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Polar Integers and Polar Integer Optimization

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Abstract This is a pioneering work, introducing a special class of complex numbers, wherein their absolute values and arguments given in a Polar coordinate system are integers, which when considered within the complex plane, constitute Unicentered Radial Lattice and similarly for quaternions and Euclidean $\mathbf{R}^2$ and $\mathbf{R}^3$ Spaces. The corresponding Optimization Problems are introduced as well.

Keywords Complex plane; Integer Lattice; Optimization; Polar Coordinate System; Target Function; Quaternions

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1. Introduction

Its well-known in number theory a complex number whose real and imaginary parts are both integers: Gaussian Integer. The Gaussian integers are the set: $\mathbf{Z}[\mathbf{i}] := \{a + b\mathbf{i} \mid a, b \in \mathbf{Z}\}$, where $\mathbf{i}^2 = -1$. Gaussian integers are closed under addition and multiplication and form commutative ring, which is a subring of the field of complex numbers. When considered within the complex plane the Gaussian integers constitute the 2- dimensional integer lattice. The Gaussian integers form unique factorization domain: it is irreducible if and only if it is a prime(Gaussian primes). The field of Gaussian rationals consists of the complex numbers whose real and imaginary part are both rational (see, e.g., [11]).
The norm of a Gaussian integer is its product with its conjugate:

$$N(a + b\mathbf{i}) = (a + b\mathbf{i})(a - b\mathbf{i}) = a^2 + b^2.$$

The norm is multiplicative, that is, one has:

$$N(zw) = N(z)N(w), \quad z, w \in \mathbf{Z}[\mathbf{i}].$$

In [19] was introduced the following subset of the Gaussian Integers:
$\mathbf{Z}_\mathbf{P}[\mathbf{i}] := \{a + b\mathbf{i} \mid a, b \in \mathbf{P}\}$, where $\mathbf{P}$ is a subset of the Prime numbers, $\mathbf{Z}_\mathbf{P}[\mathbf{i}] \subset \mathbf{Z}[\mathbf{i}]$.
Another well-known integral subclass of complex numbers are Eisenshtein integers: complex numbers of the form: z = a + bω, where a and b are integers and $\omega^2 + \omega + 1 = 0$. The Eisenshtein integers form a triangular lattice in the complex plane, in contrast with Gaussian integers, which form a square lattice in the complex plane. The Eisenstein integers form a commutative ring as well and similar to Gaussian integers form a Euclidean domain,which supposes unique factorization of Eisenshtein integers into Eisenshtein primes. Similar integral subclasses can be defined for quaternions: Lipschitz and Hurwitz Integers(quaternions).
Quaternions are generally represented in the form: $q = a + b\mathbf{i} + c\mathbf{j} + d\mathbf{k}$, where, $a \in \mathbf{R},\ b \in \mathbf{R},\ c \in \mathbf{R},\ d \in \mathbf{R}$, and

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i, **j** and **k** are the fundamental quaternion units and are a number system that extends the complex numbers [2, 7]. The set of all quaternions **H** is a normed algebra, where the norm is multiplicative: $||pq\,|| = \,||p\,||\,||q\,||, p \in \mathbf{H}, q \in \mathbf{H},\ ||q\,||^2 = a^2 + b^2 + c^2 + d^2$.
This norm makes it possible to define the distance $d(p, q) = \,||p - q\,||$ which makes **H** into a metric space.
Lipschitz Integer(quaternion) is defined as:

$$\mathbf{L} := \{q : q = a + b\mathbf{i} + c\mathbf{j} + d\mathbf{k} \mid a, b, c, d \in \mathbf{Z}\}.$$

Lipschitz Integer (quaternion) is a quaternion, whose components are all integers.
In [19] was introduced the following subset of the Lipschitz Integers:

$$\mathbf{L_P} := \{q : q = a + b\mathbf{i} + c\mathbf{j} + d\mathbf{k} \mid a, b, c, d \in \mathbf{P}\}, \mathbf{L_P} \subset \mathbf{L}.$$

Hurwitz Integers(quaternion) are defined as:

$$\mathbf{HU} := \{q : q = a + b\mathbf{i} + c\mathbf{j} + d\mathbf{k} \mid a, b, c, d \in \mathbf{Z} + \frac{1}{2}\}.$$

and, correspondingly we can introduce:

$$\mathbf{HU_P} := \{q : q = a + b\mathbf{i} + c\mathbf{j} + d\mathbf{k} \mid a, b, c, d \in \mathbf{P} + \frac{1}{2}\}, \mathbf{HU_P} \subset \mathbf{HU}.$$

Thus, Hurwitz Integer (quaternion) is a quaternion, whose components are either all integers or all half-integers.
The initial motivation for the introduction of the Polar System was the study of circular and orbital motion. Polar Coordinates are used in navigation, e.g., air traffic control, modeling of radially symmetric systems, e.g., groundwater flow equation, gravitational fields, radio antennas. Radially asymmetric systems may also be modeled with Polar Coordinates: e.g., microphone's pickup pattern(see, e.g., [4]).
The Spherical Coordinate System is used in geography, astronomy, ergonomic design, 3D game development, 3-dimensional modeling of loudspeaker, partial differential equations, volume integrals, rotational matrices(see, e.g., [14]).
The Cylindrical Coordinate System is used for computation of water flow in a straight pipes with round cross-sections, heat distribution in a metal cylinder, electromagnetic fields, accretion disks in astronomy and so on(see, e.g., [13]).
It is well-known that an optimization problem can be represented in the following way: given a function $f : \mathbf{G} \to \mathbf{R}$ from some set **G** to the real numbers; sought: an element $x_0 \in \mathbf{G}$ such that $f(x_0) \leq f(x)$ for all $x \in \mathbf{G}$,("minimization"), or such that $f(x_0) \geq f(x)$ for all $x \in \mathbf{G}$ ("maximization").
Typically, **G** is some subset of the Euclidean space $\mathbf{R}^n$, specified by a set of constraints and the function f is called an objective function or target function.
Its well-known in Optimization Theory the case when **G** is some subset of integer points: Integer Optimization (see, e.g., [6, 9]). A general model of mixed-integer optimization could be written as: max/min $f(x)$ subject to $g_1(x) \leq 0, \ldots, g_m(x) \leq 0, x \in \mathbf{R}^k \times \mathbf{Z}^s$, where $f, g_1, \ldots, g_m : \mathbf{R}^n \to \mathbf{R}$ are arbitrary nonlinear functions.
In [18, 19] Complex, Quaternionic and Prime Optimization are considered.
The purpose of this paper is to introduce and describe novel subclasses of numbers: Polar Complex Integers, Polar Complex Hurwitz-like Integers, Polar Quaternionic Integers, Polar Quaternionic Hurwitz-like Integers, Polar Euclidean Integers, Polar Euclidean Hurwitz-like Integers, Spherical Euclidean Integers, Spherical Euclidean Hurwitz-like Integers, Cylindrical Euclidean Integers, Cylindrical Euclidean Hurwitz-like Integers and the corresponding optimization problems.

2. Polar Complex Integers

Let us introduce a new subclass of complex numbers and a new approach for their definition accordingly: Polar Complex Integers.

Its well-known for a complex number $z = Re(z) + Im(z)\mathbf{i} = a + b\mathbf{i},\ a \in \mathbf{R},\ b \in \mathbf{R}, \mathbf{i}^2 = -1$, to use an alternative option for coordinates in the complex plane: polar coordinate system that uses the distant of the point z from the origin and the angle, subtended between the positive real axis and the line segment in a counterclockwise sense(see, e.g., [3, 12, 16, 17]).
The absolute value of the complex number: $r = |z|$ is the distance to the origin of the point, representing the complex number z in the complex plane.
The argument of z: $\phi = arg(z)$, is the angle of the radius with the positive real axis . Note that there are two notations of angle ϕ: in degree and in radian.
Together, r and ϕ gives another way of representing complex numbers, the polar form. Recovering the original rectangular coordinates from the polar form is done by the formula called trigonometric form:

$$z = r(\cos\phi + \mathbf{i}\sin\phi).$$

Recall that addition of two complex numbers can be done geometrically by constructing the corresponding parallelogram.
Given two complex numbers: $z_1 = r_1(\cos\phi_1 + \mathbf{i}\sin\phi_1)$ and $z_2 = r_2(\cos\phi_2 + \mathbf{i}\sin\phi_2)$, multiplication of z_1 and z_2 in polar form is given by:

$$z_1 z_2 = r_1 r_2(\cos(\phi_1 + \phi_2) + \mathbf{i}\sin(\phi_1 + \phi_2)).$$

Similarly, division is given by:

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}(\cos(\phi_1 - \phi_2) + \mathbf{i}\sin(\phi_1 - \phi_2)).$$

Using polar form, let us introduce the following new subclass of complex numbers, Polar Complex Integers:

$$\mathbf{PZ}[\mathbf{i}] := \{z : z = r(\cos\phi + \mathbf{i}\sin\phi)\ |z \in \mathbf{C}, r \in \mathbf{Z}, \phi \in \mathbf{Z}, -180^\circ < \phi \le 180^\circ\}.$$

Theorem 1. *Polar Complex Integers are closed under multiplication.*
Proof. It follows from the formula:

$$z_1 z_2 = r_1 r_2(\cos(\phi_1 + \phi_2) + \mathbf{i}\sin(\phi_1 + \phi_2)).$$

Theorem 2. *Polar Complex Integers are not closed under addition.*
Proof. Let us consider $z_1 = 0 + 1\mathbf{i}$ and $z_2 = 1 + 0\mathbf{i}$. For degree notation, where $z_1 = 1(\cos 90^\circ + \mathbf{i}\sin 90^\circ)$ and $z_2 = 1(\cos 0^\circ + \mathbf{i}\sin 0^\circ)$, absolute value of $z_1 + z_2$ is an irrational number.
Theorem 3. *Polar Complex Integers are not closed under division.*
Proof. It follows from the formula:

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}(\cos(\phi_1 - \phi_2) + \mathbf{i}\sin(\phi_1 - \phi_2)).$$

Corollary 1. *Polar Complex Integers are mutually primes if and only if their absolute values are mutually primes.*
Theorem 4. *Polar Complex Integers form countable infinite set.*
Proof. It follows from the definition.
Similarly to aforementioned Hurwitz integers, let us introduce Polar Complex Hurwitz-like Integers:

$$\mathbf{PHU}[\mathbf{i}] := \{z : z = r(\cos\phi + \mathbf{i}\sin\phi)\ |z \in \mathbf{C}, r \in \mathbf{Z} + \frac{1}{2}, \phi \in \mathbf{Z} + \frac{1}{2}, -180^\circ < \phi \le 180^\circ\},$$

and similarly to aforementioned Gaussian Rationals, the corresponding set of Polar Complex Rationals can be introduced as well.
Theorem 5. *Polar Complex Hurwitz-like Integers form countable infinite set.*
Proof. It follows from the definition.
The corresponding Prime-subclasses can be introduced as well:

$$\mathbf{PZ_P}[\mathbf{i}] := \{z : z = r(\cos\phi + \mathbf{i}\sin\phi)\ |z \in \mathbf{C}, r \in \mathbf{P}, \phi \in \mathbf{P}, -180^\circ < \phi \le 180^\circ\}, \mathbf{PZ_P}[\mathbf{i}] \subset \mathbf{PZ}[\mathbf{i}],$$

$$\mathbf{PHU_P}[\mathbf{i}] := \{z : z = r(\cos\phi + \mathbf{i}\sin\phi)\ |z \in \mathbf{C}, r \in \mathbf{P} + \frac{1}{2}, \phi \in \mathbf{P} + \frac{1}{2}, -180^\circ < \phi \le 180^\circ\},$$

$\mathbf{PHU_P}[\mathbf{i}] \subset \mathbf{PHU}[\mathbf{i}]$.

3. Unicentered Radial Lattices of Polar Complex Integers and Polar Complex Hurwitz-like Integers

As we mentioned above, when considered within the complex plane, the Gaussian integers constitute the 2-dimensional integer lattice and the Eisenshtein integers form a triangular lattice in the complex plane, in contrast with Gaussian integers, which form a square lattice in the complex plane.
As it follows from the definition:

$$\mathbf{PZ}[\mathbf{i}] := \{z : z = r(\cos\phi + \mathbf{i}\sin\phi) \,|z \in \mathbf{C}, r \in \mathbf{Z}, \phi \in \mathbf{Z}, -180^\circ < \phi \leq 180^\circ\},$$

by fixing the integer radius $r \in \mathbf{Z}$, Polar Complex Integers, when considered within the complex plane, constitute Unicentered Radial Lattice.
Accordingly, for the Polar Complex Hurwitz-like Integers, as it follows from the definition :

$$\mathbf{PHU}[\mathbf{i}] := \{z : z = r(\cos\phi + \mathbf{i}\sin\phi) \,|z \in \mathbf{C}, r \in \mathbf{Z} + \frac{1}{2}, \phi \in \mathbf{Z} + \frac{1}{2}, -180^\circ < \phi \leq 180^\circ\},$$

by fixing the integer radius $r \in \mathbf{Z}$, Polar Complex Hurwitz-like Integers, when considered within the complex plane, constitute Unicentered Radial Lattice as well.

4. Polar Quaternionic Integers

Similarly, we can introduce Polar Quaternionic Integers.
Indeed, its well known to represent quaternions as pairs of complex numbers:

$$q = a + b\mathbf{i} + c\mathbf{j} + d\mathbf{k} \iff (a + b\mathbf{i}, c + d\mathbf{i}).\text{(Cayley-Dickson construction)}$$

Correspondingly, considering each of two parts in polar form:

$$a + b\mathbf{i} = r(\cos\phi + \mathbf{i}\sin\phi), c + d\mathbf{i} = \rho(\cos\psi + \mathbf{i}\sin\psi),$$

let us introduce Polar Quaternionic Integers:

$$\mathbf{PL} := \{q : q = a + b\mathbf{i} + c\mathbf{j} + d\mathbf{k} \iff (a + b\mathbf{i}, c + d\mathbf{i}), a + b\mathbf{i} = r(\cos\phi + \mathbf{i}\sin\phi),$$
$$c + d\mathbf{i} = \rho(\cos\psi + \mathbf{i}\sin\psi) \,|q \in \mathbf{H}, r \in \mathbf{Z}, \phi \in \mathbf{Z}, \rho \in \mathbf{Z}, \psi \in \mathbf{Z}, -180^\circ < \phi \leq 180^\circ, -180^\circ < \psi \leq 180^\circ\},$$

and Polar Quaternionic Hurwitz-like Integers:
$\mathbf{PHHU} := \{q : q = a + b\mathbf{i} + c\mathbf{j} + d\mathbf{k} \iff (a + b\mathbf{i}, c + d\mathbf{i}), a + b\mathbf{i} = r(\cos\phi + \mathbf{i}\sin\phi), c + d\mathbf{i} = \rho(\cos\psi + \mathbf{i}\sin\psi) \mid q \in \mathbf{H}, r \in \mathbf{Z} + \frac{1}{2}, \phi \in \mathbf{Z} + \frac{1}{2}, \rho \in \mathbf{Z} + \frac{1}{2}, \psi \in \mathbf{Z} + \frac{1}{2}, -180^\circ < \phi \leq 180^\circ, -180^\circ < \psi \leq 180^\circ\}$.
The corresponding Prime-subclasses can be introduced as well:
$\mathbf{PL_P} := \{q : q = a + b\mathbf{i} + c\mathbf{j} + d\mathbf{k} \iff (a + b\mathbf{i}, c + d\mathbf{i}), a + b\mathbf{i} = r(\cos\phi + \mathbf{i}\sin\phi), c + d\mathbf{i} = \rho(\cos\psi + \mathbf{i}\sin\psi) \mid q \in \mathbf{H}, r \in \mathbf{P}, \phi \in \mathbf{P}, \rho \in \mathbf{P}, \psi \in \mathbf{P}, -180^\circ < \phi \leq 180^\circ, -180^\circ < \psi \leq 180^\circ\}$, $\mathbf{PL_P} \subset \mathbf{PL}$,
$\mathbf{PHHU_P} := \{q : q = a + b\mathbf{i} + c\mathbf{j} + d\mathbf{k} \iff (a + b\mathbf{i}, c + d\mathbf{i}), a + b\mathbf{i} = r(\cos\phi + \mathbf{i}\sin\phi), c + d\mathbf{i} = \rho(\cos\psi + \mathbf{i}\sin\psi) \mid q \in \mathbf{H}, r \in \mathbf{P} + \frac{1}{2}, \phi \in \mathbf{P} + \frac{1}{2}, \rho \in \mathbf{P} + \frac{1}{2}, \psi \in \mathbf{P} + \frac{1}{2}, -180^\circ < \phi \leq 180^\circ, -180^\circ < \psi \leq 180^\circ\}$,
$\mathbf{PHHU_P} \subset \mathbf{PHHU}$,
and, similarly to aforementioned Gaussian Rationals, the corresponding set of Polar Quaternion Rationals can be introduced as well.

5. Polar Euclidean Integers

Using Polar Coordinate System, let us introduce the following novel subclass of Integer numbers for Euclidean two-dimensional Space $\mathbf{R}^2$: Polar Euclidean Integers:

$$\mathbf{PR}^2\mathbf{Z} := \{(x, y) : x = r\cos\phi, y = r\sin\phi \,|(x, y) \in \mathbf{R}^2, r \in \mathbf{Z}, \phi \in \mathbf{Z}, -180^\circ < \phi \leq 180^\circ\},$$

and Polar Hurwitz-like Integers:

$$\mathbf{PR^2HU} := \{(x, y) : x = r\cos\phi, y = r\sin\phi \,|(x, y) \in \mathbf{R}^2, r \in \mathbf{Z} + \frac{1}{2}, \phi \in \mathbf{Z} + \frac{1}{2}, -180^\circ < \phi \leq 180^\circ\},$$

and the corresponding Prime-subclasses can be introduced as well:

$$\mathbf{PR^2Z_P} := \{(x, y) : x = r\cos\phi, y = r\sin\phi \,|(x, y) \in \mathbf{R}^2, r \in \mathbf{P}, \phi \in \mathbf{P}, -180^\circ < \phi \leq 180^\circ\},$$

$$\mathbf{PR^2HU_P} := \{(x, y) : x = r\cos\phi, y = r\sin\phi \,|(x, y) \in \mathbf{R}^2, r \in \mathbf{P} + \frac{1}{2}, \phi \in \mathbf{P} + \frac{1}{2}, -180^\circ < \phi \leq 180^\circ\},$$

$\mathbf{PR^2Z_P} \subset \mathbf{PR^2Z}, \mathbf{PR^2HU_P} \subset \mathbf{PR^2HU}$.
Similar to the Polar Complex Integers, by fixing the integer radius $r \in \mathbf{Z}$, Polar Euclidean Integers, when considered within $\mathbf{R}^2$ plane, constitute Unicentered Radial Lattice.

6. Spherical Euclidean Integers

Using Spherical Coordinate System , let us introduce the following novel subclass of Integer numbers for the Euclidean 3-dimensional Space: $\mathbf{R}^3$: Spherical Euclidean Integers:
$\mathbf{SR^3Z} := \{(x, y, z) : x = r\cos\phi\sin\theta, y = r\sin\phi\ \sin\theta, z = r\cos\phi\ |(x, y, z) \in \mathbf{R}^3, r \in \mathbf{Z}, \phi \in \mathbf{Z}, \theta \in \mathbf{Z}, -180^\circ < \phi \leq 180^\circ, -90^\circ < \theta \leq 90^\circ\}$, and Spherical Hurwitz-like Integers:
$\mathbf{SR^3HU} := \{(x, y, z) : x = r\cos\phi\sin\theta, y = r\sin\phi\ \ \sin\theta, z = r\cos\phi\ \ |(x, y, z) \in \mathbf{R}^3, r \in \mathbf{Z} + \frac{1}{2}, \phi \in \mathbf{Z} + \frac{1}{2}, \theta \in \mathbf{Z} + \frac{1}{2}, -180^\circ < \phi \leq 180^\circ, -90^\circ < \theta \leq 90^\circ\}$,
and the corresponding Prime-subclasses can be introduced as well:
$\mathbf{SR^3Z_P} := \{(x, y, z) : x = r\cos\phi\sin\theta, y = r\sin\phi\ \sin\theta, z = r\cos\phi\,|(x, y, z) \in \mathbf{R}^3, r \in \mathbf{P}, \phi \in \mathbf{P}, \theta \in \mathbf{P}, -180^\circ < \phi \leq 180^\circ, -90^\circ < \theta \leq 90^\circ\}$,
$\mathbf{SR^3HU_P} := \{(x, y, z) : x = r\cos\phi\sin\theta, y = r\sin\phi\ \ \sin\theta, z = r\cos\phi\ \ |(x, y, z) \in \mathbf{R}^3, r \in \mathbf{P} + \frac{1}{2}, \phi \in \mathbf{P} + \frac{1}{2}, \theta \in \mathbf{P} + \frac{1}{2}, -180^\circ < \phi \leq 180^\circ, -90^\circ < \theta \leq 90^\circ\}, \mathbf{SR^3Z_P} \subset \mathbf{SR^3Z}, \mathbf{SR^3HU_P} \subset \mathbf{SR^3HU}$.
Similar to the Polar Complex Integers, by fixing the integer radius $r \in \mathbf{Z}$, Spherical Integers, when considered within $\mathbf{R}^3$ Euclidean Space, constitute Unicentered Spherical Lattice.

7. Cylindrical Euclidean Integers

Using Cylindrical Coordinate System, let us introduce the following novel subclass of Integer numbers for the Euclidean 3-dimensional Space $\mathbf{R}^3$: Cylindrical Euclidean Integers:

$$\mathbf{CR^3Z} := \{(x, y, z) : x = r\cos\phi, y = r\sin\phi, z = z\,|(x, y, z) \in \mathbf{R}^3, r \in \mathbf{Z}, \phi \in \mathbf{Z}, z \in \mathbf{Z}, -180^\circ < \phi \leq 180^\circ\},$$

and Cylindrical Hurwitz-like Integers:

$$\mathbf{CR^3HU} := \{(x, y, z) : x = r\cos\phi, y = r\sin\phi, z = z\,|(x, y, z) \in \mathbf{R}^3, r \in \mathbf{Z} + \frac{1}{2}, \phi \in \mathbf{Z} + \frac{1}{2}, z \in \mathbf{Z} + \frac{1}{2}, -180^\circ < \phi \leq 180^\circ\},$$

and the corresponding Prime-subclasses can be introduced as well:

$$\mathbf{CR^3Z_P} := \{(x, y, z) : x = r\cos\phi, y = r\sin\phi, z = z\,|(x, y, z) \in \mathbf{R}^3, r \in \mathbf{P}, \phi \in \mathbf{P}, z \in \mathbf{P}, -180^\circ < \phi \leq 180^\circ\},$$

$$\mathbf{CR^3HU_P} := \{(x, y, z) : x = r\cos\phi, y = r\sin\phi, z = z\,|(x, y, z) \in \mathbf{R}^3, r \in \mathbf{P} + \frac{1}{2}, \phi \in \mathbf{P} + \frac{1}{2}, z \in \mathbf{P} + \frac{1}{2}, -180^\circ < \phi \leq 180^\circ\},$$

$\mathbf{CR^3Z_P} \subset \mathbf{CR^3Z}, \mathbf{CR^3HU_P} \subset \mathbf{CR^3HU}$.
By fixing the integer radius $r \in \mathbf{Z}$, Cylindrical Integers, when considered within $\mathbf{R}^3$ Euclidean Space, constitute Uniaxial(around z-axis) Cylindrical Lattice (Note that Polar Coordinate System can be generalized for the n-dimensional Euclidean Space $\mathbf{R}^n$).

8. Polar Complex Integer Optimization

Let us introduce a new class of Optimization problems, where **G** (see Section 1) is some subset of the Polar Complex Integers and target functions $f : \mathbf{C} \to \mathbf{R}$ and $f : \mathbf{C}^n \to \mathbf{R}$ are real-valued complex variable functions: "Polar Complex Integer Optimization".

8.1. Polynomial Polar Complex Integer Optimization

pcop = $\{\max \quad |c_n z^n + ... + c_1 z| \quad$ subject to
$|a_{1n} z^n + \cdots + a_{11} z| \leq b_1, \cdots, \ |a_{mn} z^n + \cdots + a_{m1} z| \leq b_m,$
$z \in$ **PZ [i]** , $a_{ij} \in \mathbf{C}, b_i \in \mathbf{R}, c_j \in \mathbf{C}, 1 \leq i \leq m, 1 \leq j \leq n, n \in \mathbf{N}, m \in \mathbf{N}\}$.
(More sophisticated examples would contain rational meromorphic complex functions).
Similar, if $z \in$ **PZ**$_{\mathrm{P}}$**[i]**, or $z \in$ **PHU[i]**, or $z \in$ **PHU**$_{\mathrm{P}}$**[i]**.

8.2. Linear Polar Complex Integer Optimization

lpcopa = $\{\max \quad |c_1 z_1 + ... + c_n z_n| \quad$ subject to
$|a_{11} z_1 + ... + a_{1n} z_n| \leq b_1, \cdots, \ |a_{m1} z_1 + ... + a_{mn} z_n| \leq b_m,$
$\mathrm{Re}(a_{11} z_1 + ... + a_{1n} z_n) \leq d_1, \cdots, \ \mathrm{Re}(a_{m1} z_1 + ... + a_{mn} z_n) \leq d_m,$
$z_j \in$ **PZ[i]**, $a_{ij} \in \mathbf{C}, b_i \in \mathbf{R}, c_j \in \mathbf{C}, d_i \in \mathbf{R}, 1 \leq i \leq m, 1 \leq j \leq n, n \in \mathbf{N}, m \in \mathbf{N}\}$.
lpcopb = $\{\max \quad |c_1 z_1 + ... + c_n z_n| \quad$ subject to
$a_{11} z_1 + ... + a_{1n} z_n = b_1, \cdots, \ a_{m1} z_1 + ... + a_{mn} z_n = b_m,$
$\arg(a_{11} z_1 + ... + a_{1n} z_n) = d_1, \cdots, \ \arg(a_{m1} z_1 + ... + a_{mn} z_n) = d_m,$
$z_j \in$ **PZ[i]**, $a_{ij} \in \mathbf{C}, b_i \in \mathbf{C}, c_j \in \mathbf{C}, (Az = b), d_i \in \mathbf{R}, 1 \leq i \leq m, 1 \leq j \leq n, n \in \mathbf{N}, m \in \mathbf{N}\}$.
Similar, if $z_j \in$ **PZ**$_{\mathrm{P}}$**[i]**, or $z_j \in$ **PHU[i]**, or $z_j \in$ **PHU**$_{\mathrm{P}}$**[i]**.

8.3. Quadratic Polar Complex Integer Optimization

qpcop = $\{\max \quad |z_1^2 + ... + z_n^2 - \mathbf{i} z_1 z_2| \quad$ subject to
$|a_{11} z_1 + ... + a_{1n} z_n| \leq b_1, \cdots, \ |a_{m1} z_1 + ... + a_{mn} z_n| \leq b_m,$
$\mathrm{Im}(a_{11} z_1 + ... + a_{1n} z_n) \leq d_1, \cdots, \ \mathrm{Im}(a_{m1} z_1 + ... + a_{mn} z_n) \leq d_m,$
$z_j \in$ **PZ[i]**, $a_{ij} \in \mathbf{C}, b_i \in \mathbf{R}, c_j \in \mathbf{C}, d_i \in \mathbf{R}, 1 \leq i \leq m, 1 \leq j \leq n, n \in \mathbf{N}, m \in \mathbf{N}\}$.
Similar, if $z_j \in$ **PZ**$_{\mathrm{P}}$**[i]**, or $z_j \in$ **PHU[i]**, or $z_j \in$ **PHU**$_{\mathrm{P}}$**[i]**.

8.4. Nonlinear Polar Complex Integer Optimization

npcop = $\{\max \ |\mathrm{e}^z - \sin(\pi z)| \quad$ subject to $\quad |\cos(\pi z)| \leq a, \ 0 \leq \mathrm{Re}(z) \leq b, \ 0 \leq \mathrm{Im}(z) \leq c$
$z \in$ **PZ [i]** , $a \in \mathbf{R}, b \in \mathbf{R}, c \in \mathbf{R}\}$.
Similar, if $z \in$ **PZ**$_{\mathrm{P}}$**[i]**, or $z \in$ **PHU[i]**, or $z \in$ **PHU**$_{\mathrm{P}}$**[i]**.

9. Polar Quaternionic Integer Optimization

Let us introduce a new class of Optimization problems, where **G** (see Section 1) is some subset of the Polar Quaternionic Integers and target functions $f : \mathbf{H} \to \mathbf{R}$ and $f : \mathbf{H}^n \to \mathbf{R}$ are real-valued complex variable functions: "Polar Quaternionic Integer Optimization".

9.1. Polynomial Polar Quaternionic Integer Optimization

qpop = $\{\max \quad ||c_n q^n + ... + c_1 q|| \quad$ subject to
$||a_{1n} q^n + ... + a_{11} q|| \leq b_1, \cdots, \ ||a_{mn} q^n + ... + a_{m1} q|| \leq b_m,$

$q \in \mathbf{PL},\ a_{ij} \in \mathbf{H}, b_i \in \mathbf{R}, c_j \in \mathbf{H}, 1 \leq i \leq m, 1 \leq j \leq n, n \in \mathbf{N}, m \in \mathbf{N}\}.$
By introducing the slack variables $w_1 \geq 0, \ldots, w_m \geq 0$ the above inequalities can be converted into the following equations:

$$||a_{1n}q^n + ... + a_{11}q\,|| + w_1 = b_1, \cdots,\ ||a_{mn}q^n + ... + a_{m1}q\,|| + w_m = b_m.$$

Similar, if $q \in \mathbf{PL_P}$, or $q \in \mathbf{PHHU}$, or $q \in \mathbf{PHHU_P}$.

9.2. Linear Polar Quaternionic Integer Optimization

lpqopa = $\{\max \quad ||c_1q_1 + ... + c_nq_n||$ subject to
$||a_{11}q_1 + ... + a_{1n}q_n|| \leq b_1, \cdots,\ ||a_{m1}q_1 + ... + a_{mn}q_n|| \leq b_m,$
$q_j \in \mathbf{PL},\ a_{ij} \in \mathbf{H}, b_i \in \mathbf{R}, c_j \in \mathbf{H}, 1 \leq i \leq m, 1 \leq j \leq n, n \in \mathbf{N}, m \in \mathbf{N}\}.$
By introducing the slack variables $w_1 \geq 0, \ldots, w_m \geq 0$ the above inequalities can be converted into the following equations:

$$||a_{11}q_1 + ... + a_{1n}q_n\,|| + w_1 = b_1, \cdots,\ ||a_{m1}q_1 + ... + a_{mn}q_n\,|| + w_m = b_m.$$

lpqopb = $\{\max \quad ||c_1q_1 + ... + c_nq_n||$ subject to
$a_{11}q_1 + ... + a_{1n}q_n = b_1, \cdots,\ a_{m1}q_1 + ... + a_{mn}q_n = b_m,$
$q_j \in \mathbf{PL},\ a_{ij} \in \mathbf{H}, b_i \in \mathbf{H}, c_j \in \mathbf{H}, 1 \leq i \leq m, 1 \leq j \leq n, n \in \mathbf{N}, m \in \mathbf{N}\}.$
Similar, if $q_j \in \mathbf{PL_P}$, or $q_j \in \mathbf{PHHU}$, or $q_j \in \mathbf{PHHU_P}$.

9.3. Nonlinear Polar Quaternionic Integer Optimization

nqpopa = $\{\max \quad ||q_1^4 + ... + q_n^4||$ subject to
$b_1 < ||a_{11}q_1 + ... + a_{1n}q_n|| \leq c_1, \cdots,\ b_m < ||a_{m1}q_1 + ... + a_{mn}q_n|| \leq c_m,$
$q_j \in \mathbf{PL},\ a_{ij} \in \mathbf{H}, b_i \in \mathbf{R}, c_i \in \mathbf{R}, b_i \geq 0, 1 \leq i \leq m, 1 \leq j \leq n, n \in \mathbf{N}, m \in \mathbf{N}\}.$
Similar, if $q_j \in \mathbf{PL_P}$, or $q_j \in \mathbf{PHHU}$, or $q_j \in \mathbf{PHHU_P}$.
nqpopb = $\{\max \ ||\mathrm{e}^p - \log(q)\,||$ subject to $||p|| \leq a,\ ||q|| \leq b,$
$p, q \in \mathbf{PL}, a, b \in \mathbf{R}\}.$
Similar, if $p, q \in \mathbf{PL_P}$, or $p, q \in \mathbf{PHHU}$, or $p, q \in \mathbf{PHHU_P}$.

10. Polar Euclidean Integer Optimization

Let us introduce a new class of Optimization problems, where **G** (see Section 1) is some subset of the Polar Euclidean Integers and target functions $f : \mathbf{R}^2 \to \mathbf{R}$: "Polar Euclidean Integer Optimization".

10.1. Linear Polar Euclidean Integer Optimization

lpeop = $\{\max \quad c_1x + c_2y$ subject to
$a_{11}x + a_{12}y \leq b_1, \qquad a_{21}x + a_{22}y \leq b_2,$
$(x, y) \in \mathbf{PR^2Z},\ a_{ij} \in \mathbf{R}, b_i \in \mathbf{R}, c_j \in \mathbf{R}, 1 \leq i \leq 2, 1 \leq j \leq 2\}.$
Similar, if $(x, y) \in \mathbf{PR^2HU}$, or $(x, y) \in \mathbf{PR^2HU_P}$, or $(x, y) \in \mathbf{PR^2Z_P}$.

10.2. Nonlinear Polar Euclidean Integer Optimization

npeop = $\{\max \quad x^4 + y^4$ subject to
$a_{11}x + a_{12}y \leq b_1, \qquad a_{21}x + a_{22}y \leq b_2,$
$(x, y) \in \mathbf{PR^2Z},\ a_{ij} \in \mathbf{R}, b_i \in \mathbf{R}, 1 \leq i \leq 2, 1 \leq j \leq 2\}.$
Similar, if $(x, y) \in \mathbf{PR^2HU}$, or $(x, y) \in \mathbf{PR^2HU_P}$, or $(x, y) \in \mathbf{PR^2Z_P}$.

11. Spherical Euclidean Integer Optimization

Let us introduce a new class of Optimization problems, where **G** (see Section 1) is some subset of the Spherical Euclidean Integers and target functions $f : \mathbf{R}^3 \to \mathbf{R}$: "Spherical Euclidean Integer Optimization".

11.1. Linear Spherical Euclidean Integer Optimization

lseop = $\{\max \quad c_1x + c_2y + c_3z$ subject to

$$a_{11}x + a_{12}y + a_{13}z \le b_1, \qquad a_{21}x + a_{22}y + a_{23}z \le b_2,$$
$$(x, y, z) \in \mathbf{SR}^3\mathbf{Z},\ a_{ij} \in \mathbf{R}, b_i \in \mathbf{R}, c_j \in \mathbf{R}, 1 \le i \le 2, 1 \le j \le 3\}.$$

Similar, if $(x, y, z) \in \mathbf{SR}^3\mathbf{HU}$, or $(x, y, z) \in \mathbf{SR}^3\mathbf{HU_P}$, or $(x, y, z) \in \mathbf{SR}^3\mathbf{Z_P}$.

11.2. Nonlinear Spherical Euclidean Integer Optimization

nseop = $\{\max \quad x^2 + y^2 + z^2 - xy$ subject to

$$a_{11}x + a_{12}y + a_{13}z \le b_1, \qquad a_{21}x + a_{22}y + a_{23}z \le b_2,$$
$$(x, y, z) \in \mathbf{SR}^3\mathbf{Z},\ a_{ij} \in \mathbf{R}, b_i \in \mathbf{R}, 1 \le i \le 2, 1 \le j \le 3\}.$$

Similar, if $(x, y, z) \in \mathbf{SR}^3\mathbf{HU}$, or $(x, y, z) \in \mathbf{SR}^3\mathbf{HU_P}$, or $(x, y, z) \in \mathbf{SR}^3\mathbf{Z_P}$.

12. Cylindrical Euclidean Integer Optimization

Let us introduce a new class of Optimization problems, where **G** (see Section 1) is some subset of the Cylindrical Euclidean Integers and target functions $f : \mathbf{R}^3 \to \mathbf{R}$: "Cylindrical Euclidean Integer Optimization".

12.1. Linear Cylindrical Euclidean Integer Optimization

lceop = $\{\max \quad c_1x + c_2y + c_3z$ subject to

$$a_{11}x + a_{12}y + a_{13}z \le b_1, \qquad a_{21}x + a_{22}y + a_{23}z \le b_2,$$
$$(x, y, z) \in \mathbf{CR}^3\mathbf{Z},\ a_{ij} \in \mathbf{R}, b_i \in \mathbf{R}, c_j \in \mathbf{R}, 1 \le i \le 2, 1 \le j \le 3\}.$$

Similar, if $(x, y, z) \in \mathbf{CR}^3\mathbf{HU}$, or $(x, y, z) \in \mathbf{CR}^3\mathbf{HU_P}$, or $(x, y, z) \in \mathbf{CR}^3\mathbf{Z_P}$.

12.2. Nonlinear Cylindrical Euclidean Integer Optimization

nceop = $\{\max \quad x^2 + y^2 + z^2 - xy$ subject to

$$a_{11}x + a_{12}y + a_{13}z \le b_1, \qquad a_{21}x + a_{22}y + a_{23}z \le b_2,$$
$$(x, y, z) \in \mathbf{CR}^3\mathbf{Z},\ a_{ij} \in \mathbf{R}, b_i \in \mathbf{R}, 1 \le i \le 2, 1 \le j \le 3\}.$$

Similar, if $(x, y, z) \in \mathbf{CR}^3\mathbf{HU}$, or $(x, y, z) \in \mathbf{CR}^3\mathbf{HU_P}$, or $(x, y, z) \in \mathbf{CR}^3\mathbf{Z_P}$.

13. Mixed Polar-Spherical-Cylindrical-Prime-Integer-Real-Complex- Quaternionic Optimization

mpscop = $\{\min \ xz^2||p^2 - pq + r^2||\ |\mathbf{i}z_1^4 - z_2^2z_3| - x^2 + y^3t^2 +$
$||p_1^2 - p_1q_1s + r_1^2||\ \mathrm{Im}(\mathbf{i}z_4^4 - z_5^2z_6z_7) + u_1^4v_1^4 - u_2^3v_2^3 +$
$u_3^2v_3^2 + u_4v_4 - u_5^4v_5^4w_5^4 + u_6^3v_6^3w_6^3 - u_7^2v_7^2w_7^2 + u_8^3v_8^3w_8^3 +$
$u_9^4v_9^4w_9^4 - u_{10}^3v_{10}^3w_{10}^3 + u_{11}^2v_{11}^2w_{11}^2 - u_{12}^3v_{12}^3w_{12}^3$

subject to

$$xy \ge N,$$
$$a_1 \le\ ||p|| \le b_1, a_2 \le ||q|| \le b_2, a_3 \le ||r|| \le b_3,\ a_4 \le |z_1| \le b_4,$$
$$a_5 \le |z_2| \le b_5, a_6 \le |z_3| \le b_6, a_7 \le x \le b_7,\ a_8 \le y \le b_8, a_9 \le z \le b_9,$$
$$a_{10} \le t \le b_{10},\ x \in \mathbf{Z},\ y \in \mathbf{Z},\ z \in \mathbf{P},\ t \in \mathbf{R},$$

$$
\begin{aligned}
& a_{11} \le ||p_1|| \le b_{11}, a_{12} \le ||q_1|| \le b_{12}, a_{13} \le ||s|| \le b_{13},\ a_{14} \le r_1 \le b_{14}, \\
& a_{15} \le z_4 \le b_{15}, a_{16} \le z_5 \le b_{16}, a_{17} \le z_6 \le b_{17},\ a_{18} \le z_7 \le b_{18}, \\
& a_i \in \mathbf{R}, b_i \in \mathbf{R}, a_i \ge 0, 1 \le i \le 18, \\
& c_i \le u_i \le d_i, e_i \le v_i \le f_i, c_i \in \mathbf{R}, d_i \in \mathbf{R}, e_i \in \mathbf{R}, f_i \in \mathbf{R}, 1 \le i \le 12, \\
& g_j \le w_j \le h_j, g_j \in \mathbf{R}, h_j \in \mathbf{R}, 5 \le j \le 12, N \in \mathbf{N}, \\
& p \in \mathbf{H}, q \in \mathbf{L}, r \in \mathbf{L_P}, z_1 \in \mathbf{C}, z_2 \in \mathbf{Z}[\mathbf{i}], z_3 \in \mathbf{Z_P}[\mathbf{i}], \\
& p_1 \in \mathbf{PL}, q_1 \in \mathbf{PHHU}, s \in \mathbf{PL}_\mathrm{P}, r_1 \in \mathbf{PHHU_P}, \\
& z_4 \in \mathbf{PZ[i]}, z_5 \in \mathbf{PHU[i]}, z_6 \in \mathbf{PZ_P[i]}, z_7 \in \mathbf{PHU_P[i]}, \\
& (u_1, v_1) \in \mathbf{PR}^2\mathbf{Z}, (u_2, v_2) \in \mathbf{PR}^2\mathbf{HU}, (u_3, v_3) \in \mathbf{PR}^2\mathbf{Z_P}, \\
& (u_4, v_4) \in \mathbf{PR}^2\mathbf{HU}_\mathrm{P}, (u_5, v_5, w_5) \in \mathbf{SR}^3\mathbf{Z}, (u_6, v_6, w_6) \in \mathbf{SR}^3\mathbf{HU}, \\
& (u_7, v_7, w_7) \in \mathbf{SR}^3\mathbf{Z_P}, (u_8, v_8, w_8) \in \mathbf{SR}^3\mathbf{HU_P}, \\
& (u_9, v_9, w_9) \in \mathbf{CR}^3\mathbf{Z}, (u_{10}, v_{10}, w_{10}) \in \mathbf{CR}^3\mathbf{HU}, \\
& (u_{11}, v_{11}, w_{11}) \in \mathbf{CR}^3\mathbf{Z_P}, (u_{12}, v_{12}, w_{12}) \in \mathbf{CR}^3\mathbf{HU_P}\}.
\end{aligned}
$$

14. Open Problems

Despite wide proliferation of Integer Optimization, it would be preferable to develop specific methods and algorithms for the Polar Integer Optimization problems. The corresponding complexity evaluation for the Polar Integer Optimization Problems would be developed as well: for example in binary encoded length of the coefficients (see, e.g., [1, 5, 9]), and, in particular, finding conditions for the polynomial-time optimization.
Recall that PRIMES is in P (see, e.g., [1, 5]).

15. Conclusion

We unveiled a special class of complex numbers, wherein their absolute values and arguments, given in a Polar Coordinate System are integers, which when considered within the complex plane, constitute Unicentered Radial Lattice and similarly for quaternions, as well as for Euclidean Polar, Spherical and Cylindrical Coordinate Systems. The corresponding Optimization problems were unveiled as well.

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