STATISTICS, OPTIMIZATION AND INFORMATION COMPUTING
Stat., Optim. Inf. Comput., Vol. 10, September 2022, pp 1072–1094.
Published online in International Academic Press (www.IAPress.org)

A New Flexible Stress-Strength Model

Alimohammad Beiranvand [1], Ramin Kazemi [2,*], Akram Kohansal [2], Farshin Hormozinejad [1]

[1]Department of Mathematics and Statistics, Islamic Azad University, Ahvaz Branch, Ahvaz, Iran
[2]Department of Statistics, Imam Khomeini International University, Qazvin, Iran

Abstract To introduce a flexible stress-strength model, statistical inference of the stress-strength parameter $R = P(X < Y)$, when stress X and strength Y are two independent two-parameter new Weibull-Fréchet variables, is considered under Type II progressive censored samples. The MLE, AMLE, asymptotic confidence intervals, Bayes estimate and HPD intervals of R are achieved in three different cases. Also, to compare the performance of three different methods, we apply the Monte Carlo simulations and also analyze a data set for illustrative aims.

Keywords Stress-strength model, two-parameter new Weibull-Fréchet distribution, type II progressive censored sample, Monte Carlo simulation.

AMS 2010 subject classifications 62F15, 62F10, 62N05.

DOI:10.19139/soic-2310-5070-1230

1. Introduction

In reliability theory, modeling of lifetime data is very vital and decisive. A number of statistical distributions such as Pareto, Weibull, Exponential, Rayleigh, Gompertz, etc., are used for modeling lifetime data. Recently, some of the authors have worked on the reliability theory. For example, Shoaee and Khorram considered stress-strength reliability of a two-parameter Bathtub-shaped lifetime distribution [15]. Also, they obtained some statistical inference of reliability parameter for Weibull distribution [16]. Rasethuntsa and Nadar [14] studied stress-strength reliability of a non-identical-component-strengths system based on upper record values from the family of Kumaraswamy generalized distributions. Recently, Kohansal [12] derived the reliability estimation in a stress strength model for Burr XII distribution. Ahmadi and Ghafouri [2] obtained the reliability estimation in a multicomponent stress-strength model under generalized half-normal distribution. Kohansal and Nadarajah [11] estimated the stress-strength parameter for a Kumaraswamy distribution. Finally, Kohansal and Shoaee [13] derived Bayesian and classical estimation of reliability in a multicomponent stress-strength model. But, in many practical areas, the classical distributions do not provide adequate fit in modeling data, and there is a clear need to the new flexible distributions.

The probability density function (PDF) and cumulative distribution function (CDF) of the one-parameter Fréchet r.v. are $f_X(x,\nu) = \nu x^{-2} e^{-\frac{\nu}{x}}$ and $F_X(x,\nu) = e^{-\frac{\nu}{x}}$ for $x, \nu > 0$, respectively, see [1]. Consider the CDF of the one-parameter Weibull distribution with shape parameter δ, given by $F_Y(y) = 1 - e^{-y^{\delta}}$ and PDF $f_Y(y) = \delta y^{\delta-1} e^{-y^{\delta}}$ for $y, \delta > 0$. If we set

$$y = \frac{-\log(1 - F_X(x,\nu))}{1 - F_X(x,\nu)},$$

*Correspondence to: Ramin Kazemi (Email: r.kazemi@sci.ikiu.ac.ir). Department of Statistics, Imam Khomeini International University, Qazvin, Iran.

ISSN 2310-5070 (online) ISSN 2311-004X (print)

then, using the T-X idea for generating families of continuous distributions [3], we can define a CDF of the new distribution by

$$F(x) = 1 - \exp\left\{ - \left(\frac{-\log(1 - e^{-\frac{\nu}{x}})}{1 - e^{-\frac{\nu}{x}}} \right)^{\delta} \right\}, \quad x, \delta, \nu > 0.$$

We call it the new two-parameter Weibull-Fréchet distribution (TNWFD). Thus, TNWFD with the shape parameter δ and scale parameter ν, respectively, which denoted by $TNWF(\delta, \nu)$, has PDF and failure rate function as follows:

$$f_{TNWF}(x) = \delta \nu x^{-2} e^{-\frac{\nu}{x}} (1 - e^{-\frac{\nu}{x}})^{-\delta - 1} \big(1 - \log(1 - e^{-\frac{\nu}{x}})\big) \big(- \log(1 - e^{-\frac{\nu}{x}})\big)^{\delta - 1} \times \exp\left\{ - \left(\frac{-\log(1 - e^{-\frac{\nu}{x}})}{1 - e^{-\frac{\nu}{x}}} \right)^{\delta} \right\},$$

$$h_{TNWF}(x) = \delta \nu x^{-2} e^{-\frac{\nu}{x}} (1 - e^{-\frac{\nu}{x}})^{-\delta - 1} \big(1 - \log(1 - e^{-\frac{\nu}{x}})\big) \big(- \log(1 - e^{-\frac{\nu}{x}})^{\delta - 1}\big), \quad x, \delta, \nu > 0,$$

respectively. The PDF and failure rate function of TNWFD are given in Figure 1. As we see, the failure rate function of TNWFD distribution is an increasing, decreasing or unimodal functions for different values of the parameters. So, if the empirical consideration suggests that the failure rate function of the prior distribution is bathtub-shaped, it can analyze the real data sets. Also, it is observed that the PDF of TNWFD distribution is unimodal function.

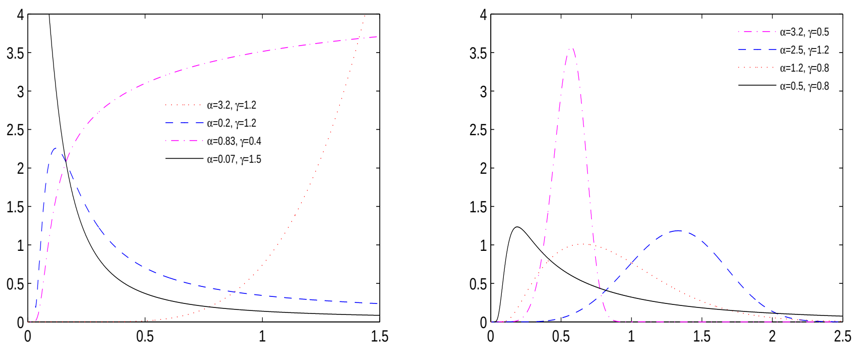

Figure 1. The PDF (right) and failure rate (left) function of TNWFD.

In the stress-strength modeling, the stress-strength parameter $R = P(X < Y)$ is a measure of component reliability when it is subjected to random stress X and has strength Y. This measure usually has been used for analyzing lifetime data with censoring. In the present paper, under Type II progressive censoring, the stress-strength parameter is estimated, when independent random variables X and Y are from the TNWF distribution (TNWFD).

2. Study of $\boldsymbol{R}$: unknown common $\boldsymbol{\nu}$

In this section, under the censoring scheme, assuming $X \sim TNWF(\delta, \nu)$ and $Y \sim TNWF(\eta, \nu)$, we obtain the point and interval estimates of $R = P(X < Y)$, from the frequentist and Bayesian viewpoints. In more details, first, we obtain the maximum likelihood estimation (MLE) of R. Because the MLEs of unknown parameters and R cannot be earned in the closed forms, we obtain the approximation maximum likelihood estimation (AMLE) of parameters and R which have the explicit forms. Also, we develop the Bayes estimates of R, using Lindley's approximation and MCMC method due to the lack of explicit forms.

Moreover, different confidence intervals such as asymptotic and HPD intervals of R are provided, in this section.

2.1. Maximum likelihood estimation

If $X \sim TNWF(\delta, \nu)$ and $Y \sim TNWF(\eta, \nu)$, then

$$\begin{aligned} R &= P(X < Y) = \int_0^\infty f_X(x)(1 - F_Y(x))dx \\ &= \int_0^1 \delta(1 - \log u)(-\log u)^{\delta-1} u^{-\delta-1} \exp\Big(-(\frac{-\log u}{u})^\delta - (\frac{-\log u}{u})^\eta\Big)du \\ &= \int_1^\infty \delta e^{\delta(t-1)} t(t-1)^{\delta-1} \exp\Big(-\big((t-1)e^{t-1}\big)^\delta - \big((t-1)e^{t-1}\big)^\eta\Big)dt \\ &= \int_0^\infty \delta z^{\delta-1} \exp\Big(-z^\delta - z^\eta\Big)dz \\ &= \int_0^\infty e^{-u-u^{\frac{\eta}{\delta}}} du. \end{aligned}$$

First, we have to give the MLEs of δ, η and ν. Let $\{X_1, \ldots, X_n\}$ be a Type II progressively censored sample from $TNWF(\delta, \nu)$ with censored scheme $(r_1, r_2, ..., r_n)$ and also $\{Y_1, \ldots, Y_m\}$ be a Type II progressively censored sample from $TNWF(\eta, \nu)$ with censored scheme $(s_1, s_2, ..., s_m)$. Therefore, the likelihood function of the unknown parameters is [4]:

$$L(\delta, \eta, \nu) = \Big\{c_1 \prod_{i=1}^n f_{TNWF}(x_i)[1 - F_{TNWF}(x_i)]^{r_i}\Big\} \times \Big\{c_2 \prod_{j=1}^m f_{TNWF}(y_j)[1 - F_{TNWF}(y_j)]^{s_j}\Big\},$$

where

$$c_1 = \prod_{j=0}^{n-1}\Big(N - j - \sum_{i=1}^j r_i\Big), \quad c_2 = \prod_{j=0}^{m-1}\Big(M - j - \sum_{i=1}^j s_i\Big).$$

Hence

$$\begin{aligned} L(\text{data}|\delta, \eta, \nu) = {} & c_1 c_2 \delta^n \eta^m \nu^{n+m} \\ & \times \prod_{i=1}^n x_i^{-2} e^{-\frac{\nu}{x_i}} (1 - e^{-\frac{\nu}{x_i}})^{-\delta-1} \big(1 - \log(1 - e^{-\frac{\nu}{x_i}})\big)\big(-\log(1 - e^{-\frac{\nu}{x_i}})\big)^{\delta-1} \\ & \quad \times \exp\Big\{-\sum_{i=1}^n (r_i + 1)\big(\frac{-\log(1 - e^{-\frac{\nu}{x_i}})}{1 - e^{-\frac{\nu}{x_i}}}\big)^\delta\Big\} \\ & \times \prod_{j=1}^m y_j^{-2} e^{-\frac{\nu}{y_j}} (1 - e^{-\frac{\nu}{y_j}})^{-\eta-1} \big(1 - \log(1 - e^{-\frac{\nu}{y_j}})\big)\big(-\log(1 - e^{-\frac{\nu}{y_j}})\big)^{\eta-1} \\ & \quad \times \exp\Big\{-\sum_{j=1}^m (s_j + 1)\big(\frac{-\log(1 - e^{-\frac{\nu}{y_j}})}{1 - e^{-\frac{\nu}{y_j}}}\big)^\eta\Big\}. \end{aligned}$$

Set

$$\begin{aligned} u(x, \nu) &:= \log\big(1 - \log(1 - e^{-\frac{\nu}{x}})\big), \\ w(x, \nu) &:= \log\big(-\log(1 - e^{-\frac{\nu}{x}})\big), \\ k(x, \nu) &:= \log(1 - e^{-\frac{\nu}{x}}), \\ z(x, \nu) &:= \frac{-\log(1 - e^{-\frac{\nu}{x}})}{1 - e^{-\frac{\nu}{x}}}. \end{aligned}$$

Then, the log-likelihood function is:

$$\begin{aligned}\ell(\delta,\eta,\nu) &= n\log(\delta)+m\log(\eta)+(n+m)\log(\nu)-2\sum_{i=1}^{n}\log x_i\\ &-\sum_{i=1}^{n}\frac{\nu}{x_i}+\sum_{i=1}^{n}u(x_i,\nu)+(\delta-1)\sum_{i=1}^{n}w(x_i,\nu)-(\delta+1)\sum_{i=1}^{n}k(x_i,\nu)\\ &-\sum_{i=1}^{n}(r_i+1)z(x_i,\nu)^{\delta}-2\sum_{j=1}^{m}\log y_j-\sum_{j=1}^{m}\frac{\nu}{y_j}+\sum_{j=1}^{m}u(y_j,\nu)\\ &+(\eta-1)\sum_{j=1}^{m}w(y_j,\nu)-(\eta+1)\sum_{j=1}^{m}k(y_j,\nu)-\sum_{j=1}^{m}(s_j+1)z(y_j,\nu)^{\eta}+\text{Constant}. \end{aligned}\tag{1}$$

To obtain the MLEs of parameters, i.e., $\widehat{\delta}$, $\widehat{\eta}$ and $\widehat{\nu}$, we must solve the three equations:

$$\frac{\partial\ell}{\partial\delta}=\frac{n}{\delta}+\sum_{i=1}^{n}w(x_i,\nu)-\sum_{i=1}^{n}k(x_i,\nu)-\sum_{i=1}^{n}(r_i+1)z(x_i,\nu)^{\delta}\log(z(x_i,\nu))=0,$$

$$\frac{\partial\ell}{\partial\eta}=\frac{m}{\eta}+\sum_{j=1}^{m}w(y_j,\nu)-\sum_{j=1}^{m}k(y_j,\nu)-\sum_{j=1}^{m}(s_j+1)z(y_j,\nu)^{\eta}\log(z(y_j,\nu))=0,$$

$$\begin{aligned}\frac{\partial\ell}{\partial\nu}&=\frac{(n+m)}{\nu}-\sum_{i=1}^{n}\frac{1}{x_i}+\sum_{i=1}^{n}u^{(1)}_{x_i,\nu}+(\delta-1)\sum_{i=1}^{n}w^{(1)}_{x_i,\nu}-(\delta+1)\sum_{i=1}^{n}k^{(1)}_{x_i,\nu}-\sum_{i=1}^{n}(r_i+1)z^{(1)}_{x_i,\delta,\nu}\\ &-\sum_{j=1}^{m}\frac{1}{y_j}+\sum_{j=1}^{m}u^{(1)}_{y_j,\nu}+(\eta-1)\sum_{j=1}^{m}w^{(1)}_{y_j,\nu}-(\eta+1)\sum_{j=1}^{m}k^{(1)}_{y_j,\nu}-\sum_{j=1}^{m}(s_j+1)z^{(1)}_{y_j,\eta,\nu}=0,\end{aligned}$$

where

$$\begin{aligned} u^{(1)}(x,\nu) &:= \frac{d}{d\nu}u(x,\nu)=\left\{x\left(e^{\frac{\nu}{x}}-1\right)\left(k(x,\nu)-1\right)\right\}^{-1},\\ w^{(1)}(x,\nu) &:= \frac{d}{d\nu}w(x,\nu)=\left\{x\left(e^{\frac{\nu}{x}}-1\right)k(x,\nu)\right\}^{-1},\\ k^{(1)}(x,\nu) &:= \frac{d}{d\nu}k(x,\nu)=\left\{x\left(e^{\frac{\nu}{x}}-1\right)\right\}^{-1},\\ z^{(1)}(x,\delta,\nu) &:= \frac{d}{d\nu}z(x,\nu)^{\delta}=\delta\big(1-k(x,\nu)\big)z(x,\nu)^{\delta}w^{(1)}(x,\nu). \end{aligned}$$

It is notable that, a numerical method such as NR algorithm is applied to obtain the roots of the above three non-linear equations. Now, by using the invariance property,

$$\widehat{R}^{MLE}=\int_0^{\infty}e^{-u-u^{\frac{\widehat{\eta}}{\widehat{\delta}}}}du.\tag{2}$$

2.2. Approximation maximum likelihood estimation

Since R does not have a closed form, we give the AMLEs of the parameters.

Theorem 1
Suppose that $Z_1\sim Weibull(\delta,\theta)$ and $Z_2\sim EV(\zeta,\vartheta)$:

$$F_{Z_1}(z)=1-e^{-\frac{z^{\delta}}{\theta}},\qquad z>0,\quad \delta,\theta>0,$$

$$F_{Z_2}(z)=1-e^{-e^{\frac{z-\zeta}{\vartheta}}},\quad z\in\mathbb{R},\quad \zeta\in\mathbb{R},\vartheta>0.$$

(i) Suppose that $Z \sim TNWF(\delta, \nu)$ and PL is the Lambert W function [5]. Then

$$Z_1 = \nu^{-\frac{1}{\delta}} PL^{-1}\Big(- \log(1 - e^{\frac{\nu}{Z}})\Big) \sim Weibull(\delta, 1/\nu).$$

(ii) Suppose that $Z_1 \sim Weibull(\delta, 1/\nu)$ and also $Z_2 = \log(Z_1)$. Then $Z_1 \sim EV(\zeta, \vartheta)$, where $\zeta = -\frac{1}{\delta}\log(\nu)$ and $\vartheta = \frac{1}{\delta}$.

Proof
Assume that $Z_1 \sim Weibull(\delta, 1/\nu)$. Then

$$F_{Z_1}(z) = 1 - e^{-\nu z^{\delta}}, \qquad z > 0, \quad \delta, \nu > 0.$$

Let g be an invertible and increasing function. We should find function g such that

$$F_Z(g^{-1}(z)) = F_{Z_1}(z) \Rightarrow \exp\left\{ -\left(\frac{-\log\left(1 - e^{-\left(\frac{\nu}{g^{-1}(z)}\right)}\right)}{1 - e^{-\left(\frac{\nu}{g^{-1}(z)}\right)}} \right)^{\delta} \right\} = \exp\{-\nu z^{\delta}\}.$$

Suppose $y = g^{-1}(z)$, $z(y) = 1 - e^{-(\nu/y)}$ and $a = \nu^{\frac{1}{\delta}}$. Then, we have

$$\frac{\log z(y)}{z(y)} = -az.$$

The solution of the above equation is $z(y) = \exp(-PL(az))$. Thus

$$y = g^{-1}(z) = \nu\Big(- \log(1 - \exp(-PL(az)))\Big)^{-1}.$$

Hence

$$g(z) = \nu^{-\frac{1}{\delta}} PL^{-1}\Big(- \log(1 - e^{\frac{\nu}{z}})\Big)$$

and proof is completed. □

Assume

$$X_i' = \nu^{-\frac{1}{\delta}} PL^{-1}\Big(- \log(1 - e^{\frac{\nu}{x_i}})\Big), \quad U_i = \log(X_i'),$$
$$Y_j' = \nu^{-\frac{1}{\eta}} PL^{-1}\Big(- \log(1 - e^{\frac{\nu}{y_j}})\Big), \quad V_j = \log(Y_j').$$

Applying Theorem 1, $U_i \sim EV(\zeta_1, \vartheta_1)$ and $V_j \sim EV(\zeta_2, \vartheta_2)$, where

$$\zeta_1 = -\frac{1}{\delta}\log(\nu), \ \zeta_2 = -\frac{1}{\eta}\log(\nu), \quad \vartheta_1 = \frac{1}{\delta}, \ \text{ and } \ \vartheta_2 = \frac{1}{\eta}.$$

Therefore, under the observed data $\{V_1, \dots, V_m\}$ and $\{U_1, \dots, U_n\}$, by ignoring the constant value,

$$\ell^*(\zeta_1, \zeta_2, \vartheta_1, \vartheta_2) \propto -n\log(\vartheta_1) + \sum_{i=1}^{n} t_i - \sum_{i=1}^{n}(r_i+1)e^{t_i} - m\log(\vartheta_2) + \sum_{j=1}^{m} z_j - \sum_{j=1}^{m}(s_j+1)e^{z_j}, \tag{3}$$

where

$$t_i = \frac{u_i - \zeta_1}{\vartheta_1}, \ z_j = \frac{v_j - \zeta_2}{\vartheta_2}.$$

Now by taking derivatives with respect to ζ_1, ζ_2, ϑ_1 and ϑ_2 from (3), we obtain the following equations:

$$\frac{\partial \ell^*}{\partial \zeta_1} = -\frac{1}{\vartheta_1}\left(n - \sum_{i=1}^{n}(r_i+1)e^{t_i}\right) = 0,$$
$$\frac{\partial \ell^*}{\partial \zeta_2} = -\frac{1}{\vartheta_2}\left(m - \sum_{j=1}^{m}(s_j+1)e^{z_j}\right) = 0,$$
$$\frac{\partial \ell^*}{\partial \vartheta_1} = -\frac{1}{\vartheta_1}\left(n + \sum_{i=1}^{n}t_i - \sum_{i=1}^{n}(r_i+1)t_i e^{t_i}\right) = 0,$$
$$\frac{\partial \ell^*}{\partial \vartheta_2} = -\frac{1}{\vartheta_2}\left(m + \sum_{j=1}^{m}z_j - \sum_{j=1}^{m}(s_j+1)z_j e^{z_j}\right) = 0.$$

To obtain the AMLEs, let

$$q_i = 1 - \prod_{t=n-i+1}^{n} \frac{t + \sum\limits_{k=n-t+1}^{n} R_k}{t+1+\sum\limits_{k=n-t+1}^{n} R_k},\ i = 1,\dots,n,$$
$$\bar{q}_j = 1 - \prod_{t=m-j+1}^{m} \frac{t + \sum\limits_{k=m-t+1}^{m} S_k}{t+1+\sum\limits_{k=m-t+1}^{m} S_k},\ j = 1,\dots,m.$$

By expanding the functions e^{t_i} and e^{z_j} around the points

$$\xi_i = \log\big(-\log(1-q_i)\big), \qquad \bar{\xi}_j = \log\big(-\log(1-\bar{q}_j)\big),$$

in Taylor series, respectively and considering the first order derivatives,

$$e^{t_i} = \delta_i + \eta_i t_i, \quad e^{z_j} = \bar{\delta}_j + \bar{\eta}_j z_j,$$

where

$$\delta_i = e^{\xi_i}(1-\xi_i), \qquad \eta_i = e^{\xi_i}, \qquad \bar{\delta}_j = e^{\bar{\xi}_j}(1-\bar{\xi}_j), \qquad \bar{\eta}_j = e^{\bar{\xi}_j}.$$

Just similar to [8], we derive the AMLEs of ζ_1, ζ_2, ϑ_1 and ϑ_2, say $\tilde{\zeta}_1$, $\tilde{\zeta}_2$, $\tilde{\vartheta}_1$ and $\tilde{\vartheta}_2$, respectively, by

$$\tilde{\zeta}_1 = A_1 - \tilde{\vartheta}_1 B_1, \qquad \tilde{\zeta}_2 = A_2 - \tilde{\vartheta}_2 B_2,$$
$$\tilde{\vartheta}_1 = \frac{-D_1 + \sqrt{D_1^2 + 4C_1E_1}}{2C_1}, \quad \tilde{\vartheta}_2 = \frac{-D_2 + \sqrt{D_2^2 + 4C_2E_2}}{2C_2},$$

where

$$A_1 = \frac{\sum_{i=1}^{n}(r_i+1)\eta_i u_i}{\sum_{i=1}^{n}(r_i+1)\eta_i}, \quad B_1 = \frac{\sum_{i=1}^{n}\delta_i - \sum_{i=1}^{n} r_i(1-\delta_i)}{\sum_{i=1}^{n}(r_i+1)\eta_i}, \quad C_1 = n,$$

$$A_2 = \frac{\sum_{j=1}^{m}(s_j+1)\bar{\eta}_j v_j}{\sum_{j=1}^{m}(s_j+1)\bar{\eta}_j}, \quad B_2 = \frac{\sum_{j=1}^{m}\bar{\delta}_j - \sum_{j=1}^{m} s_j(1-\bar{\delta}_j)}{\sum_{j=1}^{m}(s_j+1)\bar{\eta}_j}, \quad C_2 = m,$$

$$D_1 = \sum_{i=1}^{n}\delta_i u_i - A_1 B_1\left(\sum_{i=1}^{n}(r_i+1)\eta_i\right) - \sum_{i=1}^{n} r_i u_i(1-\delta_i),$$

$$D_2 = \sum_{j=1}^{J_2}\bar{\delta}_j v_j - A_2 B_2\left(\sum_{j=1}^{m}(s_j+1)\bar{\eta}_j\right) - \sum_{j=1}^{m} s_j v_j(1-\bar{\delta}_j),$$

$$E_1 = \sum_{i=1}^{n}(r_i+1)\eta_i(u_i-A_1)^2,$$

$$E_2 = \sum_{j=1}^{m}(s_j+1)\bar{\eta}_j(v_j-A_2)^2.$$

Now,

$$\tilde{\delta} = \frac{1}{\tilde{\vartheta}_1}, \quad \tilde{\eta} = \frac{1}{\tilde{\vartheta}_2}, \qquad \tilde{\nu} = \exp\left(-\frac{1}{2}\left(\frac{\tilde{\zeta}_1}{\tilde{\vartheta}_1} + \frac{\tilde{\zeta}_2}{\tilde{\vartheta}_2}\right)\right),$$

and

$$\tilde{R} = \int_0^\infty e^{-u-u^{\frac{\tilde{\eta}}{\tilde{\delta}}}}\, du, \tag{4}$$

where $\tilde{R}$ is the AMLE of R.

Suppose that $\widehat{\lambda}=(\widehat{\delta},\widehat{\eta},\widehat{\nu})$ and $I(\lambda)=[I_{ij}]=\left[-\frac{\partial^2\ell}{\partial\lambda_i\partial\lambda_j}\right]$, $i,j=1,2,3$ be the the observed Fisher information matrix. By differentiating twice from (1) with respect to parameters:

$$
\begin{aligned}
I_{11} &= \frac{n}{\delta^2}+\sum_{i=1}^{n}(r_i+1)z(x_i,\nu)^{\delta}\log^2(z(x_i,\nu)),\\
I_{22} &= \frac{m}{\eta^2}+\sum_{j=1}^{m}(s_j+1)z(y_j,\nu)^{\eta}\log^2(z(y_j,\nu)),\\
I_{12} &= I_{21}=0,\\
I_{13} &= \sum_{i=1}^{n}(r_i+1)\Big(z^{(1)}(x_i,\delta,\nu)\log(z(x_i,\nu))+z^{(1)}(x_i,\nu)z(x_i,\nu)^{\delta-1}\Big)+\sum_{i=1}^{n}k^{(1)}(x_i,\nu)-\sum_{i=1}^{n}w^{(1)}(x_i,\nu),\\
I_{23} &= \sum_{j=1}^{m}(s_j+1)\Big(z^{(1)}(y_j,\eta,\nu)\log(z(y_j,\nu))+z^{(1)}(y_j,\nu)z(y_j,\nu)^{\eta-1}\Big)+\sum_{j=1}^{m}k^{(1)}(y_j,\nu)-\sum_{j=1}^{m}w^{(1)}(y_j,\nu),\\
I_{33} &= \frac{n}{\nu^2}-\sum_{i=1}^{n}u^{(2)}(x_i,\nu)-(\delta-1)\sum_{i=1}^{n}w^{(2)}(x_i,\nu)+(\delta+1)\sum_{i=1}^{n}k^{(2)}(x_i,\nu)+\sum_{i=1}^{n}(r_i+1)z^{(2)}(x_i,\delta,\nu)\\
&\quad+\frac{m}{\nu^2}-\sum_{j=1}^{m}u^{(2)}(y_j,\nu)-(\eta-1)\sum_{j=1}^{m}w^{(2)}(y_j,\nu)+(\eta+1)\sum_{j=1}^{m}k^{(2)}(y_j,\nu)+\sum_{j=1}^{m}(s_j+1)z^{(2)}(y_j,\eta,\nu),
\end{aligned}
$$

where

$$
\begin{aligned}
z^{(1)}(x,\nu) &:= \frac{d}{d\nu}z(x,\nu)=\frac{e^{\frac{\nu}{x}}\,(k(x,\nu)-1)}{x\left(e^{\frac{\nu}{x}}-1\right)^2},\\
u^{(2)}(x,\nu) &:= \frac{d^2}{d\nu^2}u(x,\nu)=\frac{(e^{\frac{\nu}{x}}-e^{\frac{\nu}{x}}k(x,\nu)-1)k^{(1)}(x,\nu)^2}{\left(k(x,\nu)-1\right)^2},\\
w^{(2)}(x,\nu) &:= \frac{d^2}{d\nu^2}w(x,\nu)=-(e^{\frac{\nu}{x}}k(x,\nu)+1)w^{(2)}(x,\nu)^2,\\
k^{(2)}(x,\nu) &:= \frac{d^2}{d\nu^2}k(x,\nu)=-e^{\frac{\nu}{x}}k^{(1)}(x,\nu)^2,\\
z^{(2)}(x,\delta,\nu) &:= \frac{d^2}{d\nu^2}z(x,\nu)^{\delta}=\delta\big(\delta+\big(\delta+e^{\frac{\nu}{x}}\big)k(x,\nu)^2-\big(2\delta+e^{\frac{\nu}{x}}\big)k(x,\nu)-1\big)w^{(1)}(x,\nu)^2z(x,\nu)^{\delta}.
\end{aligned}
$$

Theorem 2
We have

$$[(\widehat{\delta}-\delta),\ (\widehat{\eta}-\eta),\ (\widehat{\nu}-\nu)]^T \xrightarrow{D} N_3(0,\mathbf{I^{-1}}(\delta,\eta,\nu)),$$

where

$$\mathbf{I}(\delta,\eta,\nu)=\begin{pmatrix} I_{11} & 0 & I_{13}\\ & I_{22} & I_{23}\\ & & I_{33}\end{pmatrix},\quad \mathbf{I^{-1}}(\delta,\eta,\nu)=\frac{1}{|\mathbf{I}(\delta,\eta,\nu)|}\begin{pmatrix} b_{11} & b_{12} & b_{13}\\ & b_{22} & b_{23}\\ & & b_{33}\end{pmatrix},$$

in which $|\mathbf{I}(\delta,\eta,\nu)|=I_{11}I_{22}I_{33}-I_{11}I_{23}^2-I_{13}^2I_{22}$,

$$
\begin{aligned}
b_{11}&=I_{22}I_{33}-I_{23}^2, & b_{12}&=I_{13}I_{23}, & b_{13}&=-I_{13}I_{22},\\
b_{22}&=I_{11}I_{33}-I_{13}^2, & b_{23}&=-I_{11}I_{23}, & b_{33}&=I_{11}I_{22}.
\end{aligned}
$$

Proof
It is an immediate consequence of the asymptotic normality of the MLE. □

Theorem 3
If $\widehat{R}^{MLE}$ is the MLE of R, then

$$(\widehat{R}^{MLE} - R) \xrightarrow{D} N(0, W),$$

where

$$W = \frac{1}{|\mathbf{I}(\delta,\eta,\nu)|}\left[(\frac{\partial R}{\partial \delta})^2 b_{11} + (\frac{\partial R}{\partial \eta})^2 b_{22} + 2(\frac{\partial R}{\partial \delta})(\frac{\partial R}{\partial \eta}) b_{12}\right]. \tag{5}$$

Proof
From Theorem 2 and delta method,

$$(\widehat{R}^{MLE} - R) \xrightarrow{D} N(0, W),$$

where $W = \mathbf{b}^T \mathbf{I}^{-1}(\delta,\eta,\nu)\mathbf{b}$, with $\mathbf{b} = [\frac{\partial R}{\partial \delta},\ \frac{\partial R}{\partial \eta},\ \frac{\partial R}{\partial \nu}]^T = [\frac{\partial R}{\partial \delta},\ \frac{\partial R}{\partial \eta},\ 0]^T$, in which

$$\begin{aligned}\frac{\partial R}{\partial \delta} &= \int_0^\infty \frac{\eta}{\delta^2} u^{\frac{\eta}{\delta}} \log(u) e^{-u-u^{\frac{\eta}{\delta}}} du,\\ \frac{\partial R}{\partial \eta} &= -\int_0^\infty \frac{1}{\delta} u^{\frac{\eta}{\delta}} \log(u) e^{-u-u^{\frac{\eta}{\delta}}} du.\end{aligned} \tag{6}$$

Therefore, W can be represented as (5) and proof is completed. □

The vector W is estimated by the MLEs of parameters. Then, a $100(1-a)\%$ ACI of R is:

$$(\widehat{R}^{MLE} - z_{1-\frac{a}{2}}\sqrt{\widehat{W}}, \widehat{R}^{MLE} + z_{1-\frac{a}{2}}\sqrt{\widehat{W}}),$$

where z_a is $100a$-th percentile of standard normal distribution.

2.3. Bayes estimation

Suppose that $\delta \sim \nu(a_1, b_1)$, $\eta \sim \nu(a_2, b_2)$ and $\nu \sim \nu(a_3, b_3)$ are independent random variables and the loss function is the squared error loss function. We have

$$\pi(\delta,\eta,\nu|\text{data}) \propto L(\text{data}|\delta,\eta,\nu)\pi_1(\delta)\pi_2(\eta)\pi_3(\nu), \tag{7}$$

where

$$\begin{aligned}\pi_1(\delta) &\propto \delta^{a_1-1} e^{-b_1\delta}, && \delta > 0,\ \ a_1, b_1 > 0,\\ \pi_2(\eta) &\propto \eta^{a_2-1} e^{-b_2\eta}, && \eta > 0,\ \ a_2, b_2 > 0,\\ \pi_3(\nu) &\propto \nu^{a_3-1} e^{-b_3\nu}, && \nu > 0,\ \ a_3, b_3 > 0.\end{aligned}$$

Thus, we approximate the Bayes estimates by applying Lindley's approximation and MCMC method since they cannot be obtained in the closed form.

2.3.1. Lindley's approximation

When we confront the case of three parameter $\lambda = (\lambda_1, \lambda_2, \lambda_3)$, Lindley's approximation conducts to [6]:

$$\begin{aligned}\mathbb{E}\big(u(\lambda)|\text{data}\big) = u + (u_1 d_1 + u_2 d_2 + u_3 d_3 + d_4 + d_5) + \frac{1}{2}\Big(A(u_1\vartheta_{11} + u_2\vartheta_{12} + u_3\vartheta_{13})\\ + B(u_1\vartheta_{21} + u_2\vartheta_{22} + u_3\vartheta_{23}) + C(u_1\vartheta_{31} + u_2\vartheta_{32} + u_3\vartheta_{33})\Big),\end{aligned}$$

calculated at $\widehat{\lambda} = (\widehat{\lambda}_1, \widehat{\lambda}_2, \widehat{\lambda}_3)$, where

$$
\begin{aligned}
d_i &= \rho_1\vartheta_{i1} + \rho_2\vartheta_{i2} + \rho_3\vartheta_{i3}, \quad i = 1, 2, 3,\\
d_4 &= u_{12}\vartheta_{12} + u_{13}\vartheta_{13} + u_{23}\vartheta_{23},\\
d_5 &= \frac{1}{2}(u_{11}\vartheta_{11} + u_{22}\vartheta_{22} + u_{33}\vartheta_{33}),\\
A &= \ell_{111}\vartheta_{11} + 2\ell_{121}\vartheta_{12} + 2\ell_{131}\vartheta_{13} + 2\ell_{231}\vartheta_{23} + \ell_{221}\vartheta_{22} + \ell_{331}\vartheta_{33},\\
B &= \ell_{112}\vartheta_{11} + 2\ell_{122}\vartheta_{12} + 2\ell_{132}\vartheta_{13} + 2\ell_{232}\vartheta_{23} + \ell_{222}\vartheta_{22} + \ell_{332}\vartheta_{33},\\
C &= \ell_{113}\vartheta_{11} + 2\ell_{123}\vartheta_{12} + 2\ell_{133}\vartheta_{13} + 2\ell_{233}\vartheta_{23} + \ell_{223}\vartheta_{22} + \ell_{333}\vartheta_{33}.
\end{aligned}
$$

In our case, for $(\lambda_1, \lambda_2, \lambda_3) \equiv (\delta, \eta, \nu)$ and $u = R$, we have

$$
\rho_1 = \frac{a_1 - 1}{\delta} - b_1, \quad \rho_2 = \frac{a_2 - 1}{\eta} - b_2, \quad \rho_3 = \frac{a_3 - 1}{\nu} - b_3,
$$

ϑ_{ij}, $i, j = 1, 2, 3$ are obtained by using ℓ_{ij}, $i, j = 1, 2, 3$ and

$$
\begin{aligned}
\ell_{111} &= \frac{2n}{\delta^3} - \sum_{i=1}^{n}(r_i + 1)z(x_i, \nu)^{\delta}\log^3(z(x_i, \nu)),\\
\ell_{222} &= \frac{2m}{\eta^3} - \sum_{j=1}^{m}(s_j + 1)z(y_j, \nu)^{\eta}\log^3(z(y_j, \nu)),\\
\ell_{113} = \ell_{131} = \ell_{311} &= -\sum_{i=1}^{n}(r_i + 1)\Big(z^{(1)}(x_i, \delta, \nu)\log^2(z(x_i, \nu)) + 2z^{(1)}(x_i, \nu)\log(z(x_i, \nu))z(x_i, \nu)^{\delta-1}\Big),\\
\ell_{223} = \ell_{232} = \ell_{322} &= -\sum_{j=1}^{m}(s_j + 1)\Big(z^{(1)}(y_j, \eta, \nu)\log^2(z(y_j, \nu)) + 2z^{(1)}(y_j, \nu)\log(z(y_j, \nu))z(y_j, \nu)^{\eta-1}\Big),\\
\ell_{133} = \ell_{331} = \ell_{313} &= \sum_{i=1}^{n} w^{(2)}(x_i, \nu) - \sum_{i=1}^{n} k^{(2)}(x_i, \nu) - \sum_{i=1}^{n}(r_i + 1)z^{(2,\delta)}(x_i, \delta, \nu),\\
\ell_{233} = \ell_{332} = \ell_{323} &= \sum_{j=1}^{m} w^{(2)}(y_j, \nu) - \sum_{j=1}^{m} k^{(2)}(y_j, \nu) - \sum_{j=1}^{m}(s_j + 1)z^{(2,\eta)}(y_j, \eta, \nu),\\
\ell_{333} &= \frac{2n}{\nu^3} + \sum_{i=1}^{n} u^{(3)}(x_i, \nu) + (\delta - 1)\sum_{i=1}^{n} w^{(3)}(x_i, \nu) - (\delta + 1)\sum_{i=1}^{n} k^{(3)}(x_i, \nu) - \sum_{i=1}^{n}(r_i + 1)z^{(3)}(x_i, \delta, \nu)\\
&\quad + \frac{2m}{\nu^3} + \sum_{j=1}^{m} u^{(3)}(y_j, \nu) + (\eta - 1)\sum_{j=1}^{m} w^{(3)}(y_j, \nu) - (\eta + 1)\sum_{j=1}^{m} k^{(3)}(y_j, \nu) - \sum_{j=1}^{m}(s_j + 1)z^{(3)}(y_j, \eta, \nu),
\end{aligned}
$$

where

$$
\begin{aligned}
u^{(3)}(x,\nu) &:= \frac{d^3}{d\nu^3}u(x,\nu) = \Big(-2e^{\frac{\nu}{x}} + e^{\frac{2\nu}{x}} + e^{\frac{\nu}{x}}\big(e^{\frac{\nu}{x}}+1\big)k(x,\nu)^2 + \big(e^{\frac{\nu}{x}} - 2e^{\frac{2\nu}{x}}\big)k(x,\nu) + 2\Big)u^{(1)}(x,\nu)^3,\\
w^{(3)}(x,\nu) &:= \frac{d^3}{d\nu^3}w(x,\nu) = \Big(e^{\frac{\nu}{x}}\big(e^{\frac{\nu}{x}}+1\big)k(x,\nu)^2 + 3e^{\frac{\nu}{x}}k(x,\nu) + 2\Big)w^{(1)}(x,\nu)^3,\\
k^{(3)}(x,\nu) &:= \frac{d^3}{d\nu^3}k(x,\nu) = e^{\frac{\nu}{x}}\big(e^{\frac{\nu}{x}}+1\big)k^{(1)}(x,\nu)^3,\\
z^{(3)}(x,\delta,\nu) &:= \frac{d^3}{d\nu^3}z(x,\nu)^{\delta} = -\delta z(x,\nu)^{\delta}\Bigg(\Big(-\delta^2 + 3\delta + 3(\delta-1)\big(\delta + e^{\frac{\nu}{x}}\big)k(x,\nu) - 2\Big)w^{(1)}(x,\nu)\\
&\quad - \Big(3\delta^2 + (6\delta+1)e^{\frac{\nu}{x}} + e^{\frac{2\nu}{x}}\Big)k^{(1)}(x,\nu)^2 w^{(1)}(x,\nu) + \Big(\delta^2 + (3\delta+1)e^{\frac{\nu}{x}} + e^{\frac{2\nu}{x}}\Big)k^{(1)}(x,\nu)^3\Bigg),\\
z^{(2,\delta)}(x,\delta,\nu) &:= \frac{dz^{(2)}_{x,\delta,\nu}}{d\delta} = \delta z(x,\nu)^{\delta}\Bigg(\Big(2\delta + \delta(\delta-1)\log(z(x,\nu)) - 1\Big)w^{(1)}(x,\nu)^2\\
&\quad + \Big(2\delta + \delta\big(\delta + e^{\frac{\nu}{x}}\big)\log(z(x,\nu)) + e^{\frac{\nu}{x}}\Big)k^{(1)}(x,\nu)^2\\
&\quad - \Big(4\delta + \delta\big(2\delta + e^{\frac{\nu}{x}}\big)\log(z(x,\nu)) + e^{\frac{\nu}{x}}\Big)w^{(1)}(x,\nu)k^{(1)}(x,\nu)\Bigg),
\end{aligned}
$$

and other $\ell_{ijk} = 0$. Moreover, for $i = 1,2,3$, $u_3 = u_{i3} = 0$ and u_1, u_2 are given in (6). Also,

$$
\begin{aligned}
u_{11} &= \int_0^{\infty} \frac{\eta}{\delta^4}\log(u)e^{-u-u^{\frac{\eta}{\delta}}}\Big(\eta u^{\frac{2\eta}{\delta}}\log(u) - \eta\log(u)u^{\frac{\eta}{\delta}} - 2\delta u^{\frac{\eta}{\delta}}\Big)du,\\
u_{12} = u_{21} &= -\int_0^{\infty} \frac{1}{\delta^3}\log(u)e^{-u-u^{\frac{\eta}{\delta}}}\Big(\eta\log(u)u^{\frac{2\eta}{\delta}} - \eta\log(u)u^{\frac{\eta}{\delta}} - \delta u^{\frac{\eta}{\delta}}\Big)du,\\
u_{22} &= \int_0^{\infty} \frac{1}{\delta^2}\log^2(u)e^{-u-u^{\frac{\eta}{\delta}}}\Big(u^{\frac{2\eta}{\delta}} - u^{\frac{\eta}{\delta}}\Big)du.
\end{aligned}
$$

So,

$$
\begin{aligned}
d_4 &= u_{12}\vartheta_{12},\\
d_5 &= \frac{1}{2}(u_{11}\vartheta_{11} + u_{22}\vartheta_{22}),\\
A &= \ell_{111}\vartheta_{11} + 2\ell_{131}\vartheta_{13} + \ell_{331}\vartheta_{33},\\
B &= 2\ell_{232}\vartheta_{23} + \ell_{222}\vartheta_{22} + \ell_{332}\vartheta_{33},\\
C &= \ell_{113}\vartheta_{11} + 2\ell_{133}\vartheta_{13} + 2\ell_{233}\vartheta_{23} + \ell_{223}\vartheta_{22} + \ell_{333}\vartheta_{33}.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\widehat{R}^{Lin} &= R + u_1d_1 + u_2d_2 + d_4 + d_5 + \frac{1}{2}\Big(A(u_1\vartheta_{11} + u_2\vartheta_{12})\\
&\quad + B(u_1\vartheta_{21} + u_2\vartheta_{22}) + C(u_1\vartheta_{31} + u_2\vartheta_{32})\Big), \qquad (8)
\end{aligned}
$$

where $\widehat{R}^{Lin}$ is the Bayes estimate of R. It is notable that all parameters are evaluated at $(\widehat{\delta},\widehat{\eta},\widehat{\nu})$.

Using the Lindley's approximation, it is not possible constructing the HPD credible interval. Thus, the MCMC method to approximate the Bayes estimate is applied and its HPD credible intervals is constructed.

2.3.2. MCMC method After simplifying equation (7), the posterior PDFs of parameters are:

$$\pi(\delta|\nu,\text{data}) \propto \delta^{n+a_1-1}\exp\Big(\delta\big(\sum_{i=1}^{n}w(x_i,\nu)-\sum_{i=1}^{n}k(x_i,\nu)-b_1\big)-\sum_{i=1}^{n}(r_i+1)z(x_i,\nu)^{\delta}\Big),$$
$$\pi(\eta|\nu,\text{data}) \propto \eta^{n+a_2-1}\exp\Big(\eta\big(\sum_{j=1}^{m}w(y_j,\nu)-\sum_{j=1}^{m}k(y_j,\nu)-b_2\big)-\sum_{j=1}^{m}(s_j+1)z(y_j,\nu)^{\eta}\Big),$$
$$\pi(\nu|\delta,\eta,\text{data}) \propto \nu^{n+m+a_3-1}\exp\Big(-\sum_{i=1}^{n}\frac{\nu}{x_i}-\sum_{j=1}^{m}\frac{\nu}{y_j}-b_3\nu\Big)$$
$$\times\exp\Big(\sum_{i=1}^{n}u(x_i,\nu)+(\delta-1)\sum_{i=1}^{n}w(x_i,\nu)-(\delta+1)\sum_{i=1}^{n}k(x_i,\nu)-\sum_{i=1}^{n}(r_i+1)z(x_i,\nu)^{\delta}\Big)$$
$$\times\exp\Big(\sum_{j=1}^{m}u(y_j,\nu)+(\eta-1)\sum_{j=1}^{m}w(y_j,\nu)-(\eta+1)\sum_{j=1}^{m}k(y_j,\nu)-\sum_{j=1}^{m}(s_j+1)z(y_j,\nu)^{\eta}\Big).$$

By the above posteriors PDF, we utilize the Metropolis-Hastings method to generate random samples with normal proposal distribution. Also, the proposed Gibbs sampling algorithm is:

1. Start with the begin value $(\delta_{(0)},\ \eta_{(0)},\ \nu_{(0)})$.
2. Set $t=1$.
3. Generate $\nu_{(t)}$ from $\pi(\nu|\delta_{(t-1)},\eta_{(t-1)},\text{data})$.
4. Generate $\delta_{(t)}$ from $\pi(\delta|\nu_{(t-1)},\text{data})$.
5. Generate $\eta_{(t)}$ from $\pi(\eta|\nu_{(t-1)},\text{data})$.
6. Calculate $R_t=\int_0^{\infty}e^{-u-u^{\frac{\eta_t}{\delta_t}}}du$.
7. Set $t=t+1$.
8. Repeat steps 3-7, for T times.

Thus

$$\widehat{R}^{MC}=\frac{1}{T}\sum_{t=1}^{T}R_t. \tag{9}$$

Now, a $100(1-a)\%$ HPD credible interval of R can be given (see [7]).

3. Study of $\boldsymbol{R}$: known common $\boldsymbol{\nu}$

In this section, assuming the common parameter ν is known, the MLE, asymptotic confidence interval and Bayes estimate of R, via the Lindely's approximation and MCMC method, due to the lack of explicit form, are obtained.

3.1. MLE

Let $\{X_1,\dots,X_n\}$ be a progressively Type II censored sample from $TNWF(\delta,\nu)$ with censored scheme $(r_1,r_2,\dots,r_n)$ and also $\{Y_1,\dots,Y_m\}$ be a progressively Type II censored sample from $TNWF(\eta,\nu)$ with censored scheme $(s_1,s_2,\dots,s_m)$. Thus, we must solve the two equations:

$$\frac{\partial\ell}{\partial\delta}=\frac{n}{\delta}+\sum_{i=1}^{n}w(x_i,\nu)-\sum_{i=1}^{n}k(x_i,\nu)-\sum_{i=1}^{n}(r_i+1)z(x_i,\nu)^{\delta}\log(z(x_i,\nu))=0,$$
$$\frac{\partial\ell}{\partial\eta}=\frac{m}{\eta}+\sum_{j=1}^{m}w(y_j,\nu)-\sum_{j=1}^{m}k(y_j,\nu)-\sum_{j=1}^{m}(s_j+1)z(y_j,\nu)^{\eta}\log(z(y_j,\nu))=0.$$

It is notable that, one numerical method such as NR algorithm is applied to obtain the roots of the above two non-linear equations. After obtaining the MLEs of parameters, by using the invariance property,

$$\widehat{R}^{MLE} = \int_0^\infty e^{-u-u^{\frac{\widehat{\eta}}{\widehat{\delta}}}} du. \tag{10}$$

Thus $(\widehat{R}^{MLE} - R) \xrightarrow{D} N(0, C)$, where $C = (\frac{\partial R}{\partial \delta})^2 \frac{1}{I_{11}} + (\frac{\partial R}{\partial \eta})^2 \frac{1}{I_{22}}$, and $\frac{\partial R}{\partial \delta}$ and $\frac{\partial R}{\partial \eta}$ are given in (6). Also

$$(\widehat{R}^{MLE} - z_{1-\frac{a}{2}}\sqrt{\widehat{C}}, \widehat{R}^{MLE} + z_{1-\frac{a}{2}}\sqrt{\widehat{C}}),$$

where z_a is $100a$-th percentile of the standard normal distribution.

3.2. Bayes estimation

If $\delta \sim \nu(a_1, b_1)$ and $\eta \sim \nu(a_2, b_2)$ are independent random variables, then

$$\pi(\delta, \eta|\nu, \text{data}) \propto L(\nu, \text{data}|\delta, \eta)\pi_1(\delta)\pi_2(\eta), \tag{11}$$

3.2.1. Lindley's approximation For $\lambda = (\lambda_1, \lambda_2)$, Lindley's approximation leads to

$$u + (u_1 p_1 + u_2 p_2 + p_3) + \frac{1}{2}\Big(P(u_1\vartheta_{11} + u_2\vartheta_{12}) + Q(u_1\vartheta_{21} + u_2\vartheta_{22})\Big),$$

calculated at $\widehat{\lambda} = (\widehat{\lambda}_1, \widehat{\lambda}_2)$, where

$$\begin{aligned} p_i &= \rho_1\vartheta_{i1} + \rho_2\vartheta_{i2}, \quad i = 1, 2, \\ p_3 &= \frac{1}{2}(u_{11}\vartheta_{11} + 2u_{12}\vartheta_{12} + u_{22}\vartheta_{22}), \\ P &= \ell_{111}\vartheta_{11} + 2\ell_{121}\vartheta_{12} + \ell_{221}\vartheta_{22}, \\ Q &= \ell_{112}\vartheta_{11} + 2\ell_{122}\vartheta_{12} + \ell_{222}\vartheta_{22}. \end{aligned}$$

Other expressions can be found in Section 2.3.1. Hence $P = \ell_{111}\vartheta_{11}$ and $Q = \ell_{222}\vartheta_{22}$, then, the Bayes estimator of R is

$$\widehat{R}^{Lin} = R + u_1 p_1 + u_2 p_2 + p_3 + \frac{1}{2}\Big(P(u_1\vartheta_{11} + u_2\vartheta_{12}) + Q(u_1\vartheta_{21} + u_2\vartheta_{22})\Big). \tag{12}$$

All parameters are evaluated at $(\widehat{\delta}, \widehat{\eta})$.

3.2.2. MCMC When ν is known, follows:

$$\pi(\delta|\nu, \text{data}) \propto \delta^{n+a_1-1} \exp\Big(\delta\big(\sum_{i=1}^n w(x_i, \nu) - \sum_{i=1}^n k(x_i, \nu) - b_1\big) - \sum_{i=1}^n (r_i + 1) z(x_i, \nu)^\delta\Big),$$
$$\pi(\eta|\nu, \text{data}) \propto \eta^{n+a_2-1} \exp\Big(\eta\big(\sum_{j=1}^m w(y_j, \nu) - \sum_{j=1}^m k(y_j, \nu) - b_2\big) - \sum_{j=1}^m (s_j + 1) z(y_j, \nu)^\eta\Big).$$

We generate a sample by using Gibbs sampling from the above distributions. The algorithm is as follows:

1. Start with the begin value $(\delta_{(0)}, \eta_{(0)})$.
2. Set $t = 1$.
3. Generate $\delta_{(t)}$ from $\pi(\delta|\nu_{(t-1)}, \text{data})$.
4. Generate $\eta_{(t)}$ from $\pi(\eta|\nu_{(t-1)}, \text{data})$.

5. Calculate $R_t = \int_0^\infty e^{-u-u^{\frac{\eta_t}{\delta_t}}} du$.
6. Set $t = t + 1$.
7. Repeat steps 3-6, for T times.

From this algorithm,

$$\widehat{R}^{MC} = \frac{1}{T}\sum_{t=1}^{T} R_t. \tag{13}$$

4. General case

In this section, because the assumptions which we study in Sections 2 and 3 are quite strong, we consider the statistical inference of R in general case. So, under the progressive censoring scheme, assuming $X \sim TNWF(\delta, \nu_1)$ and $Y \sim TNWF(\eta, \nu_2)$, we provide the MLE, AMLE and Bayes estimate of R.

4.1. MLE

Suppose that two independent r.v.s X and Y are from $TNWF(\delta, \nu_1)$ and $TNWF(\eta, \nu_2)$ distributions, respectively. We have

$$\begin{aligned} R = \int_0^\infty \frac{\delta\nu_1}{x^2} \exp\Big(& -\frac{\nu_1}{x} + u(x,\nu_1) + (\delta-1)w(x,\nu_1) \\ & -(\delta+1)k(x,\nu_1) - z(x,\nu_1)^\delta - z(x,\nu_2)^\eta \Big) dx. \end{aligned}$$

Then

$$\begin{aligned} L(\text{data}|\delta,\eta,\nu_1,\nu_2)\delta^n\eta^m\nu_1^n\nu_2^m & \\ \times \prod_{i=1}^{n} & x_i^{-2} e^{-\frac{\nu_1}{x_i}} (1-e^{-\frac{\nu_1}{x_i}})^{-\delta-1} \big(1-\log(1-e^{-\frac{\nu_1}{x_i}})\big)\big(-\log(1-e^{-\frac{\nu_1}{x_i}})\big)^{\delta-1} \\ & \times \exp\Big\{ -\sum_{i=1}^{n}(r_i+1)\Big(\frac{-\log(1-e^{-\frac{\nu_1}{x_i}})}{1-e^{-\frac{\nu_1}{x_i}}}\Big)^\delta \Big\} \\ \times \prod_{j=1}^{m} & y_j^{-2} e^{-\frac{\nu_2}{y_j}} (1-e^{-\frac{\nu_2}{y_j}})^{-\eta-1} \big(1-\log(1-e^{-\frac{\nu_2}{y_j}})\big)\big(-\log(1-e^{-\frac{\nu_2}{y_j}})\big)^{\eta-1} \\ & \times \exp\Big\{ -\sum_{j=1}^{m}(s_j+1)\Big(\frac{-\log(1-e^{-\frac{\nu_2}{y_j}})}{1-e^{-\frac{\nu_2}{y_j}}}\Big)^\eta \Big\}. \end{aligned}$$

Therefore,

$$\begin{aligned} \ell(\delta,\eta,\nu_1,\nu_2) = & \; n\log(\delta) + m\log(\eta) + n\log(\nu_1) + m\log(\nu_2) - 2\sum_{i=1}^{n}\log x_i \\ & -\sum_{i=1}^{n}\frac{\nu_1}{x_i} + \sum_{i=1}^{n}u(x_i,\nu_1) + (\delta-1)\sum_{i=1}^{n}w(x_i,\nu_1) - (\delta+1)\sum_{i=1}^{n}k(x_i,\nu_1) \\ & -\sum_{i=1}^{n}(r_i+1)u(x_i,\nu_1)^\delta - 2\sum_{j=1}^{m}\log y_j - \sum_{j=1}^{m}\frac{\nu_2}{y_j} + \sum_{j=1}^{m}u(y_j,\nu_2) \\ & + (\eta-1)\sum_{j=1}^{m}w(y_j,\nu_2) - (\eta+1)\sum_{j=1}^{m}k(y_j,\nu_2) - \sum_{j=1}^{m}(s_j+1)z(y_j,\nu_2)^\eta + \text{Constant}. \end{aligned}$$

Then, to obtain the MLEs of all parameters, namely, $\widehat{\delta}$, $\widehat{\eta}, \widehat{\nu}_1$ and $\widehat{\nu}_2$, respectively, we have to solve the following four equations:

$$\begin{aligned}
\frac{\partial \ell}{\partial \delta} &= \frac{n}{\delta} + \sum_{i=1}^{n} w(x_i, \nu_1) - \sum_{i=1}^{n} k(x_i, \nu_1) - \sum_{i=1}^{n} (r_i + 1) z(x_i, \nu_1)^{\delta} \log(z(x_i, \nu_1)) = 0, \\
\frac{\partial \ell}{\partial \eta} &= \frac{m}{\eta} + \sum_{j=1}^{m} w(y_j, \nu_2) - \sum_{j=1}^{m} k(y_j, \nu_2) - \sum_{j=1}^{m} (s_j + 1) z(y_j, \nu_2)^{\eta} \log(z(y_j, \nu_2)) = 0, \\
\frac{\partial \ell}{\partial \nu_1} &= \frac{n}{\nu_1} - \sum_{i=1}^{n} \frac{1}{x_i} + \sum_{i=1}^{n} u^{(1)}(x_i, \nu_1) + (\delta - 1) \sum_{i=1}^{n} w^{(1)}(x_i, \nu_1) \\
&\quad - (\delta + 1) \sum_{i=1}^{n} k^{(1)}(x_i, \nu_1) - \sum_{i=1}^{n} (r_i + 1) z^{(1)}(x_i, \delta, \nu_1) = 0, \\
\frac{\partial \ell}{\partial \nu_2} &= \frac{m}{\nu_2} - \sum_{j=1}^{m} \frac{1}{y_j} + \sum_{j=1}^{m} u^{(1)}(y_j, \nu_2) + (\eta - 1) \sum_{j=1}^{m} w^{(1)}(y_j, \nu_2) \\
&\quad - (\eta + 1) \sum_{j=1}^{m} k^{(1)}(y_j, \nu_2) - \sum_{j=1}^{m} (s_j + 1) z^{(1)}(y_j, \eta, \nu_2) = 0,
\end{aligned}$$

After estimating the MLEs of all parameters, the MLE of R is

$$\begin{aligned}
\widehat{R}^{MLE} = \int_0^{\infty} \frac{\widehat{\delta}\widehat{\nu}_1}{x^2} \exp\Big(&- \frac{\widehat{\nu}_1}{x} + u(x, \widehat{\nu}_1) + (\widehat{\delta} - 1) w(x, \widehat{\nu}_1) \\
&- (\widehat{\delta} + 1) k(x, \widehat{\nu}_1) - z(x, \widehat{\nu}_1)^{\widehat{\delta}} - z(x, \widehat{\nu}_2)^{\widehat{\eta}} \Big) dx. \qquad (14)
\end{aligned}$$

4.2. AMLE

Let $\{X_1, \ldots, X_n\}$ and $\{Y_1, \ldots, Y_m\}$ be two Type II progressive censoring samples from $TNWF(\delta, \nu_1)$ and $TNWF(\eta, \nu_2)$ distributions and

$$X_i' = \nu_1^{-\frac{1}{\delta}} PL^{-1}(-\log(1 - e^{\frac{\nu_1}{x_i}})), \quad U_i = \log(X_i'),$$
$$Y_j' = \nu_2^{-\frac{1}{\eta}} PL^{-1}(-\log(1 - e^{\frac{\nu_2}{y_j}})), \quad V_j = \log(Y_j').$$

Applying Theorem 1, $U_i \sim EV(\zeta_1, \vartheta_1)$ and $V_j \sim EV(\zeta_2, \vartheta_2)$, where

$$\zeta_1 = -\frac{1}{\delta} \log(\nu_1), \ \zeta_2 = -\frac{1}{\eta} \log(\nu_2), \quad \vartheta_1 = \frac{1}{\delta}, \quad \text{and} \quad \vartheta_2 = \frac{1}{\eta}.$$

Again, Like the previous mode,

$$\tilde{\zeta}_1 = A_1 - \tilde{\vartheta}_1 B_1, \qquad \tilde{\zeta}_2 = A_2 - \tilde{\vartheta}_2 B_2,$$
$$\tilde{\vartheta}_1 = \frac{-D_1 + \sqrt{D_1^2 + 4C_1E_1}}{2C_1}, \quad \tilde{\vartheta}_2 = \frac{-D_2 + \sqrt{D_2^2 + 4C_2E_2}}{2C_2}.$$

Now,

$$\tilde{\delta} = \frac{1}{\tilde{\vartheta}_1}, \quad \tilde{\eta} = \frac{1}{\tilde{\vartheta}_2}, \qquad \tilde{\nu}_1 = \exp\Big(- \frac{\tilde{\zeta}_1}{\tilde{\vartheta}_1} \Big), \qquad \tilde{\nu}_2 = \exp\Big(- \frac{\tilde{\zeta}_2}{\tilde{\vartheta}_2} \Big)$$

and

$$\begin{aligned}
\tilde{R} = \int_0^{\infty} \frac{\tilde{\delta}\tilde{\nu}_1}{x^2} \exp\Big(&- \frac{\tilde{\nu}_1}{x} + u(x, \tilde{\nu}_1) + (\tilde{\delta} - 1) w(x, \tilde{\nu}_1) \\
&- (\tilde{\delta} + 1) k(x, \tilde{\nu}_1) - z(x, \tilde{\nu}_1)^{\tilde{\delta}} - z(x, \tilde{\nu}_2)^{\tilde{\eta}} \Big) dx. \qquad (15)
\end{aligned}$$

4.3. Bayes estimation

If $\delta \sim \nu(a_1, b_1)$, $\eta \sim \nu(a_2, b_2)$, $\nu_1 \sim \nu(a_3, b_3)$ and $\nu_2 \sim \nu(a_4, b_4)$ are independent random variables, then

$$\pi(\delta|\nu_1, \text{data}) \propto \delta^{n+a_1-1} \exp\Big(\delta\big(\sum_{i=1}^{n} w(x_i, \nu_1) - \sum_{i=1}^{n} k(x_i, \nu_1) - b_1\big) - \sum_{i=1}^{n}(r_i+1)z(x_i, \nu_1)^{\delta}\Big),$$

$$\pi(\eta|\nu_2, \text{data}) \propto \eta^{n+a_2-1} \exp\Big(\eta\big(\sum_{j=1}^{m} w(y_j, \nu_2) - \sum_{j=1}^{m} k(y_j, \nu_2) - b_2\big) - \sum_{j=1}^{m}(s_j+1)z(y_j, \nu_2)^{\eta}\Big),$$

$$\pi(\nu_1|\delta, \text{data}) \propto \nu_1^{n+a_3-1} \exp\Big(-\sum_{i=1}^{n}\frac{\nu_1}{x_i} - b_3\nu_1\Big)$$
$$\times \exp\Big(\sum_{i=1}^{n} u(x_i, \nu_1) + (\delta-1)\sum_{i=1}^{n} w(x_i, \nu_1) - (\delta+1)\sum_{i=1}^{n} k(x_i, \nu_1) - \sum_{i=1}^{n}(r_i+1)z(x_i, \nu_1)^{\delta}\Big),$$

$$\pi(\nu_2|\eta, \text{data}) \propto \nu_2^{m+a_4-1} \exp\Big(-\sum_{j=1}^{m}\frac{\nu_2}{y_j} - b_4\nu_2\Big)$$
$$\times \exp\Big(\sum_{j=1}^{m} u(y_j, \nu_2) + (\eta-1)\sum_{j=1}^{m} w(y_j, \nu_2) - (\eta+1)\sum_{j=1}^{m} k(y_j, \nu_2) - \sum_{j=1}^{m}(s_j+1)z(y_j, \nu_2)^{\eta}\Big).$$

Now, the proposed Gibbs sampling algorithm is:

1. Start with the begin value $(\delta_{(0)}, \eta_{(0)}, \nu_{1(0)}, \nu_{2(0)})$.
2. Set $t = 1$.
3. Generate $\nu_{1(t)}$ from $\pi(\nu_1|\delta_{(t-1)}, \text{data})$.
4. Generate $\nu_{2(t)}$ from $\pi(\nu_2|\eta_{(t-1)}, \text{data})$.
5. Generate $\delta_{(t)}$ from $\pi(\delta|\nu_{1(t-1)}, \text{data})$.
6. Generate $\eta_{(t)}$ from $\pi(\eta|\nu_{2(t-1)}, \text{data})$.
7. Calculate

$$R_t = \int_0^{\infty} \frac{\delta(t)\nu_1(t)}{x^2} \exp\Big(-\frac{\nu_{1(t)}}{x} + u(x, \nu_{1(t)}) + (\delta_{(t)}-1)w(x, \nu_{1(t)})$$
$$- (\delta_{(t)}+1)k(x, \nu_{1(t)}) - z(x, \nu_{1(t)})^{\delta_{(t)}} - z(x, \nu_{2(t)})^{\eta_{(t)}}\Big)dx,$$

8. Set $t = t + 1$.
9. Repeat steps 3-8, for T times.

Thus

$$\widehat{R}^{MC} = \frac{1}{T}\sum_{t=1}^{T} R_t. \tag{16}$$

5. Simulation study

In the present section, the performance of different methods are compared by Monte Carlo simulations. The simulations were done with MATLAB R2020b software. We give the MSEs for comparing of point estimates and compute coverage percentages (C.P) and the average lengths (A.L) for comparing of interval estimates. The analysis is based 3000 replications and also the nominal level is 0.95. We use the following three censoring schemes:

$$
\begin{aligned}
&\text{Sch. 1: } R_1 = \ldots = R_n = \frac{N-n}{n},\\
&\text{Sch. 2: } R_1 = \ldots = R_{\frac{n}{2}} = 0,\ R_{\frac{n}{2}+1} = \ldots = R_n = \frac{2(N-n)}{n},\\
&\text{Sch. 3: } R_1 = \ldots = R_{n-1} = 0,\ R_n = N-n.
\end{aligned}
$$

In passing, without loss of generality, we put some values for parameters. When ν is unknown, we put $\delta = \eta = \nu = 2$. In Bayesian inference, we consider the three priors: Pri. 1: $a_j = b_j = 0$, Pri. 2: $a_j = 1,\ b_j = 0.1$, and Pri. 3: $a_j = 2,\ b_j = 0.2$ $(j = 1, 2, 3)$. We reported the outputs in Table 1. Further, average confidence/credible lengths and C.Ps for estimates of R when ν is unknown are reported in Table 2.

When ν is known, , we put $\delta = \eta = \nu = 3$. In Bayesian inference, we consider the three priors: Pri. 4: $a_j = b_j = 0$, Pri. 5: $a_j = 1,\ b_j = 0.1$, and Pri. 6: $a_j = 2,\ b_j = 0.2$ $(j = 1, 2)$. We reported the outputs in Table 3. Further, we reported the ACI and HPD credible intervals in Table 4.

In general case, we put $\delta = \eta = \nu_1 = \nu_2 = 2$. Also, Bayesian inference are given under three priors as: Pri. 7: $a_j = b_j = 0$, Pri. 8: $a_j = 1,\ b_j = 0.1$ and Pri. 9: $a_j = 2,\ b_j = 0.2$ $(j = 1, 2, 3, 4)$. We give the estimates via MCMC method in Table 5.

We presented the trace plots for three different schemes and parameters for monitoring the convergence of MCMC method (see Figures 2-4). We can conclude that MCMC method is converged in all cases.

Based on MSEs, the Bayes estimates have the best performance (see Table 1). In Bayesian case, the informative priors perform better than non-informative ones so that the best performance, based on MSEs, belong to Pri. 3. Moreover, the MCMC method performs better than Lindley's approximation. From Table 2, it is observed that the HPD credible intervals have the better performance than the AICs. Also, in Bayesian case, the informative priors perform better than non-informative ones so that the best performance belong to Pri. 3, namely, the HPD credible intervals based on Pri. 3 have the smallest A.Ls and largest C.Ps.

Based on MSEs, the Bayes estimates have the best performance (see Table 3). In Bayesian case, the informative priors perform better than non-informative ones so that the best performance, based on MSEs, belong to Pri. 6. Moreover, the MCMC method performs better than Lindley's approximation. From Table 4, it is observed that the HPD credible intervals have the better performance than the AICs. In Bayesian inference, the informative priors perform better than non-informative ones so that the best performance belong to Pri. 6, namely, the HPD credible intervals based on Pri. 6 have the smallest A.Ls and largest C.Ps.

Based on MSEs, the Bayes estimates have the best performance (see Table 5). In Bayesian case, the informative priors perform better than non-informative ones so that the best performance, based on MSEs, A.Ls and C.Ps belong to Pri. 9.

In all cases, for fixed N, with increasing n, we see the MSEs of all estimates decrease. Also, the average confidence lengths decrease and the associated C.Ps increase (see Tables 1-5).

Table 1. Estimations and MSEs when ν is unknown.

(N,n)	C.S	AMLE		MLE		Prior 1				Prior 2				Prior 3			
						MCMC		Lindley		MCMC		Lindley		MCMC		Lindley	
		Est.	MSE	Est.	MSE	Est.	MSE	Est.	MSE	Est.	MSE	Est.	MSE	Est.	MSE	Est.	MSE
(20,10)	(1,1)	0.4963	0.0178	0.5032	0.0159	0.5222	0.0129	0.4331	0.0150	0.5246	0.0119	0.4394	0.0144	0.5223	0.0115	0.4457	0.0136
	(2,2)	0.5114	0.0184	0.5099	0.0169	0.5230	0.0125	0.4765	0.0156	0.5230	0.0122	0.5200	0.0142	0.5217	0.0113	0.5234	0.0134
	(3,3)	0.5138	0.0168	0.5048	0.0160	0.5232	0.0134	0.5071	0.0154	0.5242	0.0130	0.5059	0.0151	0.5254	0.0121	0.5050	0.0147
	(1,2)	0.4907	0.0178	0.5221	0.0166	0.5197	0.0120	0.4897	0.0155	0.5244	0.0118	0.5045	0.0143	0.5060	0.0110	0.5158	0.0132
	(1,3)	0.4945	0.0169	0.5150	0.0152	0.5050	0.0138	0.5172	0.0140	0.5134	0.0126	0.5163	0.0133	0.5034	0.0115	0.5153	0.0129
	(2,3)	0.4989	0.0175	0.5157	0.0168	0.5131	0.0134	0.5153	0.0155	0.5035	0.0130	0.5237	0.0149	0.5233	0.0125	0.4720	0.0140
(40,10)	(1,1)	0.5239	0.0179	0.5191	0.0165	0.5218	0.0128	0.5022	0.0159	0.5222	0.0124	0.5019	0.0145	0.5234	0.0111	0.5016	0.0140
	(2,2)	0.4944	0.0168	0.5159	0.0157	0.5222	0.0135	0.4356	0.0146	0.5214	0.0120	0.4382	0.0141	0.5238	0.0116	0.4407	0.0138
	(3,3)	0.5166	0.0163	0.5034	0.0153	0.5231	0.0135	0.5029	0.0147	0.5230	0.0133	0.5019	0.0143	0.5227	0.0117	0.5009	0.0132
	(1,2)	0.4940	0.0173	0.5197	0.0169	0.5225	0.0137	0.5158	0.0150	0.5220	0.0129	0.5127	0.0142	0.5224	0.0115	0.5160	0.0136
	(1,3)	0.5065	0.0165	0.5225	0.0152	0.5125	0.0132	0.5229	0.0148	0.5117	0.0125	0.5223	0.0139	0.5017	0.0120	0.5216	0.0130
	(2,3)	0.5076	0.0169	0.5225	0.0157	0.5028	0.0135	0.5217	0.0146	0.5218	0.0130	0.5015	0.0137	0.5128	0.0122	0.5214	0.0129
(60,10)	(1,1)	0.5912	0.0184	0.5020	0.0161	0.5225	0.0132	0.5134	0.0144	0.5221	0.0120	0.5327	0.0135	0.5137	0.0115	0.5324	0.0123
	(2,2)	0.5298	0.0169	0.5218	0.0153	0.4998	0.0138	0.5017	0.0144	0.5233	0.0135	0.5017	0.0138	0.5199	0.0114	0.5016	0.0136
	(3,3)	0.4997	0.0162	0.6041	0.0154	0.5223	0.0125	0.5219	0.0146	0.5015	0.0129	0.5147	0.0142	0.5224	0.0113	0.5013	0.0139
	(1,2)	0.5023	0.0173	0.4991	0.0165	0.5204	0.0122	0.5247	0.0154	0.522	0.0129	0.5242	0.0142	0.5196	0.0119	0.5060	0.0146
	(1,3)	0.4976	0.0174	0.5223	0.0169	0.5127	0.0129	0.5119	0.0149	0.5118	0.0120	0.5128	0.0140	0.5139	0.0113	0.5067	0.0130
	(2,3)	0.4908	0.0179	0.5039	0.0165	0.5120	0.0134	0.5239	0.0152	0.5224	0.0129	0.5330	0.0149	0.5121	0.0120	0.5321	0.0139
(40,20)	(1,1)	0.5038	0.0145	0.5175	0.0119	0.5233	0.0078	0.5189	0.0110	0.4974	0.0068	0.5134	0.0099	0.5164	0.0035	0.5078	0.0090
	(2,2)	0.5238	0.0140	0.5055	0.0126	0.5052	0.0093	0.5107	0.0113	0.5141	0.0048	0.5038	0.0096	0.4983	0.0043	0.5028	0.0092
	(3,3)	0.4926	0.0149	0.4983	0.0131	0.4959	0.0070	0.4999	0.0110	0.4908	0.0059	0.5164	0.0097	0.5195	0.0045	0.5171	0.0095
	(1,2)	0.4960	0.0139	0.5218	0.0129	0.4843	0.0082	0.5163	0.0114	0.4912	0.0066	0.5079	0.0095	0.5208	0.0032	0.5112	0.0089
	(1,3)	0.5159	0.0135	0.4859	0.0128	0.4940	0.0092	0.5058	0.0110	0.4972	0.0064	0.5168	0.0099	0.4982	0.0030	0.5021	0.0092
	(2,3)	0.4996	0.0130	0.5123	0.0118	0.5137	0.0086	0.5105	0.0117	0.5211	0.0066	0.4894	0.0097	0.4974	0.0035	0.4990	0.0088
(60,20)	(1,1)	0.4991	0.0142	0.5206	0.0124	0.5186	0.0086	0.5053	0.0115	0.4973	0.0050	0.5088	0.0098	0.5044	0.0028	0.5147	0.0087
	(2,2)	0.5086	0.0148	0.5162	0.0131	0.5187	0.0091	0.5136	0.0112	0.5329	0.0068	0.5054	0.0099	0.5147	0.0043	0.4950	0.007
	(3,3)	0.5189	0.0146	0.5224	0.0132	0.5182	0.0077	0.5194	0.0113	0.4954	0.0063	0.4992	0.0097	0.5219	0.0038	0.5195	0.0093
	(1,2)	0.5138	0.0132	0.5182	0.0118	0.4974	0.0095	0.4965	0.0117	0.512	0.0066	0.5167	0.0098	0.5090	0.0039	0.5109	0.0088
	(1,3)	0.5243	0.0139	0.5195	0.0120	0.5056	0.0079	0.4989	0.0110	0.5188	0.0061	0.5177	0.0099	0.5168	0.0030	0.5176	0.0089
	(2,3)	0.5256	0.0137	0.4978	0.0121	0.5107	0.0077	0.5141	0.0111	0.5019	0.0056	0.5043	0.0097	0.4985	0.0042	0.4859	0.0090

Table 2. Estimates when ν is unknown.

(N,n)	C.S	AMLE		MLE		Prior 1		Prior 2		Prior 3	
		length	C.P	length	C.P	length	C.P	length	C.P	length	C.P
(20,10)	(1,1)	0.4351	0.877	0.4280	0.885	0.3764	0.900	0.3455	0.921	0.3028	0.942
	(2,2)	0.4330	0.874	0.4185	0.881	0.3760	0.906	0.3304	0.923	0.3003	0.944
	(3,3)	0.4302	0.879	0.4257	0.884	0.3713	0.910	0.3265	0.932	0.3004	0.949
	(1,2)	0.4427	0.885	0.4135	0.902	0.3669	0.910	0.3502	0.936	0.2946	0.949
	(1,3)	0.4318	0.893	0.4127	0.903	0.3887	0.915	0.3498	0.926	0.3113	0.944
	(2,3)	0.4298	0.879	0.4170	0.880	0.3763	0.915	0.3312	0.925	0.3174	0.946
(40,10)	(1,1)	0.4390	0.890	0.4228	0.903	0.3997	0.911	0.3461	0.927	0.3078	0.940
	(2,2)	0.4435	0.879	0.4156	0.888	0.3725	0.906	0.3423	0.935	0.2900	0.940
	(3,3)	0.4324	0.890	0.4129	0.895	0.3942	0.908	0.3521	0.927	0.3011	0.943
	(1,2)	0.4443	0.976	0.4148	0.900	0.3899	0.913	0.3229	0.937	0.3090	0.944
	(1,3)	0.4494	0.878	0.4260	0.893	0.3841	0.901	0.3239	0.924	0.2908	0.941
	(2,3)	0.4435	0.893	0.4369	0.903	0.3611	0.907	0.3248	0.931	0.2961	0.948
(60,10)	(1,1)	0.4238	0.892	0.4038	0.902	0.3847	0.913	0.3543	0.929	0.3113	0.940
	(2,2)	0.4352	0.885	0.4204	0.897	0.3959	0.902	0.3584	0.937	0.2980	0.948
	(3,3)	0.4313	0.880	0.4266	0.891	0.3938	0.900	0.3448	0.929	0.3199	0.948
	(1,2)	0.4380	0.883	0.4275	0.891	0.3760	0.907	0.3537	0.926	0.3092	0.946
	(1,3)	0.4396	0.877	0.4392	0.888	0.3836	0.913	0.3270	0.930	0.3041	0.940
	(2,3)	0.4381	0.893	0.4234	0.905	0.3901	0.915	0.3535	0.931	0.3073	0.946
(40,20)	(1,1)	0.4123	0.902	0.3889	0.906	0.3401	0.929	0.2903	0.942	0.2642	0.956
	(2,2)	0.4133	0.898	0.3828	0.906	0.3334	0.926	0.2984	0.940	0.2699	0.53
	(3,3)	0.4034	0.903	0.3761	0.908	0.3449	0.927	0.2944	0.945	0.2773	0.950
	(1,2)	0.3988	0.900	0.3826	0.907	0.3319	0.925	0.3176	0.949	0.2512	0.957
	(1,3)	0.4022	0.901	0.3893	0.908	0.3403	0.920	0.2969	0.948	0.2758	0.951
	(2,3)	0.4049	0.899	0.3707	0.909	0.3433	0.928	0.2953	0.935	0.2778	0.954
(60,20)	(1,1)	0.4040	0.905	0.3922	0.909	0.3371	0.926	0.2994	0.945	0.2893	0.950
	(2,2)	0.4052	0.903	0.3926	0.910	0.3378	0.919	0.2903	0.948	0.2915	0.959
	(3,3)	0.4016	0.900	0.3949	0.908	0.3425	0.921	0.2978	0.940	0.2747	0.953
	(1,2)	0.4118	0.902	0.3937	0.910	0.3320	0.918	0.3195	0.949	0.2889	0.962
	(1,3)	0.4026	0.905	0.3807	0.910	0.3492	0.917	0.3099	0.939	0.2576	0.960
	(2,3)	0.4105	0.906	0.3919	0.911	0.3302	0.925	0.2955	0.948	0.2538	0.957

Table 3. Estimations and MSEs when ν is known.

(N,n)	C.S	MLE		Prior 4				Prior 5				Prior 6			
				MCMC		Lindley		MCMC		Lindley		MCMC		Lindley	
		Est.	MSE	Est.	MSE	Est.	MSE	Est.	MSE	Est.	MSE	Est.	MSE	Est.	MSE
(20,10)	(1,1)	0.4983	0.0149	0.4737	0.0127	0.4771	0.0138	0.5070	0.0102	0.4788	0.0120	0.4986	0.0073	0.4774	0.0116
	(2,2)	0.5206	0.0140	0.5024	0.0129	0.5193	0.0135	0.5159	0.0105	0.5113	0.0123	0.4863	0.0086	0.4950	0.0115
	(3,3)	0.4782	0.0153	0.4873	0.0131	0.4981	0.0142	0.4813	0.0119	0.5145	0.0131	0.5090	0.0098	0.5220	0.0129
	(1,2)	0.5059	0.0159	0.5087	0.0132	0.4742	0.0143	0.5072	0.0117	0.5033	0.0133	0.5051	0.0095	0.4895	0.0120
	(1,3)	0.5191	0.0152	0.4746	0.0139	0.5077	0.0148	0.5192	0.0112	0.5041	0.0138	0.5212	0.0098	0.4992	0.0122
	(2,3)	0.4976	0.0159	0.5041	0.0134	0.4810	0.0147	0.5010	0.0118	0.5063	0.0134	0.5201	0.0096	0.5213	0.0124
(40,10)	(1,1)	0.4864	0.0147	0.4801	0.0128	0.4764	0.0132	0.4947	0.0106	0.5016	0.0121	0.5203	0.0077	0.5112	0.0112
	(2,2)	0.5200	0.0155	0.4901	0.0130	0.4885	0.0142	0.5201	0.0110	0.4808	0.0134	0.5066	0.0095	0.4892	0.0127
	(3,3)	0.4737	0.0143	0.4909	0.0123	0.4808	0.0134	0.5111	0.0100	0.4972	0.0121	0.5116	0.0092	0.4855	0.0110
	(1,2)	0.5198	0.0147	0.4820	0.0129	0.4963	0.0133	0.5222	0.0105	0.4927	0.0128	0.5105	0.0095	0.5096	0.0112
	(1,3)	0.5087	0.0149	0.5064	0.0123	0.5020	0.0138	0.4963	0.0109	0.4826	0.0125	0.4772	0.0084	0.5069	0.0116
	(2,3)	0.5202	0.0155	0.4806	0.0137	0.4919	0.0147	0.5000	0.0118	0.4810	0.0130	0.4854	0.0097	0.4898	0.0125
(60,10)	(1,1)	0.4785	0.0144	0.4991	0.0126	0.4737	0.0136	0.5110	0.0109	0.5164	0.0122	0.5200	0.0089	0.5074	0.0117
	(2,2)	0.5219	0.0143	0.4945	0.0128	0.5118	0.0139	0.5040	0.0102	0.5123	0.0123	0.5070	0.0084	0.4969	0.0114
	(3,3)	0.4926	0.0143	0.5135	0.0135	0.4743	0.0140	0.5058	0.0117	0.4766	0.0137	0.4876	0.0097	0.5116	0.0124
	(1,2)	0.4813	0.0145	0.4867	0.0127	0.5074	0.0135	0.4847	0.0104	0.4865	0.0129	0.4778	0.0086	0.5104	0.0116
	(1,3)	0.5166	0.0153	0.5207	0.0135	0.4825	0.0147	0.5136	0.0115	0.5101	0.0139	0.4748	0.0097	0.5055	0.0122
	(2,3)	0.4850	0.0150	0.5167	0.0136	0.4927	0.0143	0.5110	0.0111	0.5108	0.0136	0.5227	0.0099	0.4900	0.0120
(40,20)	(1,1)	0.5112	0.0116	0.4813	0.0092	0.5066	0.0112	0.5057	0.0074	0.4925	0.0109	0.4938	0.0064	0.5046	0.0081
	(2,2)	0.5162	0.0118	0.5161	0.0089	0.5161	0.0111	0.4928	0.0088	0.5011	0.0101	0.5109	0.0048	0.4878	0.0083
	(3,3)	0.4765	0.0121	0.5119	0.0083	0.5106	0.0118	0.4736	0.0075	0.5191	0.0108	0.4939	0.0053	0.5012	0.0080
	(1,2)	0.4884	0.0124	0.5162	0.0107	0.5150	0.0120	0.4948	0.0085	0.4990	0.0110	0.4740	0.0054	0.4957	0.0094
	(1,3)	0.4822	0.0129	0.4951	0.0108	0.5190	0.0120	0.5171	0.0081	0.5004	0.0112	0.4867	0.0059	0.5106	0.0099
	(2,3)	0.4744	0.0121	0.4937	0.0095	0.5077	0.0116	0.4895	0.0084	0.5166	0.0109	0.5156	0.0053	0.4869	0.0090
(60,20)	(1,1)	0.4830	0.0117	0.5146	0.0100	0.4796	0.0112	0.4875	0.0082	0.4926	0.0109	0.4881	0.0049	0.4780	0.0089
	(2,2)	0.5063	0.0118	0.4938	0.0093	0.4957	0.0115	0.5094	0.0078	0.4946	0.0103	0.4888	0.0061	0.5143	0.0090
	(3,3)	0.4828	0.0123	0.4830	0.0097	0.4739	0.0118	0.4914	0.0086	0.4857	0.0106	0.4962	0.0049	0.5007	0.0080
	(1,2)	0.5001	0.0122	0.5227	0.0084	0.4849	0.0119	0.5029	0.0070	0.5193	0.0109	0.5057	0.0069	0.5072	0.0086
	(1,3)	0.4820	0.0128	0.5180	0.0092	0.4869	0.0116	0.5135	0.0074	0.5005	0.0106	0.4767	0.0053	0.4963	0.0085
	(2,3)	0.4794	0.0126	0.5132	0.0100	0.4785	0.0120	0.4950	0.0086	0.5184	0.0112	0.5026	0.0058	0.4927	0.0097

Table 4. Estimates when ν is known.

(N,n)	C.S	MLE		Prior 4		Prior 5		Prior 6	
		length	C.P	length	C.P	length	C.P	length	C.P
(20,10)	(1,1)	0.3818	0.906	0.3447	0.929	0.2982	0.935	0.2653	0.948
	(2,2)	0.3905	0.905	0.3682	0.928	0.3107	0.935	0.2736	0.944
	(3,3)	0.4015	0.902	0.3635	0.928	0.3286	0.934	0.2883	0.949
	(1,2)	0.3889	0.899	0.3319	0.929	0.3280	0.935	0.2720	0.946
	(1,3)	0.3977	0.900	0.3508	0.928	0.3053	0.934	0.2792	0.948
	(2,3)	0.4033	0.901	0.3547	0.930	0.3237	0.936	0.2698	0.950
(40,10)	(1,1)	0.3978	0.895	0.3306	0.930	0.3074	0.937	0.2699	0.950
	(2,2)	0.3975	0.894	0.3622	0.927	0.3182	0.935	0.2884	0.948
	(3,3)	0.3777	0.903	0.3605	0.926	0.3195	0.934	0.2836	0.946
	(1,2)	0.3877	0.894	0.3410	0.926	0.3126	0.936	0.2686	0.949
	(1,3)	0.3737	0.904	0.3507	0.927	0.3237	0.935	0.2654	0.947
	(2,3)	0.3819	0.903	0.3525	0.931	0.3229	0.938	0.2675	0.951
(60,10)	(1,1)	0.3953	0.902	0.3408	0.930	0.2983	0.937	0.2653	0.950
	(2,2)	0.3778	0.890	0.3377	0.931	0.3075	0.938	0.2741	0.951
	(3,3)	0.3735	0.900	0.3303	0.927	0.3235	0.932	0.2619	0.948
	(1,2)	0.4020	0.901	0.3619	0.926	0.3208	0.932	0.2767	0.949
	(1,3)	0.3987	0.891	0.3400	0.926	0.3106	0.933	0.2874	0.948
	(2,3)	0.3861	0.901	0.3458	0.931	0.3177	0.937	0.2600	0.951
(40,20)	(1,1)	0.3417	0.919	0.3038	0.938	0.2762	0.942	0.2152	0.955
	(2,2)	0.3681	0.920	0.3124	0.938	0.2905	0.944	0.2433	0.956
	(3,3)	0.3473	0.920	0.3018	0.939	0.2900	0.943	0.2410	0.955
	(1,2)	0.3617	0.918	0.3019	0.939	0.2779	0.943	0.2334	0.956
	(1,3)	0.3630	0.918	0.3020	0.938	0.2651	0.942	0.2156	0.954
	(2,3)	0.3784	0.919	0.3047	0.939	0.2620	0.944	0.2051	0.955
(60,20)	(1,1)	0.3651	0.918	0.3163	0.938	0.2802	0.945	0.2234	0.956
	(2,2)	0.3566	0.920	0.3136	0.938	0.2814	0.942	0.2111	0.956
	(3,3)	0.3679	0.919	0.3140	0.939	0.2936	0.943	0.2395	0.955
	(1,2)	0.3415	0.919	0.3056	0.938	0.2600	0.943	0.2056	0.954
	(1,3)	0.3754	0.919	0.3181	0.939	0.2650	0.944	0.2442	0.955
	(2,3)	0.3774	0.920	0.3161	0.938	0.2726	0.945	0.2239	0.956

Table 5. Estimates in general case.

(N,n)	C.S	AMLE		MLE		Prior 7				Prior 8				Prior 9			
						MCMC		C.I		MCMC		C.I		MCMC		C.I	
		Est.	MSE	Est.	MSE	Est.	MSE	length	C.P	Est.	MSE	length	C.P	Est.	MSE	length	C.P
(20,10)	(1,1)	0.4913	0.0215	0.4818	0.0200	0.4943	0.0121	0.3810	0.924	0.4812	0.0098	0.3275	0.939	0.4884	0.0075	0.2879	0.945
	(2,2)	0.5280	0.0214	0.4955	0.0193	0.5022	0.0127	0.4093	0.919	0.4933	0.0097	0.3514	0.938	0.5109	0.0082	0.2965	0.948
	(3,3)	0.5178	0.0210	0.5142	0.0202	0.5113	0.0128	0.3856	0.922	0.5010	0.0090	0.3326	0.939	0.5118	0.0079	0.2823	0945
	(1,2)	0.4770	0.0215	0.4829	0.0194	0.4817	0.0120	0.3881	0.923	0.5081	0.0092	0.3214	0.935	0.4908	0.0080	0.2938	0.947
	(1,3)	0.4931	0.0210	0.5362	0.0192	0.5162	0.0130	0.4035	0.921	0.5018	0.0104	0.3580	0.937	0.5275	0.0081	0.2915	0.944
	(2,3)	0.4816	0.0216	0.5229	0.0190	0.4978	0.0135	0.4055	0.928	0.5173	0.0102	0.3472	0.937	0.4966	0.0083	0.2927	0.948
(40,10)	(1,1)	0.4857	0.0214	0.4834	0.0192	0.4938	0.0129	0.4043	0.922	0.5127	0.0099	0.3432	0.938	0.4955	0.0071	0.3062	0.947
	(2,2)	0.4770	0.0216	0.5192	0.0193	0.4965	0.0124	0.3756	0.927	0.5104	0.0090	0.3252	0.937	0.5122	0.0076	0.2803	0.946
	(3,3)	0.5135	0.0215	0.5176	0.0200	0.5087	0.0129	0.3837	0.922	0.4868	0.0100	0.3290	0.937	0.5141	0.0074	0.2948	0.946
	(1,2)	0.5076	0.0210	0.5111	0.0192	0.4990	0.0123	0.3884	0.923	0.4834	0.0094	0.3310	0.937	0.5070	0.0070	0.2800	0.945
	(1,3)	0.5095	0.0211	0.4889	0.0193	0.5142	0.0134	0.3833	0.920	0.4930	0.0100	0.3313	0.938	0.4850	0.0086	0.2961	0.948
	(2,3)	0.5399	0.0217	0.4850	0.0195	0.5099	0.0130	0.4047	0.924	0.4959	0.0099	0.3519	0.934	0.5122	0.0082	0.3084	0.947
(60,10)	(1,1)	0.5312	0.0213	0.5115	0.0199	0.5105	0.0127	0.4084	0.929	0.4870	0.0102	0.3483	0.937	0.4907	0.0085	0.2947	0.945
	(2,2)	0.4735	0.0215	0.4970	0.0195	0.5106	0.0131	0.3711	0.922	0.5007	0.0099	0.3232	0.934	0.4867	0.0075	0.2903	0.9457
	(3,3)	0.5294	0.0216	0.4934	0.0192	0.5151	0.0129	0.4068	0.920	0.4917	0.0095	0.3455	0.938	0.4881	0.0072	0.2925	0.947
	(1,2)	0.5061	0.0213	0.5140	0.0192	0.5197	0.0128	0.4080	0.922	0.4824	0.0098	0.3437	0.939	0.4998	0.0076	0.3054	0.949
	(1,3)	0.4920	0.0216	0.4921	0.0193	0.4929	0.0131	0.3815	0.927	0.5243	0.0099	0.3305	0.939	0.4907	0.0086	0.2939	0.944
	(2,3)	0.4721	0.0215	0.4816	0.0193	0.4990	0.0130	0.3785	0.925	0.5187	0.0105	0.3205	0.935	0.4960	0.0074	0.2900	0.945
(40,20)	(1,1)	0.5066	0.0182	0.4841	0.0152	0.4970	0.0091	0.3303	0.935	0.5051	0.0064	0.3079	0.949	0.4970	0.0025	0.2499	0.955
	(2,2)	0.5317	0.0183	0.4980	0.0150	0.5113	0.0092	0.3470	0.933	0.4934	0.0063	0.3067	0.948	0.4871	0.0029	0.2402	0.957
	(3,3)	0.5084	0.0187	0.5105	0.0154	0.5171	0.0092	0.3451	0.932	0.5195	0.0060	0.2927	0.949	0.4942	0.0020	0.2488	0.958
	(1,2)	0.4877	0.0189	0.5147	0.0142	0.5143	0.0093	0.3479	0.937	0.5158	0.0068	0.2925	0.948	0.4962	0.0024	0.2481	0.961
	(1,3)	0.5388	0.0179	0.5186	0.0142	0.4987	0.0099	0.3455	0.935	0.4826	0.0069	0.3028	0.949	0.5079	0.0028	0.2477	0.960
	(2,3)	0.5011	0.0180	0.5284	0.0148	0.5147	0.0094	0.3425	0.931	0.4898	0.0060	0.3092	0.948	0.5092	0.0020	0.2463	0.959
(60,20)	(1,1)	0.4932	0.0183	0.5221	0.0152	0.4998	0.0098	0.3400	0.934	0.4979	0.0068	0.3051	0.949	0.4917	0.0029	0.2599	0.961
	(2,2)	0.4791	0.0185	0.4925	0.0156	0.4909	0.0091	0.3404	0.936	0.4945	0.0069	0.3053	0.948	0.5075	0.0025	0.2559	0.960
	(3,3)	0.4789	0.0176	0.5145	0.0145	0.4963	0.0099	0.3302	0.936	0.5037	0.0064	0.2961	0.948	0.4818	0.0021	0.2556	0.962
	(1,2)	0.5219	0.0186	0.5291	0.0148	0.4940	0.0095	0.3599	0.934	0.5164	0.0064	0.2949	0.948	0.4954	0.0023	0.2422	0.958
	(1,3)	0.5356	0.0179	0.5066	0.0146	0.5143	0.0093	0.3423	0.935	0.5138	0.0069	0.3036	0.949	0.4963	0.0028	0.2432	0.962
	(2,3)	0.4969	0.0189	0.5139	0.0153	0.5120	0.0091	0.3486	0.936	0.5149	0.0068	0.3026	0.950	0.4834	0.0029	0.2450	0.960

Table 6. All estimates and different intervals.

		MLE	Asymp. (MLE)	AMLE	Asymp. (AMLE)	Bayes		HPD
						MCMC	Lindley	
ν	Complete	0.5039	(0.4768,0.5309)	0.5031	(0.4712,0.5317)	0.5049	0.5035	(0.4878,0.5216)
	Scheme 1	0.5092	(0.4776,0.5408)	0.5090	(0.4752,0.5421)	0.5098	0.5081	(0.4944,0.5378)
	Scheme 2	0.5062	(0.4813,0.5512)	0.5085	(0.4801,0.5576)	0.5156	0.5112	(0.4966,0.5462)
ν_1, ν_2	Complete	0.5621	-	0.5598	-	0.5603	-	(0.5294,0.6311)
	Scheme 1	0.5652	-	0.5612	-	0.5625	-	(0.5102,0.6378)
	Scheme 2	0.5665	-	0.5630	-	0.5670	-	(0.5025,0.6401)

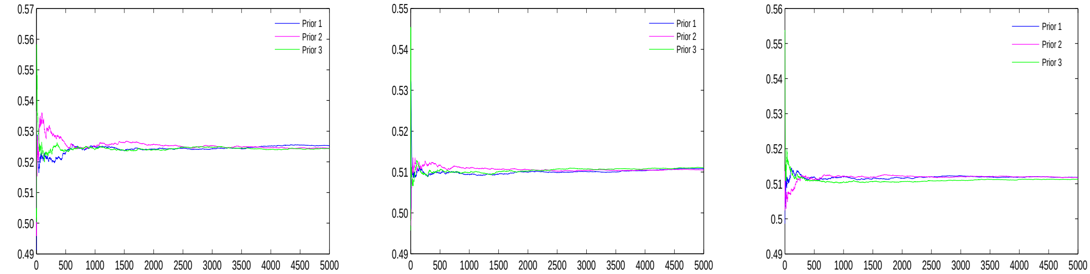

Figure 2. Trace plots with C.S $(3,3)$ and $(N,n) = (60,30)$ (right), $(2,2)$ and $(N,n) = (40,20)$ (center) and $(1,1)$ and $(N,n) = (20,10)$ (left): unknown ν.

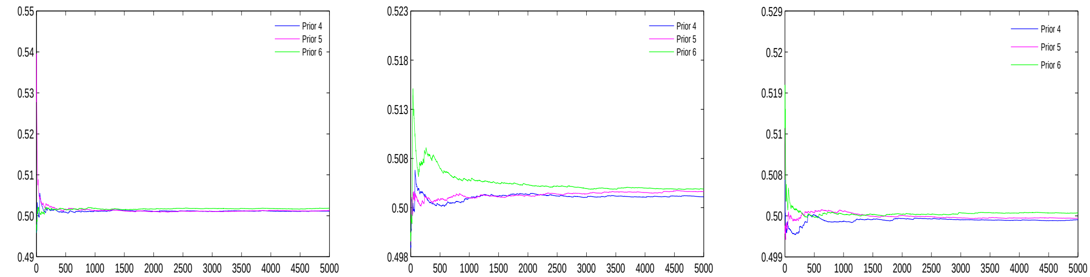

Figure 3. Trace plots with C.S $(1,1)$ and $(N,n) = (60,30)$ (right), $(2,3)$ and $(N,n) = (60,20)$ (center) and $(1,2)$ and $(N,n) = (40,10)$ (left): known ν.

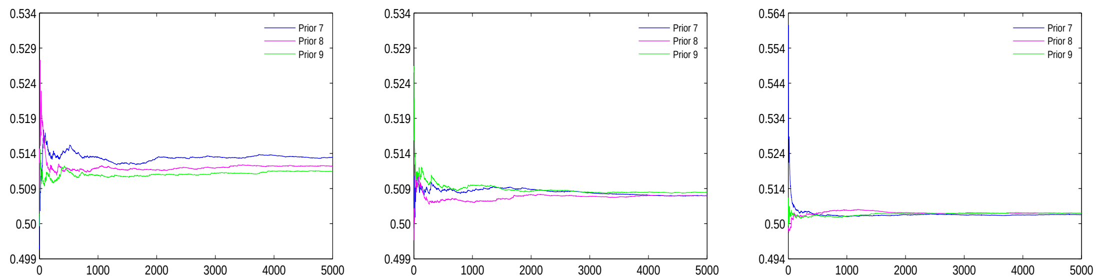

Figure 4. Trace plots with C.S $(1,3)$ and $(N,n) = (60,30)$ (right), $(1,1)$ with $(N,n) = (40,20)$ (center) and $(2,3)$ with $(N,n) = (20,10)$ (left): general case.

6. Data analysis

Here, a real data set is analyzed. For this aim, we use the monthly water capacity of the Shasta reservoir in California, USA, which can be found in http://cdec.water.ca.gov/cgi-progs/queryMonthly?SHA. (see the previous analysis on this data set in [9], [10]). We contract the excessive drought will not occur if the average water capacity on months July and August is more than the water capacity on month December (at one year). Based on our scenario, the months July, August and December from 1987 to 2016 are considered. Then, $X_1, \ldots, X_{30}$ are the capacity of December and $Y_1, \ldots, Y_{30}$ are the average capacity of months July and August from year 1987 till year 2016 and R is the non-occurrence of drought probability.

We divide all our data by the total capacity of Shasta reservoir, 4552000 acre-foot. This action do not make any change in our statistical analysis.

First, we separately fit TNWF distribution to the r.v.s X and Y data, and obtain the parameter estimate values. For r.v. X, $\widehat{\delta}$, $\widehat{\nu}$, the K-S distance and the corresponding p-value are 2.1276, 0.5264, 0.1890 and 0.2061, respectively. Also, for random variable Y, $\widehat{\eta}$, $\widehat{\nu}$, the K-S distance and the its p-value are 2.3362, 0.5566, 0.1421 and 0.5334, respectively. The above values confirm the suitable fits of the TNWF distribution to the data. See Figures 5 and 6 for the empirical cumulative distribution functions and PP-plots.

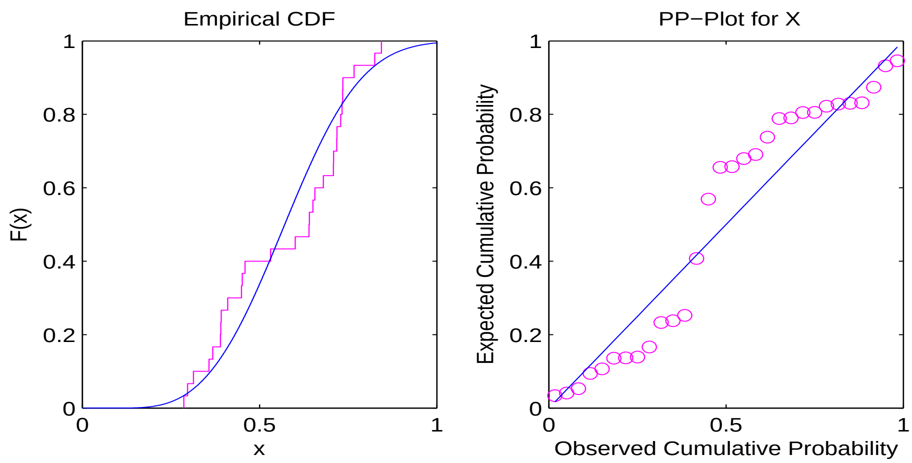

Figure 5. Empirical CDF (left) and PP-plot (right) for r.v. X.

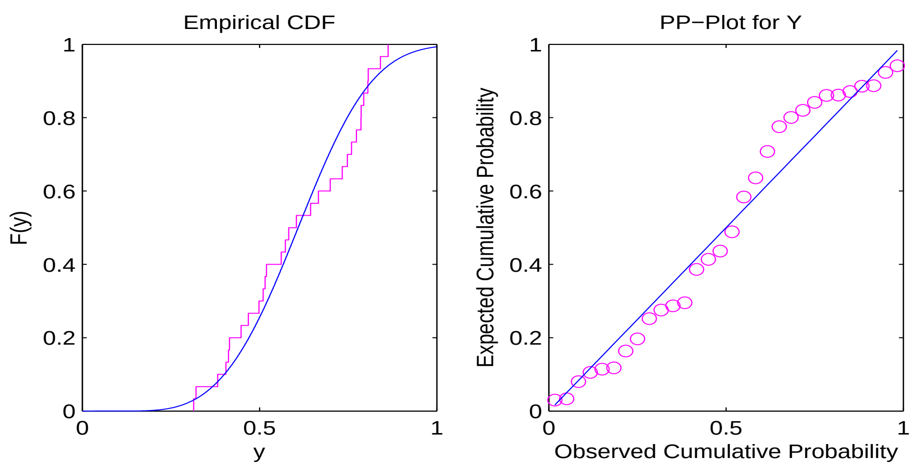

Figure 6. Empirical CDF (left) and PP-plot (right) for r.v. Y.

We consider two schemes for random variables X and Y as Sch. 1: $[1^{*10}, 0^{*10}]$ and Sch. 2: $[2^{*5}, 1^{*5}, 0^{*10}]$.

When ν is unknown, with non-informative priors $a_1 = a_2 = a_3 = b_1 = b_2 = b_3 = 0$ and when ν_1 and ν_2 are unknown and different with non-informative priors $a_1 = a_2 = a_3 = a_4 = b_1 = b_2 = b_3 = b_4 = 0$, all estimates and different intervals presented in Table 6. We can see the complete data has the smallest intervals. Furthermore, Sch. 2 has the largest intervals and the HPD intervals are the best intervals.

Acknowledgement

The authors would like to thank the editor and two anonymous referees who kindly reviewed the first version of the manuscript and provided valuable suggestions and comments.

REFERENCES

1. A.Z. Afify, M.Y. Haitham, G.M. Cordeiro, M.M.E. Ortega, and M.N. Zohdy, The Weibull Fréchet distribution and its applications, Journal of Applied Statistics vol. 43, no. 14, pp. 2608–2626. 2016.
2. K. Ahmadi, and S. Ghafouri, Reliability estimation in a multicomponent stress-strength model under generalized half-normal distribution based on progressive type-II censoring. Journal of Statistical Computation and Simulation vol. 89 pp. 2505–2548, 2019.
3. A. Alzaatreh, F. Famoye, and C. Lee, A new method for generating families of continuous distributions. Metron vol. 71, pp. 63–79, 2013.
4. N. Balakrishnan, and R. Aggarwala, Progressive Censoring: Theory, Methods and Applications, Boston: Birkhauser, 2000.
5. R.M. Corless, G.H. Gonnet, D.E.G Hare, D.J. Jeffrey, and D.E. Knuth, On the Lambert W function. Advances in Computational Mathematics vol. 5, pp. 329–359, 1996.
6. D. V. Lindley, Approximate Bayesian methods. Trabajos de Estadistica, vol. 3, pp. 281–288, 1980.
7. M.H. Chen, and Q.M. Shao, Monte Carlo estimation of Bayesian Credible and HPD intervals. Journal of Computational and Graphical Statistics vol. 8, pp. 69–92, 1999.
8. R. Kazemi, and A. Kohansal, Stress-strength parameter estimation based on Type-II progressive censored samples for a Weibull-half-logistic distribution, Bulletin of the Malaysian Mathematical Sciences Society. vol. 44, no. 4, pp. 2531–2566, 2021.
9. F. Kizilaslan, and M. Nadar, Estimation of reliability in a multicomponent stress-strength model based on a bivariate Kumaraswamy distribution. Statistical Papers. Accepted. DOI: 10.1007/s00362-016-0765-8, 2016.
10. F. Kizilaslan, and M. Nadar, Estimation and prediction of the Kumaraswamy distribution based on record values and inter-record times. Journal of Statistical Computation and Simulation, vol. 86, pp. 2471–2493, 2016.
11. A. Kohansal, and S. Nadarajah, Stress-strength parameter estimation based on Type-II hybrid progressive censored samples for a Kumaraswamy distribution. IEEE Transactions on Reliability vol. 68, pp. 1296–1310, 2019.
12. A. Kohansal, Bayesian and classical estimation of $R = P(X < Y)$ based on Burr type XII distribution under hybrid progressive censored samples. Communications in Statistics-Theory and Methods, vol. 49, pp. 1043–1081, 2020.
13. A. Kohansal, and S. Shoaee, Bayesian and classical estimation of reliability in a multicomponent stress-strength model under adaptive hybrid progressive censored data. Statistical Papers vol. 62, pp. 309–359, 2021.
14. T.R. Rasethuntsa, and M. Nadar, Stress-strength reliability of a non-identical-component-strengths system based on upper record values from the family of Kumaraswamy generalized distributions. Statistics vol. 52, pp. 684–716, 2018.
15. S. Shoaee, and E. Khorram, Stress-strength reliability of a two-parameter bathtub-shaped lifetime distribution based on progressively censored samples. Communications in Statistics-Theory and Methods, vol. 44, pp. 5306–5328, 2015.
16. S. Shoaee, and E. Khorram, Statistical inference of $\mathcal{R} = P(Y < X)$ for Weibull distribution under Type-II progressively hybrid censored data. Journal of Statistical Computation and Simulation, vol. 86, pp. 3815–3834, 2016.