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Testing the Validity of Lindley Model Based on Informational Energy with Application to Real Medical Data

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Abstract In this article, a test statistic for testing the validity of the Lindley model based on the informational energy is proposed. Consistency of our test is shown. Through a simulation study, we obtain the critical values of the test statistic and then the power of the test is computed by Monte Carlo method against various alternatives. The performance of the proposed test with some competing tests is compared. Our results show that our test is superior to the classical nonparametric tests and can apply to a testing problem in practice. A real medical data set is presented and analyzed.

Keywords Lindley distribution, Informational energy, Test of fit, Power of test, Monte Carlo simulation.

AMS 2010 subject classifications 62G10, 62P10

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1. Introduction

The density function of a Lindley distribution is given by

$$f(x;\theta) = \frac{\theta^2}{\theta+1}(1+x)e^{-\theta x}, \qquad x>0,$$

where θ is the parameter. The distribution function is given by

$$F(x;\theta) = 1 - \frac{\theta+1+\theta x}{\theta+1}e^{-\theta x}.$$

The mean and variance of the distribution are

$$\mu = E(X) = \frac{\theta+2}{\theta(\theta+1)},$$

and

$$\sigma^2 = Var(X) = \frac{\theta^2+4\theta+2}{\theta^2(\theta+1)^2}.$$

[1] conducted a detailed study about various properties of Lindley distribution including skewness, kurtosis, hazard rate function, mean residual life function, stochastic ordering, stress-strength reliability, among other things; estimation of its parameter and application to model waiting time data in a bank.

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In the literature of survival analysis and reliability theory, the exponential distribution is widely used as a model of lifetime data. However, the exponential distribution only provides a reasonable fit for modeling phenomenon with constant failure rates. Distributions like gamma, Weibull and lognormal have become suitable alternatives to the exponential distribution in many practical situations. [1] found that the Lindley distribution can be a better model than one based on the exponential distribution.

[2] discussed a comparative study of Lindley and exponential distributions for modelling various lifetime data sets from biomedical science and engineering, and concluded that there are lifetime data where exponential distribution gives better fit than Lindley distribution and in majority of data sets Lindley distribution gives better fit than exponential distribution.

Since for computing the proposed test statistic, we need to estimate the parameter θ, we apply the maximum likelihood estimate (MLE) approach to estimate the unknown parameter.

Suppose $X_1, ..., X_n$ is a random sample from the Lindley distribution, the estimator for both maximum likelihood estimate (MLE) and method of moments estimate of the parameter θ is

$$\hat{\theta} = \frac{-\left(\bar{X}-1\right)+\sqrt{\left(\bar{X}-1\right)^2+8\bar{X}}}{2\bar{X}}, \qquad \bar{X}>0.$$

[1] showed that the estimator $\hat{\theta}$ of θ is positively biased: $E(\hat{\theta})-\theta>0$, and it is consistent and asymptotically normal $\sqrt{n}\left(\hat{\theta}-\theta\right) \to N(0, 1/\sigma^2)$.

In complete sample case, [1] developed different distributional properties, reliability characteristics and some inferential procedures for the Lindley distribution. [3] discussed reliability estimation in Lindley distribution with progressively type II right censored sample. [4] gave parameter estimation of Lindley distribution with hybrid censored data. Also, [5] studied inferences on stress-strength reliability for Lindley distribution with complete sample information. [6] discussed estimation of stress-strength reliability using progressively first failure censoring. These studies suggest that in many real-life situations Lindley distribution serves as a better lifetime model than the so far popular distributions like exponential, gamma, Rayleigh, Weibull etc. Other properties of the Lindley distribution can be found in [7], [8], [9], and [10].

Therefore, it is a clear need to check whether the Lindley model is a satisfactory model for the observations.

Suppose that the random variable X has a distribution function $F(x)$ and a continuous density function $f(x)$. The informational energy $\varepsilon(f)$ of X is defined as

$$\varepsilon(f) = \int_{-\infty}^{\infty} f(x)^2\, dx.$$

Estimation of informational energy from a random sample has been considered by [11]. Based on spacing of order statistics, [11] suggested a nonparametric estimator of the informational energy. [11] expressed $\varepsilon(f)$ as

$$\varepsilon(f) = \int_0^1 \left(\frac{d}{dp}F^{-1}(p)\right)^{-1} dp$$

and then he used the empirical distribution function F_n and a difference operator and then proposed an estimator. He also estimated the derivative of $F^{-1}(p)$ by a function of the order statistics. Assuming that $X_1, \ldots, X_n$ is a random sample, Pardos estimator is

$$\varepsilon_{mn} = \frac{1}{n}\sum_{i=1}^{n} \frac{2m}{n\left(X_{(i+m)}-X_{(i-m)}\right)},$$

where m is a positive integer smaller than $n/2$, $X_{(i)} = X_{(1)}$ if $i<1$, $X_{(i)} = X_{(n)}$ if $i>n$. Also, $X_{(1)} \le X_{(2)} \le ... \le X_{(n)}$ are order statistics of a random sample of size n. Pardo showed that his estimator is consistent, i.e., $\varepsilon_{mn} \to \varepsilon(f)$ as $n\to\infty$, $m\to\infty$, $m/n\to 0$.

It is shown that among all distributions with support (0,1), that possess a density function f, the informational energy $\varepsilon(f)$ is minimized by the uniform distribution. By using this property of informational energy, Pardo introduced a test for the uniformity. Its test statistic is given as

$$\varepsilon_{mn} = \frac{1}{n}\sum_{i=1}^{n}\frac{2m}{n\left(X_{(i+m)} - X_{(i-m)}\right)},$$

where large values of ε_{mn} indicate that the sample follows a non-uniform distribution. The critical points and power values of the test are given in [11] and also a power comparison with other tests is presented.

[12] applied the informational energy and constructed a test of fit for the normal distribution. They also compared power of this test with the competing tests and found that the test based on the informational energy has a good performance.

Recently, based on the informational energy, [13] and [14] proposed tests for Laplace and Cauchy distributions, respectively. Also, [15] applied the informational energy and introduced a goodness of fit test for the exponential distribution. Then, they showed that their test has a higher power than the competitors.

The goal of this paper is to suggest a goodness of fit test for the Lindley distribution using the informational energy. In Section 2, we construct a test statistic using an estimator of the informational energy. In Section 3, the critical points and the power of the suggested test are obtained. Then power values of the test are compared with those of the competitors. Section 4 contains an application of the test in a real example. Some conclusions are presented in Section 5.

2. The test statistic

Suppose $X_1, ..., X_n$ are a random sample from a population with the cumulative distribution function F and a density function f. We interest to test the null hypothesis

$$H_0: \{X_1, ..., X_n\} \text{ is a sample from Lindley } Lin(\theta),$$

where θ is unknown. The alternative hypothesis is

$$H_1: \{X_1, ..., X_n\} \text{ is not a sample from Lindley } Lin(\theta).$$

If $f_0(x;\theta)$ denotes the density of Lindley distribution, then the hypothesis of interest is

$$H_0: f(x) = f_0(x;\theta), \qquad for\ some\ \theta \in \Omega,$$

where $\Omega = R^+$. The alternative to H_0 is

$$H_1: f(x) \neq f_0(x;\theta), \qquad for\ any\ \theta \in \Omega.$$

Let F_0 indicates the distribution function of Lindley distribution, without loss of any generality, we reduce the above testing hypothesis, by the probability integral transformation $U = F_0(X)$, to testing the hypothesis of uniformity on the interval (0,1).
Suppose that $U_i = F_0(X_i)$, $i = 1, 2, ..., n$ are the transformed sample, the hypothesis of interest becomes

$$H_0: f(u) = 1, \quad 0 < u < 1,$$

against

$$H_1: f(u) \neq 1, \quad 0 < u < 1.$$

Consequently, test for Lindley distribution converts to uniformity test on (0,1). Under the null hypothesis, each U_i has a uniform distribution, and it seems to be appropriate to use Pardos test (mentioned in pervious section) to test the uniformity of the distribution of U_is and thus Lindley assumption of the distribution of X_is. Therefore, a summary of the test procedure can be shown as

$$X_1, \ldots, X_n \rightarrow U_i = F_0(X_i) \rightarrow T = \frac{1}{n}\sum_{i=1}^{n} \frac{2m}{n\left(U_{(i+m)} - U_{(i-m)}\right)},$$

where the order statistics of transformed sample indicated by $U_{(1)} \leq U_{(2)} \leq ... \leq U_{(n)}$ and also if $i < 1$, $U_{(i)} = U_{(1)}$, if $i > n$, $U_{(i)} = U_{(n)}$.

Since large values of the test statistic T favour the alternative hypothesis to H_0, we reject the null hypothesis H_0, if the test statistic T is large, i.e. reject H_0 if $T \geq C(\alpha)$ for some critical values.
The proposed test statistic can be stated as

$$T = \frac{1}{n}\sum_{i=1}^{n} \frac{2m}{n\left(U_{(i+m)} - U_{(i-m)}\right)} = \frac{1}{n}\sum_{i=1}^{n} \frac{2m}{n\left(F_0(X_{(i+m)};\theta) - F_0(X_{(i-m)};\theta)\right)},$$

where $X_{(1)} \leq X_{(2)} \leq ... \leq X_{(n)}$ are order statistics and if $i < 1$, $X_{(i)} = X_{(1)}$, if $i > n$, $X_{(i)} = X_{(n)}$ and m is a positive integer less than $n/2$.

Since the parameters θ is unknown, we estimate θ by the maximum likelihood estimator. Then, the estimator is

$$\hat{\theta} = \frac{-\left(\bar{X} - 1\right) + \sqrt{\left(\bar{X} - 1\right)^2 + 8\bar{X}}}{2\bar{X}}, \qquad \bar{X} > 0.$$

Therefore, the proposed test statistic is

$$T = \frac{1}{n}\sum_{i=1}^{n} \frac{2m}{n\left(U_{(i+m)} - U_{(i-m)}\right)} = \frac{1}{n}\sum_{i=1}^{n} \frac{2m}{n\left(F_0(X_{(i+m)};\hat{\theta}) - F_0(X_{(i-m)};\hat{\theta})\right)},$$

where $U_i = F_0(X_i;\hat{\theta})$, $i = 1, 2, ..., n$.

Proposition 1. If the parameters of the distribution be known as $\theta = \theta_0$, (that is the null hypothesis is simple) the test statistic can be write as

$$T = \frac{1}{n}\sum_{i=1}^{n} \frac{2m}{n\left(F_0(X_{(i+m)};\theta_0) - F_0(X_{(i-m)};\theta_0)\right)}.$$

Proposition 2. Let the null hypothesis is composite, if $\hat{\theta} \rightarrow \theta_0$ as $n \rightarrow \infty$, the distribution of the test statistic T tends to the distribution of T under the simple hypothesis.

Theorem 1. Let $X_1, ..., X_n$ be a random sample from an unknown continuous distribution F with a probability density function $f(x)$. We have

$$T \geq 1 .$$

Proof. The proof of this theorem is similar to the proof of Theorem 1 in [11] and therefore it is omitted.

Theorem 2. Let F be a completely unknown continuous distribution and F_0 be the null distribution with unspecified parameter. Then under H_1 , T is a consistent test.

Proof. As $n \rightarrow \infty$, $\hat{\theta} \rightarrow \theta$, and also, by of LLN,

$$\frac{1}{n}\sum_{i=1}^{n} \frac{2m}{n\left(F_0(X_{(i+m)};\hat{\theta}) - F_0(X_{(i-m)};\hat{\theta})\right)} \rightarrow \frac{1}{n}\sum_{i=1}^{n} \frac{2m}{n\left(F_0(X_{(i+m)};\theta) - F_0(X_{(i-m)};\theta)\right)}.$$

Moreover, [11] established that

$$\varepsilon_{mn} \to \varepsilon(f) \qquad as\ \ n \to \infty,\ m \to \infty,\ m/n \to 0\,,$$

and consequently, the proof of this theorem is complete.

Theorem 3. Suppose $X_1, ..., X_n$ are a random sample from the Lindley distribution, if $m = o(n)$ and $m \neq 1$, then

$$T \to 1 \text{ as } n \to \infty\,, m \to \infty.$$

Proof. The proof is similar to the proof of Theorem 1 in [11]. Therefore, we omit it.

It is clear that under the null hypothesis, the value of the test statistic will be close to one. Therefore, for large values of the test statistic the null hypothesis H_0 will reject. Next section presents the critical points of our test statistic.

3. Critical points and power comparison

At the significance level α, we reject H_0 if the value of the test statistic is greater than $C(\alpha)$, where the critical value $C(\alpha)$ is obtained by the $\alpha-$quantile of the distribution of the test statistic under the null hypothesis H_0. For different sample sizes, we applied Monte Carlo simulations with 100,000 replicates from the standard Lindley distribution ($\theta = 1$) to compute the critical points of the proposed test statistic. The critical points of T statistic for different sample sizes are tabulated in Table 1.

Table 1. Critical points of the T statistic with $\alpha = 0.05$

	m									
n	1	2	3	4	5	6	7	8	9	10
5	15.518	4.514								
6	13.723	4.031	3.219							
7	12.259	3.690	2.881							
8	10.971	3.440	2.677	2.566						
9	10.213	3.254	2.508	2.376						
10	9.615	3.065	2.410	2.236	2.244					
15	7.154	2.580	2.054	1.894	1.837	1.841	1.884			
20	6.149	2.310	1.872	1.734	1.683	1.661	1.663	1.686	1.723	1.764
25	5.492	2.139	1.765	1.635	1.583	1.561	1.556	1.561	1.576	1.599
30	5.024	2.034	1.683	1.565	1.516	1.492	1.487	1.488	1.493	1.506
40	4.373	1.893	1.583	1.475	1.429	1.406	1.397	1.395	1.396	1.404
50	4.024	1.802	1.518	1.419	1.374	1.351	1.340	1.336	1.338	1.342

The estimated density function of our test statistic based on 100,000 Monte Carlo simulations under the Lindley hypothesis for various sample sizes is shown in Figure 1. From the figure, we found that the values of the proposed test statistic are close to one as increases. Therefore, with increasing n the bias of the test statistic T decreasing. Also, we see that variance of our statistic decreases when n increases.

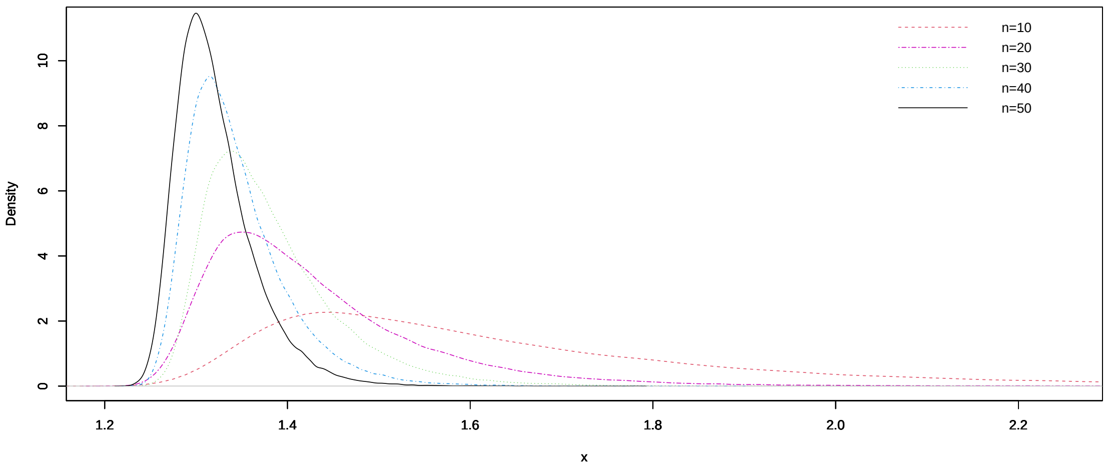

Figure 1. Empirical probability density function of the test statistic based on 100,000 simulations under the Lindley hypothesis for different sample sizes.

It is obvious that the power of the test depends on the window size and type of the considered alternative distribution. Based on a broad Monte Carlo analysis, we determine the values of the window size m which the proposed test attains good (not best) powers for all alternative distributions. These values of the window size m for different sample sizes can be obtained from the following heuristic formula.

$$m = \left[\frac{n}{3}\right] + 1,$$

where $[x]$ means the integer part of x. We can see that the optimal m increases as n increases and that the ratio m/n tends to zero.

The goodness of fit tests based on the empirical distribution function (EDF) are widely used in practice, and therefore we compare the power of the proposed test with them. Methods for assessing the tests based on the empirical distribution function (EDF) are reviewed by [16]. Here, the considered tests are Cramer von Mises test W^2, Watson test U^2, Kolmogorov-Smirnov test D, Anderson-Darling test A^2, and Kuiper test V. The test statistics of them are as follow:

$$W^2 = \frac{1}{12n} + \sum_{i=1}^{n} \left(\frac{2i-1}{2n} - p_{(i)} \right)^2,$$

$$U^2 = \frac{1}{12n} + \sum_{i=1}^{n} \left(p_{(i)} - \bar{p} + 0.5 - \frac{2i-1}{2n} \right)^2,$$

$$D = \max \left(\max_{1\le i\le n} \left\{ \frac{i}{n} - p_{(i)} \right\}, \max_{1\le i\le n} \left\{ p_{(i)} - \frac{i-1}{n} \right\} \right),$$

$$A^2 = -n - \frac{1}{n} \sum_{i=1}^{n} (2i-1) \left\{ \log(p_{(i)}) + \log(1 - p_{(n-i+1)}) \right\},$$

$$V = \max_{1\le i\le n} \left\{ \frac{i}{n} - p_{(i)} \right\} + \max_{1\le i\le n} \left\{ p_{(i)} - \frac{i-1}{n} \right\},$$

where $p_{(i)} = F_0(x_{(i)}, \hat{\theta})$, and F_0 denotes the Lindley distribution function. Also, $\hat{\theta}$ is the maximum likelihood estimator of θ.

By Monte Carlo simulations, power of the proposed test and the EDF-based tests against various alternatives are evaluated. The following alternatives are considered in power comparison.

- the Weibull distribution with density $\theta x^{\theta-1} \exp\left(-x^{\theta}\right)$, denoted by $W(\theta)$,
- the gamma distribution with density $\Gamma(\theta)^{-1} x^{\theta-1} \exp(-x)$, denoted by $\Gamma(\theta)$,
- the lognormal distribution $LN(\theta)$ with density $(\theta x)^{-1}(2\pi)^{-1/2} \exp\left(-(\log x)^2 \Big/ (2\theta^2)\right)$,
- the half-normal HN distribution with density $\Gamma(2/\pi)^{1/2} \exp\left(-x^2/2\right)$,
- the uniform distribution U with density $1, \quad 0 \leq x \leq 1$,
- the modified extreme value $EV(\theta)$, with distribution function $1 - \exp\left(\theta^{-1}(1 - e^{x})\right)$,
- the linear increasing failure rate law $LF(\theta)$, with density $(1 + \theta x) \exp\left(-x - \theta x^2/2\right)$,
- Dhillons [17] distribution $DL(\theta)$, with distribution function $1 - \exp\left(-(\log(x+1))^{\theta+1}\right)$,
- Chens [18] distribution $CH(\theta)$, with distribution function $1 - \exp\left(2\left(1 - e^{x^{\theta}}\right)\right)$.

These alternatives include densities f with decreasing failure rates (DFR), increasing failure rates (IFR) as well as models with unimodal failure rate (UFR) functions and bathtub failure rate (BFR) functions.

The powers of the considered tests under the above alternatives, are obtained as follows. For each alternative, 100,000 samples with sizes 10, 20, 30, and 50 are generated and by the frequency of the event the value of the test statistic is in the critical region, the powers of the tests are computed. Tables 2 and 3 show the powers at significance level $\alpha = 0.05$.

For each alternative, the bold values in these tables once again indicate the test achieving the maximal power.

Tables 2 shows a uniform superiority of the proposed procedure to all other competitor tests against IFR distributions. We can observe that against IFR alternatives our test based on informational energy has a reasonable and good performance in compared with competing tests. Therefore, the proposed test has the most power against IFR distributions. The difference of powers between our test and the competing tests are substantial. Consequently, our test can be confidently recommended in practice.

From Table 3, it is evident that the tests based on A^2 and T statistics have the most power against UFR alternatives and power differences between these tests and the other tests are substantial.

Table 3 reveals a superiority of the test based on A^2 statistic to all other tests as we can say that this test outperforms all other tests against DFR and BFR alternatives.

Although there is no uniformly most powerful test against all alternatives, the tests based on T and A^2 statistics can be recommended in practice. Finally, we summarized the results in Table 4. This table presents the best test in terms of power against different alternatives.

Table 2. Empirical powers of the tests against IFR alternatives at significance level 5%.

$Alternative$	n	W^2	D	V	U^2	A^2	T
$W(1.4)$	10	0.1303	0.1174	0.1104	0.1170	0.0894	**0.1590**
	20	0.2258	0.1966	0.1761	0.1884	0.1917	**0.2672**
	30	0.3237	0.2691	0.2330	0.2635	0.2967	**0.4102**
	50	0.5098	0.4231	0.3736	0.4167	0.5036	**0.5905**
$\Gamma(2)$	10	0.1175	0.1028	0.1101	0.1188	0.0810	**0.1642**
	20	0.2011	0.1754	0.1772	0.1935	0.1800	**0.2796**
	30	0.2879	0.2412	0.2369	0.2687	0.2827	**0.4392**
	50	0.4745	0.4014	0.3875	0.4408	0.5104	**0.6347**
HN	10	0.0952	0.0887	0.0844	0.0875	0.0678	**0.1071**
	20	0.1364	0.1234	0.1084	0.1149	0.1076	**0.1651**
	30	0.1835	0.1552	0.1340	0.1446	0.1492	**0.2340**
	50	0.2839	0.2321	0.1960	0.2139	0.2445	**0.3562**
U	10	0.3386	0.2647	0.3088	0.2957	0.2615	**0.4399**
	20	0.6318	0.4888	0.6071	0.5477	0.5793	**0.8354**
	30	0.8309	0.6764	0.8143	0.7416	0.8056	**0.9644**
	50	0.9756	0.9000	0.9777	0.9417	0.9756	**0.9992**
$CH(1)$	10	0.0937	0.0868	0.0772	0.0789	0.0673	**0.1041**
	20	0.1364	0.1220	0.0998	0.1061	0.1074	**0.1575**
	30	0.1826	0.1557	0.1230	0.1332	0.1477	**0.2312**
	50	0.2796	0.2301	0.1810	0.1933	0.2379	**0.3549**
$CH(1.5)$	10	0.4268	0.3505	0.3359	0.3553	0.3348	**0.4516**
	20	0.7600	0.6343	0.6239	0.6480	0.7160	**0.8066**
	30	0.9200	0.8205	0.8176	0.8370	0.9071	**0.9599**
	50	0.9943	0.9684	0.9736	0.9763	0.9943	**0.9987**
$LF(2)$	10	0.1386	0.1235	0.1113	0.1187	0.0972	**0.1543**
	20	0.2282	0.1943	0.1706	0.1802	0.1851	**0.2542**
	30	0.3292	0.2723	0.2327	0.2527	0.2828	**0.3770**
	50	0.5133	0.4204	0.3663	0.3955	0.4662	**0.5647**
$LF(4)$	10	0.2056	0.1790	0.1594	0.1700	0.1469	**0.2154**
	20	0.3777	0.3160	0.2752	0.2980	0.3192	**0.3878**
	30	0.5308	0.4386	0.3864	0.4204	0.4758	**0.5720**
	50	0.7680	0.6595	0.6067	0.6401	0.7313	**0.7922**
$EV(0.5)$	10	0.0923	0.0861	0.0749	0.0782	0.0670	**0.1021**
	20	0.1384	0.1221	0.1020	0.1074	0.1068	**0.1588**
	30	0.1833	0.1557	0.1242	0.1345	0.1467	**0.2268**
	50	0.2779	0.2262	0.1803	0.1933	0.2378	**0.3561**
$EV(1.5)$	10	0.0923	0.0861	0.0749	0.0782	0.0670	**0.2079**
	20	0.1384	0.1221	0.1020	0.1074	0.1068	**0.3989**
	30	0.1833	0.1557	0.1242	0.1345	0.1467	**0.5993**
	50	0.2779	0.2262	0.1803	0.1933	0.2378	**0.8394**

Table 3. Empirical powers of the tests against UFR, DFR and BFR alternatives at significance level 5%.

Alternative	n	W^2	D	V	U^2	A^2	T
$LN(0.8)$	10	0.1413	0.1302	0.1279	0.1403	0.1068	**0.1766**
	20	0.2221	0.1968	0.2204	0.2448	0.2110	**0.3487**
	30	0.3180	0.2720	0.3268	0.3652	0.3440	**0.5763**
	50	0.5147	0.4436	0.5541	0.6054	0.6131	**0.8363**
$LN(1.5)$	10	0.5140	0.4823	0.3849	0.4001	**0.5544**	0.2065
	20	0.8027	0.7664	0.6690	0.6869	**0.8197**	0.4925
	30	0.9257	0.9020	0.8342	0.8489	**0.9306**	0.6567
	50	0.9900	0.9842	0.9642	0.9697	**0.9905**	0.8944
$DL(1)$	10	0.0877	0.0813	0.0809	0.0862	0.0629	**0.1077**
	20	0.1185	0.1064	0.1139	0.1236	0.1041	**0.1644**
	30	0.1486	0.1274	0.1445	0.1619	0.1445	**0.2435**
	50	0.2123	0.1771	0.2245	0.2533	0.2394	**0.3590**
$DL(1.5)$	10	0.1999	0.1735	0.1751	0.1937	0.1462	**0.2536**
	20	0.3844	0.3271	0.3228	0.3634	0.3601	**0.4736**
	30	0.5568	0.4783	0.4598	0.5241	0.5677	**0.6980**
	50	0.8123	0.7363	0.7129	0.7832	0.8509	**0.8873**
$W(0.8)$	10	0.1960	0.1750	0.1288	0.1366	**0.2748**	0.0313
	20	0.3570	0.3095	0.2295	0.2438	**0.4417**	0.0542
	30	0.4933	0.4319	0.3201	0.3476	**0.5752**	0.0564
	50	0.7062	0.6330	0.5093	0.5395	**0.7720**	0.1251
$\Gamma(0.4)$	10	0.5137	0.4712	0.3701	0.3914	**0.7163**	0.1147
	20	0.8109	0.7663	0.6579	0.6850	**0.9222**	0.3173
	30	0.9354	0.9074	0.8310	0.8551	**0.9810**	0.3981
	50	0.9943	0.9894	0.9697	0.9762	**0.9990**	0.7051
$CH(0.5)$	10	0.3912	0.3546	0.2711	0.2860	**0.5728**	0.0666
	20	0.6670	0.6127	0.4979	0.5281	**0.8141**	0.1796
	30	0.8331	0.7839	0.6733	0.7102	**0.9251**	0.2156
	50	0.9669	0.9464	0.8924	0.9137	**0.9903**	0.4568

Table 4. Powerful tests against different alternatives

IFR	UFR	DFR-BFR
T	A^2 & T	A^2

4. Real data examples

In this section, we applied our proposed procedure to a real data set for illustration purpose.

Example 1. The following original data is a subset of data reported by [19] and [20] represent the survival times in years of a group of patients given chemotherapy treatment alone.

Recently, these data are analyzed by [21]. The data consisting of 46 survival times (in years) for 46 patients are:

0.047, 0.115, 0.121, 0.132, 0.164,0.197, 0.203, 0.260, 0.282, 0.296, 0.334, 0.395, 0.458, 0.466, 0.501, 0.507, 0.529, 0.534, 0.540, 0.570, 0.641, 0.644,0.696, 0.841, 0.863, 1.099, 1.219, 1.271, 1.326, 1.447, 1.485, 1.553, 1.581, 1.589, 2.178, 2.343, 2.416, 2.444, 2.825,2.830, 3.578, 3.658, 3.743, 3.978, 4.003, 4.033.

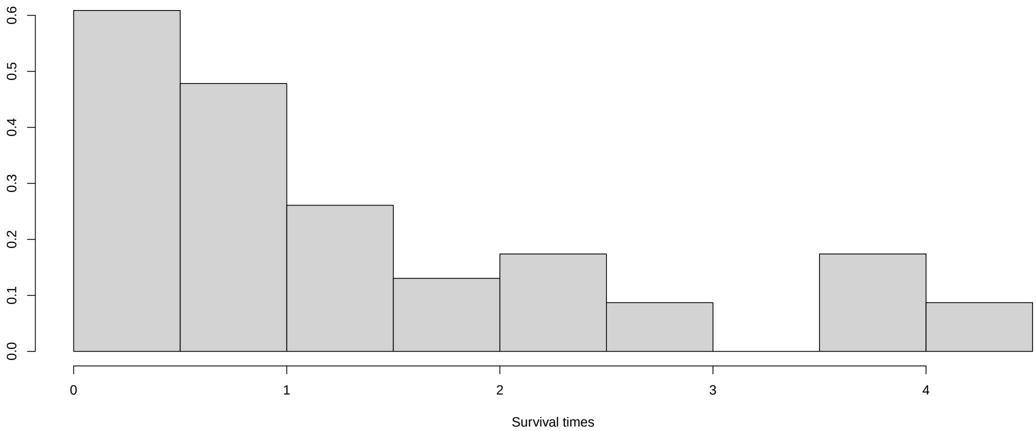

Figure 2. Histogram of survival times.

In Figure 2, we depict the histogram of this data set. We test that the data follow the Lindley model and found it is acceptable for these data. We estimate the parameter θ as

$$\hat{\theta} = 1.112$$

. Then, the value of the test statistic is computed as

$$T = 1.381,$$

and also the critical value of the test at the significance level 0.05 is obtained from a Monte Carlo simulation as 1.422. Because the value of the test statistic is smaller than the corresponding critical value, the Lindley hypothesis is accepted for these data at the significance level of 0.05. So, we conclude that the survival times in years of a group of patients follows Lindley distribution.

5. Conclusions

In this paper, we developed a simple and efficient goodness-of-fit test for Lindley distribution based on the informational energy. We computed the critical points and power of the proposed test against various alternatives and different samples sizes. A simulation study shows the performance of the considered tests against different alternatives. The results from the Monte Carlo simulations demonstrate very good performance of our test from a power perspective. Therefore, the proposed test can be applied to test for the Lindley distribution as a model for describing data in practice. Finally, we applied the proposed test in a real data example.

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