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Failure Rate, Vitality, and Residual Lifetime Measures: Characterizations Based on Stress-strength Bivariate Model with Application to an Automated Life Test Data

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Abstract In this article, we introduce some reliability concepts for the bivariate Pareto Type II distribution including joint hazard rate function, CDF for parallel and series systems, joint mean residual lifetime, and joint vitality function. The maximum likelihood and Bayesian estimation methods are utilized to estimate the model parameters. Simulation is carried out to assess the performance of the maximum likelihood and Bayesian estimators, and it is found that the two approaches work quite well in estimation process. Finally, a real lifetime data is analyzed to show the flexibility and the importance of the introduced bivariate mode.

Keywords Bivariate distributions, Failure analysis, Maximum likelihood method, Bayes theorem, Simulation, Statistics and numerical data

AMS 2010 subject classifications 60E05; 62E10; 62N05

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1. Introduction

Several continuous distributions have been extensively utilized for modeling real data in many practical fields, and consequently several classes of distributions have been proposed in the statistical literature. See, Jehhan et al. (2018), Eliwa et al. (2018, 2020, 2021), El-Morshedy et al. (2020), El-Morshedy and Eliwa (2019). The major aim of proposing various extensions of probability distributions is to get the most fit distribution to real data. Since there are types of data sets univariate (continuous/discrete) and bivariate (continuous/discrete) where this data can be generated in natural such as failure times, temperature, windspeed, daily/death cases of Covid-19 in different countries, score of teams in any game, number of students who admitted to study various sciences in any country, lifetime of any component/device, among others. To study the previous examples of date in case of bivariate observations, we should get bivariate extensions to a univariate one by utilizing several approaches like shock mode Marshall-Olkin as an example. The Marshall-Olkin technique is the most popular approach in this regard because the generated probability distribution from this method has more flexibility as compared to another one generated utilizing another technique. The authors in statistical literature aimed to propose several extensions, modifications, or generalization of bivariate model to study different types of data sets in various areas, see for

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instance, Ibrahim et al. (2019), Rafiei and Iranmanesh. (2020), El-Sherpieny et al. (2022), Eliwa and El-Morshedy (2019), and El-Morshedy et al. (2020a, 2020b). Bivariate Pareto (BP) model is popular model in many applied areas, and has many applications, see Lindley and Singpurwalla (1986), Basu (1990), Sankaran and Nair (1993), Veenus and Nair (1994), Rachev et al. (1995), Nadarajah and Kotz (2006), Nadarajah (2009), Asimit et al. (2010), Papadakis and Tsionas (2010), Hakamipour et al. (2011), among others. In this paper, we focus our efforts to derive some reliability properties of one of the most important distributions in the statistical literature, called the bivariate Pareto Type II (BPT-T2) distribution.

2. The BPT-T2 Distribution

The random variable Y is said to have Pareto Type II with parameters $\vartheta > 0$ and $\zeta > 0$, say PT-T2(ϑ, ζ), if its CDF is given by

$$F(y) = 1 - \left(\frac{\zeta}{y}\right)^{\vartheta};\ y\ \geq\ \zeta > 0. \tag{1}$$

Veenus and Nair (1994) proposed a BPT-T2 distribution with joint reliability (JR) function

$$R(x_1, x_2) = \left(\frac{\zeta}{\max(x_1, x_2)}\right)^{\vartheta_3} \prod_{i=1}^{2} \left(\frac{\zeta}{x_i}\right)^{\vartheta_i};\ x_1, x_2 \geq \zeta > 0, \tag{2}$$

where $0 < \zeta \leq min(x1, x2) < \infty$ and $\vartheta_j > 0;\ j = 1, 2, 3$. The marginal of $X_i; i = 1, 2$ is given by

$$R_{X_i}(x_i) = \left(\frac{\zeta}{x_i}\right)^{\vartheta_i+\vartheta_3};\ x_i \geq \zeta > 0. \tag{3}$$

Equation (3) represents the survival function (SF) of the PT-T2 distribution with parameters $(\vartheta_i + \vartheta_3, \zeta\ )$. Also, we find that $X1$ and $X2$ are independent iff $\vartheta 3 = 0$. Further, the marginal distribution to $X1$ and $X2$ can be written in a form

$$F_{X_i}(x_i) = \frac{(\vartheta_i + \vartheta_3)\, x_i}{x_i - [(1 - \vartheta_i - \vartheta_3)\, m_{w_i}(x_i)]}\ ; i = 1, 2, \tag{4}$$

where $m_{w_i}(x_i)$ is the marginal of mean waiting time (MWT). The corresponding joint CDF is given as

$$F(x_1, x_2) = 1 - \left(\frac{\zeta}{x_1}\right)^{\vartheta_1+\vartheta_3} - \left(\frac{\zeta}{x_2}\right)^{\vartheta_2+\vartheta_3} + \left(\frac{\zeta}{\max(x_1, x_2)}\right)^{\vartheta_3} \prod_{i=1}^{2} \left(\frac{\zeta}{x_i}\right)^{\vartheta_i}. \tag{5}$$

The corresponding joint PDF is given by

$$f(x_1, x_2) = \begin{cases} f_1(x_1, x_2) & ;\ \Psi \\ f_2(x_1, x_2) & ;\ \Psi^* \\ f_0(x) & ;\ \Psi^{**}, \end{cases} \tag{6}$$

where Ψ, Ψ^* and Ψ^{**} represent the terms $x_1 < x_2$, $x_2 < x_1$ and $x_1 = x_2 = x$, respectively, for positive domain,

$$f_1(x_1, x_2) = \vartheta_1(\vartheta_2 + \vartheta_3)\left(\frac{\zeta}{x_1}\right)^{\vartheta_1+1}\left(\frac{\zeta}{x_2}\right)^{\vartheta_2+\vartheta_3+1},$$

$$f_2(x_1, x_2) = \vartheta_2(\vartheta_1 + \vartheta_3)\left(\frac{\zeta}{x_1}\right)^{\vartheta_1+\vartheta_3+1}\left(\frac{\zeta}{x_2}\right)^{\vartheta_2+1},$$

and

$$f_0(x) = \vartheta_3\left(\frac{\zeta}{x}\right)^{\vartheta_1+\vartheta_2+\vartheta_3+1}.$$

Figure 1 shows the 3D plot of the joint PDF for some specific parameter values $\vartheta_1 = 0.5$, $\vartheta_2 = 0.5$, $\vartheta_3 = 0.5$, and $\zeta = 3$.

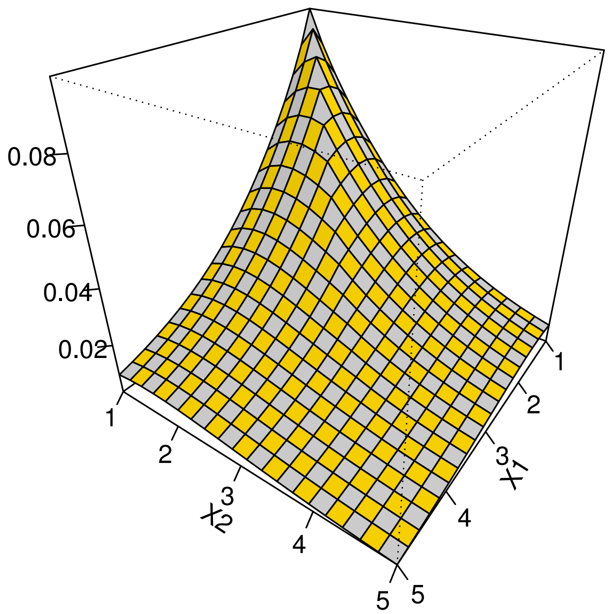

Figure 1. The joint PDF of the $BPT - T_2$ model.

Assume $X1$ and $X2$ are the lifetimes of two components in two different systems (series and parallel). Then, the distributions of $S_{\text{series}} = min(X1, X2)$ and $T_{\text{parallel}} = max(X1, X2)$ are given by

$$F_{\text{series}}(s) = 1 - \left(\frac{\zeta}{s}\right)^{\vartheta_1+\vartheta_2+\vartheta_3},$$

and

$$F_{\text{parallel}}(t) = 1 - \left(\frac{\zeta}{t}\right)^{\vartheta_1+\vartheta_3} - \left(\frac{\zeta}{t}\right)^{\vartheta_2+\vartheta_3} + \left(\frac{\zeta}{t}\right)^{\vartheta_1+\vartheta_2+\vartheta_3}, \tag{7}$$

respectively. Assume $Y \sim$ PT-T2(ϑ, ζ) distribution and Y is independent on $(X1, X2)$ then

$$R = \Pr\left[Y < T_{\text{parallel}}\right] = \left(\frac{\vartheta}{\vartheta_1 + \vartheta_3 + \vartheta} + \frac{\vartheta}{\vartheta_2 + \vartheta_3 + \vartheta} - \frac{\vartheta}{\vartheta_1 + \vartheta_2 + \vartheta_3 + \vartheta}\right). \tag{8}$$

3. Reliability Properties

3.1. Bivariate increasing (decreasing) failure on average BIFRA (BDFRA)

Assume two vectors $u = (0, u)$ and $v = (0, v)$, then the distance is

$$d\left(\underline{u}, \underline{v}\right) = \|\underline{u}, \underline{v}\| = \sqrt{u^2 + v^2},$$

see Zohdy (2013). A distribution F has BIFRA (BDFRA) if

$$\frac{1}{\sqrt{u^2+v^2}} \int_0^u \int_0^v h(x_1, x_2)\, dx_1 dx_2, \tag{9}$$

is increasing (decreasing) in $v > 0$ for all fixed $u > 0$. Moreover, if Equation (9) is increasing (decreasing) in $u > 0$ for all fixed $v > 0$. If $(X1, X2) \sim$ BPT-T2 distribution, then

$$\frac{1}{\sqrt{x_1^2+x_2^2}} logR\left(x_1, x_2\right) = \frac{1}{\sqrt{x_1^2+x_2^2}} \left\{\left(\vartheta_1+\vartheta_2+\vartheta_3\right)\log\zeta\ -\vartheta_1 \log x_1\ -\vartheta_2 \log x_2\ -\vartheta_3 \log\max\left(x_1, x_2\right)\ \right\} < 0,$$

is decreasing in $x1\ (x2) > 0$ for fixed $x2\ (x1) > 0$. Thus, BPT-T2 distribution is BDFRA.

3.2. Hazard rate (HR) components

Basu (1971) introduced the first definition of the HR in case of bivariate observations, i.e. $h(x1, x2) = f(x1, x2)/R(x1, x2)$. If $(X1, X2) \sim$ BPT-T2 distribution, then the BFR function is given by

$$h\left(x_1, x_2\right) = \begin{cases} h_1\left(x_1, x_2\right) & ;\ \Psi \\ h_2\left(x_1, x_2\right) & ;\ \Psi^* \\ h_0\left(x\right) & ;\ \Psi^{**}, \end{cases} \tag{10}$$

where

$$h_1\left(x_1, x_2\right) = \frac{\zeta^2\vartheta_1\left(\vartheta_2+\vartheta_3\right)}{x_1x_2},\ \ h_2\left(x_1, x_2\right) = \frac{\zeta^2\vartheta_2\left(\vartheta_1+\vartheta_3\right)}{x_1x_2},\ \ h_0\left(x\right) = \frac{\zeta\vartheta_3}{x}.$$

Figure 2 shows the 3D plot of the joint HRF for some specific parameter values $\vartheta_1 = 0.5$, $\vartheta_2 = 0.5$, $\vartheta_3 = 0.5$, and $\zeta = 4$.

From Equation (10), it is noted that the joint HR is decreasing in $x1$ and $x2$. Another concept related to the HR function, Johnson and Kotz (1975). If $(X1, X2) \sim$ BPT-T2 distribution, then

$$h\left(X_1 \mid X_2 > x_2\right) = \begin{cases} \frac{\vartheta_1}{x_1} & ;\ \Psi \\ \frac{\vartheta_1+\vartheta_3}{x_1} & ;\ \Psi^*, \end{cases} \tag{11}$$

and

$$h\left(X_2 \mid X_1 > x_1\right) = \begin{cases} \frac{\vartheta_2+\vartheta_3}{x_2} & ;\ \Psi \\ \frac{\vartheta_2}{x_2} & ;\ \Psi^*, \end{cases} \tag{12}$$

To test the monotonicity of the HR at $X1 < X2$, we form at first the following formula

$$f\left(x_1 \mid X_2 > x_2\right) = \frac{\vartheta_1\zeta^{\gamma_1+1}}{x_1^{\vartheta_1+2}},$$

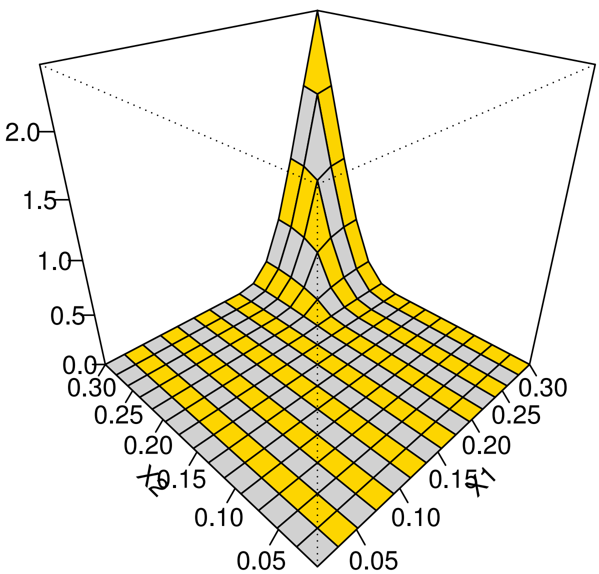

Figure 2. The joint HRF of the $BPT-T_2$ model.

it is found that $\frac{d}{dx_1}\,\eta\,(x_1) < 0$ where ($\eta(s) = -\frac{d}{ds}\ln f(s)$), hence $h_1(x1, x2)$ is DFR in x_1. Similarly, $h_2(x1, x2)$ is DFR in x_2. On the other hand, in case of $1 > X2$, we form

$$f\,(x_2 \mid X_1 > x_1) = \frac{\vartheta_2 \zeta^{\vartheta_2+1}}{x_2^{\vartheta_2+2}},$$

then $\frac{d}{dx_2}\,\eta\,(x_2) < 0$, hence $h_2(x1, x2)$ is DFR in x_2. Similarly, $h_1(x1, x2)$ is DFR in x_1. Cox (1972) defined the BVFRF as a vector

$$h\,(\underline{x}) = \left(h\,(x)\,|_{S_{\text{series}}},\; h_{12}(x1|x2)|_{X_{2:\text{Failure}}}, h_{21}(x_2|x_1)|_{X_{1:\text{Failure}}}\right). \tag{13}$$

If $(X1, X2) \sim$ BPT-T2 distribution, then the components of Cox vector can be expressed as

$$\left(\frac{\vartheta 1+\vartheta 2+\vartheta 3}{x}, \frac{\vartheta 1+\vartheta 3}{x_1}, \frac{\vartheta 2+\vartheta 3}{x_2}\right),$$

respectively. Based on Oakes (1989) and Clayton (1978) concepts, it is found that $\frac{h(x_1|X_2=x_2)}{h(x_1,x_2)}$ greater than one, by another way, for $(X1, X2) \sim$ BPT-T2, it is found that the association between $X1$ and $X2$ greater than one.

3.3. Bivariate vitality function (BVF)

The BVF of $(X1, X2)$ defined on positive domain as a binomial vector, where

$$v_j\,(z_1, z_2) = \mathbf{E}\,(Z_j \mid Z_1 > z_1,\; Z_2 > z_2)\,; j = 1, 2. \tag{14}$$

For more details, see Sankaran and Nair (1991). Under condition of $X1 < X2$, if $(X1, X2) \sim$ BPT-T2 distribution, then the binomial vector can be proposed as

$$\left(x_1 + \frac{x_1}{\vartheta_1 - 1}, x_2 + \frac{x_2}{\vartheta_2 + \vartheta_3 - 1} \right),$$

for $\vartheta 1 > 0$ and $\vartheta 2 + \vartheta 3 > 0$. Similarly, for $X1 > X2$, we get the binomial vector can be introduced as

$$\left(x_1 + \frac{x_1}{\vartheta_1 + \vartheta_3 - 1}, x_2 + \frac{x_2}{\vartheta_2 + \vartheta_3 - 1} \right),$$

for $\vartheta 2 > 0$ and $\vartheta 1 + \vartheta 3 > 0$. Another extension of BVF can be expressed

$$v(\underline{x}) = \left(v(x)|_{S_{\text{series}}},\ v_{12}(x1|x2)|_{X_{2:\text{Failure}}}, v_{21}(x_2|x_1)|_{X_{1:\text{Failure}}} \right).$$

If $(X1, X2) \sim$ BPT-T2 distribution, then

$$v(\underline{x}) = \left(\frac{(\vartheta_1 + \vartheta_2 + \vartheta_3)\, x}{\vartheta_1 + \vartheta_2 + \vartheta_3 - 1},\ \frac{\vartheta_1 x_1}{\vartheta_1 - 1}, \frac{(\vartheta_2 + \vartheta_3)\, x_2}{\vartheta_2 + \vartheta_3 - 1} \right).$$

3.4. *Bivariate mean residual lifetime (BMRL)*

The BMRL can be expressed as

$$m(x_1, x_2) = \frac{1}{R(x_1, x_2)} \int_t^\infty \int_t^\infty R(x_1, x_2)\ dx_1 dx_2. \tag{15}$$

If $(X1, X2) \sim$ BPT-T2 distribution, then

$$m(t) = \begin{cases} \frac{t^2}{(\vartheta_1 + \vartheta_3 - 1)(\vartheta_2 - 1)} & ;\ \vartheta_1 + \vartheta_3 > 1,\ \vartheta_2 > 1 \\ \\ \frac{t^2}{(\vartheta_2 + \vartheta_3 - 1)(\vartheta_1 - 1)} & ;\ \vartheta_2 + \vartheta_3 > 1,\ \vartheta_1 > 1, \end{cases} \tag{16}$$

is increasing in $t \geq 0$. A second definition for the BMRL was proposed by Arnold and Zahedi (1988) as a binomial vector as

$$m_j(z_1, z_2) = \mathbf{E}(Z_j - z_j \mid Z_1 > z_1,\ Z_2 > z_2)\, ; j = 1, 2.$$

Under condition of $X1 > X2$, if $(X1, X2) \sim$ BPT-T2 distribution, then the MRL function gradient is given by

$$\underline{m}(x_1, x_2) = \left(\frac{x_1}{\vartheta_1 + \vartheta_3 - 1}, \frac{x_2}{\vartheta_2 - 1} \right);\ \vartheta_1 + \vartheta_3 > 1,\ \vartheta_2 > 1,$$

which is increasing. Similarity, in case of $X1 < X2$, we get

$$\underline{m}(x_1, x_2) = \left(\frac{x_1}{\vartheta_1 - 1}, \frac{x_2}{\vartheta_2 + \vartheta_3 - 1} \right);\ \vartheta_1 > 1,\ \vartheta_2 + \vartheta_3 > 1,$$

which is increasing. Moreover, Asha and Jagathnath (2008) defined another definition to BMRL which takes the following form

$$m(\underline{x}) = \left(m(x)|_{S_{\text{series}}},\ m_{12}(x1|x2)|_{X_{2:\text{Failure}}}, m_{21}(x_2|x_1)|_{X_{1:\text{Failure}}} \right).$$

If $(X1, X2) \sim$ BPT-T2 distribution, then

$$m(\underline{x}) = \left(\frac{x}{\vartheta_1 + \vartheta_2 + \vartheta_3 - 1},\ \frac{x_1}{\vartheta_1 + \vartheta_3 - 1}, \frac{x_2}{\vartheta_2 + \vartheta_3 - 1} \right),$$

for $\vartheta_1 + \vartheta_2 + \vartheta_3 > 1$, $\vartheta_1 + \vartheta_3 > 1$ and $\vartheta_2 + \vartheta_3 > 1$.

4. Estimation Methods

4.1. Maximum likelihood estimation (MLE)

In this section, the parameters of the BPT-T2 model are estimated utilizing MLE approach. Suppose $(\mathrm{X}_{11}, \mathrm{X}_{21})$, $(\mathrm{X}_{12}, \mathrm{X}_{22})$, ..., $(\mathrm{X}_{1\mathrm{n}}, \mathrm{X}_{2\mathrm{n}})$ is a random sample from BPT-T2 distribution where $n_1 = (i, X_{1i} < X_{2i})$, $n_2 = (i, X_{1i} > X_{2i})$, $n_3 = (i, X_{1i} = X_{2i} = X_i)$ and $n = \sum_{j=1}^{3} n_j$. The likelihood function $l(\vartheta_1, \vartheta_2, \vartheta_3, \zeta)$ of this sample is given by

$$l(\vartheta_1, \vartheta_2, \vartheta_3, \zeta) = \prod_{\mathrm{i}=1}^{\mathrm{n}_1} \mathrm{f}_1(\mathrm{x}_{1\mathrm{i}}, \mathrm{x}_{2\mathrm{i}}) \prod_{\mathrm{i}=1}^{\mathrm{n}_2} \mathrm{f}_2(\mathrm{x}_{1\mathrm{i}}, \mathrm{x}_{2\mathrm{i}}) \prod_{i=1}^{n_3} f_0(x_i, x_i). \tag{17}$$

The log-likelihood (L) function is

$$\begin{aligned} L(\vartheta_1, \vartheta_2, \vartheta_3, \zeta) = {} & n_1 \ln(\vartheta_1 [\vartheta_2 + \vartheta_3]) - (\vartheta_1 + 1) \sum_{i=1}^{n_1} \ln\left(\frac{x_{1i}}{\zeta}\right) - (\vartheta_2 + \vartheta_3 + 1) \sum_{i=1}^{n_1} \ln\left(\frac{x_{2i}}{\zeta}\right) \\ & + n_2 \ln(\vartheta_2 [\vartheta_1 + \vartheta_3]) - (\vartheta_2 + 1) \sum_{i=1}^{n_2} \ln\left(\frac{x_{2i}}{\zeta}\right) - (\vartheta_1 + \vartheta_3 + 1) \sum_{i=1}^{n_2} \ln\left(\frac{x_{1i}}{\zeta}\right) \\ & + n_3 \ln(\vartheta_3) - (\vartheta_1 + \vartheta_2 + \vartheta_3 + 1) \sum_{i=1}^{n_3} \ln\left(\frac{x_i}{\zeta}\right). \end{aligned}$$

The normal equations with respect to $\vartheta 1, \vartheta 2, \vartheta 3$ and ζ are given by

$$\frac{\partial L}{\partial \vartheta_1} = \frac{n_1}{\vartheta_1} - \sum_{i=1}^{n_1} \ln\left(\frac{x_{1i}}{\zeta}\right) + \frac{n_2}{\vartheta_1 + \vartheta_3} - \sum_{i=1}^{n_2} \ln\left(\frac{x_{1i}}{\zeta}\right) - \sum_{i=1}^{n_3} \ln\left(\frac{x_i}{\zeta}\right), \tag{18}$$

$$\frac{\partial L}{\partial \vartheta_2} = \frac{n_1}{\vartheta_2 + \vartheta_3} - \sum_{i=1}^{n_1} \ln\left(\frac{x_{2i}}{\zeta}\right) + \frac{n_2}{\vartheta_2} - \sum_{i=1}^{n_2} \ln\left(\frac{x_{2i}}{\zeta}\right) - \sum_{i=1}^{n_3} \ln\left(\frac{x_i}{\zeta}\right), \tag{19}$$

$$\frac{\partial L}{\partial \vartheta_3} = \frac{n_1}{\vartheta_2 + \vartheta_3} - \sum_{i=1}^{n_1} \ln\left(\frac{x_{2i}}{\zeta}\right) + \frac{n_2}{\vartheta_1 + \vartheta_3} - \sum_{i=1}^{n_2} \ln\left(\frac{x_{1i}}{\zeta}\right) + \frac{n_3}{\vartheta_3} - \sum_{i=1}^{n_3} \ln\left(\frac{x_i}{\zeta}\right), \tag{20}$$

and

$$\frac{\partial L}{\partial \zeta} = \frac{(\vartheta_1 + \vartheta_2 + \vartheta_3 + 2)}{\zeta} \left\{ n_1 + n_2 + \frac{n_3 (\vartheta_1 + \vartheta_2 + \vartheta_3 + 1)}{(\vartheta_1 + \vartheta_2 + \vartheta_3 + 2)} \right\}. \tag{21}$$

By equating Equations (18) to (21) by zeros and solve them by using R package. The $(1 - \delta)100\%$ confidence intervals of the parameters $\widehat{\vartheta}_i > 0;\ i = 1,\ 2,\ 3$ and ζ can be derived as $\widehat{\vartheta}_i \pm Z_{\frac{\delta}{2}} \sqrt{var(\widehat{\vartheta}_i)}$ and $\zeta \pm Z_{\frac{\delta}{2}} \sqrt{var(\zeta)}$, respectively.

4.2. Bayesian estimation (BSE)

In order to obtain the Bayesian estimators for $\vartheta_1, \vartheta_2, \vartheta_3$ and ζ, it is necessary to derive the $l(\vartheta_1, \vartheta_2, \vartheta_3, \zeta)$ function for the model. Let the BSE under the consideration non-negative parameter vector $\Theta = (\vartheta_1, \vartheta_2, \vartheta_3, \zeta)$ is iid, which have gamma (GA) prior distribution. Thus,

$$\pi(\vartheta_1) \propto {\vartheta_1}^{a_1 - 1} e^{b_1 \vartheta_1},$$

$$\pi(\vartheta_2) \propto {\vartheta_2}^{a_2 - 1} e^{b_2 \vartheta_2},$$

$$\pi(\vartheta_3) \propto {\vartheta_3}^{a_3 - 1} e^{b_3 \vartheta_3}$$

and

$$\pi(\zeta) \propto \zeta^{a_4-1} e^{b_4 \zeta}.$$

The hyper parameters a_i and b_i are non-negative and known where $i = 1, 2, 3, 4$. The posterior distribution (PODS) of Θ is

$$G(\vartheta_1, \vartheta_2, \vartheta_3, \zeta \mid X_1, X_2) = \frac{\mathrm{l}(X_1, X_2|\vartheta_1, \vartheta_2, \vartheta_3, \zeta)\ \pi(\vartheta_1, \vartheta_2, \vartheta_3, \zeta)}{\int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty \mathrm{l}(X_1, X_2|\vartheta_1, \vartheta_2, \vartheta_3, \zeta)\ \pi(\vartheta_1, \vartheta_2, \vartheta_3, \zeta) d\vartheta_1 d\vartheta_2 d\vartheta_3 d\zeta}$$

The PODS can be expressed as

$$G(\vartheta_1, \vartheta_2, \vartheta_3, \zeta \mid X_1, X_2) \propto \mathrm{l}(X_1, X_2|\vartheta_1, \vartheta_2, \vartheta_3, \zeta)\ \pi(\vartheta_1, \vartheta_2, \vartheta_3, \zeta)$$

Thus, the BSEs of $\vartheta_1, \vartheta_2, \vartheta_3$ and ζ can be expressed as

$$\widehat{\vartheta_1} \propto \int_0^\infty \vartheta_1\, G(\vartheta_1, \vartheta_2, \vartheta_3, \zeta \mid X_1, X_2)\ d\vartheta_1,$$

$$\widehat{\vartheta_2} \propto \int_0^\infty \vartheta_2\, G(\vartheta_1, \vartheta_2, \vartheta_3, \zeta \mid X_1, X_2)\ d\vartheta_2,$$

$$\widehat{\vartheta_3} \propto \int_0^\infty \vartheta_3\, G(\vartheta_1, \vartheta_2, \vartheta_3, \zeta \mid X_1, X_2)\ d\vartheta_3,$$

and

$$\widehat{\zeta} \propto \int_0^\infty \zeta\, G(\vartheta_1, \vartheta_2, \vartheta_3, \zeta \mid X_1, X_2)\ d\zeta,$$

where $\widehat{\varphi} = \mathbf{E}(\varphi \mid X_1, X_2)$. MCMC technique is utilized to approximate/solve the previous equations.

5. Simulation

We assess the performance of estimation approaches, namely, the MLE and BSE techniques based on complete sample with different sizes $n = [50,\ 100,\ 200,\ 300]$ from $N = 1000$ replications. The assessment of the two approaches is based of two terms, namely, bias (BI) and mean square error (MESQER). For an informative BSE, we assume that all the hyper parameters are equal to 0.3. Tables 1 - 6 list the BI and the MESQER values for the BPT-T2 model under various values of $\vartheta 1, \vartheta 2, \vartheta 3$ and ζ.

Table 1. The BI and MESQER values for BPT-T_2(1.5,2.5,0.5,2.0) by using MLE method.

n →	**50**		**100**		**200**		**300**	
Parameter ↓	**BI**	**MESQER**	**BI**	**MESQER**	**BI**	**MESQER**	**BI**	**MESQER**
υ_1	0.125269	0.119101	0.121110	0.113558	0.114663	0.110221	0.1049874	0.1016580
υ_2	0.247369	0.179669	0.234697	0.175447	0.221335	0.170225	0.2014167	0.1551108
υ_3	0.167325	0.126235	0.135421	0.122214	0.128215	0.114026	0.1042145	0.1081027
ζ	0.078369	0.098659	0.074358	0.095266	0.071054	0.091120	0.0092150	0.0844157

Table 2. The BI and MESQER values for BPT-T_2(0.8, 1.5, 1.5, 0.5) by using MLE method.

n →	**50**		**100**		**200**		**300**	
Parameter ↓	**BI**	**MESQER**	**BI**	**MESQER**	**BI**	**MESQER**	**BI**	**MESQER**
υ_1	0.278236	0.148547	0.271598	0.144552	0.258220	0.113110	0.2440012	0.047221
υ_2	0.314569	0.287857	0.301110	0.245110	0.287223	0.218013	0.2110023	0.191745
υ_3	0.281102	0.075957	0.233473	0.060201	0.195647	0.053005	0.1421140	0.042368
ζ	0.187125	0.088526	0.144156	0.081504	0.127954	0.074001	0.1200354	0.065556

Table 3. The BI and MESQER values for BPT-T_2(1.5,2.5,0.5,2.0) by using BSE method.

n $\rightarrow$	**50**		**100**		**200**		**300**	
Parameter $\downarrow$	**BI**	**MESQER**	**BI**	**MESQER**	**BI**	**MESQER**	**BI**	**MESQER**
υ_1	0.122101	0.117254	0.1180017	0.114091	0.115984	0.112219	0.107101	0.102001
υ_2	0.204110	0.118118	0.1992258	0.114001	0.192186	0.111981	0.188101	0.106229
υ_3	0.170459	0.125147	0.1668746	0.127223	0.143159	0.118159	0.124001	0.111328
ζ	0.089157	0.078896	0.0801010	0.065104	0.057219	0.052697	0.019025	0.034214

Table 4. The BI and MESQER values for BPT-T_2(0.8, 1.5, 1.5, 0.5) by using BSE method.

n $\rightarrow$	**50**		**100**		**200**		**300**	
Parameter $\downarrow$	**BI**	**MESQER**	**BI**	**MESQER**	**BI**	**MESQER**	**BI**	**MESQER**
υ_1	0.177117	0.133365	0.1651287	0.121269	0.160998	0.114367	0.141783	0.035229
υ_2	0.210256	0.146559	0.1503694	0.133239	0.112210	0.132217	0.108114	0.129225
υ_3	0.290628	0.095125	0.2662104	0.071339	0.191220	0.061214	0.182004	0.031110
ζ	0.180369	0.076200	0.1352291	0.063188	0.115475	0.059100	0.106001	0.043201

Table 5. The BI and MESQER values for BPT-T_2(0.5, 1.1, 1.1, 0.8) by using MLE method.

n $\rightarrow$	**50**		**100**		**200**		**300**	
Parameter $\downarrow$	**BI**	**MESQER**	**BI**	**MESQER**	**BI**	**MESQER**	**BI**	**MESQER**
υ_1	0.159825	0.112397	0.153579	0.101367	0.149875	0.100239	0.136971	0.087699
υ_2	0.189698	0.130197	0.1887963	0.123697	0.179839	0.120369	0.169734	0.119863
υ_3	0.126979	0.110236	0.123697	0.110098	0.119876	0.106975	0.117563	0.100194
ζ	0.102698	0.100970	0.100479	0.100873	0.029796	0.100131	0.018769	0.001977

Table 6. The BI and MESQER values for BPT-T_2(0.5, 1.1, 1.1, 0.8) by using BSE method.

n $\rightarrow$	**50**		**100**		**200**		**300**	
Parameter $\downarrow$	**BI**	**MESQER**	**BI**	**MESQER**	**BI**	**MESQER**	**BI**	**MESQER**
υ_1	0.149687	0.110169	0.1486975	0.1100697	0.137639	0.110013	0.123697	0.102973
υ_2	0.183647	0.113697	0.1821369	0.1112397	0.179687	0.110067	0.172369	0.100079
υ_3	0.119873	0.114790	0.1175301	0.1132685	0.109769	0.110129	0.107469	0.102308
ζ	0.100036	0.079858	0.1000327	0.0712392	0.100113	0.043697	0.098768	0.014283

From Tables 1 - 6, it is noted that the BI and the MESQER decrease when n grows. These results indicate that the MLE and BSE are good methods to estimate the model parameters.

6. Real Data

In this section, areal data set is analyzed using the BPT-T2 model. This data represents the failure times for 36 appliances subjected to an automated life test (see Lawless, 1983). Before analyzing the bivariate data by utilizing the BPT-T2 model, we fit the marginals on this data. It is found that the p-values for the marginals lies between 0.591 and 0.754. Based on the p-values, the marginals of the BPT-T2 model fits the data. Now, we fit the BPT-T2 model on this data. The MLEs of the unknown parameters are as follows $\vartheta_1 = 0.00012,\ \ \vartheta_2 = 0.00096,\ \ \vartheta_3 = 0.00871$ and $\zeta = 0.00047$ with $L = -270.25$, while the Bayesian estimators are $\vartheta_1 = 0.00015,\ \ \vartheta_2 = 0.00078,\ \ \vartheta_3 = 0.00867$ and $\zeta = 0.00042$. The estimators for both methods approximately to be equal. Comparing to the BP distribution which was presented by Lindley and Singpurwalla (1986), we get BPT-T2 is better than BP where L of the BP equals -274.16. The approximate 95% two-sided confidence interval of $\widehat{\vartheta}_i > 0,\ i = 1,\ 2,\ 3$ are $[0,\ 0.0019]\,,\ \ [0,\ 0.0024]$ and $[0.003,\ 0.0095]$ respectively, but for $\widehat{\zeta}$ equals $[0,\ 0.001]$.

7. Conclusions

We have introduced several statistical and reliability properties of the BPT-T2 distribution including joint hazard rate function, CDF for parallel and series systems, joint mean residual lifetime, mean waiting time, and joint vitality function. The MLE and BSE methods have been used to estimate the BPT-T2 parameters. Simulation has been carried out to assess the performance of the MLE and BSE, and it was found that the two techniques work quite well for estimation the BPT-T2 parameters. Finally, a real data set has been analyzed to show the usefulness of the BPT-T2 distribution.

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