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A Novel Weighted G family of Probability Distributions with Properties, Modelling and Different Methods of Estimation

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Abstract In this work, we derive and study a new weighted G family of continuous distributions called the new weighted generated family (NW-G). We study some basic properties including quantile function, asymptotic, the mixture for cdf and pdf, residual entropy, and order statistics. Then, we study half-logistic distribution as a special case with more details. Comprehensive graphical simulations are performed under some common estimation methods. Finally, two real life data sets are analysed to demonstrate the objectives.

Keywords weighted family; Half-Logistic distribution; Moment; Quantile; Simulation.

AMS 2010 subject classifications 60Exx, 60E05

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1. Introduction

Modelling and analysing lifetimes are important in engineering, medicine, economics, etc. In many areas such as the reliability analysis, finance, insurance and engineering, we need novel extended distributions for statistical modeling. So, many new methods have been defined and studied for generating new families of distributions. A huge efforts have been made for defining new G families for expanding the well-known families. Among them, the generalized G-classes of distributions say G are used in which one or more parameter(s) are added to a baseline distribution. The exponentiated G family (Exp-G type-**I**) was introduced by Lehmann (1953). The cumulative distribution function (cdf) of Exp-G type-**I** is given by

$$F_{\mathbf{I},\alpha,\underline{\epsilon}}(y) = G^{\alpha}_{\underline{\epsilon}}(y)|_{\alpha\in\mathbf{R}^{+},y\in\mathbf{R}},$$

where $\underline{\epsilon}$ refers to the parameters vector for any baseline cdf $G(\cdot)$. Proportional hazard rate family or Exp-G type-**II** was studies by Gupta et al. (1998). The cdf of Exp-G type II can be written as

$$F_{\mathbf{II},\alpha,\underline{\epsilon}}(y) = 1 - [1 - G_{\underline{\epsilon}}(y)]^{\alpha}|_{\alpha\in\mathbf{R}^{+},y\in\mathbf{R}},$$

The various extensions of the Exp-G type-**I** and Exp-G type-**II** distribution have been studied by several authors. For example: Beta family (beta-G) by Jones (2004), Kumaraswamy family (Kw-G) by Cordeiro and Castro. (2011), Generalized beta G (GB-G) by Alexander et al. (2012), Exponentiated generalized family by Cordeiro et al. (2013) and Extended Exp-G type-**I** by Alizadeh et al. (2018). In this paper we introduce a new weighted class of distributions. The cdf of new weighted generated family is given by

$$F_{\underline{\mathfrak{B}}}(y) = G^{\alpha}_{\underline{\epsilon}}(y)/\left[2 - G^{\beta}_{\underline{\epsilon}}(y)\right] \quad |_{\alpha,\beta\in\mathbf{R}^{+},y\in\mathbf{R}}, \tag{1}$$

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where $\underline{\mathfrak{B}} = (\alpha, \beta, \underline{\epsilon})$. The related probability density function (pdf) and hazard rate function (hrf) are given by

$$f_{\underline{\mathfrak{B}}}(y) = g_{\underline{\epsilon}}(y)\, G_{\underline{\epsilon}}^{\alpha-1}(y) \frac{2\alpha + (\beta-\alpha) G_{\underline{\epsilon}}^{\beta}(y)}{\left[2 - G_{\underline{\epsilon}}^{\beta}(y)\right]^2} |_{\alpha,\beta\in\mathbf{R}^+, y\in\mathbf{R}}, \tag{2}$$

and

$$h_{\underline{\mathfrak{B}}}(y) = g_{\underline{\epsilon}}(y)\, G_{\underline{\epsilon}}^{\alpha-1}(y) \left[2 - G_{\underline{\epsilon}}^{\beta}(y)\right]^{-1} \frac{(\beta-\alpha) G_{\underline{\epsilon}}^{\beta}(y) + 2\alpha}{\left[2 - G_{\underline{\epsilon}}^{\alpha}(y) - G_{\underline{\epsilon}}^{\beta}(y)\right]} |_{\alpha,\beta\in\mathbf{R}^+, y\in\mathbf{R}}, \tag{3}$$

Let $w(y)$ be a positive function with $E(w(Y)) < \infty$, the pdf of the weighted random variable (RV) Y_w was introduced by Patil and Rao (1986) is given by

$$f_w(y) = \frac{w(y)\, f(y)}{E(w(Y))},$$

where $E(w(Y))$ denote the expectation of $w(Y)$. By taking $g_{\underline{\epsilon}}(y)$ as pdf of continuous distribution with cdf $G(\cdot)$ and

$$w(y) = \frac{G_{\underline{\epsilon}}^{\alpha-1}(y) \left[2\alpha + (\beta-\alpha) G_{\underline{\epsilon}}^{\beta}(y)\right]}{\left[2 - G_{\underline{\epsilon}}^{\beta}(y)\right]^2} |_{\alpha,\beta\in\mathbf{R}^+, y\in\mathbf{R}},$$

equation (2) define a new weighted generated family of distributions. We denote the new family by NW-G($\underline{\mathfrak{B}}$), where $\underline{\epsilon}$ denote the vector of parameters for baseline cdf G(.) and $\alpha, \beta \in \mathbf{R}^+$ are two shape parameters. The key goal of this research is to introduce two extra parameters to a family of lifetime distribution for generating more flexibility ones. Furthermore, the key motivations for using NW-G family in the practice are the followings:

1. A simple method for generating new flexible distributions.
2. For improving the flexibility of the existing distributions.
3. For introducing new extended versions whose cdf and hrf have closed form.
4. For providing better fits than the existing models.

The rest of this paper is organized as follows: In the above, new family of distributions was proposed. Various properties of the proposed distribution are explored in Section 2. These properties include quantile function, asymptotic, mixture for cdf and pdf, residual entropy and order statistics. In section 3, we consider Half-Logistic as special case and studied it with more details. The maximum likelihood estimation of parameters are compared with various methods of estimations by conducting simulation study in section 4. Real data sets are analysed to show the performance of the new family in Section 5. In Section 6, some concluding remarks are considered.

2. Properties

2.1. Quantile function (qf)

Let $Q_G(.) = G^{-1}(.)$ represent the qf of $G(.)$, for $\alpha \neq \beta$, if $U \sim U(0,1)$, $\frac{G_{\underline{\epsilon}}^{\alpha}(y)}{2-G_{\underline{\epsilon}}^{\beta}(y)} = u$ have cdf (1). For $\alpha = \beta$, if $U \sim U(0,1)$, then $Q_F(u) = Q_G\left(\left(\frac{2\,u}{1+u}\right)^{\frac{1}{\alpha}}\right)$.

2.2. Asymptotic

In this subsection we study the asymptotic of cdf, pdf and hrf of NW-G. Let $c = \inf\{y|F(y) \in \mathbf{R}^+\}$, the asymptotic of equations (1), (2) and (3) as $y \to c$ are given by

$$\begin{aligned} F_{\underline{\mathfrak{B}}}(y)|_{y\to c} &\sim \frac{1}{2}\, G^{\alpha}_{\underline{\epsilon}}(y), \\ f_{\underline{\mathfrak{B}}}(y)|_{y\to c} &\sim \frac{\alpha}{2}\, g_{\underline{\epsilon}}(y) G^{\alpha-1}_{\underline{\epsilon}}(y), \\ h_{\underline{\mathfrak{B}}}(y)|_{y\to c} &\sim \alpha \frac{g_{\underline{\epsilon}}(y) G^{\alpha-1}_{\underline{\epsilon}}(y)}{2 - G^{\alpha}_{\underline{\epsilon}}(y)}. \end{aligned} \tag{4}$$

The asymptotic of equations (1), (2) and (3) as $y \to \infty$ are given by

$$\begin{aligned} 1 - F_{\underline{\mathfrak{B}}}(y)|_{y\to\infty} &\sim \alpha \left[1 - G_{\underline{\epsilon}}(y)\right], \\ f_{\underline{\mathfrak{B}}}(y)|_{y\to\infty} &\sim \alpha\, g_{\underline{\epsilon}}(y), \\ h_{\underline{\mathfrak{B}}}(y)|_{y\to\infty} &\sim \frac{g_{\underline{\epsilon}}(y)}{1 - G_{\underline{\epsilon}}(y)}. \end{aligned} \tag{5}$$

In these equations, we see the influence of parameters on the right tail and left tail of the new model.

2.3. Mixture for cdf and pdf

For any parent distribution $G_{\underline{\epsilon}}(y)$, put $V \sim Exp^d(G)$, we represent the Exp-G type-**I** distribution if V has pdf and cdf as:

$$\pi_d(y) = d\, g_{\underline{\epsilon}}(y)\, G_{\underline{\epsilon}}(y)^{d-1} \qquad \text{and} \qquad \Pi_d(y) = G_{\underline{\epsilon}}(y)^d,$$

respectively. Now, We are able to obtain an extension for $F(y)$. Using generalized binomial expansion for any real $\alpha, \beta \in \mathbf{R}^+$, The generalized binomial expansion is written as follows

$$\begin{aligned} F_{\underline{\mathfrak{B}}}(y) &= \frac{G^{\alpha}_{\underline{\epsilon}}(y)}{2 - G^{\beta}_{\underline{\epsilon}}(y)} = \frac{G^{\alpha}_{\underline{\epsilon}}(y)}{[1 - G^{\beta}_{\underline{\epsilon}}(y)] + 1} = G^{\alpha}_{\underline{\epsilon}}(y) \sum_{\zeta_1=0}^{\infty} (-1)^{\zeta_1} \left[1 - G^{\beta}_{\underline{\epsilon}}(y)\right]^{\zeta_1} \\ &= \sum_{\zeta_1=0}^{\infty} \sum_{\zeta_2=0}^{\zeta_1} (-1)^{\zeta_1+\zeta_2} \binom{\zeta_1}{\zeta_2} G^{\varpi^*}_{\underline{\epsilon}}(y) = \sum_{\zeta_2=0}^{\infty} \sum_{\zeta_1=\zeta_2}^{\infty} (-1)^{\zeta_1+\zeta_2} \binom{\zeta_1}{\zeta_2} G^{\varpi^*}_{\underline{\epsilon}}(y) \\ &= \sum_{\zeta_2=0}^{\infty} \nu_{\zeta_2}\, G^{\varpi^*}_{\underline{\epsilon}}(y) = \sum_{\zeta_2=0}^{\infty} \nu_{\zeta_2}\, \Pi_{\varpi^*}(y) \end{aligned} \tag{6}$$

where $\varpi^* = \beta\,\zeta_2 + \alpha$, $G_{\underline{\epsilon}}(x)^{\alpha} = \Pi_{\alpha}(x)$ refers to the cdf of Exp-G type-**I** with power parameter α and

$$\nu_{\zeta_2} = \sum_{\zeta_1=\zeta_2}^{\infty} (-1)^{\zeta_1+\zeta_2} \binom{\zeta_1}{\zeta_2}.$$

The pdf of Z follows by differentiating (6) as

$$f_{\underline{\mathfrak{B}}}(y) = \sum_{\zeta_2=0}^{\infty} \nu_{\zeta_2}\, \pi_{\varpi^*}(y), \tag{7}$$

where $\pi_{\varpi^*}(y) = d\Pi_{\varpi^*}(y)/dy$ is the Exp-G density function with power parameter ϖ^*. The NW-G density function is a simple linear combination of Exp-G type-**I** densities, as shown by Equation (7).

2.4. Residual Entropy

Residual Entropy is important measure of information. The Residual entropy of Y is given by

$$\mathcal{E}(Y) = -\int_0^\infty F_{\underline{\mathfrak{B}}}(y)\log(F_{\underline{\mathfrak{B}}}(y))dy$$

After some simple algebra using geometric expansion and generalized binomial expansion, for NW-G($\underline{\mathfrak{B}}$) we can obtain,

$$\begin{aligned}
-F_{\underline{\mathfrak{B}}}(y)\log(F_{\underline{\mathfrak{B}}}(y)) &= \frac{G_{\underline{\epsilon}}^{\alpha}(y)}{2-G_{\underline{\epsilon}}^{\beta}(y)}\left[\alpha\,\log(G_{\underline{\epsilon}}(y)) + \log(2-G_{\underline{\epsilon}}^{\beta}(y))\right] \\
&= \frac{G_{\underline{\epsilon}}^{\alpha}(y)}{2-G_{\underline{\epsilon}}^{\beta}(y)}\left[\alpha\sum_{\zeta_1=0}^{\infty}\frac{\overline{G}_{\underline{\epsilon}}(y)^{\zeta_1+1}}{\zeta_1+1} + \sum_{\zeta_1=0}^{\infty}\frac{(-1)^{\zeta_1}\left[1-G_{\underline{\epsilon}}^{\beta}(y)\right]^{\zeta_1}}{\zeta_1+1}\right] \\
&= G_{\underline{\epsilon}}^{\alpha}(y)\left[\alpha\sum_{\zeta_1,\zeta_2=0}^{\infty}\frac{(-1)^{\zeta_2}\overline{G}_{\underline{\epsilon}}(y)^{\zeta_1+1}\left[1-G_{\underline{\epsilon}}^{\beta}(y)\right]^{\zeta_2}}{\zeta_1+1} + \sum_{\zeta_1,\zeta_2=0}^{\infty}\frac{(-1)^{\zeta_1+\zeta_2}\left[1-G_{\underline{\epsilon}}^{\beta}(y)\right]^{\zeta_1+\zeta_2}}{\zeta_1+1}\right] \\
&= \alpha\sum_{\zeta_1,\zeta_2=0}^{\infty}\sum_{\zeta_3=0}^{\zeta_1+1}\frac{(-1)^{\zeta_2+\zeta_3}\binom{\zeta_1+1}{\zeta_3}G_{\underline{\epsilon}}^{\alpha+\zeta_3}(y)\left[1-G_{\underline{\epsilon}}^{\beta}(y)\right]^{\zeta_2}}{\zeta_1+1} \\
&\quad + \sum_{\zeta_1,\zeta_2=0}^{\infty}\sum_{\zeta_3=0}^{\zeta_1+1}\frac{(-1)^{\zeta_1+\zeta_2+\zeta_3}\binom{\zeta_1+\zeta_2}{\zeta_3}G_{\underline{\epsilon}}^{\beta\zeta_3+\alpha}(y)}{\zeta_1+1} \\
&= \alpha\sum_{\zeta_1,\zeta_2=0}^{\infty}\sum_{\zeta_3=0}^{\zeta_1+1}\sum_{l=0}^{\zeta_2}\frac{(-1)^{\zeta_2+\zeta_3}\binom{\zeta_1+1}{\zeta_3}\binom{\zeta_2}{l}G_{\underline{\epsilon}}^{\alpha+\zeta_3+l}(y)}{\zeta_1+1} \\
&\quad + \sum_{\zeta_1,\zeta_2=0}^{\infty}\sum_{\zeta_3=0}^{\zeta_1+1}\frac{(-1)^{\zeta_1+\zeta_2+\zeta_3}\binom{\zeta_1+\zeta_2}{\zeta_3}G_{\underline{\epsilon}}^{\beta\zeta_3+\alpha}(y)}{\zeta_1+1} \\
&= \alpha\sum_{\zeta_1,\zeta_2=0}^{\infty}\sum_{\zeta_3=0}^{\zeta_1+1}\sum_{l=0}^{\zeta_2}V_{\zeta_1,\zeta_2,\zeta_3,l}G_{\underline{\epsilon}}^{\alpha+\zeta_3+l}(y) + \sum_{\zeta_1,\zeta_2=0}^{\infty}\sum_{\zeta_3=0}^{\zeta_1+1}U_{\zeta_1,\zeta_2,\zeta_3}\,G_{\underline{\epsilon}}^{\beta\zeta_3+\alpha}(y)
\end{aligned} \tag{8}$$

where

$$V_{\zeta_1,\zeta_2,\zeta_3,l} = \frac{(-1)^{\zeta_2+\zeta_3}}{\zeta_1+1}\binom{\zeta_1+1}{\zeta_3}\binom{\zeta_2}{l},$$

and

$$U_{\zeta_1,\zeta_2,\zeta_3} = \frac{(-1)^{\zeta_1+\zeta_2+\zeta_3}}{\zeta_1+1}\binom{\zeta_1+\zeta_2}{\zeta_3}.$$

Then

$$\mathcal{E}(Y) = \alpha\sum_{\zeta_1,\zeta_2=0}^{\infty}\sum_{\zeta_3=0}^{\zeta_1+1}\sum_{l=0}^{\zeta_2}V_{\zeta_1,\zeta_2,\zeta_3,l}\times\zeta(\alpha+\zeta_3+l) + \sum_{\zeta_1,\zeta_2=0}^{\infty}\sum_{\zeta_3=0}^{\zeta_1+1}U_{\zeta_1,\zeta_2,\zeta_3}\times\mathbf{I}_{-\infty}^{+\infty}(\beta\,\zeta_3+\alpha,Y),$$

where

$$\mathbf{I}_{-\infty}^{+\infty}(\lambda, Y) = \int_{-\infty}^{\infty} G_{\underline{\epsilon}}^{\lambda}(y)dy.$$

2.5. Order Statistics

Let $Y_1, Y_2, \cdots, Y_{\mathfrak{n}}$ is a random sample (RS) from Eq. (1). Suppose $Y_{1:\mathfrak{n}} \leq Y_{2:\mathfrak{n}} \leq \cdots \leq Y_{\mathfrak{n}:\mathfrak{n}}$ indicates to the corresponding order statistics. Then, the pdf corresponding to the i^{th} order statistic can be derived from

$$F_{i:\mathfrak{n}}(y) = \sum_{\zeta_1=i}^{\mathfrak{n}} \binom{\mathfrak{n}}{\zeta_1} F(y)^{\zeta_1} \left[1 - F_{\underline{\mathfrak{B}}}(y)\right]^{\mathfrak{n}-\zeta_1} = \sum_{\zeta_1=i}^{\mathfrak{n}} \sum_{\zeta_2-0}^{\mathfrak{n}-\zeta_1} (-1)^{\zeta_2} \binom{\mathfrak{n}}{\zeta_1} \binom{\mathfrak{n}-\zeta_1}{\zeta_2} F_{\underline{\mathfrak{B}}}(y)^{\zeta_1+\zeta_2} \tag{9}$$

We can reformulate it using generalized binomial expansion as follows:

$$\begin{aligned} F_{\underline{\mathfrak{B}}}(y)^{\zeta_1+\zeta_2} &= \frac{G_{\underline{\epsilon}}^{\alpha(\zeta_1+\zeta_2)}(y)}{[2 - G_{\underline{\epsilon}}^{\beta}(y)]^{\zeta_1+\zeta_2}} = G_{\underline{\epsilon}}^{\alpha(\zeta_1+\zeta_2)}(y) \sum_{l=0}^{\infty} \binom{-\zeta_1-\zeta_2}{l} \left[1 - G_{\underline{\epsilon}}^{\beta}(y)\right]^{l} \\ &= \sum_{l=0}^{\infty} \sum_{\zeta_3=0}^{l} (-1)^{\zeta_3} \binom{-\zeta_1-\zeta_2}{l} \binom{l}{\zeta_3} G_{\underline{\epsilon}}^{\beta\,\zeta_3+\alpha(\zeta_1+\zeta_2)}(y). \end{aligned} \tag{10}$$

Then

$$\begin{aligned} F_{i:\mathfrak{n}}(y) &= \sum_{\zeta_1=i}^{\mathfrak{n}} \sum_{\zeta_2=0}^{\mathfrak{n}-\zeta_1} \sum_{l=0}^{\infty} \sum_{\zeta_3=0}^{l} (-1)^{\zeta_2+\zeta_3} \binom{\mathfrak{n}}{\zeta_1} \binom{\mathfrak{n}-\zeta_1}{\zeta_2} \binom{-\zeta_1-\zeta_2}{l} \binom{l}{\zeta_3} G_{\underline{\epsilon}}^{\omega^*}(y) \\ &= \sum_{\zeta_1=i}^{\mathfrak{n}} \sum_{\zeta_2=0}^{\mathfrak{n}-\zeta_1} \sum_{l=0}^{\infty} \sum_{\zeta_3=0}^{l} w_{\zeta_1,\zeta_2,\zeta_3} \Pi_{\omega^*}(y) \end{aligned} \tag{11}$$

where $\omega^* = \alpha(\zeta_1+\zeta_2) + \beta\,\zeta_3$ and

$$w_{\zeta_1,\zeta_2,\zeta_3} = (-1)^{\zeta_2+\zeta_3} \binom{\mathfrak{n}}{\zeta_1} \binom{\mathfrak{n}-\zeta_1}{\zeta_2} \binom{-\zeta_1-\zeta_2}{l} \binom{l}{\zeta_3}.$$

By differentiating from equation 11 with respect to Y, the density function of the ith order statistic of any NW-G distribution can be expressed as

$$f_{i:\mathfrak{n}}(y) = \sum_{\zeta_1=i}^{\mathfrak{n}} \sum_{\zeta_2-0}^{\mathfrak{n}-\zeta_1} \sum_{\zeta_3=0}^{\infty} w_{\zeta_1,\zeta_2,\zeta_3} \pi_{\omega^*}(y),$$

where $\pi_{\omega^*}(y)$ is the exp-G type-**I** density function with parameter ω^*.

3. Half-Logistic case

In this section we study Half-Logistic by taking $G_{\underline{\epsilon}}(y) = \frac{1-\exp[-y]}{1+\exp[-y]}$ and $g_{\underline{\epsilon}}(y) = \frac{2\exp[-y]}{(1+\exp[-y])^2}$ for $y \in \mathbf{R}^+$ as half-logistic cdf and pdf in equations (1,2 and 3). We denote it by NW-HL(α, β). The cdf, pdf and hrfof NW-HL(α, β) are given by

$$F_{\alpha,\beta}(y) = \frac{\left[\frac{1-\exp[-y]}{1+\exp[-y]}\right]^{\alpha}}{2 - \left[\frac{1-\exp[-y]}{1+\exp[-y]}\right]^{\beta}},$$

$$f_{\alpha,\beta}(y)=\frac{2\exp\left[-y\right]\left(1-\exp\left[-y\right]\right)^{\alpha-1}\left[2\alpha+(\beta-\alpha)\left[\frac{1-\exp[-y]}{1+\exp[-y]}\right]^{\beta}\right]}{\left(1+\exp\left[-y\right]\right)^{\alpha+1}\left[2-\left[\frac{1-\exp[-y]}{1+\exp[-y]}\right]^{\beta}\right]^{2}},$$

and

$$h_{\alpha,\beta}(y)=\frac{2\exp\left[-y\right]\left(1-\exp\left[-y\right]\right)^{\alpha-1}\left[2\alpha+(\beta-\alpha)\left[\frac{1-\exp[-y]}{1+\exp[-y]}\right]^{\beta}\right]}{\left(1+\exp\left[-y\right]\right)^{\alpha+1}\left[2-\left[\frac{1-\exp[-y]}{1+\exp[-y]}\right]^{\beta}\right]\left[2-\left[\frac{1-\exp[-y]}{1+\exp[-y]}\right]^{\beta}-\left[\frac{1-\exp[-y]}{1+\exp[-y]}\right]^{\alpha}\right]}$$

Figures 1 and 2. provide the pdf and the hrf of NW-HL(α,β) for selected parameter values. These graphs show that the pdf of NW-HL(α,β) is unimodal, right skew or almost symmetric. The hrf of NW-HL(α,β) can be decreasing, increasing and bathtub shape.

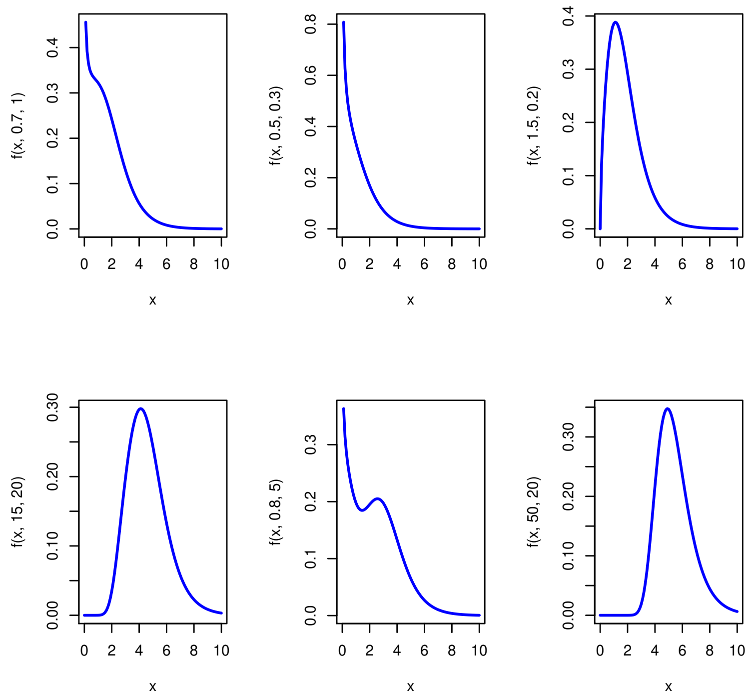


Figure 1. Plots of density function of NW-HL(α,β) for some subset value of parameters.

3.1. Moments

Here, we give lemma, which will be used later.

Lemma 1
For $\varsigma_1,\varsigma_2,\varsigma_4\in\mathbf{R}^+$ and $\varsigma_3>-1$, let

$$Q(\varsigma_1,\varsigma_2,\varsigma_3,\varsigma_4)=\int_0^{\infty}\frac{y^{\varsigma_1}\,e^{-\varsigma_2\,y}\,(1-e^{-\,y})^{\varsigma_3}}{(1+e^{-\,y})^{\varsigma_4}}\,dy.$$

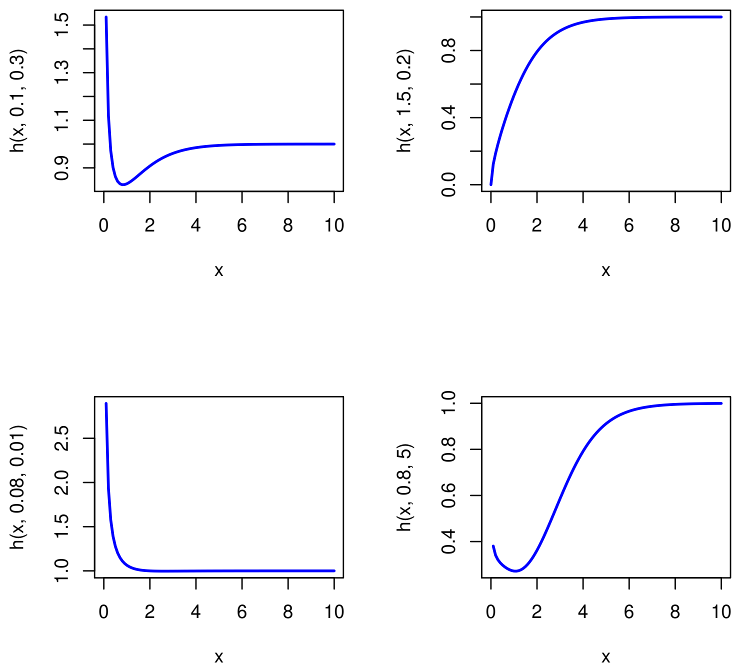


Figure 2. Plots of hazard rate function of NW-HL(α, β) for some subset value of parameters.

Then with using some algebra, we obtain

$$
\begin{aligned}
Q(\varsigma_1, \varsigma_2, \varsigma_3, \varsigma_4) &= \int_0^\infty \frac{y^{\varsigma_1}\, e^{-\varsigma_2\, y}\, (1 - e^{-\, y})^{\varsigma_3}}{(1 + e^{-\, y})^{\varsigma_4}}\, dy \\
&= \sum_{\zeta_1 = 0}^{\infty} \binom{-\varsigma_4}{\zeta_1} \int_0^\infty y^{\varsigma_1}\, e^{-(\varsigma_2 + \zeta_1)\, y}\, (1 - e^{-\, y})^{\varsigma_3} dy \\
&= \sum_{\zeta_1, \zeta_2 = 0}^{\infty} (-1)^{\zeta_2} \binom{-\varsigma_4}{\zeta_1} \binom{\varsigma_3}{\zeta_2} \int_0^\infty y^{\varsigma_1}\, e^{-(\varsigma_2 + \zeta_1 + \zeta_2)\, y} dy \\
&= \sum_{\zeta_1, \zeta_2 = 0}^{\infty} (-1)^{\zeta_2} \binom{-\varsigma_4}{\zeta_1} \binom{\varsigma_3}{\zeta_2} \frac{\Gamma(\varsigma_1 + 1)}{(\zeta_1 + \zeta_2 + \varsigma_2)^{\varsigma_1 + 1}}.
\end{aligned}
$$

where $\Gamma(\cdot)$ is the well-know gamma function.

Next, the $\mathfrak{n}$-th moment of the NW-HL model is given as follows:

$$
E(Y^{\mathfrak{n}}) = 2 \sum_{\zeta_2 = 0}^{\infty} \varpi^* \nu_{\zeta_2}\; Q(\mathfrak{n}, 1, \varpi^* - 1, \varpi^* + 1). \tag{12}
$$

For integer values of $\mathfrak{n}$, Let $\mu'_{\mathfrak{n},Y} = E(Y^{\mathfrak{n}})$ and $\mu'_{1,Y} = E(Y)$, then one can also find the $\mathfrak{n}$-th central moment of the NW-HL distribution as

$$\mu_{\mathfrak{n},Y} = E(Y - \mu'_{1,Y})^{\mathfrak{n}} = \sum_{\zeta_1=0}^{\mathfrak{n}} \binom{\mathfrak{n}}{\zeta_1} \mu'_{\zeta_1,Y} (-\mu'_{1,Y})^{\mathfrak{n}-\zeta_1}.$$

Using (12), the measures of skewness and kurtosis of the NW-HL model can be obtained as

$$Skew(Y) = \frac{-3\mu'_{2,Y}\mu'_{1,Y} + \mu'_{3,Y} + 2{\mu'_{1,Y}}^3}{\left(\mu'_{2,Y} - {\mu'_{1,Y}}^2\right)^{\frac{3}{2}}},$$

and

$$Kurt(Y) = \frac{\mu'_{4,Y} - 4\mu'_{1,Y}\mu'_{3,Y} + 6{\mu'_{1,Y}}^2\mu'_{3,Y} - 3{\mu'_{1,Y}}^4}{\mu'_{2,Y} - {\mu'_{1,Y}}^2},$$

respectively. The moment generating function of NW-HL model can then be written as

$$E[\mathrm{e}^{\mathrm{t}\,Y}] = 2\sum_{\zeta_2=0}^{\infty} (\varpi^* + 1)\nu_{\zeta_2}\ \mathrm{Q}(0, 1 - \mathrm{t}, \varpi^* - 1, \varpi^* + 1).$$

Figures 3 shows the behaviour of skewness and kurtosis of NW-HL model.

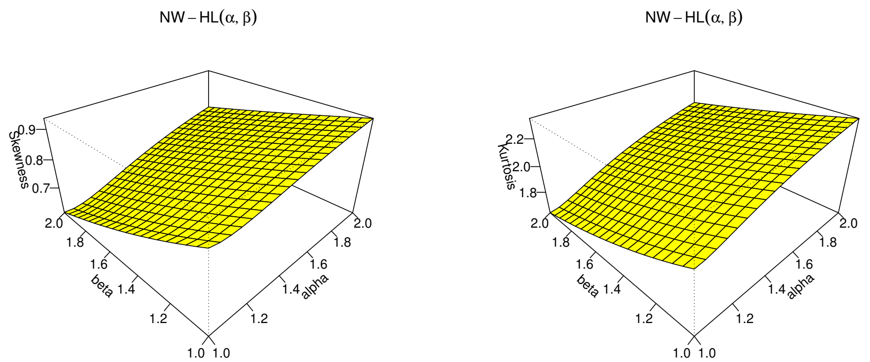


Figure 3. 3D plots of Skewness and Kurtosis of NW-HL(α, β) .

3.2. Conditional moments

Here, we intend to determine the conditional moments of the new family. Let

$$B(\varsigma_1, \varsigma_2, \varsigma_3, \varsigma_4, y) = \int_0^y \frac{y^{\varsigma_1}\, e^{-\varsigma_2\, y}\, (1 - e^{-\,y})^{\varsigma_3}}{(1 + e^{-\,y})^{\varsigma_4}}\, dy,$$

For $\varsigma_1, \varsigma_2, \varsigma_4 \in \mathbf{R}^+$ and $\varsigma_3 > -1$. Then, we obtain

$$B(\varsigma_1, \varsigma_2, \varsigma_3, \varsigma_4, y) = \sum_{\zeta_1,\zeta_2=0}^{\infty} (-1)^{\zeta_2} \binom{-\varsigma_4}{\zeta_1} \binom{\varsigma_3}{\zeta_2} \frac{\gamma(y(\varsigma_2 + \zeta_1 + \zeta_2), \varsigma_1 + 1)}{(\zeta_1 + \zeta_2 + \varsigma_2)^{\varsigma_1+1}},$$

where

$$\gamma(y_1, y_2) = \int_0^{y_1} t^{y_2-1}\, e^{-t}\, dt$$

is the incomplete gamma (lower case function). So, the $\mathfrak{n}$-th conditional moments of Y can be expressed as

$$E(Y^{\mathfrak{n}}|y < Y) = \frac{2}{1-F(y)} \sum_{\zeta_2=0}^{\infty} \varpi^* \nu_{\zeta_2} \begin{bmatrix} Q(\mathfrak{n}, 1, \varpi^*-1, \varpi^*+1) \\ -B(\mathfrak{n}, 1, \varpi^*-1, \varpi^*+1; y) \end{bmatrix} \tag{13}$$

Therefore

$$E(Y^{\mathfrak{n}}|y \geq Y) = \frac{2}{F(y)} \sum_{\zeta_2=0}^{\infty} \varpi^*\ \nu_{\zeta_2} B(\mathfrak{n}, 1, \varpi^*-1, \varpi^*+1; y).$$

3.3. Asymptotic

The asymptotic of cdf,pdf and hrf of the NW-HL(α, β) distribution as $y \to 0^+$ and the asymptotic of cdf, pdf and hrf of the NW-HL(α, β) distribution as $y \to \infty$ are, respectively, given as

$$f_{\alpha,\beta}(y) \sim \frac{1}{2}\alpha\, y^{\alpha-1}, F_{\alpha,\beta}(y) \sim \frac{1}{2}\, y^{\alpha} \text{ and } h_{\alpha,\beta}(y) \sim \frac{1}{2-y^{\alpha}}\alpha\, y^{\alpha-1}.$$

$$1 - F_{\alpha,\beta}(y) \sim 2\alpha\, \exp\left[-y\right], f_{\alpha,\beta}(y) \sim 2\alpha\, \exp\left[-y\right] \text{ and } h_{\alpha,\beta}(y) \sim 1.$$

3.4. Extreme Value

If $y = (y_1 + y_2 ... + y_{\mathfrak{n}})/\mathfrak{n}$ indicates to the mean of the sample. Then, using the central limit (CL) theorem, $\mathfrak{n}^{0.5}(y - E(Y))/\left[\text{Var}(y)\right]^{0.5}$ is the standard normal model as $\mathfrak{n} \to +\infty$. Then, the converges of the relevant extreme values $\mathcal{M}_{1,\mathfrak{n}}^{[Y]} = \max(y_1, y_2, ..., y_{\mathfrak{n}})$ and $m_{1,\mathfrak{n}}^{[Y]} = \min(y_1, y_2, ..., y_{\mathfrak{n}})$ could be useful. Then for the (1), we get

$$y^{\alpha} = \lim_{t \to 0} \frac{1}{F_{\underline{\mathfrak{B}}}(t)} F_{\underline{\mathfrak{B}}}(ty),$$

and

$$\mathrm{e}^{-y} = \lim_{t \to +\infty} \left[1 - F_{\underline{\epsilon}}(y\, \tau(t)) + t\right] \frac{1}{1 - F_{\underline{\mathfrak{B}}}(t)}.$$

Thus, according to Leadbetter et al. (2012) (Theorem 1.6.2), there must be constants $a_{\mathfrak{n}}, \nu_{\mathfrak{n}}, c_{\mathfrak{n}} \in \mathbf{R}^+$ and $d_{\mathfrak{n}} \in \mathbf{R}^+$ in order for

$$\Pr\left(\frac{1}{a_{\mathfrak{n}}^{-1}}\left(\mathcal{M}_{1,\mathfrak{n}}^{[Y]} - \nu_{\mathfrak{n}}\right) \leq y\right) \to e^{-\exp[-y]},$$

and

$$\Pr\left(\frac{1}{c_{\mathfrak{n}}^{-1}}\left(\mathcal{M}_{1,\mathfrak{n}}^{[Y]} - d_{\mathfrak{n}}\right) \leq y\right) \to 1 - \exp\left[-y\right],$$

as $\mathfrak{n} \to +\infty$. It is possible also to determine the shape of the norming constants.

4. Estimation

This Section is divided into three subsections. The first subsection describes the maximum-likelihood estimation (MLE) method. The second one describes the least estimation (LS) method, weighted least squares estimation (WLS) method, Cramer-von-Mises estimation (CM) method, Anderson-Darling (AD) and right-tailed estimations (RAD). In the third subsection, a comprehensive graphical simulations are performed under some common estimation methods.

4.1. MLE method

Suppose $y_1, y_2, \ldots, y_{\mathfrak{n}}$ be a any RS of size $\mathfrak{n}$ from the NW-HL($\underline{\mathbf{\Phi}}$) model. The log-likelihood function ($l_{\mathfrak{n}}(\underline{\mathbf{\Phi}})$) for a vector of parameters can be written as

$$\begin{aligned} l_{\mathfrak{n}}(\underline{\mathbf{\Phi}}) &= \mathfrak{n}\log(2) - \sum_{\tau=1}^{\tau} y_\tau + (\alpha-1)\sum_{\tau=1}^{\mathfrak{n}}\log(1-\exp[-y_\tau]) - (\alpha+1)\sum_{\tau=1}^{\mathfrak{n}}\log(1+\exp[-y_\tau]) \\ &+ \sum_{\tau=1}^{\mathfrak{n}}\log\left[2\alpha+(\beta-\alpha)q_\tau^\beta\right] - 2\sum_{\tau=1}^{\mathfrak{n}}\log(2-q_\tau^\beta) \end{aligned} \tag{14}$$

where

$$q_\tau = \frac{1-\exp[-y_\tau]}{1+\exp[-y_\tau]}$$

The function $l_{\mathfrak{n}}(\underline{\mathbf{\Phi}})$ can then be maximized directly or by resolving the non-linear equations By differentiating (14), the components of the common score vector $U(\underline{\mathbf{\Phi}})$ can be derived as

$$U_\alpha(\underline{\mathbf{\Phi}}) = \sum_{\tau=1}^{\mathfrak{n}}\log(q_\tau) + \sum_{\tau=1}^{\mathfrak{n}}\frac{2-q_\tau^\beta}{\beta q_\tau^\beta - \alpha q_\tau^\beta + 2\alpha},$$

and

$$U_\beta(\underline{\mathbf{\Phi}}) = \sum_{\tau=1}^{\mathfrak{n}}\frac{q_\tau^\beta + \beta q_\tau^\beta\log(q_\tau)}{\beta q_\tau^\beta - \alpha q_\tau^\beta + 2\alpha} + 2\sum_{\tau=1}^{\mathfrak{n}}\frac{q_\tau^\beta\log(q_{\zeta_1})}{2-q_\tau^\beta}$$

4.2. Different methods

•LS & WLS

Swain et al. (1988) first developed "the LS estimations (LSEs) and WL estimations (WLSEs)" by minimizing the following two functions, respectively,

$$S_{\text{LS}}(\underline{\mathbf{\Phi}}) = \sum_{\tau=1}^{\mathfrak{n}}\left(F_{NW-HL}(t_{\tau:\mathfrak{n}};(\underline{\mathbf{\Phi}})) - \frac{1}{\mathfrak{n}+1}\tau\right)^2 |_{\tau=1,2,\ldots,\mathfrak{n}}$$

and

$$S_{\text{WLS}}(\underline{\mathbf{\Phi}}) = \sum_{\tau=1}^{\mathfrak{n}}\xi_\tau(\mathfrak{n})\left(F_{NW-HL}(t_{\tau:\mathfrak{n}};\underline{\mathbf{\Phi}}) - \frac{1}{\mathfrak{n}+1}\tau\right)^2,$$

where

$$\xi_\tau(\mathfrak{n}) = \frac{(\mathfrak{n}+1)^2(\mathfrak{n}+2)}{\tau(\mathfrak{n}-\tau+1)}$$

•CM method

The CM estimations (CMEs) (see Choi and Bulgren (1968)) are determined by minimizing

$$S_{\text{CM}}(\underline{\mathbf{\Phi}}) = \frac{1}{12\mathfrak{n}} + \sum_{\tau=1}^{\mathfrak{n}}\left(F_{NW-HL}(t_{\tau:\mathfrak{n}};\underline{\mathbf{\Phi}}) - \frac{1}{2\mathfrak{n}}(2\tau-1)\right)^2.$$

•AD and RAD methods

The AD and RTAD estimations (ADEs & RTADEs) (Anderson and Darling (1952)) can be obtained by minimizing

$$S_{\text{AD}}(\underline{\mathbf{\Phi}}) = -\mathfrak{n} - \frac{1}{\mathfrak{n}}\sum_{\tau=1}^{\mathfrak{n}}(2\tau-1)\left[\log F(t_\tau;\underline{\mathbf{\Phi}}) + \log\overline{F_{NW-HL}}(t_{\mathfrak{n}+1-\tau};\underline{\mathbf{\Phi}})\right]$$

and

$$S_{\text{RAD}}\left(\underline{\mathbf{\Phi}}\right)=\frac{\mathfrak{n}}{2}-2\sum_{\tau=1}^{\mathfrak{n}}F_{NW-HL}\left(t_{\tau};\underline{\mathbf{\Phi}}\right)-\frac{1}{\mathfrak{n}}\sum_{\tau=1}^{\mathfrak{n}}(2\tau-1)\log\overline{F_{NW-HL}}\left(t_{\mathfrak{n}+1-\tau};\underline{\mathbf{\Phi}}\right),$$

where $\overline{F}\left(\cdot\right)=1-F\left(\cdot\right)$.

4.3. Simulation study

To examine the above-mentioned estimators, we look at the one model that was applied in this part and look at the MSE of those estimators for various samples. as example, given what has been described before, for $(\underline{\mathbf{\Phi}})=(0.9,0.6),(2,1),(3.1,0.4)$.

The review of each way of parameter estimations for the NW-HL model with regards to sample of size of n is supposed. To do this, a simulation analysis is made based on the steps that follow:

1. **Step 1.**Create 10^4 samples of size n from (1) for HL case. This operation is carried out using the quantile function and generated data of uniform distribution.
2. **Step 2.** Calculate the estimates for the 10^4 samples, say $\left(\hat{\alpha}_{\tau},\hat{\beta}_{\tau}\right)$ for $\tau=1,2,...,10^4$.
3. **Step 3.** Calculate the biases and mean squared errors as follows

$$Bias_{\varepsilon}(\mathfrak{n})=\frac{1}{10^4}\sum_{\tau=1}^{10^4}\left(\hat{\varepsilon}_{\tau}-\varepsilon\right)$$

and

$$MSE_{\varepsilon}(\mathfrak{n})=\frac{1}{10^4}\sum_{\tau=1}^{10^4}\left(\hat{\varepsilon}_{\tau}-\varepsilon\right)^2$$

We computed the $bias_{\varepsilon}(\mathfrak{n})$ and the $MSE_{\varepsilon}(\mathfrak{n})$ for $\varepsilon=\underline{\mathbf{\Phi}}$ and $\mathfrak{n}=30,50,\cdots 500$ using the optim function and the well-known Nelder-Mead method by statistical package R. Figures 4-6 display the results.

As can be seen in the figures 4-6, MSE plots for two parameters, as the sample size grows, all methods approach zero, proving the truth of these numerical computations and estimation methods for the NW-HL model parameters. Moreover,

- for estimating α, the LSE approach has the least amount of bias, however for large sample size, all methods have almost same behaviour.
- for estimating β, the RTADE approach has the least amount of bias, however for large sample size, all methods have almost same behaviour.
- for estimating α,the RTADE approach has the least amount of bias, however for large sample size, all methods have almost same behaviour.
- for estimating β, the RTADE approach has the least amount of bias, however for large sample size, all methods have almost same behaviour.

5. Applications

We introduce two applications to real data sets. For the first two examples, the Cramér–von Mises (W*) (Balakrishnan and Chen, 1985), Anderson-Darling (A*) and p-value for Kolmogorov-Smirnov test were chosen to comparison.

The exponentiated half-logistic (ESHL) distribution (Kang and Seo, 2011), Kumaraswamy standard Half-Logistic distribution (KwSHL) (Cordeiro and de Castro, 2011) , the Beta standard Half-Logistic (BSHL) (Jones,

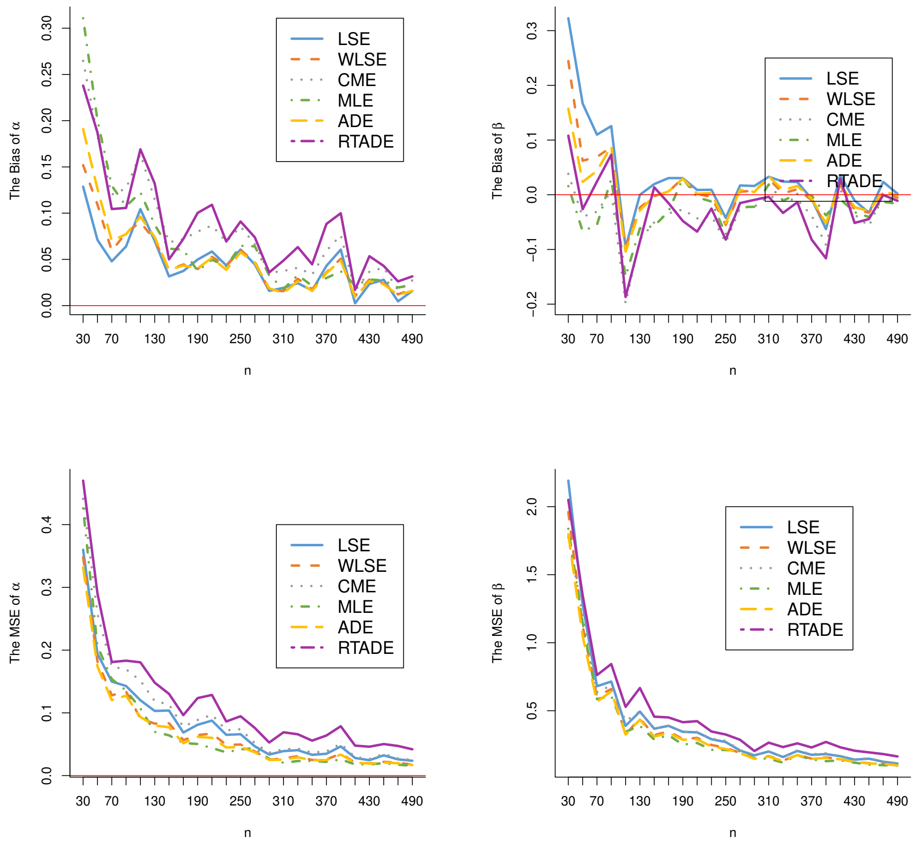


Figure 4. Bias and MSE of estimations for parameter values $(\alpha, \beta) = (0.9, 0.6)$

2004), McDonald standard Half-Logistic (McSHL)distributin (Oliveria et.al, 2016), New Odd log-logistic standard Half-Logistic (NOLL-SHL) distribution (Alizadeh et al. , 2019), weibull distrbution (W), Generalized Exponential (GE) distribution (Gupta and Kundu, 1999), Log Normal (LN) distribution, Gamma (Ga) distribution, Lindley (Li) distribution (Ghitany et al., 2008), Power Lindley (PL) distribution (Ghitany et al., 2013) and Nadarajah-Haghighi (NH) distribution (Nadarajah and Haghighi, 2011) have been selected for comparison in two examples. The cdf of these models are given below:

$$\mathcal{H}\left(y\right) = \frac{1 - \exp\left[-y\right]}{\exp\left[-y\right] + 1}$$

$$F_{SHL}(y;\alpha) = \left(\mathcal{H}(y)\right)^{\alpha}|_{y,\alpha\in\mathbf{R}^{+}},$$

$$F_{NOLL-HL}(y;\alpha,\beta) = \frac{\left(\mathcal{H}(y)\right)^{\alpha}}{\left(\mathcal{H}(y)\right)^{\alpha} + \left(1 - \mathcal{H}(y)\right)^{\beta}}|_{y,\beta,\alpha\in\mathbf{R}^{+}},$$

$$F_{KwSHL}(y;\alpha,\beta) = 1 - \left[1 - \left(\mathcal{H}(y)\right)^{\alpha}\right]^{\beta}|_{y,\beta,\alpha\in\mathbf{R}^{+}},$$

$$F_{BSHL}(y;\alpha,\beta) = \frac{1}{B(\beta,\alpha)}\int_{0}^{\mathcal{H}(y)} t^{-1+\alpha}(1-t)^{\beta-1}dt|_{y,\beta,\alpha\in\mathbf{R}^{+}},$$

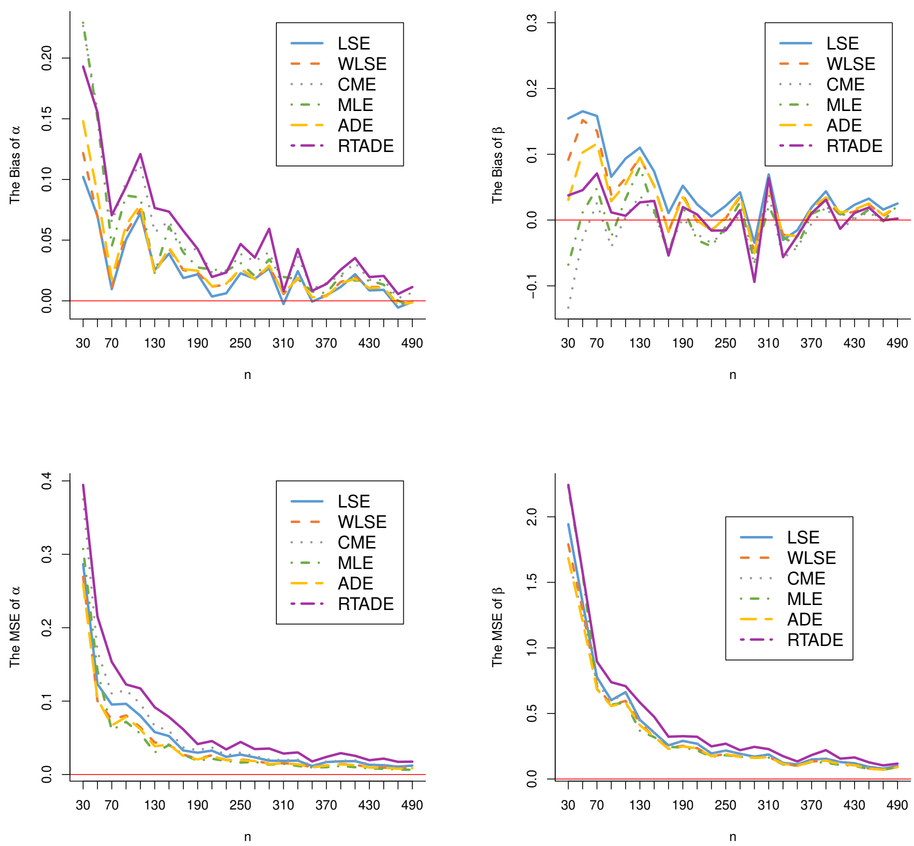


Figure 5. Bias and MSE of estimations for parameter values $(\alpha, \beta) = (2, 1)$

where $B(\vartheta_1, \vartheta_2) = \int_0^1 u^{\vartheta_1 - 1}(1-u)^{\vartheta_2 - 1} du$ denote the beta function.

$$F_{McSHL}(y; \alpha, \beta, c) = \frac{1}{B(\beta, \alpha)} \int_0^{(\mathcal{H}(y))^c} t^{-1+\alpha}(1-t)^{\beta-1} dt|_{y,\beta,\alpha,c \in \mathbf{R}^+},$$

$$F_{Li}(y; \alpha) = 1 - \left(1 + \frac{1}{1+\alpha} \alpha\, y\right) e^{-\alpha\, y}|_{y,\alpha \in \mathbf{R}^+}$$

$$F_{PL}(y; \alpha, \beta) = 1 - \left(1 + \frac{1}{1+\alpha} \alpha\, y^{\beta}\right) e^{-\alpha\, y^{\beta}}|_{y,\beta,\alpha \in \mathbf{R}^+},$$

$$F_{GE}(y; \alpha, \beta) = (1 - e^{-\alpha\, y})^{\beta}|_{y,\beta,\alpha \in \mathbf{R}^+},$$

$$F_{NH}(y; \alpha, \beta) = 1 - e^{1-(1+\alpha\, y)^{\beta}}|_{y,\beta,\alpha \in \mathbf{R}^+},$$

$$F_{LN}(y; \alpha, \beta) = N\left(\frac{1}{\beta}(\log y - \alpha)\right)|_{y,\beta,\alpha \in \mathbf{R}^+},$$

where $N(y) = \int_{-\infty}^{y} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}t^2} dt$ denote the cdf of standard Normal RV.

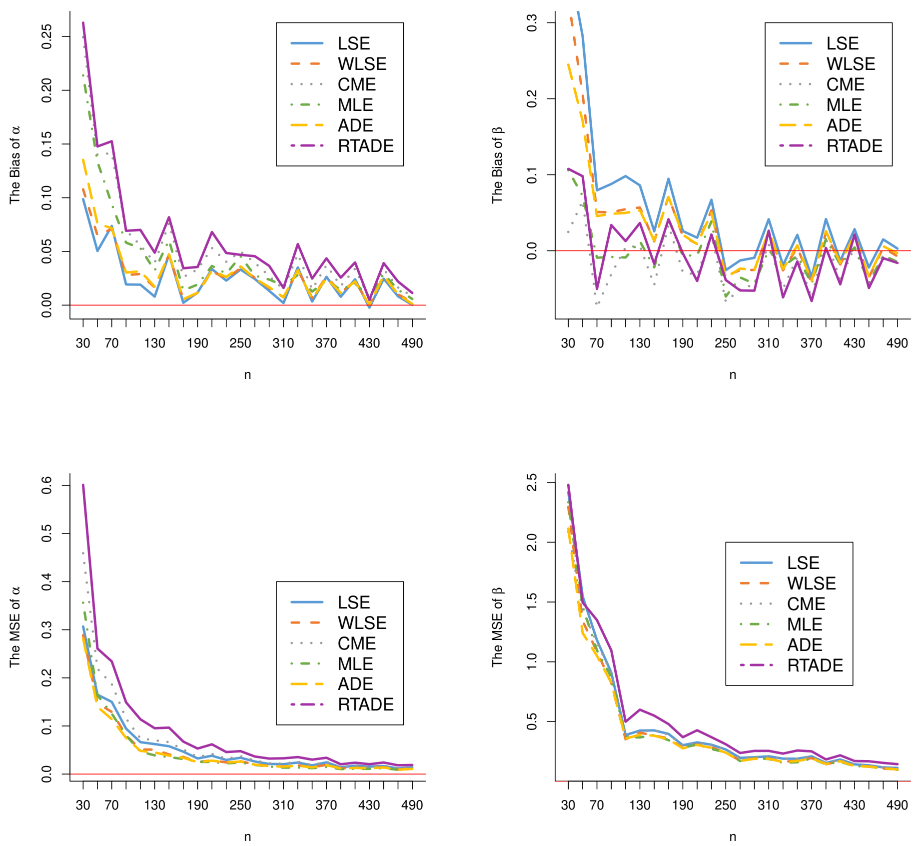


Figure 6. Bias and MSE of estimations for parameter values $(\alpha, \beta) = (3.1, 0.4)$

$$F_{Ga}(y; \alpha, \beta) = \frac{1}{\Gamma(\alpha)} \int_0^y z^{-1+\alpha} e^{-\beta\, z} dz|_{y,\beta,\alpha \in \mathbf{R}^+},$$

$$F_W(y; \alpha, \beta) = 1 - e^{-\alpha\, y^{\beta}}|_{y,\beta,\alpha \in \mathbf{R}^+}.$$

We used MLE method to estimate the model parameters.

5.1. lifetime data

A first data set is related to lifetimes of 20 electronic components given by Murty (2004, p100). The data are: 0.03, 0.12, 0.22, 0.35, 0.73, 0.79, 1.25, 1.41, 1.52, 1.79, 1.80, 1.94, 2.38, 2.40, 2.87, 2.99, 3.14, 3.17, 4.72, 5.09 .
Table 1 summaries the results of the fitted information criteria and estimated MLEs. One can see, the NW-HL model is chosen as the best extension. The histogram of data set I, as well as the fitted pdf plots, are shown in Figure 7 .

5.2. failure time data

A second real data set is failure time of 50 items given by Murty (2004, p195). The data are: 0.008, 0.017, 0.058, 0.061, 0.084, 0.090, 0.134, 0.238, 0.245, 0.353, 0.374, 0.480, 0.495, 0.535, 0.564, 0.681, 0.686, 0.688, 0.921,

Table 1. Results for lifetime data

model	estimatted parameters (se)			W^*	A^*	$p-value$
NW-HL (α, β)	0.711 (0.250)	2.271 (1.412)		$\mathbf{0.022}$	$\mathbf{0.164}$	$\mathbf{0.981}$
NOLL-SHL (α, β)	0.939 (0.296)	0.697 (0.712)		0.067	0.396	0.338
ESHL (α)	1.228 (0.274)			0.057	0.345	0.292
KwSHL (α, β)	0.904 (0.302)	0.644 (0.177)		0.058	0.351	0.816
BSHL (α, β)	0.920 (0.278)	0.647 (0.180)		0.058	0.354	0.820
McSHL (α, β, c)	168.082 (159.066)	0.652 (0.174)	0.047 (0.041)	0.056	0.339	0.848
Li(α)	0.803 (0.133)			0.064	0.381	0.848
PL(α, β)	0.761 (0.174)	1.063 (0.187)		0.059	0.354	0.939
GE(α, β)	0.559 (0.154)	1.139 (0.332)		0.084	0.493	0.648
NH(α, β)	0.071 (0.030)	4.863 (2.154)		0.041	0.256	0.968
LN(α, β)	0.712 (0.286)	1.279 (0.202)		0.203	1.152	0.268
Ga(α, β)	1.162 (0.328)	0.600 (0.210)		0.083	0.486	0.680
W(α, β)	0.426 (0.135)	1.196 (0.224)		0.071	0.418	0.863

0.959, 1.022, 1.092, 1.260, 1.284, 1.295, 1.373, 1.395, 1.414, 1.760, 1.858, 1.892, 1.921, 1.926, 1.933, 2.135, 2.169, 2.301, 2.320, 2.405, 2.506, 2.598, 2.808, 2.971, 3.087, 3.492, 3.669, 3.926, 4.446, 5.119, 8.596 .

Table 2 lists the results of the fitted information criteria and estimated MLEs. One can see, the NW-HL model is chosen as the best model. The histogram of failure time data, as well as the fitted pdf plots, are shown in Figure 8. Figures 9-10 show the unimodality of profile likelihood functions of parameters for lifetime data and failure time data.

Table 2. Results for failure time data

model	estimatted parameters (se)			W^*	A^*	$p-value$
NW-HL (α, β)	0.592 (0.128)	1.233 (0.627)		**0.019**	**0.161**	**0.983**
NOLL-SHL (α, β)	0.796 (0.147)	0.743 (0.126)		0.041	0.256	0.715
ESHL (α)	0.946 (0.133)			0.035	0.224	0.127
KwSHL (α, β)	0.719 (0.149)	0.668 (0.116)		0.037	0.230	0.910
BSHL (α, β)	0.738 (0.136)	0.664 (0.119)		0.038	0.239	0.900
McSHL (α, β, c)	91.722 (78.442)	0.677 (0.115)	0.007 (0.005)	0.034	0.218	0.921
Li(α)	0.910 (0.096)			0.049	0.300	0.865
PL(α, β)	0.994 (0.127)	0.882 (0.096)		0.063	0.375	0.818
GE(α, β)	0.560 (0.105)	0.903 (0.162)		0.076	0.448	0.653
NH(α, β)	0.463 (0.276)	1.179 (0.454)		0.063	0.379	0.873
LN(α, β)	-0.123 (0.205)	1.452 (0.145)		0.329	1.902	0.155
Ga(α, β)	0.914 (0.159)	0.546 (0.125)		0.077	0.455	0.661
W(α, β)	0.610 (0.105)	0.976 (0.111)		0.079	0.468	0.723

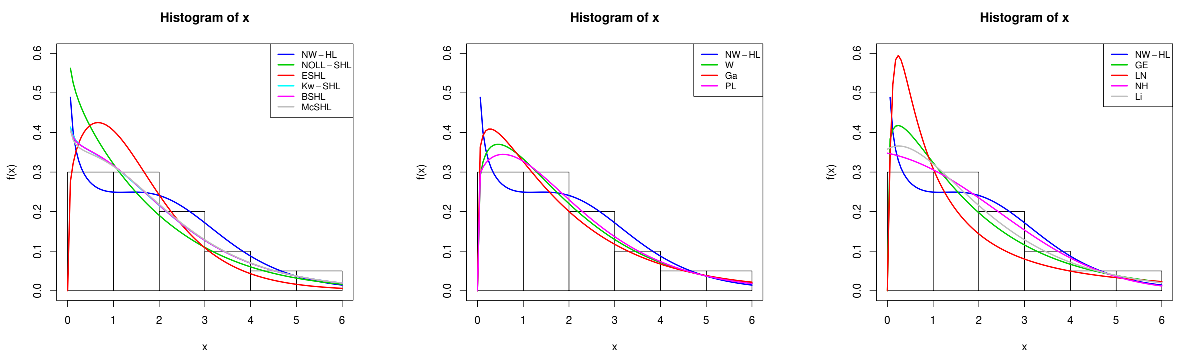


Figure 7. Histogram and fitted pdfs for lifetime data.

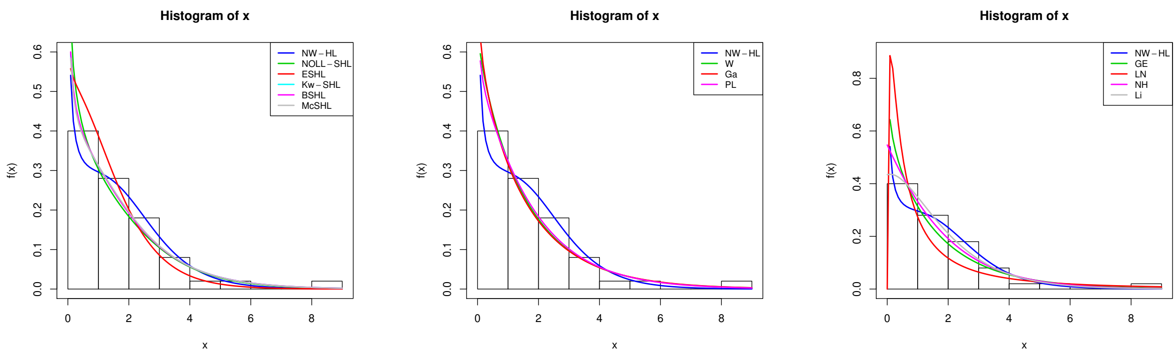


Figure 8. Histogram and fitted pdfs for failure time data.

Figure 9. Unimodality of profile likelihood functions of parameters for lifetime data.

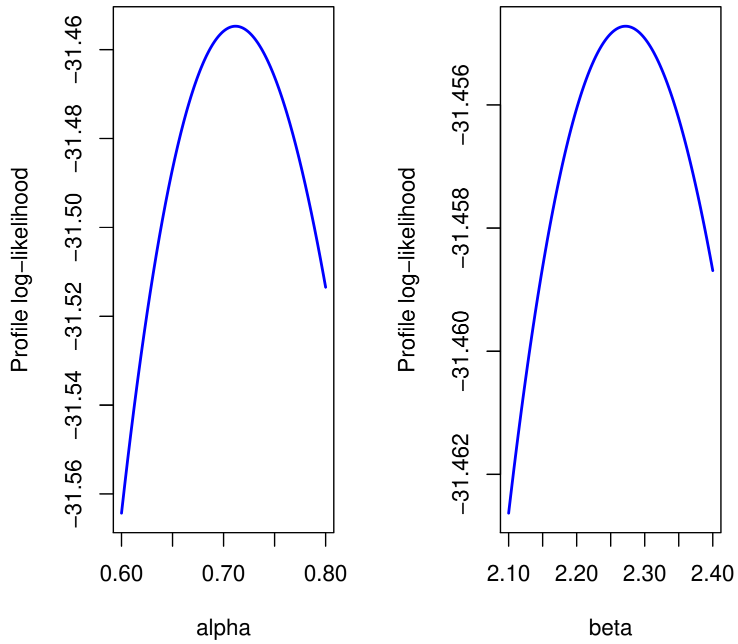
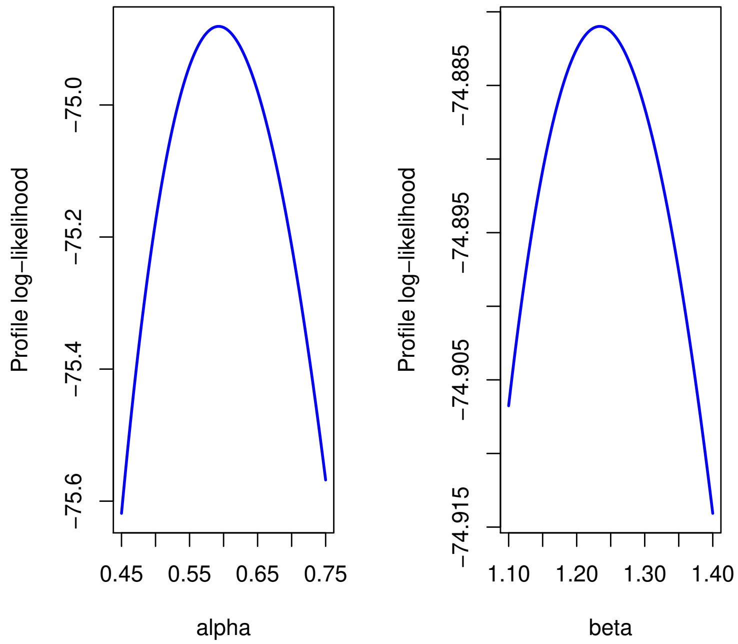


Figure 10. Unimodality of profile likelihood functions of parameters for failure time data.



6. Conclusions

We introduced a novel weighted G family of distributions with only two parameters called the new weighted generated (NW-G) family. Some properties of the new family, such as quantile function, asymptotic, mixture for cdf and pdf, residual entropy and order statistics are obtained. Then, the half -logistic case is studied with more details. The failure rate of the new family accommodated the ”decreasing-constant-increasing-constant”, ”monotonically increasing-constant”, ”monotonically decreasing” and ”bathtub-constant” shapes. The maximum-likelihood estimation method, the least square estimation method, weighted least squares estimation method, Cramer-von-Mises estimation method, Anderson-Darling, right-tailed Anderson-Darling estimations and maximum product of spacings method are considered and assessed. The Bias and MSE plots of parameters for all methods are given, The MSE values approach to zero with the increase in the size of the sample which confirms the validity of the used estimation methods. The flexibility of novel distribution is illustrated by applying it to two real data sets. The new model is compared with many other relevant extensions. The results of tables and figures illustrate the new family provide the best fits against other common competitive extensions. So Applications demonstrate the wide applicability and importance of the new family.

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