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Dynamics of the Predator-Prey Model with Beddington-DeAngelis Functional Response Perturbed by Lévy Noise

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Abstract We study the non-autonomous stochastic predator-prey model with Beddington-DeAngelis functional response driven by the system of stochastic differential equations with white noise, centered and non-centered Poisson noises. It is proved the existence and uniqueness of the global positive solution of considered system. We obtain sufficient conditions of stochastic ultimate boundedness, stochastic permanence, non-persistence in the mean, weak and strong persistence in the mean and extinction of the population densities in the considered stochastic predator-prey model.

Keywords Non-autonomous, Stochastic Predator-Prey Model, Beddington-DeAngelis Functional Response, Stochastic Ultimate Boundedness, Stochastic Permanence, Non-Persistence in the Mean, Weak and Strong Persistence in the Mean, Extinction

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1. Introduction

The study of dynamics of predator-prey systems is one of the important subjects in population dynamics. The predator-prey model usually described by the system of differential equations

$$\begin{aligned} \frac{dx(t)}{dt} &= x(t)(a_1 - b_1 x(t)) - f(x(t), y(t))y(t), \\ \frac{dy(t)}{dt} &= -y(t)a_2 + cf(x(t), y(t))y(t), \end{aligned}$$

where $x(t)$, $y(t)$ represent the population density of prey and predator respectively at time t, a_1 is the growth rate of prey, b_1 measures the strength of competition among individuals of species x, a_2 is the death rate of predator, c denotes the conversion coefficient, $f(x, y)$ is the functional response of the predator. In [1] and [2] authors proposed the Beddington − DeAngelis functional response of the form $f(x, y) = x/(m_1 x + m_2 y + m_3)$. There are considerable evidences in nature that predator species may be density dependent. So we need to take into account levels of predator density dependence.

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In [3] the authors study the deterministic model of the density dependent predator-prey system with Beddington-DeAngelis functional response. This model is driven by the system of differential equations

$$\begin{aligned} dx(t) &= x(t)\left(a_1 - b_1 x(t) - \frac{c_1 y(t)}{m_1 x(t) + m_2 y(t) + m_3}\right) dt, \\ dy(t) &= y(t)\left(-a_2 - b_2 y(t) + \frac{c_2 x(t)}{m_1 x(t) + m_2 y(t) + m_3}\right) dt. \end{aligned} \tag{1}$$

All parameters in system (1) are positive and b_2 is the predator density dependence rate. In [3] the authors study the conditions under which the model (1) has a positive equilibrium and when this equilibrium is globally asymptotically stable.

In the paper [4] it is considered the stochastic version of the model (1) in the following form

$$\begin{aligned} dx(t) &= x(t)\left(a_1 - b_1 x(t) - \frac{c_1 y(t)}{m_1 x(t) + m_2 y(t) + m_3}\right) dt + \alpha x(t) dw_1(t), \\ dy(t) &= y(t)\left(-a_2 - b_2 y(t) + \frac{c_2 x(t)}{m_1 x(t) + m_2 y(t) + m_3}\right) dt - \beta y(t) dw_2(t), \end{aligned} \tag{2}$$

where $w_1(t)$ and $w_2(t)$ are mutually independent Wiener processes. The authors proved that there is a unique positive solution to the system (2). It is shown that there is a stationary distribution of the solution of system (2) and it has ergodic property under small white noise. The sufficient conditions under which the system (2) is nonpersistent are obtained.

Population systems may suffer abrupt environmental perturbations, such as epidemics, fires, earthquakes, etc. It is natural to introduce centered and non-centered Poisson noises into the population model for describing such discontinuous systems. So, we take into account not only "small" jumps, corresponding to the centered Poisson measure, but also the "large" jumps, corresponding to the non-centered Poisson measure. It is worth noting that the impact of centered and non-centered Poisson noises to the stochastic non-autonomous logistic model and to the stochastic two-species mutualism model is studied in the papers [5] – [8].

In this paper we deal with the non-autonomous stochastic predator-prey model driven by the system of stochastic differential equations

$$\begin{aligned} dx_i(t) &= x_i(t)\left[(-1)^{i-1}\left(a_i(t) - \frac{c_i(t) x_{3-i}(t)}{m_1(t)x_1(t) + m_2(t)x_2(t) + m_3(t)}\right) - b_i(t)x_i(t)\right] dt + \sigma_i(t)x_i(t)dw_i(t) \\ &+ \int_{\mathbb{R}} \gamma_i(t,z)x_i(t-)\tilde{\nu}_1(dt,dz) + \int_{\mathbb{R}} \delta_i(t,z)x_i(t-)\nu_2(dt,dz),\ x_i(0) = x_{i0} > 0,\ i = 1,2, \end{aligned} \tag{3}$$

where $x_1(t)$ and $x_2(t)$ are the prey and predator population densities at time t, respectively, $w_i(t), i = 1,2$ are independent standard one-dimensional Wiener processes, $\nu_i(t,A), i = 1,2$ are independent Poisson measures, which are independent on $w_i(t), i = 1,2,$ $\tilde{\nu}_1(t,A) = \nu_1(t,A) - t\Pi_1(A),$ $E[\nu_i(t,A)] = t\Pi_i(A), i = 1,2,$ $\Pi_i(A), i = 1,2$ are a finite measures on the Borel sets A in $\mathbb{R}$.

In the following we will use the notations $X(t) = (x_1(t), x_2(t))$, $X_0 = (x_{10}, x_{20})$, $|X(t)| = \sqrt{x_1^2(t) + x_2^2(t)}$, $\mathbb{R}^2_+ = \{X \in \mathbb{R}^2 :\ x_1 > 0, x_2 > 0\}$,

$$\begin{aligned} \alpha_i(t) &= a_i(t) + \int_{\mathbb{R}} \delta_i(t,z)\Pi_2(dz), \\ \beta_i(t) &= \frac{\sigma_i^2(t)}{2} + \int_{\mathbb{R}} [\gamma_i(t,z) - \ln(1+\gamma_i(t,z))]\Pi_1(dz) - \int_{\mathbb{R}} \ln(1+\delta_i(t,z))]\Pi_2(dz), \end{aligned}$$

$i = 1,2$. For the bounded, continuous functions $f_i(t), t \in [0,+\infty), i = 1,2,$ let us denote

$$f_{i\sup} = \sup_{t\geq 0} f_i(t),\ \ f_{i\inf} = \inf_{t\geq 0} f_i(t),\ \ i = 1,2.$$

We prove that system (3) has a unique, positive, global (no explosion in a finite time) solution for any positive initial value and that this solution is stochastically ultimate bounded. The sufficient conditions for stochastic permanence, extinction, non-persistence in the mean, strong and weak persistence in the mean of solution are derived.

The rest of this paper is organized as follows. In Section 2, we prove the existence of the unique global positive solution of system (3) and derive some auxiliary results. In Section 3, we prove the stochastic ultimate boundedness of the solution of system (3), obtain conditions under which the solution is stochastically permanent. The sufficient conditions for extinction, non-persistence in the mean, strong and weak persistence in the mean of the solution are derived.

2. Existence of global solution and some auxiliary lemmas

Let $(\Omega,\mathcal{F},\mathrm{P})$ be a probability space, $w_i(t), i=1,2, t\geq 0$ are independent standard one-dimensional Wiener processes on $(\Omega,\mathcal{F},\mathrm{P})$, and $\nu_i(t,A), i=1,2$ are independent Poisson measures defined on $(\Omega,\mathcal{F},\mathrm{P})$ independent on $w_i(t), i=1,2$. Here $\mathrm{E}[\nu_i(t,A)]=t\Pi_i(A), i=1,2$, $\tilde{\nu}_i(t,A)=\nu_i(t,A)-t\Pi_i(A), i=1,2$, $\Pi_i(\cdot), i=1,2$ are finite measures on the Borel sets in $\mathbb{R}$. On the probability space $(\Omega,\mathcal{F},\mathrm{P})$ we consider an increasing, right continuous family of complete sub-σ-algebras $\{\mathcal{F}_t\}_{t\geq 0}$.

We need the following assumption.

Assumption 1
It is assumed, that $a_i(t), b_i(t), c_i(t), \sigma_i(t), \gamma_i(t,z), \delta_i(t,z), i=1,2, m_j(t), j=\overline{1,3}$ are bounded, continuous on t functions, $a_i(t)>0$, $b_{i\inf}>0$, $c_{i\inf}>0, i=1,2$, $\min\{m_{j\inf}, j=\overline{1,3}\}>0$, and $\ln(1+\gamma_i(t,z)), \ln(1+\delta_i(t,z)), i=1,2$ are bounded, $\Pi_i(\mathbb{R})<\infty, i=1,2$.

In what follows in this paper we will assume that Assumption 1 holds.

Theorem 1
There exists a unique global solution $X(t)$ of system (3) for any initial value $X(0)=X_0\in\mathbb{R}^2_+$, and $\mathrm{P}\{X(t)\in\mathbb{R}^2_+\}=1$, $\forall t\geq 0$.

Proof. Let us consider the system of stochastic differential equations

$$\begin{aligned} dv_i(t) &= \left[(-1)^{i-1}\left(a_i(t)-\frac{c_i(t)e^{v_{3-i}(t)}}{m_1(t)e^{v_1(t)}+m_2(t)e^{v_2(t)}+m_3(t)}\right)-b_i(t)e^{v_i(t)}-\beta_i(t)\right]dt \\ &+\sigma_i(t)dw_i(t)+\int_{\mathbb{R}}\ln(1+\gamma_i(t,z))\tilde{\nu}_1(dt,dz)+\int_{\mathbb{R}}\ln(1+\delta_i(t,z))\tilde{\nu}_2(dt,dz), \\ v_i(0) &= \ln x_{i0},\ \ i=1,2. \end{aligned} \quad (4)$$

The coefficients of system (4) are local Lipschitz continuous. So, for any initial value $(v_1(0), v_2(0))$ there exists a unique local solution $\Xi(t)=(v_1(t),v_2(t))$ on $[0,\tau_e)$, where $\sup_{t<\tau_e}|\Xi(t)|=+\infty$ (cf. Theorem 6, p.246, [9]). Therefore, from the Itô's formula we derive that the process $X(t)=(\exp\{v_1(t)\},\exp\{v_2(t)\})$ is a unique, positive local solution to system (3). To show this solution is global, we need to show that $\tau_e=+\infty$ a.s. Let $n_0\in\mathbb{N}$ be sufficiently large for $x_{i0}\in[1/n_0,n_0], i=1,2$. For any $n\geq n_0$ we define the stopping time

$$\tau_n=\inf\left\{t\in[0,\tau_e):\ X(t)\notin\left(\frac{1}{n},n\right)\times\left(\frac{1}{n},n\right)\right\}.$$

It is easy to see that τ_n is increasing as $n\to+\infty$. Denote $\tau_\infty=\lim_{n\to\infty}\tau_n$, whence $\tau_\infty\leq\tau_e$ a.s. If we prove that $\tau_\infty=\infty$ a.s., then $\tau_e=\infty$ a.s. and $X(t)\in\mathbb{R}^2_+$ a.s. for all $t\in[0,+\infty)$. So we need to show that $\tau_\infty=\infty$ a.s. If it is not true, there are constants $T>0$ and $\varepsilon\in(0,1)$, such that $\mathrm{P}\{\tau_\infty<T\}>\varepsilon$. Hence, there is $n_1\geq n_0$ such that

$$\mathrm{P}\{\tau_n<T\}>\varepsilon,\quad \forall n\geq n_1. \quad (5)$$

For the non-negative function $V(X)=\sum\limits_{i=1}^{2}k_i(x_i-1-\ln x_i),\ x_i>0,\ k_i>0,\ i=1,2$ by the Itô's formula we obtain

$$dV(X(t))=\sum_{i=1}^{2}k_i\Bigg\{(-1)^{i-1}(x_i(t)-1)\left[a_i(t)-\frac{c_i(t)x_{3-i}(t)}{m_1(t)x_1(t)+m_2(t)x_2(t)+m_3(t)}\right]$$
$$-b_i(t)x_i(t)(x_i(t)-1)+\beta_i(t)+\int\limits_{\mathbb{R}}\delta_i(t,z)x_i(t)\Pi_2(dz)\Bigg\}dt+\sum_{i=1}^{2}k_i\Bigg\{(x_i(t)-1)\sigma_i(t)dw_i(t) \tag{6}$$
$$+\int\limits_{\mathbb{R}}[\gamma_i(t,z)x_i(t-)-\ln(1+\gamma_i(t,z))]\tilde{\nu}_1(dt,dz)+\int\limits_{\mathbb{R}}[\delta_i(t,z)x_i(t-)-\ln(1+\delta_i(t,z))]\tilde{\nu}_2(dt,dz)\Bigg\}.$$

For the function

$$f(t,x_1,x_2)=-k_1b_1(t)x_1^2+k_1(\alpha_1(t)+b_1(t))x_1-k_2b_2(t)x_2^2+k_2\left(-a_2(t)+b_2(t)+\int_{\mathbb{R}}\delta_2(t,z)\Pi_2(dz)\right)x_2$$
$$+\frac{k_1c_1(t)x_2-k_1c_1(t)x_1x_2+k_2c_2(t)x_1x_2-k_2c_2(t)x_1}{m_1(t)x_1+m_2(t)x_2+m_3(t)}+k_1(\beta_1(t)-a_1(t))+k_2(\beta_2(t)+a_2(t)),$$

$x_i>0, i=1,2,$ we have

$$f(t,x_1,x_2)\le\sum_{i=1}^{2}k_i(-b_{i\inf}x_i^2+(\alpha_{i\sup}+b_{i\sup})x_i)+\frac{(k_2c_{2\sup}-k_1c_{1\inf})x_1x_2}{m_1(t)x_1+m_2(t)x_2+m_3(t)}$$
$$+\left[\frac{k_1c_{1\sup}}{m_{2\inf}}+k_1(\beta_{1\sup}-a_{1\inf})+k_2(\beta_{2\sup}+a_{2\sup})\right],x_i>0,i=1,2.$$

If we put $k_1=c_{2\sup}$, $k_2=c_{1\inf}$, then there is a constant $L=L(k_1,k_2)>0$, such that $f(t,x_1,x_2)\le L$. So from (6) we obtain by integrating

$$V(X(T\wedge\tau_n))\le V(X_0)+L(T\wedge\tau_n)+\sum_{i=1}^{2}k_i\Bigg\{\int\limits_0^{T\wedge\tau_n}(x_i(t)-1)\sigma_i(t)dw_i(t)$$
$$+\int\limits_0^{T\wedge\tau_n}\int\limits_{\mathbb{R}}[\gamma_i(t,z)x_i(t-)-\ln(1+\gamma_i(t,z))]\,\tilde{\nu}_1(dt,dz)+\int\limits_0^{T\wedge\tau_n}\int\limits_{\mathbb{R}}[\delta_i(t,z)x_i(t-)-\ln(1+\delta_i(t,z))]\tilde{\nu}_2(dt,dz)\Bigg\}. \tag{7}$$

Taking the expectation we derive from (7)

$$\mathrm{E}\left[V(X(T\wedge\tau_n))\right]\le V(X_0)+LT. \tag{8}$$

Set $\Omega_n=\{\tau_n\le T\}$ for $n\ge n_1$. Then by (5), $\mathrm{P}(\Omega_n)=\mathrm{P}\{\tau_n\le T\}>\varepsilon$, $\forall n\ge n_1$. Note that for every $\omega\in\Omega_n$ there is some $i=1,2$ such that $x_i(\tau_n,\omega)$ equals either n or $1/n$. So

$$V(X(\tau_n))\ge 2\min\{c_{1\inf},c_{2\sup}\}\min\{n-1-\ln n,\frac{1}{n}-1+\ln n\}.$$

It then follows from (8) that

$$V(X_0)+LT\ge\mathrm{E}[\mathbf{1}_{\Omega_n}V(X(\tau_n))]\ge 2\varepsilon\min\{c_{1\inf},c_{2\sup}\}\min\{n-1-\ln n,\frac{1}{n}-1+\ln n\},$$

where $\mathbf{1}_{\Omega_n}$ is the indicator function of Ω_n.

Letting $n\to\infty$ leads to the contradiction $\infty>V(X_0)+L(k_1,k_2)T=\infty$. This completes the proof of the theorem.

Lemma 1
The density of the population $x_i(t), i = 1, 2$ obeys

$$\limsup_{t\to\infty} \frac{\ln x_i(t)}{t} \le 0,\ i = 1, 2 \qquad \text{a.s.}$$

Proof. By the Itô's formula we have for $i = 1, 2$

$$e^t \ln x_i(t) - \ln x_{i0} = \int_0^t e^s \left\{ \ln x_i(s) + (-1)^{i-1} \left[a_i(s) - \frac{c_i(s)x_{3-i}(s)}{m_1(s)x_1(s) + m_2(s)x_2(s) + m_3(s)} \right] \right.$$
$$\left. -b_i(s)x_i(s) - \frac{\sigma_i^2(s)}{2} + \int_{\mathbb{R}} [\ln(1+\gamma_i(s,z)) - \gamma_i(s,z)]\Pi_1(dz) \right\} ds + \psi_i(t), \tag{9}$$

where

$$\psi_i(t) = \int_0^t e^s \sigma_i(s) dw_i(s) + \int_0^t\int_{\mathbb{R}} e^s \ln(1+\gamma_i(s,z))\tilde{\nu}_1(ds,dz) + \int_0^t\int_{\mathbb{R}} e^s \ln(1+\delta_i(s,z))\nu_2(ds,dz),\ i = 1, 2.$$

By virtue of the exponential inequality ([6], Lemma 2.2) we have

$$\mathrm{P}\left\{ \sup_{0\le t\le T} \zeta_i(\kappa, t) > \beta \right\} \le e^{-\kappa\beta},\ \forall 0 < \kappa \le 1,\ \beta > 0,\ i = 1, 2$$

where

$$\zeta_i(\kappa, t) = \psi_i(t) - \frac{\kappa}{2}\int_0^t e^{2s}\sigma_i^2(s)ds - \frac{1}{\kappa}\int_0^t\int_{\mathbb{R}} \left[(1+\gamma_i(s,z))^{\kappa e^s} - 1 - \kappa e^s \ln(1+\gamma_i(s,z)) \right] \Pi_1(dz)ds$$
$$-\frac{1}{\kappa}\int_0^t\int_{\mathbb{R}} \left[(1+\delta_i(s,z))^{\kappa e^s} - 1 \right] \Pi_2(dz)ds,\ i = 1, 2.$$

Choose $T = k\tau, k \in \mathbb{N}, \tau > 0, \kappa = e^{-k\tau}, \beta = \theta e^{k\tau}\ln k,\ \theta > 1$ we get

$$\mathrm{P}\left\{ \sup_{0\le t\le k\tau} \zeta_i(\kappa, t) > \theta e^{k\tau}\ln k \right\} \le \frac{1}{k^\theta},\ i = 1, 2.$$

By Borel-Cantelli lemma for almost all $\omega \in \Omega$, there is a random integer $k_0(\omega)$, such that $\forall k \ge k_0(\omega)$ and $0 \le t \le k\tau$

$$\psi_i(t) \le \frac{1}{2e^{k\tau}}\int_0^t e^{2s}\sigma_i^2(s)ds + e^{k\tau}\int_0^t\int_{\mathbb{R}} \left[(1+\gamma_i(s,z))^{e^{s-k\tau}} - 1 - e^{s-k\tau}\ln(1+\gamma_i(s,z)) \right] \Pi_1(dz)ds$$
$$+e^{k\tau}\int_0^t\int_{\mathbb{R}} \left[(1+\delta_i(s,z))^{e^{s-k\tau}} - 1 \right] \Pi_2(dz)ds + \theta e^{k\tau}\ln k,\ i = 1, 2. \tag{10}$$

By using the inequality $x^r \le 1 + r(x-1),\ \forall x \ge 0,\ 0 \le r \le 1$ for $x = 1 + \gamma_i(s,z),\ r = e^{s-k\tau}$, then for $x = 1 + \delta_i(s,z),\ r = e^{s-k\tau}$, we derive from (10) the estimates

$$\psi_i(t) \le \frac{1}{2e^{k\tau}}\int_0^t e^{2s}\sigma_i^2(s)ds + \int_0^t\int_{\mathbb{R}} e^s\left[\gamma_i(s,z) - \ln(1+\gamma_i(s,z))\right]\Pi_1(dz)ds + \int_0^t\int_{\mathbb{R}} e^s\delta_i(s,z)\Pi_2(dz)ds \quad (11)$$
$$+\theta e^{k\tau}\ln k,\ i=1,2.$$

So from (9) and (11) we get for $i=1,2$

$$e^t\ln x_i(t) \le \ln x_{i0} + \int_0^t e^s\left\{\ln x_i(s) + (-1)^{i-1}\left[a_i(s) - \frac{c_i(s)x_{3-i}(s)}{m_1(s)x_1(s)+m_2(s)x_2(s)+m_3(s)}\right]\right.$$
$$\left. -b_i(s)x_i(s) - \frac{\sigma_i^2(s)}{2}\left(1-e^{s-k\tau}\right) + \int_{\mathbb{R}}\delta_i(s,z)\Pi_2(dz)\right\}ds + \theta e^{k\tau}\ln k \le \ln x_{i0}$$
$$+\int_0^t e^s[\ln x_i(s) - b_{i\,\inf}x_i(s) + K_i]ds + \theta e^{k\tau}\ln k \le \ln x_{i0} + L(e^t-1) + \theta e^{k\tau}\ln k, \forall k \ge k_0(\omega), 0\le t\le k\tau,$$

for some constant $L>0$, where $K_1 = \alpha_{1\,\sup}, K_2 = \alpha_{2\,\sup} + c_{2\,\sup}/m_{1\,\inf}$.
So for any $(k-1)\tau \le t \le k\tau$, $\forall k \ge k_0(\omega)$ we have

$$\frac{\ln x_i(t)}{\ln t} \le e^{-t}\frac{\ln x_{i0}}{\ln t} + \frac{L}{\ln t}(1-e^{-t}) + \frac{\theta e^{k\tau}\ln k}{e^{(k-1)\tau}\ln(k-1)\tau}, i=1,2 \quad \text{a.s.}$$

Therefore

$$\limsup_{t\to\infty}\frac{\ln x_i(t)}{\ln t} \le \theta e^{\tau},\ i=1,2,\ \forall\theta>1,\ \forall\tau>0, \quad \text{a.s.}$$

If $\theta\downarrow 1$, $\tau\downarrow 0$, then we obtain

$$\limsup_{t\to\infty}\frac{\ln x_i(t)}{\ln t} \le 1,\ i=1,2 \quad \text{a.s.}$$

So

$$\limsup_{t\to\infty}\frac{\ln x_i(t)}{t} \le 0,\ i=1,2 \quad \text{a.s.}$$

Lemma 2
Let $p>0$.Then for any initial value $x_{i0}>0, i=1,2$ we have

$$\limsup_{t\to\infty}\mathrm{E}\left[x_i^p(t)\right] \le K_i(p),\ i=1,2,$$

where $K_i(p)>0, i=1,2$ are some constants depending on p.

Proof. Let τ_n be the stopping time defined in Theorem 1. Applying the Itô's formula to the process $V(t, x_i(t)) = e^t x_i^p(t),\ i = 1, 2, p > 0$, we obtain for $i = 1, 2$

$$V(t\wedge\tau_n, x_i(t\wedge\tau_n)) = x_{i0}^p + \int_0^{t\wedge\tau_n} e^s x_i^p(s)\Bigg\{1 + p\Bigg[(-1)^{i-1}\left(a_i(s) - \frac{c_i(s)x_{3-i}(s)}{m_1(s)x_1(s)+m_2(s)x_2(s)+m_3(s)}\right) - b_i(s)x_i(s)\Bigg] + \frac{p(p-1)\sigma_i^2(s)}{2} + \int_{\mathbb{R}}[(1+\gamma_i(s,z))^p - 1 - p\gamma_i(s,z)]\,\Pi_1(dz)$$
$$+ \int_{\mathbb{R}}[(1+\delta_i(s,z))^p - 1]\,\Pi_2(dz)\Bigg\}ds + \int_0^{t\wedge\tau_n} pe^s x_i^p(s)\sigma_i(s)dw_i(s) + \int_0^{t\wedge\tau_n}\int_{\mathbb{R}} e^s x_i^p(s-)\,[(1+\gamma_i(s,z))^p - 1]\,\tilde{\nu}_1(ds,dz)$$
$$+ \int_0^{t\wedge\tau_n}\int_{\mathbb{R}} e^s x_i^p(s-)\,[(1+\delta_i(s,z))^p - 1]\,\tilde{\nu}_2(ds,dz). \tag{12}$$

Under Assumption 1 there are a constants $K_i(p) > 0, i = 1, 2$, such that

$$e^s x_i^p(s)\Bigg\{1 + p\Bigg[(-1)^{i-1}\left(a_i(s) - \frac{c_i(s)x_{3-i}(s)}{m_1(s)x_1(s)+m_2(s)x_2(s)+m_3(s)}\right) - b_i(s)x_i(s)\Bigg] + \frac{p(p-1)\sigma_i^2(s)}{2}$$
$$+ \int_{\mathbb{R}}[(1+\gamma_i(s,z))^p - 1 - p\gamma_i(s,z)]\,\Pi_1(dz) + \int_{\mathbb{R}}[(1+\delta_i(s,z))^p - 1]\,\Pi_2(dz)\Bigg\} \le e^s K_i(p) \tag{13}$$

From (12) and (13), taking the expectation, we obtain

$$\mathrm{E}\,[V(t\wedge\tau_n, x_i(t\wedge\tau_n))] \le x_{i0}^p + K_i(p)e^t, i = 1, 2.$$

If $n \to \infty$, then we get

$$e^t\mathrm{E}\,[x_i^p(t)] \le x_{i0}^p + K_i(p)e^t,\ \ i = 1, 2.$$

Hence $\limsup_{t\to\infty}\mathrm{E}\,[x_i^p(t)] \le K_i(p),\ \ i = 1, 2.$

Lemma 3

Under condition $p_{i\,\inf} > 0, i = 1, 2,$ where $p_1(t) = a_1(t) - c_1(t)/m_2(t) - \beta_1(t),\ p_2(t) = -a_2(t) - \beta_2(t),$ we have

$$\limsup_{t\to\infty}\mathrm{E}\left[\left(\frac{1}{x_i(t)}\right)^\theta\right] \le L_i(\theta),\ \ 0 < \theta < 1,\ \ i = 1, 2,$$

where $L_i(\theta) > 0, i = 1, 2$ are some constants depending on θ.

Proof. For the processes $U_i(t) = 1/x_i(t),\ i = 1, 2$ by the Itô's formula we derive

$$U_i(t) = U_i(0) + \int_0^t U_i(s)\Bigg[(-1)^i\left(a_i(s) - \frac{c_i(s)x_{3-i}(s)}{m_1(s)x_1(s)+m_2(s)x_2(s)+m_3(s)}\right)$$
$$+ b_i(s)x_i(s) + \sigma_2^2(s) + \int_{\mathbb{R}}\frac{\gamma_i^2(s,z)}{1+\gamma_i(s,z)}\Pi_1(dz)\Bigg]ds - \int_0^t U_i(s)\sigma_i(s)dw_i(s)$$
$$- \int_0^t\int_{\mathbb{R}} U_i(s-)\frac{\gamma_i(s,z)}{1+\gamma_i(s,z)}\tilde{\nu}_1(ds,dz) - \int_0^t\int_{\mathbb{R}} U_i(s-)\frac{\delta_i(s,z)}{1+\delta_i(s,z)}\nu_2(ds,dz).$$

Then, applying again the Itô's formula we get for $0<\theta<1, i=1,2$

$$
\begin{aligned}
&(1+U_i(t))^\theta \le (1+U_i(0))^\theta + \int_0^t \theta(1+U_i(s))^{\theta-2}\Bigg\{(1+U_i(s))U_i(s) \\
&\times\Bigg[(-1)^i\left(a_i(s)-\frac{c_i(s)x_{3-i}(s)}{m_1(s)x_1(s)+m_2(s)x_2(s)+m_3(s)}\right)+b_i(s)x_i(s)+\sigma_i^2(s)+\int_{\mathbb{R}}\frac{\gamma_i^2(s,z)}{1+\gamma_i(s,z)}\Pi_1(dz)\Bigg] \\
&+\frac{\theta-1}{2}U_i^2(s)\sigma_i^2(s)+\frac{1}{\theta}\int_{\mathbb{R}}\Bigg[(1+U_i(s))^2\left(\left(\frac{1}{1+\gamma_i(s,z)}+\frac{1}{1+U_i(s)}\right)^\theta-1\right) \\
&+\theta(1+U_i(s))\frac{U_i(s)\gamma_i(s,z)}{1+\gamma_i(s,z)}\Bigg]\Pi_1(dz)+\frac{1}{\theta}\int_{\mathbb{R}}(1+U_i(s))^2\left[\left(\frac{1}{1+\delta_i(s,z)}+\frac{1}{1+U_i(s)}\right)^\theta-1\right]\Pi_2(dz)\Bigg\}ds \qquad (14) \\
&-\int_0^t \theta(1+U_i(s))^{\theta-1}U_i(s)\sigma_i(s)dw_i(s)+\int_0^t\int_{\mathbb{R}}\left[\left(1+\frac{U_i(s-)}{1+\gamma_i(s,z)}\right)^\theta-(1+U_i(s-))^\theta\right]\tilde\nu_1(ds,dz) \\
&+\int_0^t\int_{\mathbb{R}}\left[\left(1+\frac{U_i(s-)}{1+\delta_i(s,z)}\right)^\theta-(1+U_i(s-))^\theta\right]\tilde\nu_2(ds,dz)=(1+U_i(0))^\theta+\int_0^t\theta(1+U_i(s))^{\theta-2}J_i(s)ds \\
&-I_{1,stoch}(t)+I_{2,stoch}(t)+I_{3,stoch}(t),
\end{aligned}
$$

where $I_{j,stoch}(t), j=\overline{1,3}$ are corresponding stochastic integrals in (14). Under the Assumption 1 there exists constants $|K_1(\theta)|<\infty$, $|K_2(\theta)|<\infty$ such, that for the functions $J_i(t), i=1,2$ we have the estimate

$$
\begin{aligned}
J_i(t) \le -U_i^2(t)\Bigg\{&\tilde a_i(t)-\frac{\sigma_i^2(t)}{2}-\int_{\mathbb{R}}\gamma_i(t,z)\Pi_1(dz)-\theta\frac{\sigma_i^2(t)}{2}-\frac{1}{\theta}\int_{\mathbb{R}}\left[(1+\gamma_i(t,z))^{-\theta}-1\right]\Pi_1(dz) \\
&-\frac{1}{\theta}\int_{\mathbb{R}}\left[(1+\delta_i(t,z))^{-\theta}-1\right]\Pi_2(dz)\Bigg\}+U_i(t)K_{i1}(\theta)+K_{i2}(\theta)=-U_i^2(t)K_{i0}(t,\theta)+U_i(t)K_{i1}(\theta)+K_{i2}(\theta),
\end{aligned}
$$

where $\tilde a_1(t)=a_1(t)-c_1(t)/m_2(t)$, $\tilde a_2(t)=-a_2(t)$. Here we use the inequality $(x+y)^\theta\le x^\theta+\theta x^{\theta-1}y$, $0<\theta<1$, $x,y>0$.

Due to

$$
\begin{aligned}
\lim_{\theta\to0+}\Bigg[&\frac{\theta}{2}\sigma_i^2(t)+\frac{1}{\theta}\int_{\mathbb{R}}[(1+\gamma_i(t,z))^{-\theta}-1]\Pi_1(dz)+\frac{1}{\theta}\int_{\mathbb{R}}[(1+\delta_i(t,z))^{-\theta}-1]\Pi_2(dz) \\
&+\int_{\mathbb{R}}\ln(1+\gamma_i(t,z))\Pi_1(dz)+\int_{\mathbb{R}}\ln(1+\delta_i(t,z))\Pi_2(dz)\Bigg]=\lim_{\theta\to0+}\Delta_i(t,\theta)=0,
\end{aligned}
$$

and condition $p_{i\inf}>0, i=1,2$ we can choose a sufficiently small $0<\theta<1$ to satisfy

$$K_{i0}(\theta)=\inf_{t\ge0}K_{i0}(t,\theta)=\inf_{t\ge0}[p_i(t)-\Delta_i(t,\theta)]>0,\quad i=1,2.$$

Therefore from (14) and estimate for $J_i(t), i = 1, 2$ we get

$$
\begin{aligned}
&d\left[(1+U_i(t))^\theta\right] \le \theta(1+U_i(t))^{\theta-2}[-U_i^2(t)K_{i0}(\theta)+U_i(t)K_{i1}(\theta)+K_{i2}(\theta)]dt \\
&-\theta(1+U_i(t))^{\theta-1}U_i(t)\sigma_i(t)dw_i(t)+\int_{\mathbb{R}}\left[\left(1+\frac{U_i(t-)}{1+\gamma_i(t,z)}\right)^\theta-(1+U_i(t-))^\theta\right]\tilde{\nu}_1(dt,dz) \\
&+\int_{\mathbb{R}}\left[\left(1+\frac{U_i(t-)}{1+\delta_i(t,z)}\right)^\theta-(1+U_i(t-))^\theta\right]\tilde{\nu}_2(dt,dz),\ i=1,2.
\end{aligned}
\tag{15}
$$

By the Itô's formula and (15) we have

$$
\begin{aligned}
&d\left[e^{\lambda t}(1+U_i(t))^\theta\right] = \lambda e^{\lambda t}(1+U_i(t))^\theta dt + e^{\lambda t}d\left[(1+U_i(t))^\theta\right] \\
&\le e^{\lambda t}\theta(1+U_i(t))^{\theta-2}\left[-U_i^2(t)\left(K_{i0}(\theta)-\frac{\lambda}{\theta}\right)+U_i(t)\left(K_{i1}(\theta)+\frac{2\lambda}{\theta}\right)+K_{i2}(\theta)+\frac{\lambda}{\theta}\right]dt \\
&-\theta e^{\lambda t}(1+U_i(t))^{\theta-1}U_i(t)\sigma_i(t)dw_i(t)+e^{\lambda t}\int_{\mathbb{R}}\left[\left(1+\frac{U_i(t-)}{1+\gamma_i(t,z)}\right)^\theta-(1+U_i(t-))^\theta\right]\tilde{\nu}_1(dt,dz) \\
&+e^{\lambda t}\int_{\mathbb{R}}\left[\left(1+\frac{U_i(t-)}{1+\delta_i(t,z)}\right)^\theta-(1+U_i(t-))^\theta\right]\tilde{\nu}_2(dt,dz),\ i=1,2.
\end{aligned}
\tag{16}
$$

Let us choose $\lambda = \lambda(\theta) > 0$ such that $K_{i0}(\theta) - \lambda/\theta > 0$, $i = 1, 2$. Then we have

$$
(1+U_i(t))^{\theta-2}\left[-U_i^2(t)\left(K_{i0}(\theta)-\frac{\lambda}{\theta}\right)+U_i(t)\left(K_{i1}(\theta)+\frac{2\lambda}{\theta}\right)+K_{i2}(\theta)+\frac{\lambda}{\theta}\right]\le K, i=1,2 \tag{17}
$$

for some constant $K > 0$. Let τ_n be the stopping time defined in Theorem 1. Then by integrating (16), applying (17) and taking the expectation, we obtain

$$
\mathrm{E}\left[e^{\lambda(t\wedge\tau_n)}(1+U_i(t\wedge\tau_n))^\theta\right]\le\left(1+\frac{1}{x_{i0}}\right)^\theta+\frac{\theta}{\lambda}K\left(e^{\lambda t}-1\right), i=1,2.
$$

Letting $n \to \infty$ leads to the estimate

$$
e^{\lambda t}\mathrm{E}\left[(1+U_i(t))^\theta\right]\le\left(1+\frac{1}{x_{i0}}\right)^\theta+\frac{\theta}{\lambda}K\left(e^{\lambda t}-1\right), i=1,2. \tag{18}
$$

From (18) we derive

$$
\limsup_{t\to\infty}\mathrm{E}\left[\left(\frac{1}{x_i(t)}\right)^\theta\right]=\limsup_{t\to\infty}\mathrm{E}\left[U_i^\theta(t)\right]\le\limsup_{t\to\infty}\mathrm{E}\left[(1+U_i(t))^\theta\right]\le\frac{\theta}{\lambda(\theta)}K, i=1,2.
$$

This completes the proof of lemma.

3. The long time behavior

We need the following definitions.

Definition 1 (see [10])
The solution $X(t)$ to the system (3) is said to be stochastically ultimately bounded if for any $\varepsilon \in (0, 1)$, there is a positive constant $\chi = \chi(\varepsilon) > 0$, such that for any initial value $X_0 \in \mathbb{R}^2_+$, the solution of system (3) has the property that

$$
\limsup_{t\to\infty}\mathrm{P}\{|X(t)|>\chi\}<\varepsilon.
$$

Definition 2 (see [11])
The solution $X(t) = (x_1(t), x_2(t))$ to the system (3) is said to be stochastically permanent if for any $\varepsilon > 0$, there are positive constants $H = H(\varepsilon)$, $h = h(\varepsilon)$ such that for $i = 1, 2$

$$\liminf_{t\to\infty} \mathrm{P}\{x_i(t) \le H\} \ge 1 - \varepsilon, \quad \liminf_{t\to\infty} \mathrm{P}\{x_i(t) \ge h\} \ge 1 - \varepsilon,$$

for any inial value $X_0 \in \mathbb{R}^2_+$.

Theorem 2
The solution $X(t)$ to the system (3) is stochastically ultimately bounded for any initial value $X_0 \in \mathbb{R}^2_+$.

The proof is a simple application of the Chebyshev's inequality and Lemma 2.

Theorem 3
Under conditions of Lemma 3 the solution $X(t)$ of system (3) is stochastically permanent for any initial value $X_0 \in \mathbb{R}^2_+$.

The proof follows from the Chebyshev's inequality, Lemma 2 and Lemma 3.

Remark 1
The presence of non-centered Poisson noise in the model (3) is crucial for the predator population, because if $\delta_2(t, z) \equiv 0$, then $p_2(t) = -a_2(t) - \beta_2(t) < 0$, $\forall t \in [0, \infty)$.

Theorem 4
If for $i = 1, 2$ we have

$$\bar{q}_i^* = \limsup_{t\to\infty} \frac{1}{t} \int_0^t q_i(s)ds < 0, \text{ where } q_1(t) = a_1(t) - \beta_1(t),\ q_2(t) = -a_2(t) + \frac{c_2(t)}{m_1(t)} - \beta_2(t),$$

then solution $X(t) = (x_1(t), x_2(t))$ of system (3) with the initial value $X_0 \in \mathbb{R}^2_+$ will be extinct: $\lim_{t\to\infty} x_i(t) = 0$, $i = 1, 2$ almost surely (a.s.)

Proof. By the Itô's formula we obtain

$$\begin{aligned} \ln x_i(t) = \ln x_{i0} + \int_0^t \left\{ (-1)^{i-1} \left[a_i(s) - \frac{c_i(s)x_{3-i}(s)}{m_1(s)x_1(s) + m_2(s)x_2(s) + m_3(s)} \right] - \beta_i(s) - b_i(s)x_i(s) \right\} ds \\ + M_i(t) \le \ln x_{i0} + \int_0^t q_i(s)ds + M_i(t), \ i = 1, 2 \end{aligned} \tag{19}$$

where the martingales

$$M_i(t) = \int_0^t \sigma_i(s)dw_i(s) + \int_0^t\int_{\mathbb{R}} \ln(1 + \gamma_i(s, z))\tilde{\nu}_1(ds, dz) + \int_0^t\int_{\mathbb{R}} \ln(1 + \delta_i(s, z))\tilde{\nu}_2(ds, dz), \ i = 1, 2 \tag{20}$$

has quadratic characteristics (Meyer's angle bracket process)

$$\langle M_i, M_i\rangle(t) = \int_0^t \sigma_i^2(s)ds + \int_0^t\int_{\mathbb{R}} \ln^2(1 + \gamma_i(s, z))\Pi_1(dz)ds + \int_0^t\int_{\mathbb{R}} \ln^2(1 + \delta_i(s, z))\Pi_2(dz)ds \le Kt, \ i = 1, 2$$

for some constant $K > 0$. Then the strong law of large numbers for local martingales ([12]) yields $\lim_{t\to\infty} M_i(t)/t = 0, i = 1, 2$ a.s. Therefore, from (19) we derive

$$\limsup_{t\to\infty} \frac{\ln x_i(t)}{t} \le \limsup_{t\to\infty} \frac{1}{t} \int_0^t q_i(s)ds = \bar{q}_i^* < 0, i = 1, 2 \quad \text{a.s.}$$

So $\lim_{t\to\infty} x_i(t) = 0, i = 1, 2$ a.s.

Theorem 5
If $\bar{q}_i^* = 0$, $i = 1, 2$, then the solution $X(t) = (x_1(t), x_2(t))$ of system (3) with the initial value $X_0 \in \mathbb{R}_+^2$ will be non-persistent in the mean:

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t x_i(s)ds = 0, \ i = 1, 2 \quad \text{a.s.}$$

Proof. From the first equality in (19) we derive

$$\ln x_i(t) \le \ln x_{i0} + \int_0^t q_i(s)ds - b_{i\,\inf} \int_0^t x_i(s)ds + M_i(t), \ i = 1, 2, \tag{21}$$

where martingales $M_i(t), i = 1, 2$ are defined in (20). From the definition of $\bar{q}_i^*, i = 1, 2$ and the strong law of large numbers for $M_i(t), i = 1, 2$ it follows, that $\forall \varepsilon > 0$, $\exists t_0 \ge 0$, and $\exists \Omega_\varepsilon \subset \Omega$, $\mathrm{P}(\Omega_\varepsilon) \ge 1 - \varepsilon$ such that

$$\frac{1}{t} \int_0^t q_i(s)ds \le \bar{q}_i^* + \frac{\varepsilon}{2}, \ \frac{M(t)}{t} \le \frac{\varepsilon}{2}, \ i = 1, 2, \ \forall t \ge t_0, \ \omega \in \Omega_\varepsilon.$$

Hence, from (21) we derive for $i = 1, 2$

$$\ln x_i(t) - \ln x_{i0} \le t(\bar{q}_i^* + \varepsilon) - b_{i\,\inf} \int_0^t x_i(s)ds = t\varepsilon - b_{i\,\inf} \int_0^t x_i(s)ds, \ \forall t \ge t_0, \ \omega \in \Omega_\varepsilon. \tag{22}$$

Let $y_i(t) = \int_0^t x_i(s)ds$, $i = 1, 2$ then from (22) we have

$$\ln\left(\frac{dy_i(t)}{dt}\right) \le \varepsilon t - b_{i\,\inf}\ y_i(t) + \ln x_{i0}, \ i = 1, 2, \ \forall t \ge t_0, \ \omega \in \Omega_\varepsilon.$$

So

$$\exp\{b_{i\,\inf}\ y_i(t)\} \frac{dy_i(t)}{dt} \le x_{i0} \exp\{\varepsilon t\}, \ i = 1, 2, \ \forall t \ge t_0, \ \omega \in \Omega_\varepsilon.$$

By integrating last inequality from t_0 to t we obtain

$$\exp\{b_{i\,\inf}\ y_i(t)\} \le \frac{b_{i\,\inf} x_{i0}}{\varepsilon} \left(\exp\{\varepsilon t\} - \exp\{\varepsilon t_0\}\right) + \exp\{b_{i\,\inf}\ y_i(t_0)\}, \ i = 1, 2, \ \forall t \ge t_0, \ \omega \in \Omega_\varepsilon.$$

Therefore

$$y_i(t) \le \frac{1}{b_{i\,\inf}} \ln\left[\exp\{b_{i\,\inf}\ y_i(t_0)\} + \frac{b_{i\,\inf} x_{i0}}{\varepsilon} \left(\exp\{\varepsilon t\} - \exp\{\varepsilon t_0\}\right)\right], \ i = 1, 2, \ \forall t \ge t_0, \ \omega \in \Omega_\varepsilon.$$

So

$$\limsup_{t\to\infty} \frac{1}{t} \int_0^t x_i(s)ds \le \frac{\varepsilon}{b_{i\,\inf}}, \ i = 1, 2, \ \omega \in \Omega_\varepsilon.$$

Since $\varepsilon > 0$ is arbitrary and $X(t) \in \mathbb{R}_+^2$ almost surely, we have

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t x_i(s)ds = 0, \ i = 1, 2, \qquad \text{a.s.}$$

Theorem 6
If for $i=1,2$ we have

$$\bar{p}_i^* = \limsup_{t\to\infty} \frac{1}{t}\int\limits_0^t p_i(s)ds > 0, \text{ where } p_1(t) = a_1(t) - \frac{c_1(t)}{m_2(t)} - \beta_1(t), p_2(t) = -a_2(t) - \beta_2(t), \quad (23)$$

then the solution $X(t) = (x_1(t), x_2(t))$ of system (3) with initial value $X_0 \in \mathbb{R}^2_+$ will be weakly persistent in the mean:

$$\bar{x}_i^* = \limsup_{t\to\infty} \frac{1}{t}\int\limits_0^t x_i(s)ds > 0, \ i=1,2, \text{ a.s.}$$

Proof. If the theorem statement is not true, then $\mathrm{P}\{\bar{x}_i^* = 0\} > 0$, $i=1,2$. From the first equality in (19) we derive

$$\frac{\ln x_i(t)}{t} \geq \frac{\ln x_{i0}}{t} + \frac{1}{t}\int\limits_0^t p_i(s)ds - \frac{b_{i\sup}}{t}\int\limits_0^t x_i(s)ds + \frac{M_i(t)}{t}, \ i=1,2, \quad (24)$$

where martingales $M_i(t), i=1,2$ are defined in (20). For $\forall\omega \in \{\omega\in\Omega: \ \bar{x}_i^* = 0\}$ due to the strong law of large numbers for martingales $M_i(t), i=1,2$ we obtain

$$\limsup_{t\to\infty} \frac{\ln x_i(t)}{t} \geq \bar{p}_i^* > 0, \ i=1,2.$$

Therefore

$$\mathrm{P}\left\{\omega\in\Omega: \ \limsup_{t\to\infty} \frac{\ln x_i(t)}{t} > 0\right\} > 0, \ i=1,2.$$

But from Lemma 1 we have

$$\mathrm{P}\left\{\omega\in\Omega: \ \limsup_{t\to\infty} \frac{\ln x_i(t)}{t} \leq 0\right\} = 1, \ i=1,2.$$

This is a contradiction.

Theorem 7
If $\bar{p}_{i*} = \liminf_{t\to\infty} \frac{1}{t}\int_0^t p_i(s)ds > 0$, $i=1,2$, where $p_i(t), i=1,2$ are defined in (23), then

$$\bar{x}_{i*} = \liminf_{t\to\infty} \frac{1}{t}\int_0^t x_i(s)ds \geq \frac{\bar{p}_{i*}}{b_{i\inf}}, \ i=1,2, \quad \text{a.s.}$$

Therefore the solution $X(t) = (x_1(t), x_2(t))$ of system (3) with initial value $X_0 \in \mathbb{R}^2_+$ will be strongly persistent in the mean: $\bar{x}_{i*} > 0$ almost surely, $i=1,2$.

Proof. From (24) we get

$$\ln x_i(t) \geq \ln x_{i0} + \int\limits_0^t p_i(s)ds - b_{i\sup}\int\limits_0^t x_i(s)ds + M_i(t), \ i=1,2, \quad (25)$$

where martingales $M_i(t), i=1,2$ are defined in (20). From the definition of $\bar{p}_{i*}, i=1,2$ and the strong law of large numbers for $M_i(t), i=1,2$ it follows, that $\forall\varepsilon>0$, $\exists t_0 \geq 0$, and $\exists\Omega_\varepsilon \subset \Omega$, such that $\mathrm{P}(\Omega_\varepsilon) \geq 1-\varepsilon$,

$$\frac{1}{t}\int\limits_0^t p_i(s)ds \geq \bar{p}_{i*} - \frac{\varepsilon}{2}, \ \frac{M(t)}{t} \geq -\frac{\varepsilon}{2}, \ i=1,2, \ \forall t\geq t_0, \ \omega\in\Omega_\varepsilon.$$

Hence, from (25) we have

$$\ln x_i(t) \geq \ln x_{i0} + t(\bar{p}_{i*} - \varepsilon) - b_{i\,\sup} \int_0^t x_i(s)ds, \; i = 1, 2, \; \forall t \geq t_0, \; \omega \in \Omega_\varepsilon.$$

Applying the same arguments for the processes $y_i(t) = \int_0^t x_i(s)ds$, $i = 1, 2$ as in the proof of Theorem 5, we derive the estimate

$$y_i(t) \geq \frac{1}{b_{i\,\sup}} \ln \left[e^{b_{i\,\sup}\; y_i(t_0)} + \frac{b_{i\,\sup} x_{i0}}{\bar{p}_{i*} - \varepsilon} \left(e^{(\bar{p}_{i*}-\varepsilon)t} - e^{(\bar{p}_{i*}-\varepsilon)t_0} \right) \right], \; i = 1, 2, \; \forall t \geq t_0, \; \omega \in \Omega_\varepsilon.$$

So

$$\liminf_{t\to\infty} \frac{1}{t} \int_0^t x_i(s)ds \geq \frac{\bar{p}_{i*} - \varepsilon}{b_{i\,\sup}}, \; i = 1, 2, \; \omega \in \Omega_\varepsilon.$$

Using the arbitrariness of $\varepsilon > 0$, we complete the proof of the theorem.

Remark 2
If in the model (3) we have

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t \frac{c_2(s)}{m_1(s)} ds = 0,$$

then $\bar{p}_2^* = \bar{q}_2^*$ and under the absence of non-centered Poisson noise in the predator model $\delta_2(t, z) \equiv 0$, we obtain $\bar{p}_2^* = \bar{q}_2^* \leq 0$. Therefore by Theorems 4 and 5 predator population will be extinct or will be non-persistent in the mean. But for sufficiently large $\delta_2(t, z) > 0$ we have $\bar{p}_{2*} > 0$ and by Theorem 7 the predator population will be strongly persistent in the mean. So, the presence of non-centered Poisson noise in the model (3) is crucial for the persistence in the mean of predator population.

4. Conclusions

In this paper, we consider the non-autonomous stochastic predator-prey model with Beddington-DeAngelis functional response driven by the system of stochastic differential equations with white noise, centered and non-centered Poisson noises. So, we take into account not only "small" jumps, corresponding to the centered Poisson measure, but also the "large" jumps, corresponding to the non-centered Poisson measure. For the considered system, sufficient criteria for the existence and uniqueness of a global positive solution are obtained. We derive sufficient conditions of stochastic ultimate boundedness, stochastic permanence, non-persistence in the mean, weak and strong persistence in the mean and extinction of the population densities in the considered stochastic predator-prey model. The results strictly generalize the existing results, so it is meaningful and important.

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