}\widehat{l}_{\theta\beta\theta}+\widehat{\sigma}_{\beta\beta}\widehat{\sigma}_{\theta\theta}\widehat{l}_{\beta\theta\theta}+3\widehat{\sigma}_{\theta\beta}\widehat{\sigma}_{\beta\beta}\widehat{l}_{\beta\beta\theta}\right]}{2},$$

so, $\widehat{\beta}_{PRE}=\sqrt{E_\pi(\beta^2|\underline{x})}$.

## 5. Simulation Study

The purpose of the simulation study is to compare the performance of the $MLE$'s and the Bayesian estimates, based on symmetric and asymmetric loss functions, using independent gamma priors for the shape and scale parameters. Progressively censored samples (Balakrishnan and Aggarwala (2000)) are randomly generated from the Lomx distribution as follows.

1. Generate $m$ independent $U(0,1)$ random variables $W_1, W_2, ..., W_m$.
2. For given values of progressive censoring scheme $R_1, R_2, ..., R_m$, we set $E_i=(i+\sum_{j=m-i+1}^{m} R_j), i = 1, ..., m$. $V_i=W_i^{1/E_i}$, for $i=1,...,m$.
3. Consider $U_i=1-V_m\times V_{m-1}\times...\times V_{m-i+1},\ \ i=1,...,m,$ then $U_1,...,U_m$ is a Progressive Type-II censored sample of size $m$ from $U(0,1)$.
4. For given values of $\theta$ and $\beta$ we set $X_i=F^{-1}(U_i)=\frac{(1-U_i)^{\frac{-1}{\theta}}-1}{\beta}, i=1,...,m$. Finally, $x_1, x_2, ..., x_m$ is the required Progressive Type-II censored sample of size $m$ from the Lomax$(\theta,\beta)$ distribution.
5. Determine the value of $j$, where $X_j<T<X_{j+1}$ and discard the sample $X_{j+2},\ldots,X_m$.
6. Generate the first $m-j-1$ order statistics from a truncated distribution $\frac{f(x)}{1-F(x_{j+1n})}$ with sample size $n-\sum_{i=1}^{j}R_{i-j-1}$ as $X_{j+2},\ldots,X_m$.
7. Obtain the maximum likelihood estimates of $\theta$ and $\beta$ using iterative process.
8. Values of $\theta$ and $\beta$ are generated from $\pi_1$ and $\pi_2$ as given in (14) with specified parameters "$a$", "$b$", "$c$", and "$d$". The resulted values of $\theta$ and $\beta$ are considered to be the true values that will be used to generate the $Adaptive$-IIPH censored sample.
9. We generate 5000 $Adaptive$-IIPH censored samples from the Lomax distribution (with $a=0.5, b=0.2, c=1, d=1$). Three different $T$ values: $T_1=X_{\lfloor\frac{m}{2}\rfloor}$; $T_2=X_{\lfloor\frac{4*m}{5}\rfloor}$; $T_3=(X_m+2)$ and different combinations of sample sizes and effective sample sizes $(n,m):(n,m)=(20,6);(20,14);(40,30);(60,40)$ are conducted with three censoring schemes $(R_1,...,R_m)$. Note that, when $n=20, m=14$, the censoring scheme $R=(2,3,0^{*12})$ means that after the first failure, two items are removed at random from the remaining 19 items, then after the second failure, three items are removed at random from the remaining 16 items. The next twelve failure times are observed. For simplicity of notations, $R=(0^{*4})$ indicates $R=(0,0,0,0)$. The three censoring schemes are shown below:
   - Censoring scheme (Cs) I: $R_1=n-m, R_i=0$, for $i\neq 1$.
   - Censoring scheme (Cs) II: $R_m=n-m, R_i=0$ for $i\neq m$.
   - Censoring scheme (Cs) III: $R_1=R_m=(n-m)/2, R_i=0$ for $i\neq 1$ & $i\neq m$.

Table 1. Average estimates and **MSEs** of $\theta$, for $T_1 = X_{\frac{m}{2}}$ and different choices of $n, m,$and $R$

| n | m | CS | $\widehat{\theta}_{MLE}$(**MSE**) | $\widehat{\theta}_{SEL}$(**MSE**) | $\widehat{\theta}_{\text{Pre}}$(**MSE**) | $\widehat{\theta}_{LIN(\lambda=2)}$(**MSE**) | $\widehat{\theta}_{LIN(\lambda=-1)}$(**MSE**) | $\widehat{\theta}_{GE(\lambda=2)}$(**MSE**) | $\widehat{\theta}_{GE(\lambda=-1)}$(**MSE**) |
|---|---|---|---|---|---|---|---|---|---|
| 20 | 6 | I | 0.5440(**0.000888**) | 0.564(**0.000952**) | 0.614(**0.001388**) | 0.502(**0.000541**) | 0.596(**0.001268**) | 0.541(**0.000851**) | 0.642(**0.001889**) |
| | | II | 0.1010(**0.004756**) | 0.130(**0.005052**) | 0.136(**0.004925**) | 0.121(**0.004997**) | 0.13(**0.004851**) | 0.102(**0.004762**) | 0.162(**0.007169**) |
| | | III | 0.167(**0.003645**) | 0.195(**0.003646**) | 0.209(**0.003594**) | 0.18(**0.003489**) | 0.2(**0.003696**) | 0.167(**0.003647**) | 0.226(**0.004526**) |
| 20 | 14 | I | 0.5254(**0.000108**) | 0.5333(**0.000109**) | 0.5538(**0.000128**) | 0.5094(**0.000087**) | 0.5452(**0.000124**) | 0.5242(**0.000106**) | 0.5610(**0.000147**) |
| | | II | 0.2120(**0.000456**) | 0.219(**0.000437**) | 0.227(**0.0004188**) | 0.215(**0.000433**) | 0.221(**0.000444**) | 0.212(**0.000456**) | 0.224(**0.000429**) |
| | | III | 0.2980(**0.000277**) | 0.307(**0.000262**) | 0.318(**0.0002479**) | 0.299(**0.000259**) | 0.311(**0.000268**) | 0.298(**0.000276**) | 0.317(**0.000258**) |
| 40 | 30 | I | 0.5178(**0.000016**) | 0.5215(**0.000016**) | 0.531(**0.0000169**) | 0.511(**0.000014**) | 0.5268(**0.000017**) | 0.5174(**0.000016**) | 0.5335(**0.000018**) |
| | | II | 0.2020(**0.000101**) | 0.205(**0.000099**) | 0.209(**0.0000967**) | 0.204(**0.0000987**) | 0.206(**0.0000994**) | 0.202(**0.0001008**) | 0.207(**0.000098**) |
| | | III | 0.2770(**0.000061**) | 0.281(**0.000059**) | 0.286(**0.0000567**) | 0.278(**0.0000588**) | 0.283(**0.0000592**) | 0.277(**0.0000605**) | 0.285(**0.000058**) |
| 60 | 40 | I | 0.5115(**4E-6**) | 0.5137(**4E-6**) | 0.5194(**4E-6**) | 0.5077(**4E-6**) | 0.5168(**4E-6**) | 0.5112(**4E-6**) | 0.5205(**4E-6**) |
| | | II | 0.2363(**0.000028**) | 0.2383(**0.000028**) | 0.2408(**0.000027**) | 0.2371(**0.000028**) | 0.2389(**0.000028**) | 0.2371(**0.000028**) | 0.2397(**0.000028**) |
| | | III | 0.3087(**0.000016**) | 0.3111(**0.000016**) | 0.3143(**0.000015**) | 0.309(**0.000016**) | 0.3121(**0.000015**) | 0.3087(**0.000016**) | 0.3134(**0.000015**) |

Table 2. Average estimates and **MSEs** of $\theta$, for $T_2 = X_{\frac{(4\times m)}{5}}$ and different choices of $n, m$, and $R$

| n | m | CS | $\widehat{\theta}_{MLE}$(**MSE**) | $\widehat{\theta}_{SEL}$(**MSE**) | $\widehat{\theta}_{\text{Pre}}$(**MSE**) | $\widehat{\theta}_{LIN(\lambda=2)}$(**MSE**) | $\widehat{\theta}_{LIN(\lambda=-1)}$(**MSE**) | $\widehat{\theta}_{GE(\lambda=2)}$(**MSE**) | $\widehat{\theta}_{GE(\lambda=-1)}$(**MSE**) |
|---|---|---|---|---|---|---|---|---|---|
| 20 | 6 | I | 0.5061(**0.000836**) | 0.5272(**0.000867**) | 0.5732(**0.001189**) | 0.4722(**0.000568**) | 0.5548(**0.001108**) | 0.5031(**0.000808**) | 0.5959(**0.001603**) |
| | | II | 0.1047(**0.004587**) | 0.1298(**0.004651**) | 0.1363(**0.004527**) | 0.1226(**0.004504**) | 0.1303(**0.004618**) | 0.1051(**0.00459**) | 0.152(**0.005917**) |
| | | III | 0.1733(**0.003462**) | 0.2035(**0.003426**) | 0.2172(**0.003353**) | 0.1888(**0.003294**) | 0.208(**0.00346**) | 0.1736(**0.003464**) | 0.2336(**0.00424**) |
| 20 | 14 | I | 0.5063(**0.000104**) | 0.5144(**0.000104**) | 0.5341(**0.000118**) | 0.4923(**0.000086**) | 0.5255(**0.000115**) | 0.5053(**0.000102**) | 0.5402(**0.000134**) |
| | | II | 0.2407(**0.000389**) | 0.2496(**0.00037**) | 0.2584(**0.000353**) | 0.2442(**0.000368**) | 0.2523(**0.000377**) | 0.2407(**0.000388**) | 0.2567(**0.000363**) |
| | | III | 0.3196(**0.000237**) | 0.3294(**0.000224**) | 0.3412(**0.000211**) | 0.3199(**0.000220**) | 0.334(**0.000231**) | 0.3195(**0.000237**) | 0.3411(**0.00022**) |
| 40 | 30 | I | 0.5092(**0.000015**) | 0.513(**0.000015**) | 0.5223(**0.000016**) | 0.5028(**0.000014**) | 0.518(**0.000016**) | 0.5088(**0.000015**) | 0.5245(**0.000017**) |
| | | II | 0.2285(**0.000085**) | 0.2321(**0.000083**) | 0.236(**0.000081**) | 0.2302(**0.000084**) | 0.2331(**0.000082**) | 0.2285(**0.000085**) | 0.2344(**0.000082**) |
| | | III | 0.3021(**0.000049**) | 0.3063(**0.000048**) | 0.3115(**0.000046**) | 0.3028(**0.000049**) | 0.308(**0.000047**) | 0.3021(**0.000049**) | 0.3103(**0.000046**) |
| 60 | 40 | I | 0.5071(**4E-6**) | 0.5099(**4E-6**) | 0.5169(**4E-6**) | 0.5024(**4E-6**) | 0.5137(**4E-6**) | 0.5068(**4E-6**) | 0.5184(**4E-6**) |
| | | II | 0.1796(**0.000065**) | 0.1819(**0.000064**) | 0.1842(**0.000063**) | 0.181(**0.000064**) | 0.1823(**0.000064**) | 0.1796(**0.000065**) | 0.1829(**0.000064**) |
| | | III | 0.2483(**0.000041**) | 0.2511(**0.00004**) | 0.2543(**0.000039**) | 0.2494(**0.000041**) | 0.2519(**0.00004**) | 0.2483(**0.000041**) | 0.253(**0.00004**) |

Table 3. Average estimates and **MSEs** of $\theta$, for $T_3 = (X_m + 2)$ and different choices of $n, m$, and $R$

| n | m | CS | $\widehat{\theta}_{MLE}$(**MSE**) | $\widehat{\theta}_{SEL}$(**MSE**) | $\widehat{\theta}_{\text{Pre}}$(**MSE**) | $\widehat{\theta}_{LIN(\lambda=2)}$(**MSE**) | $\widehat{\theta}_{LIN(\lambda=-1)}$(**MSE**) | $\widehat{\theta}_{GE(\lambda=2)}$(**MSE**) | $\widehat{\theta}_{GE(\lambda=-1)}$(**MSE**) |
|---|---|---|---|---|---|---|---|---|---|
| 20 | 6 | I | 0.556(**0.000422**) | 0.5714(**0.000452**) | 0.6097(**0.000625**) | 0.5236(**0.000275**) | 0.5953(**0.000575**) | 0.5533(**0.000407**) | 0.6301(**0.000802**) |
| | | II | 0.5768(**0.000562**) | 0.7521(**0.001986**) | 0.7991(**0.002526**) | 0.6376(**0.00085**) | 0.7834(**0.002362**) | 0.5804(**0.000579**) | 0.8216(**0.007263**) |
| | | III | 0.5695(**0.000509**) | 0.6511(**0.000975**) | 0.6980(**0.001350**) | 0.5726(**0.000461**) | 0.6824(**0.001246**) | 0.5693(**0.000506**) | 0.801(**0.002713**) |
| 20 | 14 | I | 0.5452(**0.000109**) | 0.5532(**0.000112**) | 0.5746(**0.000136**) | 0.5276(**0.000085**) | 0.566(**0.000129**) | 0.5439(**0.000107**) | 0.5831(**0.000156**) |
| | | II | 0.5574(**0.000131**) | 0.5836(**0.000164**) | 0.6089(**0.000204**) | 0.5499(**0.000114**) | 0.5995(**0.000193**) | 0.5567(**0.00013**) | 0.6382(**0.000283**) |
| | | III | 0.5521(**0.000121**) | 0.5679(**0.000136**) | 0.5912(**0.000167**) | 0.5387(**0.000098**) | 0.5822(**0.000158**) | 0.551(**0.000119**) | 0.6086(**0.00021**) |
| 40 | 30 | I | 0.5349(**0.000017**) | 0.5386(**0.000017**) | 0.5485(**0.000018**) | 0.5274(**0.000015**) | 0.2146(**0.000015**) | 0.5344(**0.000016**) | 0.2145(**0.000015**) |
| | | II | 0.5445(**0.000021**) | 0.5543(**0.000023**) | 0.5657(**0.000025**) | 0.5404(**0.000019**) | 0.561(**0.000025**) | 0.5441(**0.000021**) | 0.575(**0.000029**) |
| | | III | 0.5403(**0.000019**) | 0.5467(**0.00002**) | 0.5572(**0.000022**) | 0.5342(**0.000017**) | 0.5528(**0.000021**) | 0.5399(**0.000019**) | 0.5629(**0.000024**) |
| 60 | 50 | I | 0.5265(**5E-6**) | 0.5288(**5E-6**) | 0.5346(**5E-6**) | 0.5223(**5E-6**) | 0.532(**5E-6**) | 0.5263(**5E-6**) | 0.5361(**5E-6**) |
| | | II | 0.5314(**6E-6**) | 0.5355(**6E-6**) | 0.5419(**6E-6**) | 0.5283(**5E-6**) | 0.5391(**6E-6**) | 0.5312(**6E-6**) | 0.5451(**7E-6**) |
| | | III | 0.5291(**5E-6**) | 0.5321(**5E-6**) | 0.5382(**6E-6**) | 0.5253(**5E-6**) | 0.5355(**5E-6**) | 0.5288(**5E-6**) | 0.5405(**5E-6**) |

Table 4. Average estimates and **MSEs** of $\beta$, for $T_1 = X_{\frac{m}{2}}$ and different choices of $n, m$, and $R$

| n | m | CS | $\widehat{\beta}_{\text{MLE}}$(**MSE**) | $\widehat{\beta}_{\text{SEL}}$(**MSE**) | $\widehat{\beta}_{\text{Pre}}$(**MSE**) | $\widehat{\beta}_{\text{LIN}(\lambda=2)}$(**MSE**) | $\widehat{\beta}_{\text{LIN}(\lambda=-1)}$(**MSE**) | $\widehat{\beta}_{\text{GE}(\lambda=2)}$(**MSE**) | $\widehat{\beta}_{\text{GE}(\lambda=-1)}$(**MSE**) |
|---|---|---|---|---|---|---|---|---|---|
| 20 | 6 | I | 0.212(**0.000704**) | 0.207(**0.000625**) | 0.237(**0.000836**) | 0.198(**0.000543**) | 0.212(**0.000675**) | 0.212(**0.000697**) | 0.211(**0.000660**) |
| | | II | 0.466(**0.002256**) | 0.436(**0.001808**) | 0.5(**0.0027905**) | 0.409(**0.001440**) | 0.452(**0.002034**) | 0.464(**0.002212**) | 0.443(**0.001901**) |
| | | III | 0.430(**0.001979**) | 0.405(**0.001608**) | 0.463(**0.002455**) | 0.380(**0.001289**) | 0.418(**0.001803**) | 0.428(**0.001942**) | 0.411(**0.001697**) |
| 20 | 14 | I | 0.228(**0.000110**) | 0.225(**0.000104**) | 0.239(**0.000120**) | 0.221(**0.000097**) | 0.2273(**0.000108**) | 0.228(**0.000110**) | 0.227(**0.000107**) |
| | | II | 0.447(**0.000363**) | 0.437(**0.000336**) | 0.461(**0.000402**) | 0.427(**0.000307**) | 0.443(**0.000352**) | 0.446(**0.000361**) | 0.441(**0.000345**) |
| | | III | 0.387(**0.000280**) | 0.379(**0.000260**) | 0.401(**0.000310**) | 0.37(**0.000238**) | 0.384(**0.000272**) | 0.386(**0.000278**) | 0.382(**0.000267**) |
| 40 | 30 | I | 0.218(**0.000016**) | 0.217(**0.000015**) | 0.223(**0.000017**) | 0.215(**0.000015**) | 0.2179(**0.000016**) | 0.218(**0.000016**) | 0.218(**0.000016**) |
| | | II | 0.480(**0.000090**) | 0.475(**0.000087**) | 0.486(**0.000094**) | 0.469(**0.000084**) | 0.478(**0.000089**) | 0.479(**0.000089**) | 0.476(**0.000088**) |
| | | III | 0.422(**0.000067**) | 0.419(**0.000065**) | 0.429(**0.000071**) | 0.414(**0.000063**) | 0.421(**0.000067**) | 0.422(**0.000067**) | 0.42(**0.000066**) |
| 60 | 40 | I | 0.2165(**4E-6**) | 0.216(**4E-6**) | 0.2197(**4E-6**) | 0.2151(**4E-6**) | 0.2165(**4E-6**) | 0.2165(**4E-6**) | 0.2164(**4E-6**) |
| | | II | 0.4751(**0.000031**) | 0.4723(**0.000031**) | 0.4792(**0.000032**) | 0.469(**0.000030**) | 0.474(**0.000031**) | 0.4748(**0.000031**) | 0.4733(**0.000031**) |
| | | III | 0.4035(**0.000021**) | 0.4015(**0.00002**) | 0.4075(**0.000021**) | 0.3989(**0.00002**) | 0.4028(**0.000021**) | 0.4035(**0.000021**) | 0.4023(**0.000021**) |

Table 5. Average estimates and **MSEs** of $\beta$, for $T_2 = X_{\frac{(4\times m)}{5}}$ and different choices of $n, m$, and $R$

| n | m | CS | $\widehat{\beta}_{\text{MLE}}$(**MSE**) | $\widehat{\beta}_{\text{SEL}}$(**MSE**) | $\widehat{\beta}_{\text{Pre}}$(**MSE**) | $\widehat{\beta}_{\text{LIN}(\lambda=2)}$(**MSE**) | $\widehat{\beta}_{\text{LIN}(\lambda=-1)}$(**MSE**) | $\widehat{\beta}_{\text{GE}(\lambda=2)}$(**MSE**) | $\widehat{\beta}_{\text{GE}(\lambda=-1)}$(**MSE**) |
|---|---|---|---|---|---|---|---|---|---|
| 20 | 6 | I | 0.2559(**0.000895**) | 0.2479(**0.000776**) | 0.2834(**0.001104**) | 0.2361(**0.000649**) | 0.2545(**0.000852**) | 0.2551(**0.000883**) | 0.2531(**0.000828**) |
| | | II | 0.4748(**0.002299**) | 0.4394(**0.001777**) | 0.5099(**0.002874**) | 0.4097(**0.001384**) | 0.4565(**0.002026**) | 0.4718(**0.002248**) | 0.4454(**0.001867**) |
| | | III | 0.4415(**0.002055**) | 0.4123(**0.001633**) | 0.4759(**0.002573**) | 0.3858(**0.001288**) | 0.4273(**0.001848**) | 0.439(**0.002014**) | 0.4189(**0.001724**) |
| 20 | 14 | I | 0.2507(**0.000123**) | 0.2476(**0.000116**) | 0.2625(**0.000136**) | 0.2428(**0.000107**) | 0.2501(**0.000121**) | 0.2504(**0.000122**) | 0.2496(**0.000119**) |
| | | II | 0.4455(**0.000363**) | 0.4349(**0.000333**) | 0.4603(**0.000403**) | 0.4235(**0.000303**) | 0.4409(**0.00035**) | 0.4445(**0.00036**) | 0.438(**0.000342**) |
| | | III | 0.3879(**0.000276**) | 0.3799(**0.000255**) | 0.4019(**0.000308**) | 0.3706(**0.000233**) | 0.3847(**0.000268**) | 0.3872(**0.000274**) | 0.3828(**0.000262**) |
| 40 | 30 | I | 0.2303(**0.000017**) | 0.2292(**0.000017**) | 0.2357(**0.000018**) | 0.2274(**0.000016**) | 0.2302(**0.000017**) | 0.2302(**0.000017**) | 0.23(**0.000017**) |
| | | II | 0.4725(**0.000087**) | 0.4674(**0.000084**) | 0.4795(**0.000091**) | 0.4618(**0.00008**) | 0.4703(**0.000086**) | 0.472(**0.000087**) | 0.469(**0.000085**) |
| | | III | 0.4113(**0.000063**) | 0.4075(**0.000061**) | 0.418(**0.000067**) | 0.4029(**0.000059**) | 0.4098(**0.000062**) | 0.411(**0.000063**) | 0.4089(**0.000062**) |
| 60 | 40 | I | 0.2224(**4E-6**) | 0.2219(**4E-6**) | 0.2256(**4E-6**) | 0.2209(**4E-6**) | 0.2224(**4E-6**) | 0.2224(**4E-6**) | 0.2223(**4E-6**) |
| | | II | 0.4589(**0.000029**) | 0.4562(**0.000028**) | 0.4631(**0.00003**) | 0.453(**0.000029**) | 0.4578(**0.000029**) | 0.4587(**0.000029**) | 0.4571(**0.000029**) |
| | | III | 0.3869(**0.000019**) | 0.384(**0.000019**) | 0.3909(**0.00002**) | 0.3861(**0.000019**) | 0.3861(**0.000019**) | 0.3857(**0.000019**) | 0.3857(**0.000019**) |

Table 6. Average estimates and **MSEs** of $\beta$, for $T_3 = (X_m + 2)$ and different choices of $n, m$, and $R$

| n | m | CS | $\widehat{\beta}_{\text{MLE}}$(**MSE**) | $\widehat{\beta}_{\text{SEL}}$(**MSE**) | $\widehat{\beta}_{\text{Pre}}$(**MSE**) | $\widehat{\beta}_{\text{LIN}(\lambda=2)}$(**MSE**) | $\widehat{\beta}_{\text{LIN}(\lambda=-1)}$(**MSE**) | $\widehat{\beta}_{\text{GE}(\lambda=2)}$(**MSE**) | $\widehat{\beta}_{\text{GE}(\lambda=-1)}$(**MSE**) |
|---|---|---|---|---|---|---|---|---|---|
| 20 | 6 | I | 0.2398(**0.000424**) | 0.2346(**0.000381**) | 0.2609(**0.000499**) | 0.226(**0.000333**) | 0.2392(**0.000409**) | 0.2393(**0.00042**) | 0.2384(**0.000401**) |
| | | II | 0.2355(**0.000397**) | 0.2072(**0.000262**) | 0.2771(**0.000484**) | 0.1898(**0.000213**) | 0.2178(**0.000304**) | 0.234(**0.000385**) | 0.2097(**0.000259**) |
| | | III | 0.2360(**0.0004**) | 0.2211(**0.000315**) | 0.2672(**0.000482**) | 0.208(**0.000261**) | 0.2285(**0.000352**) | 0.2351(**0.000393**) | 0.2246(**0.000325**) |
| 20 | 14 | I | 0.2285(**0.000103**) | 0.2261(**0.000098**) | 0.2401(**0.000113**) | 0.2218(**0.000091**) | 0.2282(**0.000101**) | 0.2283(**0.000103**) | 0.2279(**0.00010**) |
| | | II | 0.2249(**0.000103**) | 0.2197(**0.000094**) | 0.2394(**0.000112**) | 0.2141(**0.000086**) | 0.2226(**0.000098**) | 0.2246(**0.000102**) | 0.2214(**0.000096**) |
| | | III | 0.2257(**0.000102**) | 0.2221(**0.000095**) | 0.2387(**0.000112**) | 0.2173(**0.000088**) | 0.2246(**0.000099**) | 0.2255(**0.000102**) | 0.2239(**0.000097**) |
| 40 | 30 | I | 0.2147(**0.000015**) | 0.2138(**0.000015**) | 0.2199(**0.000016**) | 0.2121(**0.000014**) | 0.2146(**0.000015**) | 0.2146(**0.000015**) | 0.2145(**0.000015**) |
| | | II | 0.2141(**0.000016**) | 0.2122(**0.000015**) | 0.2204(**0.000017**) | 0.2101(**0.000015**) | 0.2133(**0.000016**) | 0.2140(**0.000016**) | 0.2129(**0.000016**) |
| | | III | 0.2138(**0.000016**) | 0.2125(**0.000015**) | 0.2196(**0.000016**) | 0.2106(**0.000015**) | 0.2134(**0.000015**) | 0.2137(**0.000016**) | 0.2132(**0.000015**) |
| 60 | 40 | I | 0.2127(**4E-6**) | 0.2122(**4E-6**) | 0.2158(**4E-6**) | 0.2113(**4E-6**) | 0.2127(**4E-6**) | 0.2127(**4E-6**) | 0.2126(**4E-6**) |
| | | II | 0.2121(**4E-6**) | 0.2113(**4E-6**) | 0.2156(**4E-6**) | 0.2102(**4E-6**) | 0.2119(**4E-6**) | 0.2121(**4E-6**) | 0.2117(**4E-6**) |
| | | III | 0.2122(**4E-6**) | 0.2116(**4E-6**) | 0.2155(**4E-6**) | 0.2106(**4E-6**) | 0.2121(**4E-6**) | 0.2122(**4E-6**) | 0.212(**4E-6**) |

Due to the length of the article, we present only some of the simulation results, the other results are similar. In this simulation, we computed the average values and the MSEs of the $MLE$ and the Bayesian estimates for $\theta$ and $\beta$. All values are reported in Tables 1-6.

- It is clearly obvious that the MSE values for all censoring schemes are consistently small. Moreover, as the failure rate $\frac{m}{n}$ increases, the MSE values decrease and approach zero.
- In terms of MSE values, the Bayesian estimates based on $Linex$ and $GE$ loss functions indicate that the choice of $\lambda = 2$ is no different than the choice of $\lambda = -1$ except when m is small $(= 6\&14)$.
- For fixed n and m, there is no significant distinction in terms of the MSE values between the estimates with different predetermined termination time T.
- Bayes estimates outperform the $MLEs$ in terms of bias and MSE values.
- The Bayes estimate based on $Linex$ loss function with $\lambda = 2$ outperforms all competing estimates including the $MLE$, in terms of absolute bias and MSE values. Moreover, the Bayesian estimates based on $GE$ with $\lambda = 2$ is quite close to the $MLE$ in terms of absolute bias and MSE values.
- For fixed $n, m$ and $T$, estimates based on Scheme I perform better compared to Schemes II and III, in terms of bias and MSE values.

## 6. Real life data

In this section, we consider a real life data to demonstrate the proposed method and verify how our estimates work in practice. The data for this application was obtained from a meteorological study by Simpson (1972). The data represents radar-evaluated rainfall from 52 south Florida cumulus clouds. The data were utilized by Helu et al. (2015). The legitimacy of Lomax model is checked based on $\theta = 1.3055$ and $\beta = 0.00664$ using Kolmogrov-Smirnov ($K$-$S$) test, as well as Anderson-Darling ($A$-$D$) and chi-square tests. It is observed that $K$-$S = 0.0952$ with $p_{value} = 0.6979$, $A$-$D = 2.5018$ and chi-square distance $= 4.4278$ with a corresponding $p_{value} = 0.48961$. This indicates that the Lomax model provides a good fit to the data. In addition, Figure 2 gives the histogram of the data-set and the plots of the fitted density. The QQ plot in Figure 3 suggests that $Lomax$ is very suitable for the meteorological data.

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 0 | 4.1 | 4.9 | 4.9 | 7.7 | 11.5 | 17.3 | 17.5 | 21.7 | 24.4 |
| 26.1 | 26.3 | 28.6 | 29.0 | 31.4 | 32.7 | 36.6 | 40.6 | 41.1 | 47.3 |
| 68.5 | 81.2 | 87 | 92.4 | 95 | 115.3 | 118.3 | 119 | 129.6 | 147.8 |
| 163 | 198.6 | 200.7 | 242.5 | 244.3 | 255 | 274.7 | 274.7 | 302.8 | 321.2 |
| 334.1 | 345.5 | 372.4 | 430 | 489.1 | 703.4 | 830.1 | 978 | 1202.6 | 1656 |
| 1697.8 | 2745.6 | | | | | | | | |

From the given data we created three different artificial $Adaptive$-IIHP censored data. The three applied censoring schemes are the same as those in Section 6 and they are as follows:

- **Scheme I**: $n = 52$, $m = 30$, $R = \{22, 0^{*29}\}, T_1 = X_{26}$.
- **Scheme II**: $n = 52$, $m = 30$, $R = \{0^{*29}, 22\}, T_2 = X_{42}$.
- **Scheme III**: $n = 52$, $m = 30$, $R = \{11, 0^{*28}, 11\}, T_3 = 2X_{30}$.

For producing the Bayes estimators, a non-informative prior is applied since we have no information about the priors, thus we choose $a = b = c = d = 0$. The generated data can be found in Table 7.

Meanwhile, the influence of overestimation and underestimation is explicit, so it is decided to use $\lambda = -1\&2$ for the $Linex$ and the general entropy loss functions. The estimates based on the censored data can be found in Table 8. In Table 8, the values of the proposed estimates for $\beta$ are almost identical to the $MLE$ and coincide with the $MLE$ of $\beta$ based on complete data. In addition, for fixed scheme, the Bayes estimates of $\theta$ are close to each other and comparable with the $MLE$, except for the general entropy when $\lambda = -1$.

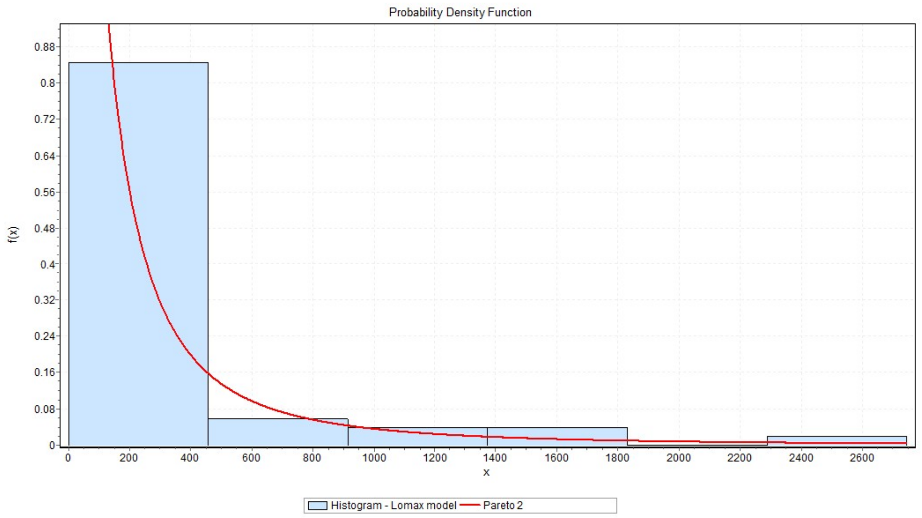


Figure 2. The histogram of the data set and its fitted density function to the meteorological data.

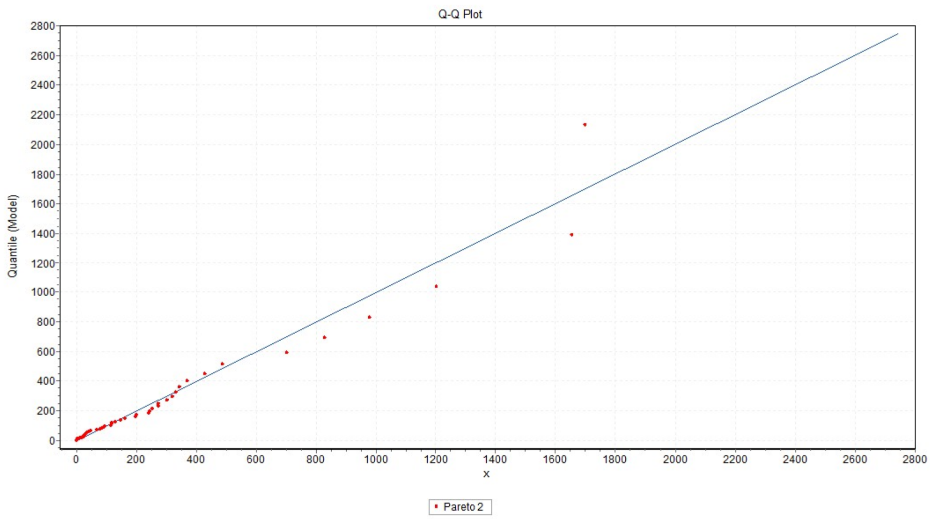


Figure 3. Plot of the empirical quantile of Lomax distribution fitted to the meteorological data.

It is of great importance to notice through this analysis that the Bayes estimates and the $MLE$ based on Scheme I, are very close to those of the complete data set. Moreover, we prefer the $Linex$ and the $GE$ when $\lambda = 2$ to estimate the parameters of the Lomax distribution based on adaptive type II progressive hybrid censored data of the radar-evaluated rainfall from 52 south Florida cumulus clouds.

## 7. Conclusions and recommendations

In this article, the authors considered the Adaptive Type-II Progressive Hybrid censoring data from Lomax distribution. This type of censoring satisfies the experiment time limitation and can also be used to simulate different practical situations.

Table 7. The meteorological study data

| scheme | censored data |
|---|---|
| $I$ | 0, 4.1, 4.9, 4.9, 7.7, 11.5, 17.3, 17.5, 21.7, 26.3, 31.4, 32.7, 115.3,118.3, 119, 129.6, 147.8, 163,198.6, 200.7,242.5, 244.3, 255, 274.7, 274.7, 302.8, 321.2, 334.1, 345.5, 372.4 |
| $II$ | 0, 4.1, 4.9, 4.9, 7.7, 11.5, 17.3, 17.5, 21.7, 24.4,26.1, 26.3, 28.6, 29.0, 31.4, 32.7, 36.6, 40.6, 41.1, 47.3, 68.5, 81.2, 87, 92.4, 95, 115.3, 118.3, 119, 129.6, 147.8 |
| $III$ | 0, 28.6, 29.0, 31.4, 32.7, 36.6, 40.6, 41.1, 47.3, 68.5, 81.2, 87, 92.4, 95, 115.3, 118.3, 119, 129.6, 147.8, 163, 198.6, 200.7, 242.5, 244.3, 255, 274.7, 274.7, 302.8, 321.2, 334.1 |

Table 8. The meteorological study data

| | scheme I | | scheme II | | scheme III | |
|---|---|---|---|---|---|---|
| Method | $\theta$ | $\beta$ | $\theta$ | $\beta$ | $\theta$ | $\beta$ |
| $MLE$ | 1.32907 | 0.01 | 0.69270 | 0.01654 | 0.75799 | 0.01 |
| $SEL$ | 1.39084 | 0.00980 | 0.73875 | 0.01601 | 0.79203 | 0.00979 |
| $PRE$ | 1.42460 | 0.01041 | 0.75801 | 0.01732 | 0.81084 | 0.01038 |
| $LIN\,(\lambda = 2)$ | 1.29326 | 0.00979 | 0.70803 | 0.01599 | 0.76073 | 0.00978 |
| $LIN\,(\lambda = -1)$ | 1.43452 | 0.00980 | 0.75241 | 0.01602 | 0.80649 | 0.00979 |
| $GE\,(\lambda = 2)$ | 1.32300 | 0.01 | 0.69396 | 0.01654 | 0.75825 | 0.01 |
| $GE\,(\lambda = -1)$ | 1.74916 | 0.00980 | 0.81570 | 0.01601 | 0.86404 | 0.00979 |

Two types of inference procedures are considered; the $MLE$ and the Bayesians ($SQR$, $PRE$, $Linex$, $GE$) to estimate the unknown parameters of the Lomax distribution. The Bayes estimates of the unknown parameters are computed based on symmetrical and asymmetrical loss functions. As the Bayes estimates are difficult to obtain by direct calculations, Lindley's approximation method is applied under the assumption of gamma priors.

It is clear from the simulation that Scheme I is superior to the other schemes as it provides the smallest MSE and the smallest bias. It is also observed that the estimates under the $Linex$ loss function outperform all other estimates.

On the whole, the Bayes estimates under the $Linex$ loss function are recommended for estimating the shape and scale parameters of the Lomax distribution based on Adaptive Type-II Progressive Hybrid censoring and based on Scheme I.

## 1. Appendix

The entries for Lindley's approximation are given by the following equations

$$\widehat{\sigma} = \begin{bmatrix} \widehat{\sigma}_{\theta\theta} & \widehat{\sigma}_{\theta\beta} \\ \widehat{\sigma}_{\beta\theta} & \widehat{\sigma}_{\beta\beta} \end{bmatrix} = \begin{bmatrix} -\frac{\partial^2 l}{\partial\theta^2}\big|_{\theta=\widehat{\theta},\beta=\widehat{\beta}} & -\frac{\partial^2 l}{\partial\theta\partial\beta}\big|_{\theta=\widehat{\theta},\beta=\widehat{\beta}} \\ -\frac{\partial^2 l}{\partial\beta\partial\theta}\big|_{\theta=\widehat{\theta},\beta=\widehat{\beta}} & -\frac{\partial^2 l}{\partial\beta^2}\big|_{\theta=\widehat{\theta},\beta=\widehat{\beta}} \end{bmatrix}$$

$$\begin{aligned}
\widehat{l}_{\theta\theta} &= \frac{m\widehat{\beta}}{\widehat{\theta}^2} - \\
&\sum_{i=1}^{m}\left[\frac{(\widehat{\beta}+1)\widehat{\beta}x_i}{(\widehat{\theta}x_i)^{2+\widehat{\beta}}} + \frac{R_i(\widehat{\beta}+1)\widehat{\beta}e^{-(\widehat{\theta}x_{i:})^{-\widehat{\beta}}}x_i}{(\widehat{\theta}x_i)^{2+\widehat{\beta}}\left(1-e^{-(\widehat{\theta}x_{i:})^{-\widehat{\beta}}}\right)} - \frac{R_i\widehat{\beta}^2e^{-(\widehat{\theta}x_i)^{-\widehat{\beta}}}x_i}{(\widehat{\theta}x_i)^{2+2\widehat{\beta}}\left(1-e^{-(\widehat{\theta}x_{i:})^{-\widehat{\beta}}}\right)^2}\right],
\end{aligned}$$

$$\begin{aligned}
\widehat{l}_{\theta\beta} &= \frac{-m}{\widehat{\theta}} - \\
&\sum_{i=1}^{m}\left[\frac{x_i(\widehat{\beta}\log(\widehat{\theta}x_i)-1)}{(\widehat{\theta}x_i)^{1+\widehat{\beta}}} - \frac{R_ix_ie^{-(\widehat{\theta}x_i)^{-\widehat{\beta}}}\widehat{\beta}\log(\widehat{\theta}x_i)}{(\widehat{\theta}x_i)^{1+2\widehat{\beta}}\left(1-e^{-(\widehat{\theta}x_i)^{-\widehat{\beta}}}\right)^2} + \frac{R_ix_ie^{-(\widehat{\theta}x_i)^{-\widehat{\beta}}}(\widehat{\beta}\log(\widehat{\theta}x_i)-1)}{(\widehat{\theta}x_i)^{1+\widehat{\beta}}\left(1-e^{-(\widehat{\theta}x_i)^{-\widehat{\beta}}}\right)}\right]
\end{aligned}$$

$$\begin{aligned}
\widehat{l}_{\theta\theta\theta} &= \frac{-2m\widehat{\beta}}{\widehat{\theta}^3} + \sum_{i=1}^{m}\left[\frac{(\widehat{\beta}+1)\widehat{\beta}(\widehat{\beta}+2)x_i^3}{(\widehat{\theta}x_i)^{3+\widehat{\beta}}} - \frac{R_i\widehat{\beta}^2(\widehat{\beta}+1)e^{-(\widehat{\theta}x_i)^{-\widehat{\beta}}}x_i}{(\widehat{\theta}x_i)^{3+2\widehat{\beta}}\left(1-e^{-(\widehat{\theta}x_i)^{-\widehat{\beta}}}\right)^2}\right] \\
&-\sum_{i=1}^{m}\left[\frac{e^{-(\widehat{\theta}x_i)^{-\widehat{\beta}}}R_ix_{i:}^3\widehat{\beta}(\widehat{\beta}+1)(\widehat{\beta}+2)}{(\widehat{\theta}x_i)^{3+\widehat{\beta}}\left(1-e^{-(\widehat{\theta}x_i)^{-\widehat{\beta}}}\right)} + \frac{R_i\widehat{\beta}^3e^{-(\widehat{\theta}x_{i:})^{-\widehat{\beta}}}x_i^3(1-3e^{-(\widehat{\theta}x_{i:})^{-\widehat{\beta}}})}{(\widehat{\theta}x_i)^{3+3\widehat{\beta}}\left(1-e^{-(\widehat{\theta}x_i)^{-\widehat{\beta}}}\right)^3}\right].
\end{aligned}$$

$$\widehat{l}_{\beta\beta} = \frac{-m}{\widehat{\beta}^2} - \sum_{i=1}^{m}\left[\frac{\log(\widehat{\theta}x_i)^2}{(\widehat{\theta}x_i)^{\widehat{\beta}}} - \frac{R_ie^{-(\widehat{\theta}x_i)^{-\widehat{\beta}}}\log(\widehat{\theta}x_i)^2(1-e^{-(\widehat{\theta}x_i)^{-\widehat{\beta}}}-(\widehat{\theta}x_i)^{-\widehat{\beta}})}{(\widehat{\theta}x_i)^{\widehat{\beta}}\left(1-e^{-(\widehat{\theta}x_i)^{-\widehat{\beta}}}\right)^2}\right]$$

$$\begin{aligned}
\widehat{l}_{\theta\theta\beta} &= \frac{m}{\widehat{\theta}^2} - \\
&\sum_{i=1}^{m}\left[\frac{x_i[1+2\widehat{\beta}-\widehat{\beta}(\widehat{\beta}+1)\log(\widehat{\theta}x_i)]}{(\widehat{\theta}x_i)^{2+\widehat{\beta}}} + \frac{R_ix_i\widehat{\beta}^2e^{-(\widehat{\theta}x_i)^{-\widehat{\beta}}}(1-3e^{-(\widehat{\theta}x_i)^{-\widehat{\beta}}})\log(\widehat{\theta}x_i)}{(\widehat{\theta}x_i)^{2+3\widehat{\beta}}(1-e^{-(\widehat{\theta}x_i)^{-\widehat{\beta}}})^3}\right] \\
&+\sum_{i=1}^{m}\left[\frac{R_ie^{-(\widehat{\theta}x_i)^{-\widehat{\beta}}}x_i(1+2\widehat{\beta}-\widehat{\beta}(\widehat{\beta}+1)\log(\widehat{\theta}x_i))}{(\widehat{\theta}x_i)^{2+\widehat{\beta}}\left(1-e^{-(\widehat{\theta}x_i)^{-\widehat{\beta}}}\right)} + \frac{R_ix_i\widehat{\beta}e^{-(\widehat{\theta}x_i)^{-\widehat{\beta}}}(2+(1-\widehat{\beta})\log(\widehat{\theta}x_i))}{(\widehat{\theta}x_i)^{2+2\widehat{\beta}}(1-e^{-(\widehat{\theta}x_i)^{-\widehat{\beta}}})^2}\right]
\end{aligned}$$


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REFERENCES


1. Arnold, B.C., *The pareto distributions*, International Co-operative publishing house, Fairland, MD., 1983.
2. Balakrishnan, N., & Asgharzadeh, A., *Inference for the scaled half-logistic distribution based on progressively Type-II censored samples*, Communications in Statistics-Theory and Methods, 34(1), 73-87, 2005.
3. Balakrishnan, N., & Cramer, E., *The art of progressive censoring*, New York: Springer, 2014.
4. Basu, A. P., & Ebrahimi, N., *Bayesian approach to life testing and reliability estimation using asymmetric loss function*, Journal of statistical planning and inference, 29(1-2), 21-31, 1991.
5. Bryson, M.C., *Heavy-tailed distributions: properties and tests*, Technometrics. 16, 61-68, 1974.
6. Calabria, R. and Pulcini, G., *Point estimation under asymmetric loss functions for left-truncated exponential samples*, Communications in Statistics Theory and Methods, 25, 585-600, 1996.
7. Chahkandi, M., and Ganjali, M, *On some lifetime distributions with decreasing failure rate*, Comput. Statist. Data Anal, 53, 4433-4440, 2009.
8. Chen, S., & Gui, W., *Statistical analysis of a lifetime distribution with a bathtub-shaped failure rate function under adaptive progressive type-II censoring*, Mathematics, 8(5), 670, 2020.
9. Childs, A., Balakrishnan, N. and Moshref, M., *Order statistics from non-identical right-truncated Lomax random variables with applications*, Statistical Papers, 42, 187-206, 2001.
10. Cui, W., Yan, Z., & Peng, X., *Statistical Analysis for Constant-Stress Accelerated Life Test With Weibull Distribution Under Adaptive Type-II Hybrid Censored Data*, IEEE Access, 7, 165336-165344, 2019.
11. Feynman, R.P., *Mr. Feynman goes to Washigton. Engineering and science. California Institute of Technology*, Pasadena, CA, 6-22, 1987.
12. David, H. A., & Nagaraja, H. N., *Order statistics, third edition*, Wiley: New York, NY., 2003.

13. Elfattah, A., Alaboud, F. and Alharby, A., *On Sample Size Estimation For Lomax Disrtibution*, Australian Journal of Basic and Applied Sciences, 4, 373-378, 2007.
14. El-Sherpieny, E. S. A., Almetwally, E. M., & Muhammed, H. Z., *Progressive Type-II hybrid censored schemes based on maximum product spacing with application to Power Lomax distribution*, Physica A: Statistical Mechanics and its Applications, 553, 124251, 2020.
15. Helu, A., & Samawi, H., *On Marginal Distributions under Progressive Type II Censoring: Similarity/Dissimilarity Properties*, Open Journal of Statistics, 7(4), 633-644, 2017.
16. Helu, A., Samawi, H., Rochani, H., Yin, J., & Vogel, R., *Kernel density estimation based on progressive type-II censoring*, Journal of the Korean Statistical Society, 49(2), 475-498, 2020.
17. Helu, A., Samawi, H., & Raqab, M. Z., *Estimation on Lomax progressive censoring using the EM algorithm*, Journal of Statistical Computation and Simulation, 85(5), 1035-1052, 2015.
18. Howlader, and H. Hossain, A., *Bayesian survival estimation of Pareto distribution of the second kind based on failure-censored data*, Computational statistics and data analysis, 38, 301-314, 2002.
19. Kohansal, A., & Shoaee, S., *Bayesian and classical estimation of reliability in a multicomponent stress-strength model under adaptive hybrid progressive censored data*, Statistical Papers, 1-51. 2019.
20. Kundu, D., & Joarder, A., *Analysis of Type-II progressively hybrid censored data*, Computational Statistics & Data Analysis, 50(10), 2509-2528, 2006.
21. Lindley, D. V., *Approximate bayesian methods*, Trabajos de estadística y de investigación operativa, 31(1), 223-245, 1980.
22. Lomax, K., *Business failures. Another example of the analysis of failure data*, J. Amer. Statist. Assoc. 49, 847-852. 1954.
23. Nassar, M., Abo-Kasem, O., Zhang, C., & Dey, S., *Analysis of Weibull distribution under adaptive type-II progressive hybrid censoring scheme*, Journal of the Indian Society for Probability and Statistics, 19(1), 25-65, 2018.
24. Ng, Hon Keung Tony, Debasis Kundu, and Ping Shing Chan, *Statistical analysis of exponential lifetimes under an adaptive Type-II*, progressive censoring scheme." Naval Research Logistics (NRL) 56, no. 8: 687-698, 2009.
25. Norstrom, J. G., *The use of precautionary loss functions in risk analysis*, IEEE Transactions on reliability, 45(3), 400-403, 1996.
26. Panahi, H., *Estimation methods for the generalized inverted exponential distribution under type ii progressively hybrid censoring with application to spreading of micro-drops data*, Communications in Mathematics and Statistics, 5(2), 159-174, 2017.
27. Panahi, H., & Asadi, S., *On adaptive progressive hybrid censored Burr type III distribution: application to the nano droplet dispersion data*, Quality Technology & Quantitative Management, 18(2), 179-201, 2021.
28. Simpson J., *Use of the gamma distribution in single-cloud rainfall analysis*, MonthlyWeather Rev., 100:309–312, 1972.
29. Varian, H. R., *A Bayesian approach to real estate assessment*, Studies in Bayesian econometric and statistics in Honor of Leonard J. Savage, 195-208, 1975.
30. Wang, L., *Inference for Weibull competing risks data under generalized progressive hybrid censoring*, IEEE Transactions on Reliability, 67(3), 998-1007, 2018.
31. Yan, Z., & Wang, N., *Statistical analysis based on adaptive progressive hybrid censored sample from alpha power generalized exponential distribution*, IEEE Access, 8, 54691-54697, 2020.
32. Ye, Z. S., Chan, P. S., Xie, M., & Ng, H. K. T. *Statistical inference for the extreme value distribution under adaptive Type-II progressive censoring schemes*, Journal of Statistical Computation and Simulation, 84(5), 1099-1114, 2014.
33. Zellner, A., *On assessing prior distributions and Bayesian regression analysis with g-prior distributions*, Bayesian inference and decision techniques, 1986.
34. Zheng, G., & Shi, Y. M., *Statistical analysis in constant-stress accelerated life tests for generalized exponential distribution based on adaptive type-II progressive hybrid censored data*, Chinese Journal of Applied Probability and Statistics, 29(4), 363-80, 2013.