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The Odd Log-Logistic Transmuted-G family of distributions: Properties, Characterizations, Applications and Different Methods of Estimation

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Abstract In this work, we propose a new class of lifetime distributions called the odd log-logistic transmuted-G family. The proposed family of distributions is constructed by compounding the odd log-logistic distribution with the transmuted distribution. It can provide better fits than some of the known lifetime models and this fact represents a good characterization of this new family. Some characterizations for the new family are presented as well as some of its mathematical properties including. The maximum likelihood, Least squares and weighted least squares, Cramér−von−Mises, Anderson-Darling and right-tailed Anderson-Darlingare and maximum product of spacings methods are used for estimating the model parameters. The importance and flexibility of the new family are illustrated by means of an application to a real data set.

Keywords Odd log-logistic-G family; Transmuted-G family; Estimation; Characterization

AMS 2010 subject classifications 60E05, 62E15.

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1. Introduction

The statistical literature contains a good number of new families of distributions that extend classical distributions which are very important for statisticians due to their flexible properties. These new families have been extensively used in modelling data in several applied areas such as reliability, engineering and life testing. In recent years there has been an increased interest in developing more flexible generators for univariate continuous distributions by adding extra shape parameter(s) to the baseline distribution. Recently, many extensions of odd log-logistic-G families have been developed by Cordeiro et al. (2016a,b,c), Alizadeh et al. (2015) and Haghbin et al. (2017), among others. On the other hand many transmuted-G extensions have been developed by Yousof et al. (2015), Merovc et al. (2016), Alizadeh et al. (2018), Afify et al. (2016a,2016b,2017), Nofal et al. (2017) and Yousof et al. (2017), among others.

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Let $G(x;\psi)=G(x)$ be a baseline cumulative distribution function (cdf) and $g(x;\psi)=g(x)$ be the associated probability density function (pdf), where $\psi=(\psi_1,\psi_2,\ldots)$ is a parameter vector. Then, the cdf and pdf of the transmuted-G (T-G) family of distributions are, respectively, given by

$$F_{T-G}(x;\lambda,\psi)=G(x;\psi)\left[1+\lambda-\lambda G(x;\psi)\right], \tag{1}$$

and

$$f_{T-G}(x;\lambda,\psi)=g(x;\psi)\left[1+\lambda-2\lambda G(x;\psi)\right]. \tag{2}$$

where $|\lambda|\leq 1$. It is noted that the T-G family is a mixture of the baseline and exponentiated-G (exp-G) distributions, the last one with power parameter equal to two. Further, we obtain the baseline distribution when $\lambda=0$. For more details about the T-G family, see Shaw and Buckley (2007). The odd log-logistic (OLL) family of distribution, originally developed by Gleaton and Lynch (2004) and (2006). They called this family as generalized log-logistic (GLL) family. The cdf of this family is given by

$$F_{OLL-G}(x;\alpha,\psi)=G(x,\psi)^{\alpha}\left[G(x,\psi)^{\alpha}+\overline{G}(x,\psi)^{\alpha}\right]^{-1}, \tag{3}$$

and the pdf given by

$$f_{OLL-G}(x;\alpha,\psi)=\alpha g(x;\psi)\left[G(x;\psi)\overline{G}(x;\psi)\right]^{\alpha-1}\left[G(x;\psi)^{\alpha}+\overline{G}(x;\psi)^{\alpha}\right]^{-2}, \tag{4}$$

where $\alpha>0$ is a shap parameter. In this paper, we propose and study a new flexible extension of the T-G family by adding one parameter in equation of $H_{T-G}(x;\lambda,\psi)$ to provide more flexibility to the generated family. We construct a new generator called the odd log-logistic transmuted-G (OLLT-G) family by taking the T-G cdf as the baseline cdf H in the last two equations. Further, we give a comprehensive description of the mathematical properties of the new family. In fact, the OLLT-G family is motivated by its important flexibility in applications. By means of an application, we show that the OLLT-G class provides better fits than at least five other families.

In this work, we generalize the T-G family by incorporating one additional parameter to yield a more flexible generator. The OLLT-G family is given by the cdf (for $x>0$)

$$F(x)=\frac{\left\{G(x;\psi)\left[1+\lambda\overline{G}(x;\psi)\right]\right\}^{\alpha}}{\left\{G(x;\psi)\left[1+\lambda\overline{G}(x;\psi)\right]\right\}^{\alpha}+\left\{\overline{G}(x;\psi)\left[1-\lambda G(x;\psi)\right]\right\}^{\alpha}},\quad x\in\mathbb{R}. \tag{5}$$

The pdf corresponding to (5) is

$$f(x)=\frac{\alpha\,g(x;\psi)\left[1+\lambda-2\lambda\,G(x;\psi)\right]\left\{G(x;\psi)\left[1+\lambda\,\bar{G}(x;\psi)\right]\right\}^{\alpha-1}\left\{1-G(x;\psi)\left[1+\lambda\,\bar{G}(x;\psi)\right]\right\}^{\alpha-1}}{\left\{\left\{G(x;\psi)\left[1+\lambda\,\bar{G}(x;\psi)\right]\right\}^{\alpha}+\left\{1-G(x;\psi)\left[1+\lambda\,\bar{G}(x;\psi)\right]\right\}^{\alpha}\right\}^{2}}. \tag{6}$$

Henceforth, we denote by $X\sim$OLLT-G (α,λ,ψ) a random variable having pdf (6). The hazard rate function (hrf) of X given by

$$\tau(x)=\frac{\alpha\,g(x;\psi)\left[1+\lambda-2\lambda\,G(x;\psi)\right]\left\{G(x;\psi)\left[1+\lambda\,\bar{G}(x;\psi)\right]\right\}^{\alpha-1}}{\left\{1-G(x;\psi)\left[1+\lambda\,\bar{G}(x;\psi)\right]\right\}\left\{\left\{G(x;\psi)\left[1+\lambda\,\bar{G}(x;\psi)\right]\right\}^{\alpha}+\left\{1-G(x;\psi)\left[1+\lambda\,\bar{G}(x;\psi)\right]\right\}^{\alpha}\right\}}. \tag{7}$$

An interpretation of the OLLT-G family can be given as follows. Let T be a random variable describing a stochastic system by the cdf $G(x)$. If the random variable X represents the odds, the risk that the system following the lifetime T will be not working at time x is given by $F_{OLL-G}(x;\psi)/[1-F_{OLL-G}(x;\psi)]$. If we are interested in modeling the randomness of the odds by the exponentiated half-logistic cdf $R(t)=\frac{t^{\alpha}}{1+t^{\alpha}}$ (for $t>0$), the cdf of X is given by

$$Pr(X\le x)=R\left[\frac{F_{OLL-G}(x;\psi)}{1-F_{OLL-G}(x;\psi)}\right]=\frac{\left\{G\left(x;\psi\right)\left[1+\lambda\overline{G}\left(x;\psi\right)\right]\right\}^{\alpha}}{\left\{G\left(x;\psi\right)\left[1+\lambda\overline{G}\left(x;\psi\right)\right]\right\}^{\alpha}+\left\{\overline{G}\left(x;\psi\right)\left[1-\lambda G\left(x;\psi\right)\right]\right\}^{\alpha}}.$$

Let $G(.)$ be identifiable cdf, it is easy to show that T-G and OLL-G are identifiable, then one can prove easily that OLLT-G is identifiable.

This paper is organized as follows. Some useful characterizations are presented in Section 2. In Section 3, we derive some of the mathematical properties of the new family. Maximum likelihood and other methods of estimation for the model parameters under uncensored data is addressed in Section 4. In Section 5, a simulation studies are presented to assess the performance of the estimators in Section 6, potentiality of the proposed class is illustrated by means of a real data set. Finally, Section 7 provides some conclusions.

2. Characterizations

In this section we present certain characterizations of OLLT-G distribution. These characterizations are in terms of: (i) a simple relationship between two truncated moments and (ii) the hazard function. One of the advantages of characterization (i) is that the cdf is not required to have a closed form. We present our characterizations $(i)-(ii)$ in three subsections.

2.1. Characterizations based on a simple relationship between two truncated moments

In this subsection we present characterizations of OLLT-G distribution in terms of the ratio of two truncated moments. This characterization result employs a theorem due to Glänzel (1987), see Theorem 1 of Appendix A. Note that the result holds also when the interval H is not closed. Moreover, as mentioned above, it could also be applied when the cdf F does not have a closed form. As shown in Glänzel (1990), this characterization is stable in the sense of weak convergence.

Proposition 2.1. Let $X:\Omega\rightarrow\mathbb{R}$ be a continuous random variable and let $q_1\left(x\right)=\left\{\left\{G\left(x\right)\left[1+\lambda\overline{G}\left(x\right)\right]\right\}^{\alpha}+\left\{\overline{G}\left(x\right)\left[1-\lambda G\left(x\right)\right]\right\}^{\alpha}\right\}^{2}\left\{\overline{G}\left(x\right)\left[1-\lambda G\left(x\right)\right]\right\}^{-\alpha}$ and $q_2\left(x\right)=q_1\left(x\right)\left\{G\left(x\right)\left[1+\lambda\overline{G}\left(x\right)\right]\right\}^{\alpha}$ for $x\in\mathbb{R}$. The random variable X has pdf (2) if and only if the function η defined in Theorem 1 has the form

$$\eta\left(x\right)=\frac{1}{2}\left\{1+\left\{G\left(x\right)\left[1+\lambda\overline{G}\left(x\right)\right]\right\}^{\alpha}\right\},\quad x\in\mathbb{R}$$

Proof. Let X be a random variable with pdf (2), then

$$\left(1-F\left(x\right)\right)E\left[q_1\left(X\right)\mid X\ge x\right]=1-\left\{G\left(x\right)\left[1+\lambda\overline{G}\left(x\right)\right]\right\}^{\alpha},\quad x\in\mathbb{R},$$

and

$$\left(1-F\left(x\right)\right)E\left[q_2\left(X\right)\mid X\ge x\right]=\frac{1}{2}\left\{1-\left\{G\left(x\right)\left[1+\lambda\overline{G}\left(x\right)\right]\right\}^{2\alpha}\right\},\quad x\in\mathbb{R},$$

and finally

$$\eta\left(x\right)q_1\left(x\right)-q_2\left(x\right)=\frac{q_1\left(x\right)}{2}\left\{1-\left\{G\left(x\right)\left[1+\lambda\overline{G}\left(x\right)\right]\right\}^{\alpha}\right\}\neq0\quad for\quad x\in\mathbb{R}.$$

Conversely, if η is given as above, then

$$s'(x)=\frac{\eta'(x)\,q_1(x)}{\eta(x)\,q_1(x)-q_2(x)}=\frac{\alpha g(x)\,(1+\lambda-2\lambda G(x))\left\{G(x)\left[1+\lambda\overline{G}(x)\right]\right\}^{\alpha-1}}{1-\left\{G(x)\left[1+\lambda\overline{G}(x)\right]\right\}^{\alpha}}\quad x\in\mathbb{R},$$

and hence

$$s(x)=\log\left(1-\left\{G(x)\left[1+\lambda\overline{G}(x)\right]\right\}^{\alpha}\right),\quad x\in\mathbb{R}.$$

Now, in view of Theorem 1, X has density (2).

Corollary 2.1. Let $X:\Omega\rightarrow\mathbb{R}$ be a continuous random variable and let $q_1(x)$ be as in Proposition 2.1. The pdf of X is (2) if and only if there exist functions q_2 and η defined in Theorem 1 satisfying the differential equation

$$\frac{\eta'(x)\,q_1(x)}{\eta(x)\,q_1(x)-q_2(x)}=\frac{\alpha g(x)\,(1+\lambda-2\lambda G(x))\left\{G(x)\left[1+\lambda\overline{G}(x)\right]\right\}^{\alpha-1}}{1-\left\{G(x)\left[1+\lambda\overline{G}(x)\right]\right\}^{\alpha}},\quad x\in\mathbb{R}.$$

The general solution of the differential equation in Corollary 2.1 is

$$\eta(x)=\left\{1-\left\{G(x)\left[1+\lambda\overline{G}(x)\right]\right\}^{\alpha}\right\}^{-1}\times$$
$$\left[-\int\alpha g(x)\,(1+\lambda-2\lambda G(x))\left\{G(x)\left[1+\lambda\overline{G}(x)\right]\right\}^{\alpha-1}(q_1(x))^{-1}\,q_2(x)+D\right],$$

where D is a constant. Note that a set of functions satisfying the above differential equation is given in Proposition A.1 with $D=\frac{1}{2}$. However, it should be also noted that there are other triplets (q_1,q_2,η) satisfying the conditions of Theorem 1.

2.2. Characterization based on hazard function

It is known that the hazard function, h_F, of a twice differentiable distribution function, F, satisfies the first order differential equation

$$\frac{f'(x)}{f(x)}=\frac{h_F'(x)}{h_F(x)}-h_F(x).$$

For many univariate continuous distributions, this is the only characterization available in terms of the hazard function. The following characterization establish a non-trivial characterization of OLLT-G distribution, for $\alpha=1$, which is not of the above trivial form.

Proposition 2.2. Let $X:\Omega\rightarrow\mathbb{R}$ be a continuous random variable. The pdf of X is (2), for $\alpha=1$, if and only if its hazard function $h_F(x)$ satisfies the differential equation

$$h_F'(x)-\frac{g'(x)}{g(x)}h_F(x)=g(x)\,\frac{d}{dx}\left\{\frac{1+\lambda-2\lambda G(x)}{\overline{G}(x)\left[1-\lambda G(x)\right]}\right\},\quad x\in\mathbb{R}.$$

Proof. If X has pdf (2), for $\alpha=1$, then clearly the above differential equation holds. Now, if this differential equation holds, then

$$\frac{d}{dx}\left\{g(x)^{-1}\,h_F(x)\right\}=\frac{d}{dx}\left\{\frac{1+\lambda-2\lambda G(x)}{\overline{G}(x)\left[1-\lambda G(x)\right]}\right\},\quad x\in\mathbb{R},$$

from which, we obtain

$$h_F(x)=g(x)\left\{\frac{1+\lambda-2\lambda G(x)}{\overline{G}(x)\left[1-\lambda G(x)\right]}\right\},\quad x\in\mathbb{R},$$

which is the hazard function of OLLT-G distribution.

3. Properties

The formulae derived throughout the paper can be easily handled in most symbolic computation software platforms such as Maple, Mathematica and Matlab because of their ability to deal with analytic expressions of formidable size and complexity. Established explicit expressions to calculate statistical measures can be more efficient than computing them directly by numerical integration.

3.1. Linear combination for cdf and pdf

First using generalized binomial expansion we can write

$$\left\{G(x)\left[1+\lambda\,\bar{G}(x)\right]\right\}^{\alpha}=\sum_{j=0}^{\infty}a_j\,\left\{G(x)\left[1+\lambda\,\bar{G}(x)\right]\right\}^{j},$$

where $a_j=\sum_{i=j}^{\infty}(-1)^{i+j}\binom{\alpha}{i}\binom{i}{j}$ and

$$\left\{G(x)\left[1+\lambda\,\bar{G}(x)\right]\right\}^{\alpha}+\left\{1-G(x)\left[1+\lambda\,\bar{G}(x)\right]\right\}^{\alpha}=\sum_{j=0}^{\infty}b_j\,\left\{G(x)\left[1+\lambda\,\bar{G}(x)\right]\right\}^{j},$$

where $b_j=a_j+(-1)^J\binom{\alpha}{j}$. Then using the ratio of two power series we can write

$$F(x)=\frac{\sum_{j=0}^{\infty}a_j\,\left\{G(x)\left[1+\lambda\,\bar{G}(x)\right]\right\}^{j}}{\sum_{j=0}^{\infty}b_j\,\left\{G(x)\left[1+\lambda\,\bar{G}(x)\right]\right\}^{j}}=\sum_{j=0}^{\infty}c_j\,\left\{G(x)\left[1+\lambda\,\bar{G}(x)\right]\right\}^{j},$$

where $c_0=\frac{a_0}{b_0}$ and for $j\geq 1$ we have

$$c_j=\frac{1}{b_0}\left[a_j-\frac{1}{b_0}\sum_{r=1}^{j}b_r\,c_{j-r}\right].$$

Again using binomial expansion we can write

$$\begin{aligned}F(x)&=\sum_{j=0}^{\infty}\sum_{l=j}^{\infty}c_j\,\lambda^l\binom{j}{l}G(x)^j\,\bar{G}(x)^l=\sum_{j=0}^{\infty}\sum_{l=0}^{j}\sum_{r=0}^{l}c_j\,\lambda^l\,(-1)^r\binom{j}{l}\binom{l}{r}G(x)^{j+r}\\&=\sum_{j=0}^{\infty}\sum_{l=0}^{j}\sum_{r=0}^{l}w_{j,l,r}\,G(x)^{j+r}=\sum_{j=0}^{\infty}\sum_{l=0}^{j}\sum_{r=0}^{l}w_{j,l,r}H_{j+r}(x),\end{aligned}\tag{8}$$

where $w_{j,l,r}=c_j\,\lambda^l\,(-1)^r\binom{j}{l}\binom{l}{r}$. Then by differentiating (8) we get

$$f(x)=\sum_{j=0}^{\infty}\sum_{l=0}^{j}\sum_{r=0}^{l}w_{j,l,r}h_{j+r}(x),\tag{9}$$

where $H_a(x)=G(x)^a$ and $h_a(x)=a\,g(x)\,G(x)^{a-1}$ denote the cdf and pdf of Exp-G with power parameter a.

3.2. Quantile function

For $\lambda \neq 0$, if $U \sim U(0,1)$, then

$$X_u = Q_G \left\{ \frac{1+\lambda - \sqrt{(1+\lambda)^2 - \frac{4\lambda\, u^{\frac{1}{\alpha}}}{u^{\frac{1}{\alpha}}+(1-u)^{\frac{1}{\alpha}}}}}{2\lambda} \right\}$$

has cdf (5). For $\lambda = 0$, $X_u = Q_G \left\{ \frac{u^{\frac{1}{\alpha}}}{u^{\frac{1}{\alpha}}+(1-u)^{\frac{1}{\alpha}}} \right\}$ has cdf (5).

3.3. Asymptotics

Let $a = \inf\{x | G(x) > 0\}$, then, the asymptotics of cdf, pdf and hrf as $x \to a$ are given by

$$\begin{array}{lcllc} F(x) & \sim & [(1+\lambda)G(x)]^{\alpha} & \text{as} & x \to a, \\ f(x) & \sim & \alpha\,(1+\lambda)^{\alpha}\, g(x)\, G(x)^{\alpha-1} & \text{as} & x \to a, \\ h(x) & \sim & \alpha\,(1+\lambda)^{\alpha}\, g(x)\, G(x)^{\alpha-1} & \text{as} & x \to a. \end{array}$$

The asymptotics of cdf, pdf and hrf as $x \to \infty$ are given by

$$\begin{array}{lcllc} 1 - F(x) & \sim & \bar{G}(x)^{\alpha} & \text{as} & x \to \infty, \\ f(x) & \sim & \alpha\, g(x)\, \bar{G}(x)^{\alpha-1} & \text{as} & x \to \infty, \\ h(x) & \sim & \frac{\alpha\, g(x)}{\bar{G}(x)} & \text{as} & x \to \infty. \end{array}$$

These equations show the effect of parameters on tails of OLLT-G.

3.4. Moments and generating function

The nth ordinary moment of X is given by

$$\mu_n' = E(X^n) = \sum_{j=0}^{\infty} \sum_{l=0}^{j} \sum_{r=0}^{l} w_{j,l,r} E(Y_{j+r}^n), \tag{9}$$

where Y_{j+r} denotes the Exp-G distribution with power parameter $j+r$. By setting $n = 1$ in (12), we have the mean of X. The last integration can be computed numerically for most parent distributions.The nth central moment of X, the skewness and kurtosis measures can be calculated from the ordinary moments using well-known relationships. The moment generating function (mgf), $M_X(t) = E\left(e^{t\,X}\right)$ of X, can be derived from equation (9) as $M_X(t) = \sum_{j=0}^{\infty} \sum_{l=0}^{j} \sum_{r=0}^{l} w_{j,l,r}\, M_{j+r}(t)$, where $M_{j+r}(t)$ is the mgf of Y_{j+r}. Hence, $M_X(t)$ can be determined from the Exp-G generating function. The main applications of the first incomplete moment refer to the mean deviations and the Bonferroni and Lorenz curves. These curves are very useful in economics, reliability, demography, insurance and medicine. The sth incomplete moment, say $\varphi_s(t)$, of X can be expressed from (9) as

$$\varphi_s(t) = \int_{-\infty}^{t} x^s f(x)\, dx = \sum_{j=0}^{\infty} \sum_{l=0}^{j} \sum_{r=0}^{l} w_{j,l,r} \int_{-\infty}^{t} x^s\, \pi_{j+r}(x)\, dx. \tag{10}$$

3.5. Special case: graphical presentation and statistical properties

For the OLLTWeibull (OLLTW) model-with cdf $G\left(x;a,b\right)=1-\mathrm{e}^{-(x/a)^{b}}$ as a special OLLT-G model- we have the following results

$$F_{OLLTW}(x)=\frac{\left\{\left[1-\mathrm{e}^{-(x/a)^{b}}\right]\left[1+\lambda\mathrm{e}^{-(x/a)^{b}}\right]\right\}^{\alpha}}{\left\{\left[1-\mathrm{e}^{-(x/a)^{b}}\right]\left[1+\lambda\mathrm{e}^{-(x/a)^{b}}\right]\right\}^{\alpha}+\left(\mathrm{e}^{-(x/a)^{b}}\left\{1-\lambda\left[1-\mathrm{e}^{-(x/a)^{b}}\right]\right\}\right)^{\alpha}},$$

$$\mu_n'=E(X^n)=\sum_{j,m=0}^{\infty}\sum_{l=0}^{j}\sum_{r=0}^{l}w_{j,l,r,m}^{(j+r,n)}\Gamma\left(1+n/b\right),\ \forall\ n>-b,$$

$$\mu_1'=E(X)=\sum_{j,m=0}^{\infty}\sum_{l=0}^{j}\sum_{r=0}^{l}w_{j,l,r,m}^{(j+r,1)}\Gamma\left(1+1/b\right),\ \forall\ 1>-b,$$

$$\mu_2'=E(X^2)=\sum_{j,m=0}^{\infty}\sum_{l=0}^{j}\sum_{r=0}^{l}w_{j,l,r,m}^{(j+r,2)}\Gamma\left(1+2/b\right),\ \forall\ 2>-b,$$

$$M_X\left(t\right)=\sum_{j,m,n=0}^{\infty}\sum_{l=0}^{j}\sum_{r=0}^{l}\omega_{j,l,r,m,n}^{(j+r,n)}\Gamma\left(1+n/b\right),\ \forall\ n>-b,$$

$$\varphi_s\left(t\right)=\sum_{j,m=0}^{\infty}\sum_{l=0}^{j}\sum_{r=0}^{l}w_{j,l,r,m}^{(j+r,s)}\,\gamma\left(1+s/b,at^{-b}\right),\ \forall\ s>-b$$

and

$$\varphi_1\left(t\right)=\sum_{j,m=0}^{\infty}\sum_{l=0}^{j}\sum_{r=0}^{l}w_{j,l,r,m}^{(j+r,1)}\,\gamma\left(1+1/b,at^{-b}\right),\ \forall\ 1>-b,$$

where

$$w_{j,l,r,m}^{(j+r,n)}=w_{j,l,r}\nu_m^{(j+r,n)},\ \omega_{j,l,r,m,n}^{(j+r,n)}=t^n\left(n!\right)^{-1}w_{j,l,r,m}^{(j+r,n)}\ \text{and}\ \nu_m^{(j+r,n)}=\frac{(k+1)\left(-1\right)^m}{\left(1/a\right)^n\left(m+1\right)^{(n+b)/b}}\binom{k}{m}.$$

Figure 1 gives the some plot of the OLLTW pdf and hrf From Figure 1 we notice that pdf can exhibit various shapes like increasing, decreasing, unimodal and bimodal, that pdf can exhibit various shapes like increasing, unimodal then increasing and bathtub.

4. Parameter estimation

4.1. Maximum likelihood estimation

Let $x_1,\ldots,x_n$ be a random sample from the OLLT-G distribution with parameters δ,a and ψ. Let $\Theta=(\alpha,\lambda,\psi^{\intercal})^{\intercal}$ be the $(p+2)\times 1$ parameter vector. To determine the Maximum likelihood estimations (MLE) of Θ, we have the log-likelihood function

$$\begin{aligned}\ell\ =\ \ell(\Theta)&=n\log\alpha+\sum_{i=1}^{n}\log g\left(x_i;\psi\right)+\sum_{i=1}^{n}\log\left[1+\lambda-2\lambda\,G(x_i;\psi)\right]\\&+(\alpha-1)\sum_{i=1}^{n}\log s_{i,r}-2\sum_{i=1}^{n}\log\left(s_{i,r}^{\alpha}+z_{i,r}^{\alpha}\right)+(\alpha-1)\sum_{i=1}^{n}\log z_{i,r}^{\alpha}.\end{aligned}$$

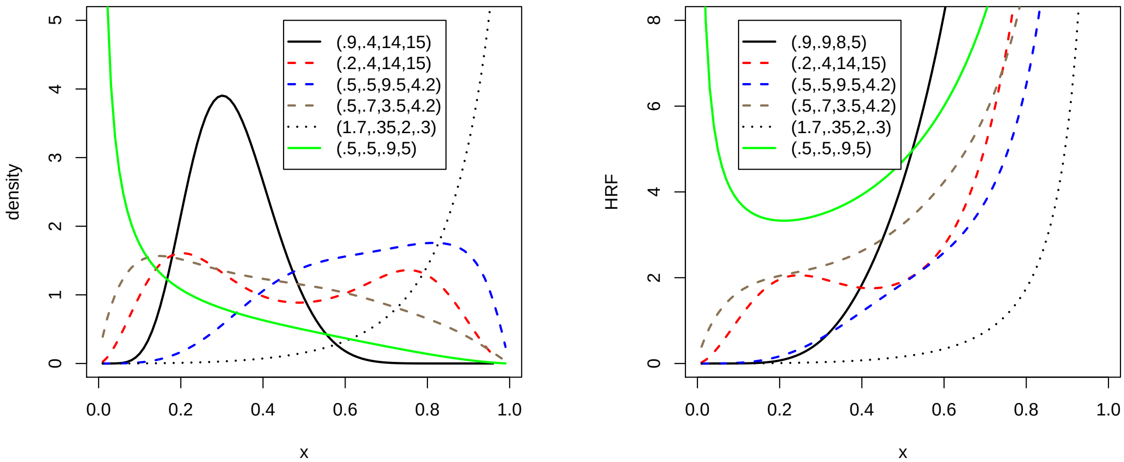

Figure 1. The OLLTW Distribution (α, λ, b, c):pdf (left), hrf (right).

The components of the score vector, $\boldsymbol{U}(\Theta) = \frac{\partial \ell}{\partial \Theta} = \left(\frac{\partial \ell}{\partial \alpha}, \frac{\partial \ell}{\partial \lambda}, \frac{\partial \ell}{\partial \psi}\right)^{\mathsf{T}}$, are

$$U_\alpha = \frac{n}{\alpha} + \sum_{i=1}^{n} \log s_{i,r} + \sum_{i=1}^{n} \log z_{i,r} - 2\sum_{i=1}^{n} \frac{\log(s_{i,r})\, s_{i,r}^{\alpha} + \log(z_{i,r})\, z_{i,r}^{\alpha}}{s_{i,r}^{\alpha} + z_{i,r}^{\alpha}},$$

$$\begin{aligned} U_\lambda \;=\; & \sum_{i=1}^{n} \frac{1 - 2\,G(x_i;\psi)}{1 + \lambda - 2\lambda\, G(x_i;\psi)} + (\alpha - 1)\sum_{i=1}^{n} (s_{i,r})^{-1} G(x;\psi)\left[1 + \bar{G}(x_i;\psi)\right] \\ & -2\sum_{i=1}^{n} \frac{\alpha G(x_i;\psi)\left(s_{i,r}^{\alpha-1} - z_{i,r}^{\alpha-1}\right)}{\left(s_{i,r}^{\alpha} + z_{i,r}^{\alpha}\right)\left[1 + \bar{G}(x_i;\psi)\right]^{-1}} - (\alpha - 1)\sum_{i=1}^{n} (z_{i,r})^{-1} G(x_i;\psi)\left[1 + \bar{G}(x_i;\psi)\right] \end{aligned}$$

and (for $r = 1, 2, \ldots, p$)

$$\begin{aligned} U_{\psi_{\boldsymbol{r}}} \;=\; & +\sum_{i=1}^{n} \frac{g_r'(x_i;\psi)}{g(x_i;\psi)} + \sum_{i=1}^{n} \frac{-2\lambda G_r'(x_i;\psi)}{1 + \lambda - 2\lambda\, G(x_i;\psi)} \\ & + (\alpha - 1)\sum_{i=1}^{n} (s_{i,r})^{-1} G_r'(x_i;\psi)\left[1 + \lambda\,\bar{G}(x_i;\psi) - \lambda G(x;\psi)\right] \\ & -2\alpha \sum_{i=1}^{n} G_r'(x_i;\psi)\left\{1 + \lambda\,\bar{G}(x_i;\psi) - \lambda G(x_i;\psi)\right\}\left(s_{i,r}^{\alpha-1} - z_{i,r}^{\alpha-1}\right)\left(s_{i,r}^{\alpha} + z_{i,r}^{\alpha}\right)^{-1}, \end{aligned}$$

where $g_r'(x_i;\psi) = \partial g(x_i;\psi)/\partial\psi_{\boldsymbol{r}}, G_r'(x_i;\psi) = \partial G(x_i;\psi)/\partial\psi_{\boldsymbol{r}},\; s_{i,r} = G(x_i;\psi)\left[1 + \lambda\,\bar{G}(x_i;\psi)\right]$ and $z_{i,r} = 1 - G(x_i;\psi)\left[1 + \lambda\,\bar{G}(x_i;\psi)\right]$. Setting the nonlinear system of equations $U_\delta = U_a = 0$ and $U_\psi = \mathbf{0}$ and solving them simultaneously yields the MLE $\widehat{\Theta} = \left(\widehat{\alpha}, \widehat{\lambda}, \widehat{\psi}^{\mathsf{T}}\right)^{\mathsf{T}}$. Usually, it is more efficient to obtain the MLEs by maximizing ℓ directly. We used the routine optim in the R software for direct numerical maximization of ℓ. optim is based on a quasi-Newton algorithm. The initial values for numerical maximization were determined by the method of moments; that is, by equating the first $(p+2)$ moments of the OLLT-G distribution with the corresponding empirical moments. The simultaneous roots of these $(p+2)$ equations were determined by the routine multiroot in the R software. The optim routine always converged when the method of moments estimates were used as initial values.

For interval estimation of the parameters, we obtain the $(p+2)\times(p+2)$ observed information matrix $J(\Theta)=\left\{\frac{\partial^2\ell}{\partial r\,\partial s}\right\}$ (for $r,s=\alpha,\lambda,\psi$), whose elements can be computed numerically. Under standard regularity conditions when $n\to\infty$, the distribution of $\widehat{\Theta}$ can be approximated by a multivariate normal $N_{p+2}\left(\Theta, J\left(\widehat{\Theta}\right)^{-1}\right)$ distribution to construct approximate confidence intervals for the parameters, where $J\left(\widehat{\Theta}\right)$ is the total observed information matrix evaluated at $\widehat{\Theta}$. For example, 95 percent confidence intervals for α and λ are

$$\left(\widehat{\alpha}\pm 1.96\sqrt{J^{11}}\right),$$

and

$$\left(\widehat{\lambda}\pm 1.96\sqrt{J^{22}}\right),$$

respectively, where J^{11} denotes the $(1,1)$th element of $J\left(\widehat{\Theta}\right)^{-1}$ and J^{22} denotes the $(2,2)$th element of $J\left(\widehat{\Theta}\right)^{-1}$. A test of $H_0:\alpha=\alpha_0$ versus $H_1:\alpha\neq\alpha_0$ at the five percent significance level is to reject H_0 if

$$\left(|\alpha_0-\widehat{\alpha}|\right)\left(J^{11}\right)^{-\frac{1}{2}}>1.96.$$

Similarly, a test of $H_0:\lambda=\lambda_0$ versus $H_1:\lambda\neq\lambda_0$ at the five percent significance level is to reject H_0 if

$$\left(|\lambda_0-\widehat{\lambda}|\right)\left(J^{22}\right)^{-\frac{1}{2}}>1.96.$$

The method of re-sampling bootstrap can be used for correcting the biases of the MLEs of the model parameters. Good interval estimates may also be obtained using the bootstrap percentile method. The elements of $J(\Theta)$ can be obtained from the authors upon request.

4.2. Other methods of estimation

There are several approaches to estimate the parameters of distributions that each of them has its characteristic features and benefits. In this subsection five of those methods are briefly introduced and will be numerically investigated in the simulation study. A useful summary of these methods can be seen in Dey et al. (2017). Here $\{t_i; i=1,2,...,n\}$ and $\{t_{i:n}; i=1,2,...,n\}$ is the random sample and associated order statistics and F is the distribution function of OLLTWeibull distribution.

4.2.1. Least squares and weighted least squares estimators

The Least Squares (LSE) and weighted Least Squares Estimators (WLSE) are introduced by Swain et al., (1988). The LSE's and WLSE's are obtained by minimizing the following functions:

$$S_{\mathrm{LSE}}(\alpha,\lambda,b,c)=\sum_{i=1}^{n}\left(F(t_{i:n};\alpha,\lambda,b,c)-\frac{i}{n+1}\right)^2$$

and

$$S_{\mathrm{WLSE}}(\alpha,\lambda,b,c)=\sum_{i=1}^{n}\frac{(n+1)^2(n+2)}{i(n-i+1)}\left(F(t_{i:n};\alpha,\lambda,b,c)-\frac{i}{n+1}\right)^2.$$

4.2.2. Cramér–von–Mises estimator Cramér–von–Mises Estimator (CME) is introduced by Choi and Bulgren (1968). The CMEs is obtained by minimizing the following function:

$$S_{\text{CME}}(\alpha, \lambda, b, c) = \frac{1}{12n} + \sum_{i=1}^{n} \left(F(t_{i:n}; \alpha, \lambda, b, c) - \frac{2i-1}{2n} \right)^2.$$

4.2.3. Anderson–Darling and right-tailed Anderson–Darling The Anderson Darling (ADE) and Right-Tailed Anderson Darling Estimators (RTADE) are introduced by Anderson and Darling (1952). The ADE's and RTADE's are obtained by minimizing the following functions:

$$S_{\text{ADE}}(\alpha, \lambda, b, c) = -n - \frac{1}{n} \sum_{i=1}^{n} (2i-1) \{ \log F(t_i; \alpha, \lambda, b, c) + \log \overline{F}(t_{n+1-i}; \alpha, \lambda, b, c) \}$$

and

$$S_{\text{RTADE}}(\alpha, \lambda, b, c) = \frac{n}{2} - 2 \sum_{i=1}^{n} F(t_i; \alpha, \lambda, b, c) - \frac{1}{n} \sum_{i=1}^{n} (2i-1) \log \overline{F}(t_{n+1-i}; \alpha, \lambda, b, c),$$

where $\overline{F}(\cdot) = 1 - F(\cdot)$.

4.2.4. Method of maximum product of spacings Chenga nd Amin (1979 , 1983) introduced the maximum product of spacings (MPS) method as an alternative to MLEs for estimating parameters of continuous univariate distributions. Ranneby (1984) independently developed the same method as an approximation to the Kullbackâ Leibler measure of information. The MPS's are obtained by maximizing the following functions:

$$G(a, b, c, \alpha) = \left[\prod_{i=1}^{n+1} D_i(a, b, c, \alpha) \right]^{\frac{1}{n+1}},$$

where $D_i(\alpha, \lambda, b, c) = F(t_{i:n}; \alpha, \lambda, b, c) - F(t_{i-1:n}; \alpha, \lambda, b, c)$, $i = 1, ..., n,$, $F(t_{0:n}; \alpha, \lambda, b, c) = 0$ and $F(t_{n+1:n}; \alpha, \lambda, b, c) = 1$.

4.3. Simulation study

In order to comprise the estimators introduced in previous section, we consider an special case of the proposed model (OLLT-Weibull) and investigate the MSE of those estimators for different samples. For instance according to what has been mentioned above, for $(\alpha, \lambda, b, c) = (2, 0.5, 1.5, 3), (.25, .7, 8, 2.5), (.2, -.5, 4, 1.5)$. The shapes of OLLTWeibull pdf for these choices are unimodal and bimodal shapes, respectively (Figure 2).

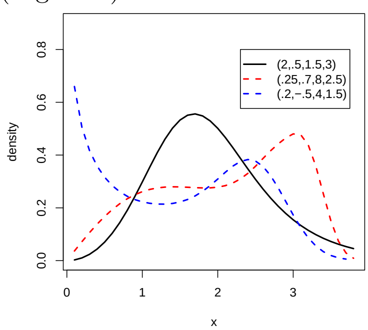

Figure 2. The shape of OLLT-Weibull pdfs for simulation study

The performance of each method of parameters estimations for the OLLTWeibull distribution with respect to sample size n is considered. For this aim, simulation study is done based on following steps:
1. generate one thousand samples of size n from (5) for weibull case. This work is done simply by inverse method via the quantile function and generating data from uniform distribution.
2. compute the estimates for the one thousand samples, say $\left(\hat{\alpha}_i, \hat{\lambda}_i, \hat{b}_i, \hat{c}_i\right)$ for $i = 1, 2, ..., 1000$.
3. compute the biases and mean squared errors given by

$$Bias_{\varepsilon}(n) = \frac{1}{1000}\sum_{i=1}^{1000}(\hat{\varepsilon}_i - \varepsilon)$$

and

$$MSE_{\varepsilon}(n) = \frac{1}{1000}\sum_{i=1}^{1000}(\hat{\varepsilon}_i - \varepsilon)^2,$$

for $\varepsilon = \alpha, \lambda, b, c$.
We repeated these steps for $n = 30, 60, 90, \cdots 400$ with mentioned special case of parameters. So computing $bias_{\varepsilon}(n)$ and $MSE_{\varepsilon}(n)$ for $\varepsilon = \alpha, \lambda, b, c$ and $n = 30, 60, 90, \cdots 400$. To obtain the value of the estimators, we have used the optima function and Nelder-Mead method in R. The result of the simulations of this subsection is shown in Figures 3 to 7. A general result about above figures is that MSE plot for four parameters with the increase in the volume of the sample all methods will approach to zero and this verifies the validity of the these estimation methods and numerical calculations for the distribution parameters OLLTW.
Some of other results are mentioned as follow:

- In unimodal case of parameters and all estimation methods, the bias of parameter α larger than other parameters (Figure 3).
- For first selected values of parameters, MLE method better works than other methods in all parameters except c. For parameter c, CME estimation has smaller bias and MSE when sample size tend to infinity (Figure 3 and 4).
- For second selected values of parameters, MLE estimation has better perfomance than other estimation methods in all parameters when sample size tend to infinity (Figures 5 and 6).
- For last selected values of parameters, MLE method better works than other methods in all parameters except λ and c. For parameters λ and c, WLSE estimation has smaller bias and MSE when sample size tend to infinity (Figures 7 and 8).

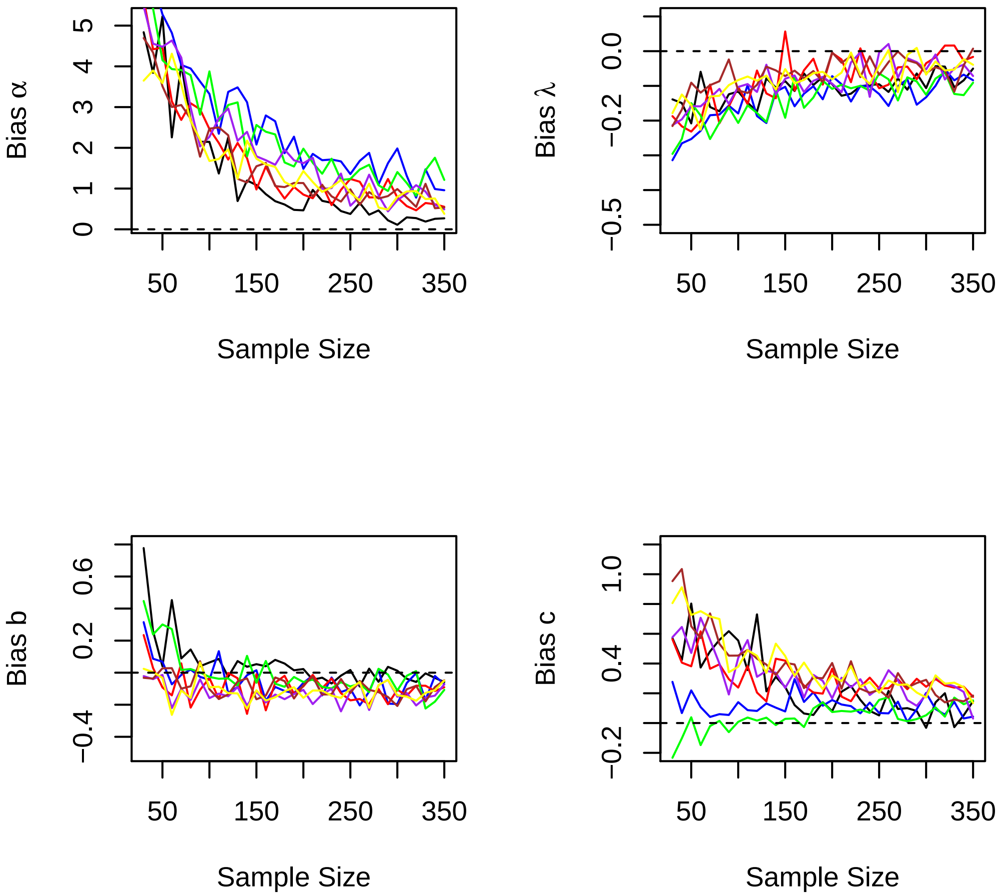

Figure 3. Bias of estimations for parameter values $2, 0.5, 1.5, 3$ (Black:MLE, Blue:LSE, Red:WLSE, Green:CME, Purple:ADE, Brown:RTADE, yellow:MPSE)

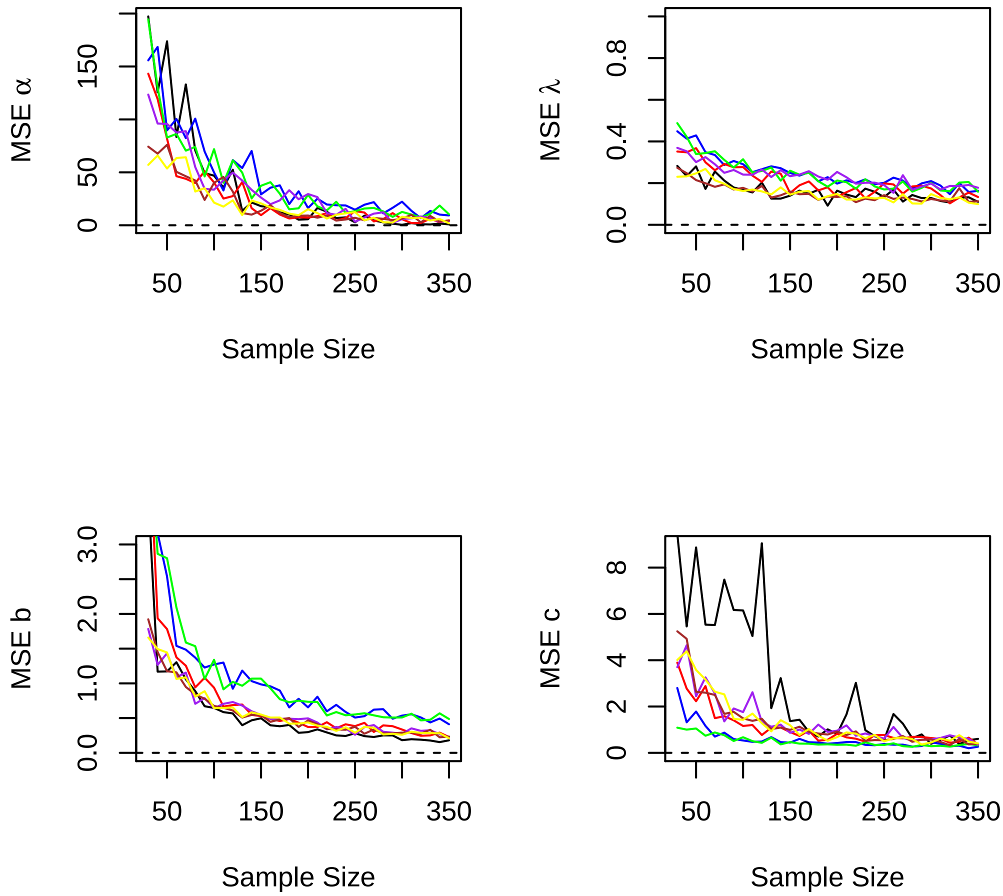

Figure 4. MSE of estimations for parameter values $2, 0.5, 1.5, 3$(Black:MLE, Blue:LSE, Red:WLSE, Green:CME, Purple:ADE, Brown:RTADE, yellow:MPSE)

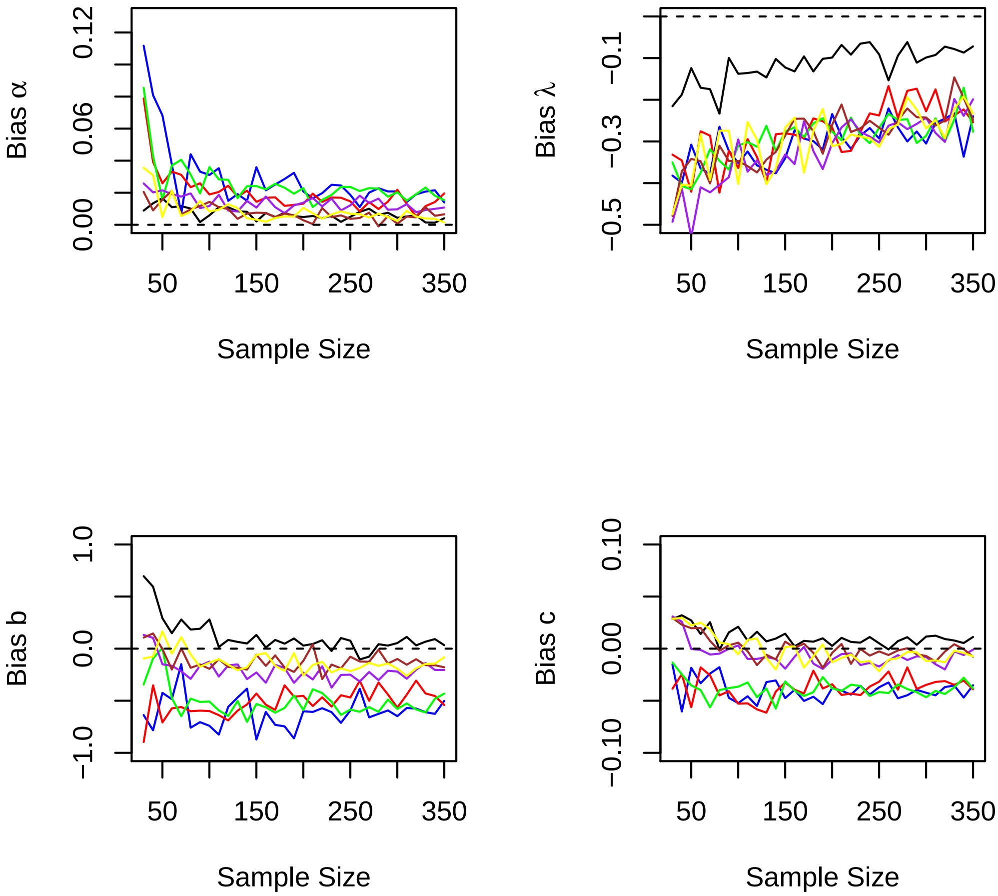

Figure 5. Bias of estimations for parameter values $0.25, 0.7, 8, 2.5$(Black:MLE, Blue:LSE, Red:WLSE, Green:CME, Purple:ADE, Brown:RTADE, yellow:MPSE)

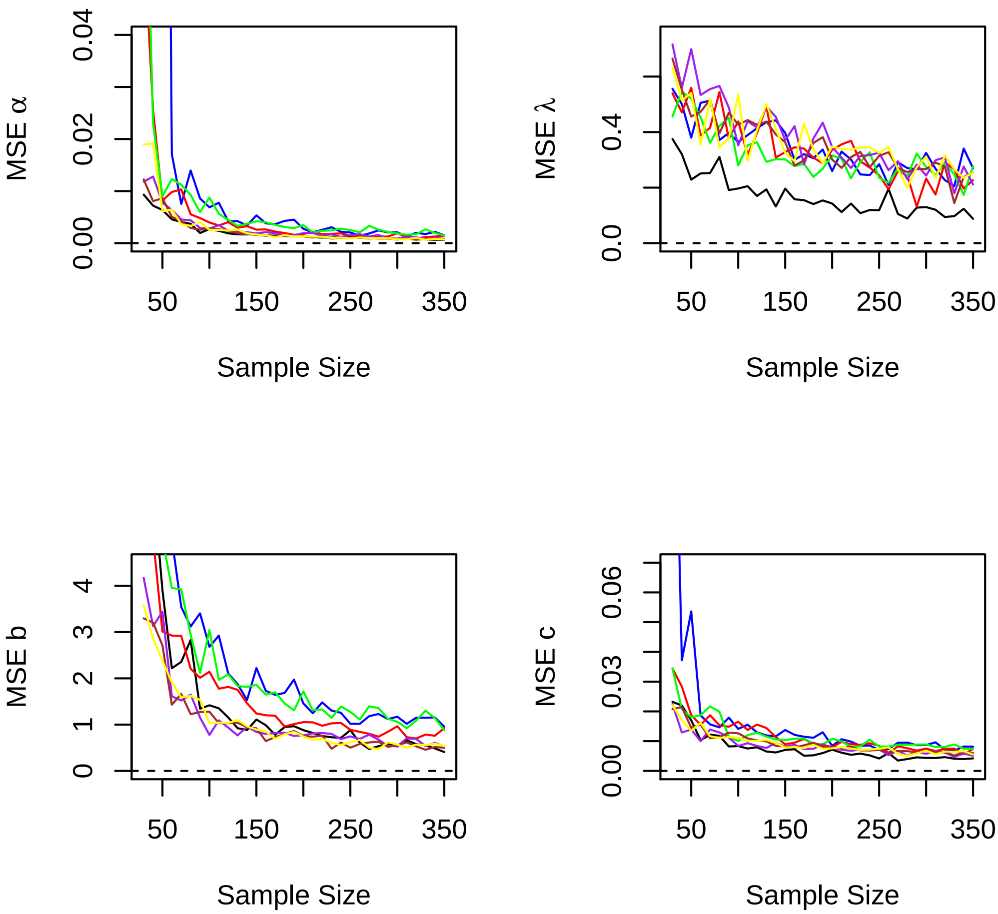

Figure 6. MSE of estimations for parameter values $0.25, 0.7, 8, 2.5$(Black:MLE, Blue:LSE, Red:WLSE, Green:CME, Purple:ADE, Brown:RTADE, yellow:MPSE)

5. Application

In this section, the OLLTWeibull distribution is applied to a data set from Sylwia (2007) on the lifetime of a certain device. The OLLTWeibull distribution is fitted to the data set and compared the results with submodels OLL-Weibull, Transmuted Weibull (Khan et al., 2017) and Weibull distributions by likelihood ratio test. Also this model is compared with some well-known four parameter generalization of weibull distributions such as beta Weibull (Lee et al., 2007), kumaraswamy Weibull (Cordeiro et al., 2010), exponentiated modified Weibull extension (Sarhan and Apaloo, 2013), P-A-L extended Weibull (Al-Zahrani et al., 2016) and gamma modified Weibull (Cordeiro et al., 2015) distributions.
The maximum likelihood estimates, the log-likelihood value, the AIC (Akaike Information Criterion), the BIC (Bayesian Information Criterion), the CAIC (Consistent Akaike Information Criterion) and the HQIC (Hannan-Quinn Information Criterion) for the fitted distributions are reported in Tables 1 and 2. Each distribution was fitted to the datasets using the optim() function in R program. In Figure 9, the

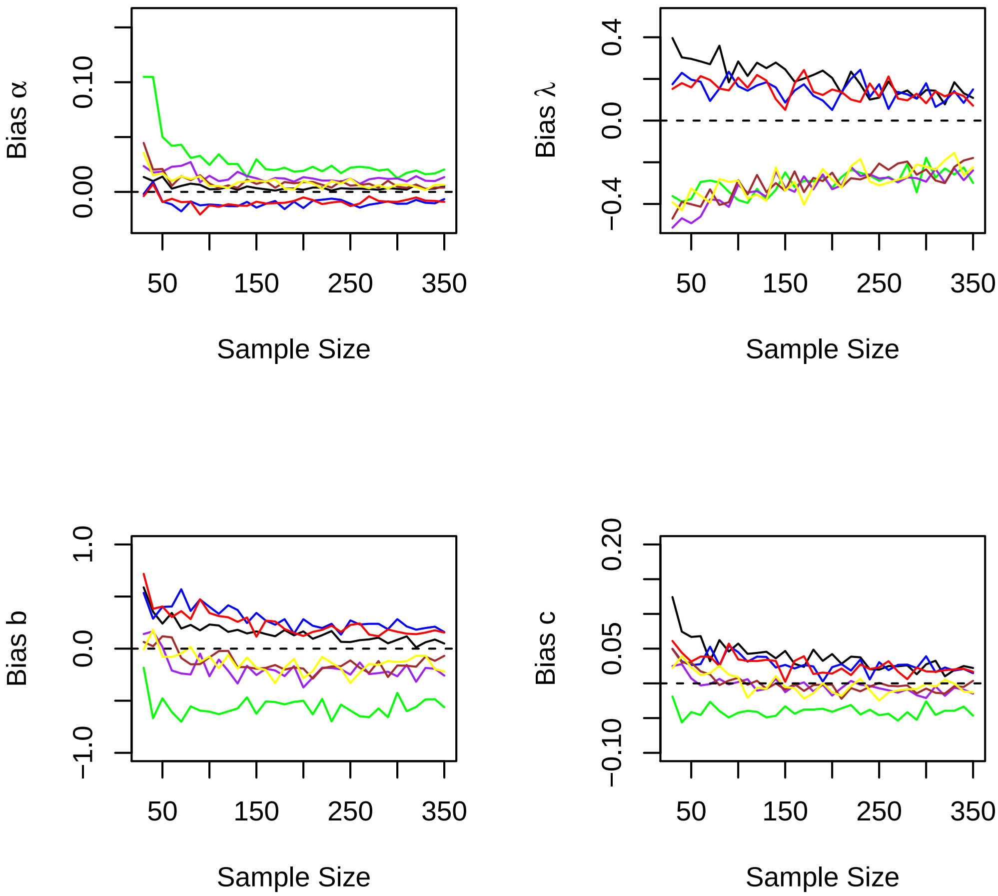

Figure 7. Bias of estimations for parameter values $0.2, -0.5, 4, 1.5$(Black:MLE, Blue:LSE, Red:WLSE, Green:CME, Purple:ADE, Brown:RTADE, yellow:MPSE)

TTT plot (Aarset, 1987) of this set of data displays bathtube hazards rate function that indicates the appropriateness of the OLLTW distribution to fit the data sets. We see that OLLTW distribution fitted on data better than other rivals. Also likelihood ratio test is performed for comparing our model with well-known submodels(Table 3). Figure 10 is illustrated fitted proposed model with submodels and other rivals on this datasets.

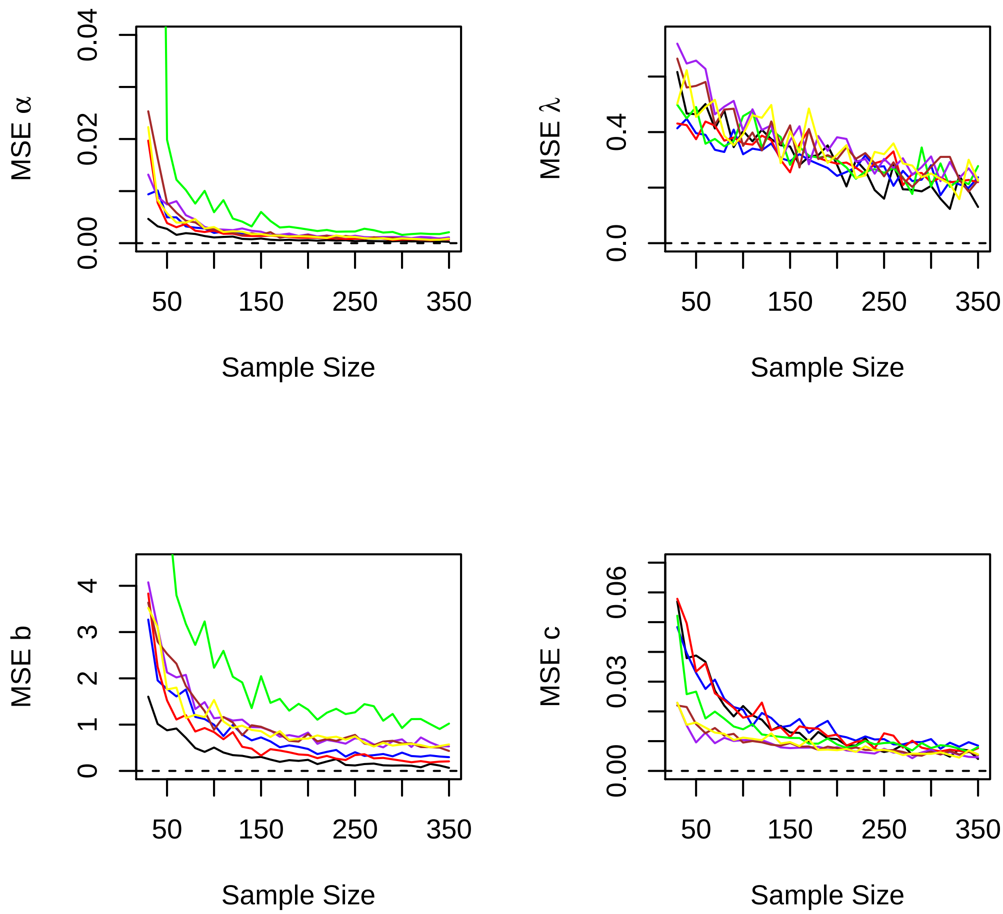

Figure 8. MSE of estimations for parameter values $0.2, -0.5, 4, 1.5$(Black:MLE, Blue:LSE, Red:WLSE, Green:CME, Purple:ADE, Brown:RTADE, yellow:MPSE)

6. Conclusion

In this work, we propose a new class of lifetime distributions called the odd log-logistic transmuted-G family. The proposed family of distributions is constructed by compounding the odd log-logistic distribution with the transmuted distribution. It can provide better fits than some of the known lifetime models and this fact represents a good characterization of this new family. Some characterizations for the new family are presented as well as some of its mathematical properties including. The maximum likelihood, Least squares and weighted least squares, Cramér-von-Mises, Anderson-Darling and right-tailed Anderson-Darlingare and maximum product of spacings methods are used for estimating the model parameters. The importance and flexibility of the new family are illustrated by means of an application to a real data set. Some usefull results are mentioned as follow: In unimodal case of parameters and all estimation methods, the bias of parameter α larger than other parameters (as shown in Figure 3); For first selected values of parameters, MLE method better works than other methods in all parameters except

Table 1. Parameters estimates and log likelihood values for the lifetime of a certain device dataset.

Model	Estimates (Standard Error)	Log Likelihood
OLLTW	0.110, -0.993, 7.407, 8.529	-73.925
(α, λ, a, b)	(0.017), (0.013), (0.002), (0.002)	
OLLW(Submodel)	0.113, 6.334, 7.856	-75.953
(α, a, b)	(0.017), (0.002), (0.002)	
TW(Submodel)	-0.659, 1.500, 7.976	-90.347
(λ, a, b)	(0.198), (0.260), (0.950)	
W(Submodel)	1.619, 9.585	-92.729
(a, b)	(0.277), (1.096)	
B-W	0.012, 173.126, 0.067, 91.720	-79.148
$(\alpha, \beta, \lambda, c)$	0.005, 1.203, 0.831, 3.638	
Kw-W	0.087, 0.058, 0.292, 2.501	-78.071
(a, b, λ, c)	(0.003), (0.010), (0.004), (0.004)	
EMWE	0.094, 14.531, 4.783, 0.233	-81.318
$(\lambda, \alpha, \beta, \gamma)$	(0.010), (0.005), (0.005), (0.043)	
PALEW	1.986, 0.967, 61.257, 2.197	-81.897
(α, β, ν, p)	(1.004), (0.210), (78.954), (5.287)	
GAMW	1.836, 0.143, 0.217, 0.131	-74.502
$(a, \alpha, \lambda, \beta)$	(1.014), (0.197), (0.065), (0.110)	

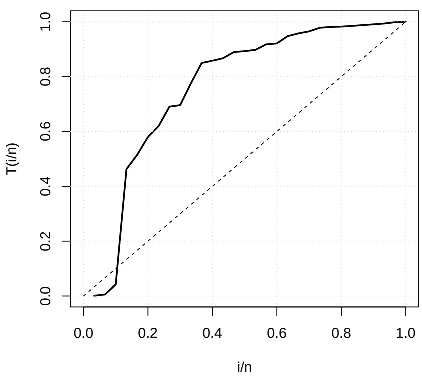

Figure 9. TTT plot of the lifetime of a certain device data set

c, for parameter c, Cramer–von–Mises estimation has smaller bias and MSE when sample size tend to infinity (see Figure 3 and 4); For second selected values of parameters, maximum likelihood method has better perfomance than other estimation methods in all parameters when sample size tend to infinity (as illustrated in Figures 5 and 6); For last selected values of parameters, maximum likelihood method better works than other methods in all parameters except λ and c. For parameters λ and c, weighted least squares method has smaller bias and maximum likelihood method when sample size tend to infinity (as shown in Figure 7 and 8).

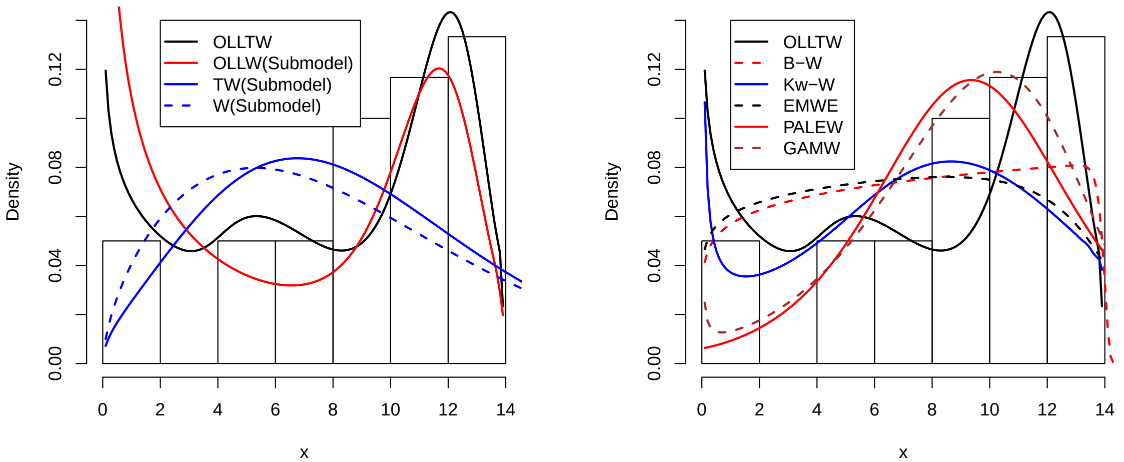

Figure 10. Fitted pdfs on histogram: (Left) with Submodels, (Right) Other rivals

Table 2. Formal goodness of fit statistics

Model	Goodness of fit criteria			
	AIC	*BIC*	*HQIC*	*CAIC*
OLLTW	155.850	161.455	157.643	157.450
B-W	166.297	171.902	168.090	167.897
Kw-W	164.142	169.746	165.935	165.742
EMWE	170.637	176.242	172.430	172.237
PALEW	171.795	177.400	173.588	173.395
GAMW	157.004	162.609	158.797	158.604

Table 3. Likelihood Ratio test for Submodels

$Hypothesis$	LR	$P-Value$
$H_0 : OLLW\ versus\ H_1 : OLLTW$	0.131	0.044
$H_0 : TW\ versus\ H_1 : OLLTW$	7.38×10^{-8}	9.98×10^{-9}
$H_0 : W\ versus\ H_1 : OLLTW$	6.81×10^{-9}	6.18×10^{-9}

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Appendix A

Theorem 1. Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a given probability space and let $H = [a, b]$ be an interval for some $d < b$ $(a = -\infty,\ b = \infty$ might as well be allowed$)$. Let $X : \Omega \to H$ be a continuous random variable with the distribution function F and let q_1 and q_2 be two real functions defined on H such that

$$\mathbf{E}\left[q_2(X) \mid X \geq x\right] = \mathbf{E}\left[q_1(X) \mid X \geq x\right]\eta(x), \quad x \in H,$$

is defined with some real function η. Assume that $q_1, q_2 \in C^1(H)$, $\eta \in C^2(H)$ and F is twice continuously differentiable and strictly monotone function on the set H. Finally, assume that the equation $\eta q_1 = q_2$

has no real solution in the interior of H. Then F is uniquely determined by the functions q_1, q_2 and η , particularly

$$F(x) = \int_a^x C \left| \frac{\eta'(u)}{\eta(u) q_1(u) - q_2(u)} \right| \exp(-s(u)) \, du ,$$

where the function s is a solution of the differential equation $s' = \frac{\eta' \, q_1}{\eta q_1 - q_2}$ and C is the normalization constant, such that $\int_H dF = 1$.