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Optimal Multi Zones Search Technique to Detect a Lost Target by Using k Sensors

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Abstract This paper presents the discrete search technique on multi zones to detect a lost target by using k sensors. The search region is divided into k zones. These zones contain an equal number of states (cells) not necessarily identical. Each zone has a one sensor to detect the target. The target moves over the cells according to a random process. We consider the searching effort as a random variable with a known probability distribution. The detection function with the discounted reward function in a certain state j and time interval i are given. The distribution of the optimal effort that minimizes the probability of undetection is obtained after solving a discrete stochastic optimization problem. An algorithm is constructed to obtain the optimal solution as in the numerical application.

Keywords Detection Model, Nonlinear Stochastic Programming Problem, Discrete Search Problem, Discounted Effort Reward Search

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1. Introduction

The searching for the lost targets which are either fixed or moving, is vital in numerous regular citizen and military applications. For example, the decision of penetrating profundities in the quest for an underground mineral, looking for the lost submarines and boats on or under the ocean, looking for a school of fish and searching for a plane in the sky [1–3]. The search theory has been studied in many variations. When the target is fixed or moves randomly on the real line, we get the so called linear search problem. This problem has an important application in our life. It might emerge in numerous true circumstances, for example, looking for a broken unit in a huge straight framework (electrical cables, phone lines and mining framework), searching for some data in a memory of PC tapes, etc., (see [4–12]. When the target is very important, we use the cooperative search technique. In an earlier work, this technique is used to maximize the probability of the target detection as in [13–21]. Many authors have been succeeded to apply the idea of the cooperative techniques on the plane and space, see [22–27, 41, 42]. The readers can find different techniques and models to detect the lost target in [28–36]. The main purpose of these different techniques is finding the target in minimum cost and with maximum probability. More recent search models to detect the lost targets can be found in [44, 45].

Sometimes the search region has various natural factors and we have a difficulty to apply the above techniques. Thus, the experts in this field divided the search region to a set of states (may be identical or not). In [38–40], the search region is divided into a finite set of square and identical cells. The lost target is randomly moving over these

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cells. They also studied a special case when the target is hidden in one cell of them. On the other hand, in [41] the search area is divided into hexagonal cells. They introduced an algorithm for the optimal search path.

In this paper, the search region is divided into k zones. These zones contain an equal number of cells not necessarily identical. Each zone contains m cells. There are k sensors, where each zone has one sensor $\hat{S}$. The lost target is assumed to be in one of several km-states (cells). The sensor searches for this target during the time intervals $i = 1, 2, \cdots, X$. After the searching in any state, the sensor switch without any delay to another state in its zone. Also, we apply the discounted effort reward function which used in [38]. The main objective is to minimize the searching effort and maximize the probability of detection.

This paper is organized as follows: Section 2 presents the formulation of our model. Section 3 gives the minimum search effort after solving a difficult discrete stochastic optimization problem under the effect of the discounted effort reward function. We construct an algorithm to get this solution numerically as in Section 4. This numerical solution appears in Section 5 for a real life application. Section 6 discusses the results and the future work.

2. Model Formulation

To accelerate the target finding of the lost target in a region of varied terrain, we should divided it into a group of zones.

The space of search: The search region is divided into k zones. These zones contain an equal number of states (cells) not necessarily identical, see Figure 1. Each zone contains m cells.

The means of search: To conduct the search technique within the cells, we need k sensors where each zone has one sensor $\hat{S}$ to explore it.

The target: It is assumed to be in one of several km-states (cells) not necessarily identical states. The sensor searches for this target during the time interval i. After the search in any state, the sensor switch without any delay to another state in its zone. Any sensor of them must distribute his effort among its states in such a manner as to minimize the probability of undetection. The target occupies one state during each of X times intervals. The probabilities that the target exists in state j in zone s at time interval i is denoted by p_{ijs}, $j = 1, 2, \cdots, m$, $s = 1, 2, ..., k$. The Effort is given by $0 \leq L_i(W) \leq \sum_{\hat{S}=1}^{k} V_i^{(\hat{S})}$ which will be distributed among the states and its value bounded by a random variable $\sum_{\hat{S}=1}^{k} V_i^{(\hat{S})}$. The allocation of all search effort is $W_{ijs}^{(\hat{S})}$, where $i = 1, 2, \cdots, X$, $j = 1, 2, \cdots, m$, $\hat{S} = s = 1, 2, \cdots, k$, which gives the effort to be put in state j in zone s at time interval i by the sensor $\hat{S}$.

We call $W = (W_{ij1}^{(1)}, W_{ij2}^{(2)}, \cdots, W_{ijk}^{(k)})$ be a search plan. The conditional probability of detecting the target at time i with $W_{ijs}^{(\hat{S})}$ amount of effort given that the target is located in state j in zone s, is given by the detection function $(1 - b(i, j, s, W_{ijs}^{(\hat{S})}))$, $i = 1, 2, \cdots, X$, $j = 1, 2, \cdots, m$, $s = 1, 2, \cdots, k$. We assume that the searches at distinct time intervals are independent and the motion of the target is independent of the sensors's actions.

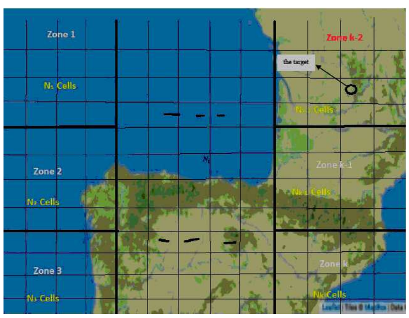


Figure 1. The Search region divided into k zones with km cells.

Theorem 1
The probability of undetection of the lost target over the whole time is given by:

$$H(W)=\prod_{1=1}^{X}\Big[\sum_{s=\hat{S}=1}^{k}\sum_{j=1}^{m}p_{ijs}b(i,j,s,W_{ijs}^{(\hat{S})})\Big]. \tag{1}$$

Proof

$$\begin{aligned}H(W)=&[p_{111}b(1,1,1,W_{111}^{(1)})+p_{121}b(1,2,1,W_{121}^{(1)})+\cdots+p_{1m1}b(1,m,1,W_{1m1}^{(1)})+p_{112}b(1,1,2,W_{112}^{(2)})\\&+p_{122}b(1,2,2,W_{122}^{(2)})+\cdots+p_{1m2}b(1,m,2,W_{1m2}^{(2)})+\cdots+p_{11k}b(1,1,k,W_{11k}^{(k)})+p_{12k}b(1,2,k,\\&W_{12k}^{(k)})+\cdots+p_{1mk}b(1,m,k,W_{1mk}^{(k)})]\times[p_{211}b(2,1,1,W_{211}^{(1)})+p_{221}b(2,2,1,W_{221}^{(1)})+\cdots\\&+p_{2m1}b(2,m,1,W_{2m1}^{(1)})+p_{212}b(2,1,2,W_{212}^{(2)})+p_{222}b(2,2,2,W_{222}^{(2)})+\cdots+p_{2m2}b(2,m,2,W_{2m2}^{(2)})\\&+\cdots+p_{21k}b(2,1,k,W_{21k}^{(k)})+p_{22k}b(2,2,k,W_{22k}^{(k)})+\cdots+p_{2mk}b(2,m,k,W_{2mk}^{(k)})]\times\cdots\times\\&[p_{x11}b(x,1,1,W_{x11}^{(1)})+p_{x21}b(x,2,1,W_{x21}^{(1)})+\cdots+p_{xm1}b(x,m,1,W_{xm1}^{(1)}))p_{x12}b(x,1,2,W_{x12}^{(2)})\\&+p_{x22}b(x,2,2,W_{x22}^{2)})+\cdots+p_{xm2}b(x,m,2,W_{xm2}^{(2)})+\cdots+p_{x1k}b(x,1,k,W_{x1k}^{(k)})+p_{x2k}b(x,2,k,\\&W_{x2k}^{(k)})+\cdots+p_{xmk}b(x,m,k,W_{xmk}^{(k)})],\end{aligned}$$

which can be written as $H(W)=\prod_{i=1}^{X}\Big[\sum_{s=\hat{S}=1}^{k}\sum_{j=1}^{m}p_{ijs}b(i,j,s,W_{ijs}^{(\hat{S})})\Big]$. □

In addition, we obtain the total effort from the following theorem.

Theorem 2
The total effort is given by:

$$L(W)=\sum_{i=1}^{x}\sum_{s=\hat{S}=1}^{k}\sum_{j=1}^{m}W_{ijs}^{(\hat{S})}. \tag{2}$$

Proof

$$\begin{aligned}L(W)=&W_{111}^{(1)}+W_{121}^{(1)}+\cdots+W_{1m1}^{(1)}+W_{112}^{(2)}+W_{122}^{(2)}+\cdots+W_{1m2}^{(2)}+\cdots+W_{11k}^{(k)}+W_{12k}^{(k)}+\cdots+W_{1mk}^{(k)}\\&+W_{211}^{(1)}+W_{221}^{(1)}+\cdots+W_{2m1}^{(1)}+W_{212}^{(2)}+W_{222}^{(2)}+\cdots+W_{2m2}^{(2)}+\cdots+W_{21k}^{(k)}+W_{22k}^{(k)}+\cdots+W_{2mk}^{(k)}\\&+\cdots+W_{x11}^{(1)}+W_{x21}^{(1)}+\cdots+W_{xm1}^{(1)}+W_{x12}^{(2)}+W_{x22}^{(2)}+\cdots+W_{xm2}^{(2)}+\cdots+W_{x1k}^{(k)}+W_{x2k}^{(k)}\\&+\cdots+W_{xmk}^{(k)}.\end{aligned}$$

Hence, $L(W)=\sum_{i=1}^{x}\sum_{s=\hat{S}=1}^{k}\sum_{j=1}^{m}W_{ijs}^{(\hat{S})}$. □

The detection function: We consider the setection function is exponential, that is $1-b(i,j,s,W_{ijs}^{(\hat{S})})=\frac{1}{k}\Big(1-e^{\frac{-W_{ijs}^{(\hat{S})}}{T_{js}}}\Big)$, $i=1,2,\cdots,X$, $j=1,2,\cdots,m$, $s=\hat{S}=1,2,\cdots,k$, where T_{js} is a factor due to the search in cell j in zone s, and the dimensions of it, then the probability of undetection of the target over the whole time is given by,

$$H(W)=\prod_{i=1}^{X}\Big[\sum_{s=\hat{S}=1}^{k}\sum_{j=1}^{m}p_{ijs}\Big(1-\frac{1}{k}\Big(1-e^{\frac{-W_{ijs}^{(\hat{S})}}{T_{js}}}\Big)\Big)\Big], \tag{3}$$

and the unrestricted effort is given by:

$$L(W)=\sum_{i=1}^{X}\sum_{s=\hat{S}=1}^{k}\sum_{j=1}^{m}W_{ijs}^{(\hat{S})}\leq\sum_{i=1}^{X}\sum_{s=\hat{S}=1}^{k}V_i^{(\hat{S})}=V,\quad L_i(W)\leq\sum_{s=\hat{S}=1}^{k}V_i^{(\hat{S})}. \tag{4}$$

Let $\sum\limits_{s=\hat{S}=1} V_i^{(\hat{S})}$ be a random variable with normal distribution and it has a probability density function $f\Bigg(\sum\limits_{s=\hat{S}=1}^{k} V_i^{(\hat{S})}\Bigg)$ and the distribution function $F\Bigg(\sum\limits_{s=\hat{S}=1}^{k} V_i^{(\hat{S})}\Bigg)$, $i=1,2,3,\cdots,X$.

Our aim is to find $W=(W_{ij1}^{(1)},W_{ij2}^{(2)},\cdots,W_{ijk}^{(k)})$ which minimize $H(W)$ subject to the constraints: $L_i(W)\leq\sum\limits_{s=\hat{S}=1}^{k}V_i^{(\hat{S})}$, $0\leq W_{ijs}^{(k)})$ and $\sum\limits_{s=1}^{k}\sum\limits_{j=1}^{m}p_{ijs}=1$, where W is a function of $\Bigg(\sum\limits_{s=\hat{S}=1}^{k}V_i^{(\hat{S})}\Bigg)$. Since, the detection function is exponential then the problem will become a convex nonlinear programming problem (NLP) as follows:

NLP :

$$\min_{W_{ijs}^{(\hat{S})}} H(W)=\prod_{i=1}^{X}\Bigg[\sum_{s=\hat{S}=1}^{k}\sum_{j=1}^{m}p_{ijs}\Bigg(1-\frac{1}{k}\Bigg(1-e^{\frac{-W_{ijs}^{(\hat{S})}}{Tjs}}\Bigg)\Bigg)\Bigg]$$

subject to (5)

$$W\Bigg(\sum_{s=\hat{S}=1}^{k}V_i^{(\hat{S})}\Bigg)=\Bigg\{W\in R^{Xkm}/L_i(W)\leq W\Bigg(\sum_{s=\hat{S}=1}^{k}V_i^{(\hat{S})}\Bigg),$$

$$L(W)=\sum_{i=1}^{X}\sum_{s=\hat{S}=1}^{k}\sum_{j=1}^{m}W_{ijs}^{(\hat{S})}\leq\sum_{i=1}^{X}\sum_{s=\hat{S}=1}^{k}V_i^{(\hat{S})}\Bigg\},$$

$$0\leq W_{ijs}^{(\hat{S})}\ \forall\ i=1,2,\cdots,x, j=1,2,\cdots,m, s=\hat{S}=1,2,\cdots,k \text{ and } \sum_{s=\hat{S}=1}^{k}\sum_{j=1}^{m}p_{ijs}=1$$

where R^{Xkm} is the feasible set of constrained decisions. The unique solution is guaranteed by the convexisty of $H(W)$ and $W\Bigg(\sum\limits_{s=\hat{S}=1}^{k}V_i^{(\hat{S})}\Bigg)$.

Definition 1

$\bar{W}\in W\Bigg(\sum\limits_{s=\hat{S}=1}^{k}V_i^{(\hat{S})}\Bigg)$ is said to be an optimal solution for problem (5) if there does not exists $W\in W\Bigg(\sum\limits_{s=\hat{S}=1}^{k}V_i^{(\hat{S})}\Bigg)$ such that $H(W)\leq H(\bar{W})$ with at least one strict inequality holds, with $\Bigg(L_i(W)\leq\sum\limits_{s=\hat{S}=1}^{k}V_i^{(\hat{S})}\Bigg)\leq\beta$, $\beta\in[0,1]$.

The corresponding nonlinear stochastic programming problem (NLP) is:

NLP :

$$\min_{W_{ijs}^{(\hat{S})}} H(W) = \prod_{i=1}^{X}\Big[\sum_{s=\hat{S}=1}^{k}\sum_{j=1}^{m} p_{ijs}\Big(1-\frac{1}{k}\Big(1-e^{\frac{-w_{ijs}^{(\hat{S})}}{Tjs}}\Big)\Big)\Big],$$

subject to (6)

$$P(L_i(W) \leq \sum_{s=\hat{S}=1}^{k} V_i^{(\hat{S})}\Big) \leq \beta, \quad \beta \in [0,1],$$

$$0 \leq W_{ijs}^{(\hat{S})} \ \forall\ i = 1,2,\cdots,X, \quad j = 1,2,\cdots,m, \quad s = \hat{S} = 1,2,\cdots,k \quad \text{and} \quad \sum_{s=1}^{k}\sum_{j=1}^{m} p_{ijs} = 1.$$

The constraints $\widetilde{P}(L_i(W) \leq \sum_{s=\hat{S}=1}^{k} V_i^{(\hat{S})} \geq 1-\beta$ has to satisfied with unprobability of at least $(1-\beta)$ and can be restated as:

$$\widetilde{P}\left(\frac{L_i(W) - E\Big(\sum_{s=\hat{S}=1}^{k} V_i^{(\hat{S})}\Big)}{\sqrt{var\Big(\sum_{s=\hat{S}=1}^{k} V_i^{(\hat{S})}\Big)}} \leq \frac{\Big(\sum_{s=\hat{S}=1}^{k} V_i^{(\hat{S})}\Big) - E\Big(\sum_{s=\hat{S}=1}^{k} V_i^{(\hat{S})}\Big)}{\sqrt{var\Big(\sum_{s=\hat{S}=1}^{k} V_i^{(\hat{S})}\Big)}}\right) \geq 1-\beta,$$

and for the complement probability we have:

$$\widetilde{P}\left(\frac{L_i(W) - E\Big(\sum_{s=\hat{S}=1}^{k} V_i^{(\hat{S})}\Big)}{\sqrt{var\Big(\sum_{s=\hat{S}=1}^{k} V_i^{(\hat{S})}\Big)}} \geq \frac{\Big(\sum_{s=\hat{S}=1}^{k} V_i^{(\hat{S})}\Big) - E\Big(\sum_{s=\hat{S}=1}^{k} V_i^{(\hat{S})}\Big)}{\sqrt{var\Big(\sum_{s=\hat{S}=1}^{k} V_i^{(\hat{S})}\Big)}}\right) \leq \beta,$$

where $\frac{\Big(\sum_{s=\hat{S}=1}^{k} V_i^{(\hat{S})}\Big) - E\Big(\sum_{s=\hat{S}=1}^{k} V_i^{(\hat{S})}\Big)}{\sqrt{var\Big(\sum_{s=\hat{S}=1}^{k} V_i^{(\hat{S})}\Big)}}$ is a standard normal random variable.

If K_p represent the value of the standard normal random variable at which $\varphi(K_p) = \beta$, then this constraint can be expressed as:

$$\varphi\left(\frac{L_i(W) - E\Big(\sum_{s=\hat{S}=1}^{k} V_i^{(\hat{S})}\Big)}{\sqrt{var\Big(\sum_{s=\hat{S}=1}^{k} V_i^{(\hat{S})}\Big)}}\right) \leq \varphi(K_p),$$

this inequality will be satisfied only if:

$$L_i(W) - E\Big(\sum_{s=\hat{S}=1}^{k} V_i^{(\hat{S})}\Big) \leq K_p\sqrt{var\Big(\sum_{s=\hat{S}=1}^{k} V_i^{(\hat{S})}\Big)}.$$

Suppose that the probability of undetection (3) is combined with the discounted effort function $0 < \delta^i_{js} < 1$ which used in El-Hadidy [38, 39] to develop the final discounted effort reward function:

$$H(W) = \prod_{i=1}^{X}\Big[\sum_{s=\hat{S}=1}^{k}\sum_{j=1}^{m}(1-\delta^i_{js})p_{ijs}\Big(1-\frac{1}{k}\Big(1-e^{\frac{-W^{(\hat{S})}_{ijs}}{T_{js}}}\Big)\Big)\Big], \tag{7}$$

thus, the NLSP problem (6) is equivalent to the following deterministic nonlinear stochastic programming problem (DNLSP),

DNLSP :

$$\min_{W^{(\hat{S})}_{ijs}} H(W) = \prod_{i=1}^{X}\Big[\sum_{s=\hat{S}=1}^{k}\sum_{j=1}^{m}(1-\delta^i_{js})p_{ijs}\Big(1-\frac{1}{k}\Big(1-e^{\frac{-W^{(\hat{S})}_{ijs}}{T_{js}}}\Big)\Big)\Big],$$

subject to (8)

$$L_i(W) - E\Big(\sum_{s=\hat{S}=1}^{k} V^{(\hat{S})}_i\Big) \leq K_p\sqrt{var\Big(\sum_{s=\hat{S}=1}^{k} V^{(\hat{S})}_i\Big)}$$

$$0 \leq W^{(\hat{S})}_{ijs}, \quad 0 < \delta^i_{js} < 1 \ \forall\ i = 1, 2, \cdots, X, j = 1, 2, \cdots, m, s = 1, 2, \cdots, k \text{ and } \sum_{s=1}^{k}\sum_{j=1}^{m} p_{ijs} = 1.$$

which equivalent to:

$$\min_{W^{(\hat{S})}_{ijs}, \delta^i_{js}} H(W) = \prod_{i=1}^{X}\Big[\sum_{s=\hat{S}=1}^{k}\sum_{j=1}^{m}(1-\delta^i_{js})p_{ijs}\Big(1-\frac{1}{k}\Big(1-e^{\frac{-W^{(\hat{S})}_{ijs}}{T_{js}}}\Big)\Big)\Big],$$

subject to (9)

$$W\Big(\sum_{s=\hat{S}=1}^{k} V^{(\hat{S})}_i\Big) = \Big\{W \in R^{Xkm}/g(w) = \sum_{s=\hat{S}=1}^{k}\sum_{j=1}^{m} W^{(\hat{S})}_{ijs} - E\Big(\sum_{s=\hat{S}=1}^{k} V^{(\hat{S})}_i\Big)$$
$$- K_p\sqrt{var\Big(\sum_{s=\hat{S}=1}^{k} V^{(\hat{S})}_i\Big)} \leq 0\Big\},$$

$$- W^{(\hat{S})}_{ijs} \leq 0,\ -\delta^i_{js} < 0, p_{ijs} - 1 \leq 0\ \forall i = 1, 2, \cdots, X, j = 1, 2, \cdots, m, s = 1, 2, \cdots, k.$$

3. The Minimum Search Effort

The main purpose here is to minimize the searching effort under the above constraints. Since, the detection function is convex then one can apply the necessary Kuhn-Tucker conditions to solve the problem (9). Consequently, we get:

$$-\frac{(1-\delta^{\sigma}_{js})p_{\sigma js}}{kT_{js}}e^{-(W^{(\hat{S})}_{\sigma js}/T_{js})}\prod_{\substack{i=1\\ i\neq\sigma}}^{X}\sum_{s=\hat{S}=1}^{k}\sum_{j=1}^{m}(1-\delta^i_{js})p_{ijs}\Big(1-\frac{1}{k}\Big(1-e^{-\frac{W^{(\hat{S})}_{ijs}}{T_{js}}}\Big)\Big) + U_{1\sigma} - U_{2\sigma} = 0, \tag{10}$$

$$-\sigma\delta^{\sigma-1}_{js}\prod_{\substack{i=1\\ i\neq\sigma}}^{X}\sum_{s=\hat{S}=1}^{k}\sum_{j=1}^{m}(1-\delta^i_{js})p_{ijs}\Big(1-\frac{1}{k}\Big(1-e^{-\frac{W^{(\hat{S})}_{ijs}}{T_{js}}}\Big)\Big) - U_{3\sigma} = 0, \tag{11}$$

$$U_{1\sigma}\left\{\sum_{s=\hat{S}=1}^{k}\sum_{j=1}^{m}W_{\sigma js}^{(\hat{S})}-E\left(\sum_{s=\hat{S}=1}^{k}v_{\sigma}^{(\hat{S})}\right)-K_p\sqrt{Var\left(\sum_{s=\hat{S}=1}^{k}v_{\sigma}^{(\hat{S})}\right)}\right\}=0, \tag{12}$$

$$U_{2\sigma}\{-W_{\sigma js}^{(\hat{S})}\}=0, \tag{13}$$

$$U_{3\sigma}\{-\delta_{js}^{\sigma}\}=0, \tag{14}$$

where $U_{\xi\sigma,\xi}=1,2,3$ is the Lagrange multipliers. Since $W_{\sigma js}^{(\hat{S})}>0$ and $\delta_{js}^{\sigma}>0$ then from (13) and (14) we have $U_{2\sigma}=U_{3\sigma}=0$. Also, $\sum\limits_{j=1}^{m}\sum\limits_{s=\hat{S}=1}^{k}W_{\sigma js}^{(\hat{S})}-E\left(\sum\limits_{s=\hat{S}=1}^{k}v_{\sigma}^{(\hat{S})}\right)-K_p\sqrt{Var\left(\sum\limits_{s=\hat{S}=1}^{k}v_{\sigma}^{(\hat{S})}\right)}<0$, then from (12), one can get $U_{1\sigma}\neq 0$. Thus, the case $U_{1\sigma}>0$, $U_{2\sigma}=U_{3\sigma}=0$ is the only case which consider to get the minimum searching effort $(W_{\sigma js}^{(\hat{S})})^{*}$ and the optimal value $(\delta_{js}^{\sigma})^{*}$. Thus, from (12) we have:

$$U_{1\sigma}=\frac{(1-\delta_{js}^{\sigma})p_{\sigma js}}{kT_{js}}e^{-\frac{W_{\sigma js}^{(\hat{S})}}{T_{js}}}\prod_{\substack{i=1\\ i\neq\sigma}}^{X}\sum_{s=\hat{S}=1}^{k}\sum_{j=1}^{m}(1-\delta_{js}^{i})p_{ijs}\left(1-\frac{1}{k}\left(1-e^{-\frac{W_{ijs}^{(\hat{S})}}{T_{js}}}\right)\right). \tag{15}$$

From (15) in (12), we get:

$$\left(\frac{(1-\delta_{js}^{\sigma})p_{\sigma js}}{kT_{js}}e^{-\frac{W_{\sigma js}^{(\hat{S})}}{T_{js}}}\prod_{\substack{i=1\\ i\neq\sigma}}^{X}\sum_{s=\hat{S}=1}^{k}\sum_{j=1}^{m}(1-\delta_{js}^{i})p_{ijs}\left(1-\frac{1}{k}\left(1-e^{-\frac{W_{ijs}^{(\hat{S})}}{T_{js}}}\right)\right)\right)\times\left\{\sum_{s=\hat{S}=1}^{k}\sum_{j=1}^{m}W_{\sigma js}^{(\hat{S})}\right.$$
$$\left.-E\left(\sum_{s=\hat{S}=1}^{k}v_{\sigma}^{(\hat{S})}\right)-K_p\sqrt{Var\left(\sum_{s=\hat{S}=1}^{k}v_{\sigma}^{(\hat{S})}\right)}\right\}=0. \tag{16}$$

Thus,

$$\frac{(1-\delta_{js}^{\sigma})p_{\sigma js}}{kT_{js}}e^{-\frac{W_{\sigma js}^{(\hat{S})}}{T_{js}}}=0 \tag{17}$$

or

$$\prod_{\substack{i=1\\ i\neq\sigma}}^{X}\sum_{s=\hat{S}=1}^{k}\sum_{j=1}^{m}(1-\delta_{js}^{i})p_{ijs}\left(1-\frac{1}{k}\left(1-e^{-\frac{W_{ijs}^{(\hat{S})}}{T_{js}}}\right)\right)=0 \tag{18}$$

or

$$\left\{\sum_{s=\hat{S}=1}^{k}\sum_{j=1}^{m}W_{\sigma js}^{(\hat{S})}-E\left(\sum_{s=\hat{S}=1}^{k}v_{\sigma}^{(\hat{S})}\right)-K_p\sqrt{Var\left(\sum_{s=\hat{S}=1}^{k}v_{\sigma}^{(\hat{S})}\right)}\right\}=0. \tag{19}$$

Put $r_i=E\left(\sum\limits_{s=\hat{S}=1}^{k}V_i^{(\hat{S})}\right)+K_p\sqrt{Var\left(\sum\limits_{s=\hat{S}=1}^{k}V_i^{(\hat{S})}\right)}$, then from Equation (12) we obtain

$$\sum_{s=\hat{S}=1}^{k}\sum_{j=1}^{m}W_{\sigma js}^{(\hat{S})}-r_i=0$$

at least one of these boundaries satisfies that,

$$W_{ijs}^{(\hat{S})} - r_i = 0. \tag{20}$$

Also, from (12), we conclude that at least one of these boundaries satisfies such that

$$(1-\delta_{js}^{i})p_{ijs}\left(1-\frac{1}{k}\left(1-e^{-\frac{W_{ijs}^{(\hat{S})}}{T_{js}}}\right)\right) = 0. \tag{21}$$

Hence,

$$\begin{aligned}
&\left(\frac{(1-\delta_{js}^{\sigma})p_{\sigma js}}{kT_{js}}e^{-\frac{W_{\sigma js}^{(\hat{S})}}{T_{js}}}\prod_{\substack{i=1\\ i\neq\sigma}}^{X}\sum_{s=\hat{S}=1}^{k}\sum_{j=1}^{m}(1-\delta_{js}^{i})p_{ijs}\left(1-\frac{1}{k}\left(1-e^{-\frac{W_{ijs}^{(\hat{S})}}{T_{js}}}\right)\right)\right)\cdot\left\{\sum_{s=\hat{S}=1}^{k}\sum_{j=1}^{m}W_{\sigma js}^{(\hat{S})}\right.\\
&\left.-E\left(\sum_{s=\hat{S}=1}^{k}v_{\sigma}^{(\hat{S})}\right)-K_p\sqrt{Var\left(\sum_{s=\hat{S}=1}^{k}v_{\sigma}^{(\hat{S})}\right)}\right\}\\
&=-\sigma\delta_{js}^{\sigma-1}\prod_{\substack{i=1\\ i\neq\sigma}}^{X}\sum_{s=\hat{S}=1}^{k}\sum_{j=1}^{m}(1-\delta_{js}^{i})p_{ijs}\left(1-\frac{1}{k}\left(1-e^{-\frac{W_{ijs}^{(\hat{S})}}{T_{js}}}\right)\right)=0
\end{aligned}$$

This leads to,

$$\left(\frac{(1-\delta_{js}^{\sigma})p_{\sigma js}}{kT_{js}}e^{-\frac{W_{\sigma js}^{(\hat{S})}}{T_{js}}}\cdot\left\{\sum_{s=\hat{S}=1}^{k}\sum_{j=1}^{m}W_{\sigma js}^{(\hat{S})}-E\left(\sum_{s=\hat{S}=1}^{k}v_{\sigma}^{(\hat{S})}\right)-K_p\sqrt{Var\left(\sum_{s=\hat{S}=1}^{k}v_{\sigma}^{(\hat{S})}\right)}\right\}+\sigma\delta_{js}^{\sigma-1}=0. \tag{22}$$

By solving (22) numerically, one can get the minimum value $(W_{\sigma js}^{(\hat{S})})^*$ and the optimal value $(\delta_{js}^{\sigma})^*$.

4. Algorithm

We construct the following algorithm to calculate the minimum search effort. The steps of the algoritm can be summarized as follows:

Step 1. Input the values of the following:

- k is the number of zones,
- m is the number of cells in each zone,
- X is the total time intervals,
- p the probability that the target exists in state j in zone s at time interval i,
- T_{js} is a factor due to the search in cell j in zone s and the dimensions of it, where $j = 1, 2, \cdots, m$, $s = \hat{S} = 1, 2, \cdots, k$,
- K_p is the value of the standard normal random variable,
- $E\left(\sum_{s=\hat{S}=1}^{k} V_i^{(\hat{S})}\right)$ is the expected value of $\sum_{s=\hat{S}=1}^{k} V_i^{(\hat{S})}$,
- $var\left(\sum_{s=\hat{S}=1}^{k} V_i^{(\hat{S})}\right)$ is the variance of $\sum_{s=\hat{S}=1}^{k} V_i^{(\hat{S})}$.

Step 2. Compute the values of δ_{js}^{i} from Equation (17),

Step 3. Compute the values of r_i from the relation $r_i = E\left(\sum_{s=\hat{S}=1}^{k} V_i^{(\hat{S})}\right) + K_p\sqrt{var\left(\sum_{s=\hat{S}=1}^{k} V_i^{(\hat{S})}\right)}$, $\forall\ i = 1, 2, \cdots, X$, $s = \hat{S} = 1, 2, \cdots, k$, otherwise, go to step 8,

Step 4. Compute the values of $W_{js}^{(\hat{S})}$ from Equation (20) otherwise, go to step 8,

Step 5. Substitute with the value of δ_{js}^{i}, $W_{ijs}^{(\hat{S})}, p_{ijs}$ and r_i in Equation (7) to compute the value of $H(W)$,

Step 6. Replace j by $j+1$, if $j \leq m$, then return to step 2, and replace i by $i+1$ and test the condition $i \leq X$, if yes then go step 2, otherwise, go to step 7,

Step 7. Give the total value of $H(W)$ and then stop.

Step 8. End.

5. Application

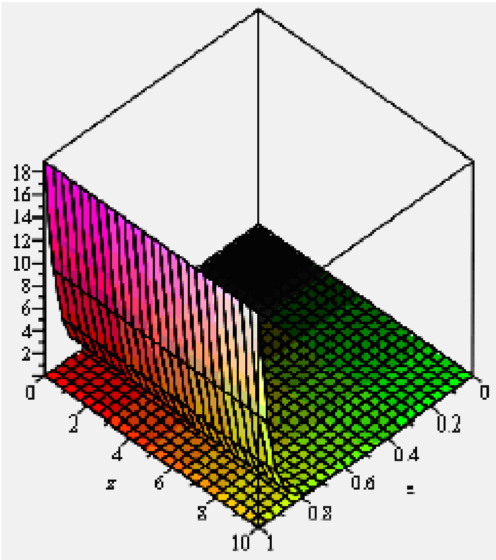
In this section, we apply the above algorithm to formulate a probabilistic search model to detect the randomly moving target in one of several different zones. Assume that, we have 5 zones that are divided into a total number of $N = 30, 40, 50$ and 60 cells. For example, in the case of 30 cells (total number of cells), each zone will have 6 cells. In addition, each zone contains one sensor and there is no interference between them. The target moves randomly between these zones. At a random time interval, we consider the probability $p_{\sigma js}$ that the target exists in state j in zone s at time interval σ is randomly generated by using Maple 13. Also the parameters T_{js} and r_i are randomly generated, see Table 1. After applying the above algorithm, the minimum value of the search effort $(W_{\sigma js}^{(\hat{S})})^*$ and the optimal value $(\delta_{js}^{\sigma})^*$ are appearing in Table (1) in each case.

If we substitute with these random generated values in (22), then we have an infinite number of solutions which give the optimal values $(\delta_{js}^{\sigma})^*$ and $(W_{\sigma js}^{(\hat{S})})^*$. This appears in Figure (2). It presents the plotting $3D$ of the relationship between δ_{js}^{σ} and $W_{\sigma js}^{(\hat{S})}$. The solution of (22) is an interested curve produced from the two intersecting planes and it is satisfied (22) (i.e., an infinite number of solution). We use Maple 13 to deduce the equation of the intersected curve as in Figure (3). This figure gives the value of $(\delta_{js}^{\sigma})^*$ which satisfies (22) and hence we deduce the minimum value $W_{\sigma js}^{(\hat{S})}$ for each N.

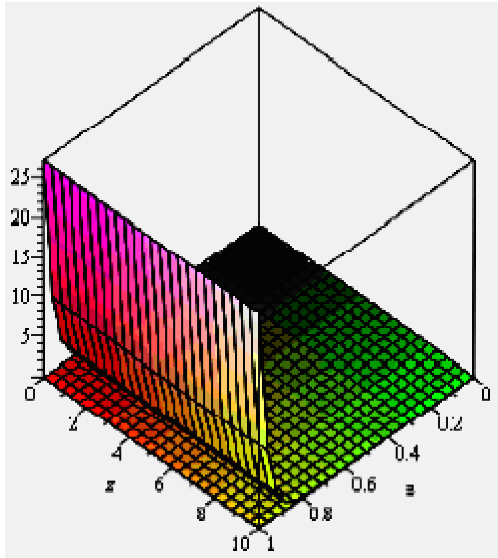
Table 1. The randomly generated values of σ, r_σ, T_{js} and $p_{\sigma js}$ which give the optimal values of $(\delta_{js}^{\sigma})^*$ and $(W_{\sigma js}^{(\hat{S})})^*$ in $k = 5, 10$ zones.

N	σ	r_σ	T_{js}	$p_{\sigma js}$	5 zones		10 zones	
					$(\delta_{js}^{\sigma})^* \cong$	$W_{\sigma js}^{(\hat{S})} \cong$	$(\delta_{js}^{\sigma})^* \cong$	$W_{\sigma js}^{(\hat{S})} \cong$
30	19	5.279190642	747.1503308	0.0334672756	0.484	0.04071571836	4.66	0.0381033075
40	27	8.668184741	497.2015204	0.9669052975	0.705	0.03057684590	0.688	0.0130959880
50	4	4.301371693	866.8184741	0.8675579182	0.0573	0.1353252176	0.0458	0.1153107240
60	50	7.596612973	867.5579182	0.4301371693	0.796	0.01119397377	0.781	0.0411131617

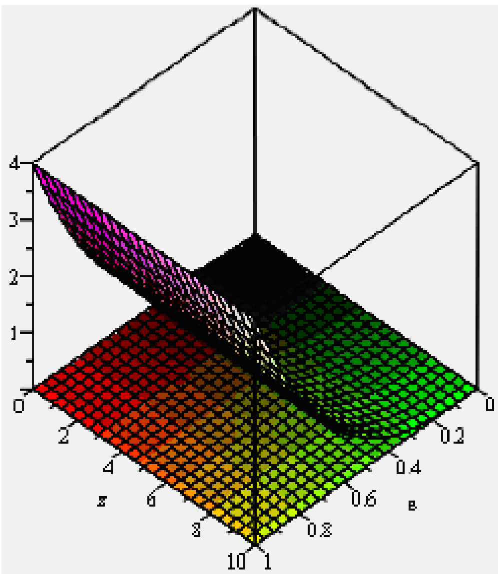
From Table (1), one can conclude that the region which divided into 5 zones has the maximum discounted effort reward parameters rather than 10 zones for the same number of cells. In general, we notice that, when the region is divide to 10 zones then the searching effort equals approximately the searching effort in the 5 zones case. In the 5 zones case, we used a small number of sensors. All of these results lead us to the result; that is the case of 5 zones will give the minimum value of $H(W)$ with maximum value $(\delta_{js}^{\sigma})^*$.



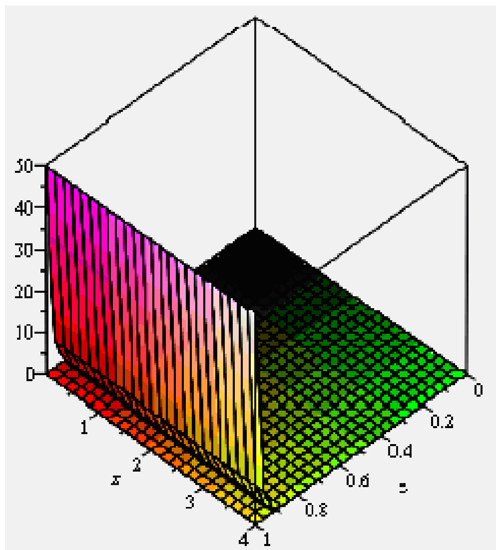
(a) $N = 30$



(b) $N = 40$

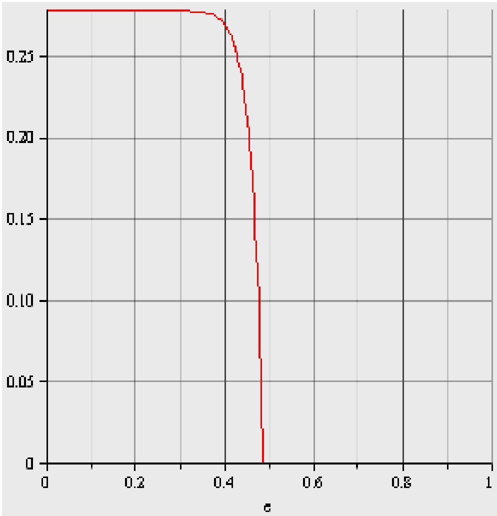


(c) $N = 50$

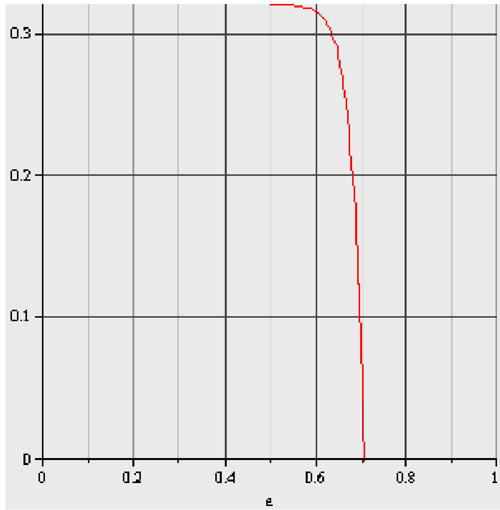


(d) $N = 60$

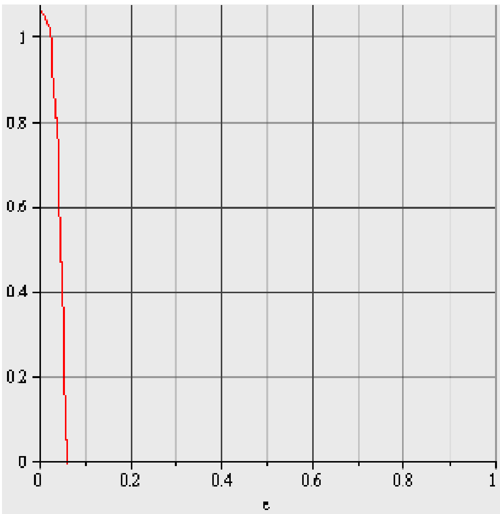
Figure 2. Plotting $3D$ which presents the solution of (22) in 5 zones and different values of N.



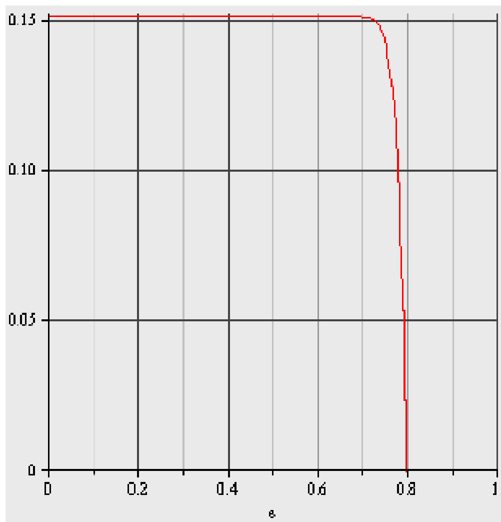
(a) $N = 30$



(b) $N = 40$



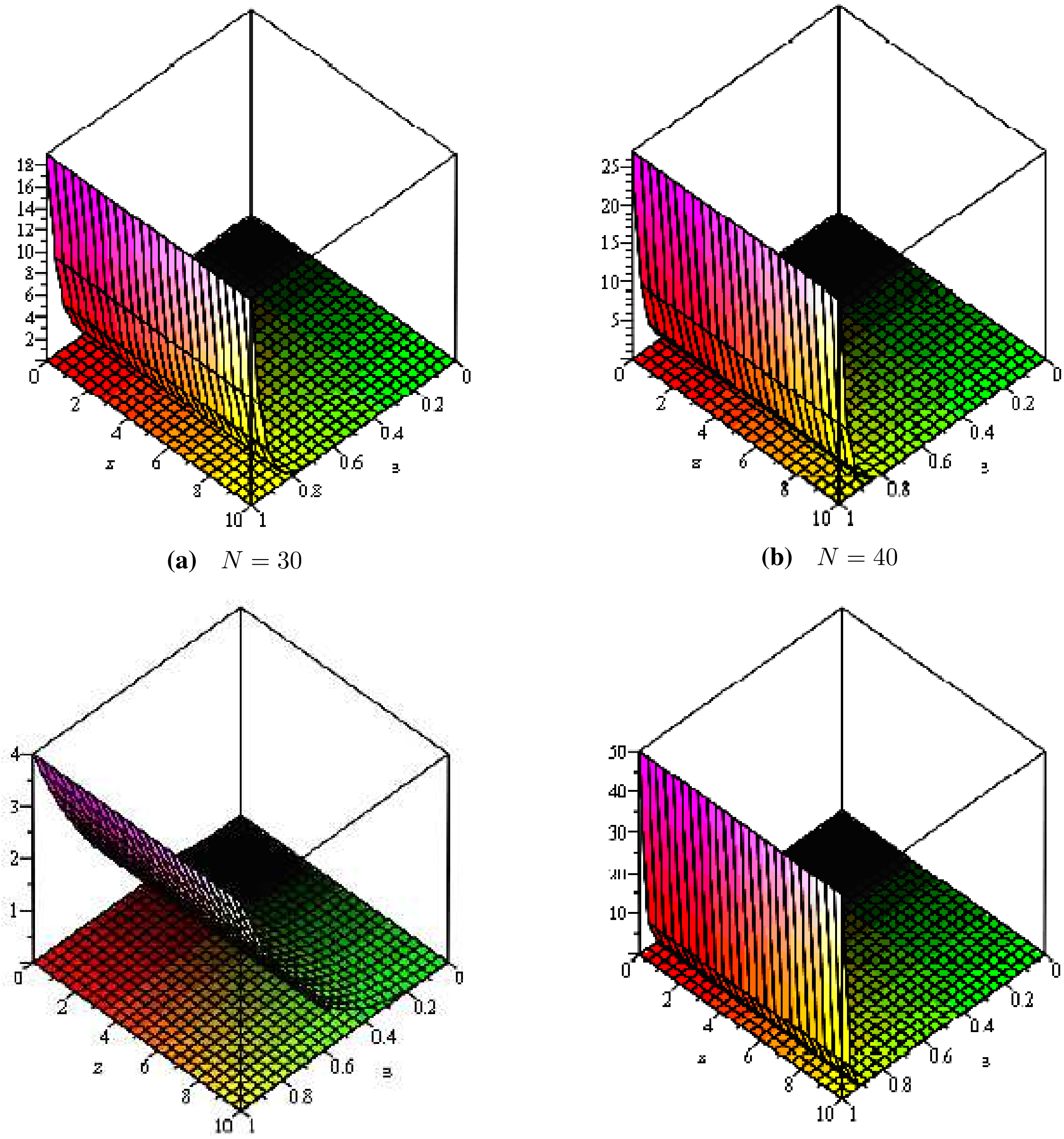
(c) $N = 50$



(d) $N = 60$

Figure 3. The optimal values of $(\delta^{\sigma}_{js})^*$ and $(W^{(\hat{S})}_{\sigma js})^*$ in 5 zones and different values of N.

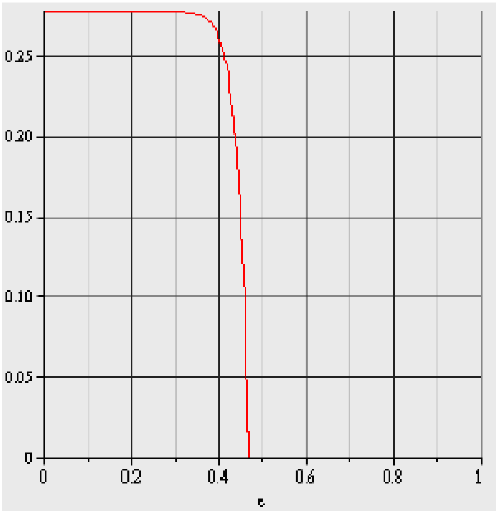
If we divide the region into 10 zones, but with the same total number $N = 30, 40, 50$ and 60 cells, then the optimal values $(\delta^{\sigma}_{js})^*$ and $(W^{(\hat{S})}_{\sigma js})^*$ are given in Table (1). These values obtained by the same method which used in the 5 zones case. Figure (4) presents the plotting $3D$ of (22) and Figure (5) shows the optimal value of $(\delta^{\sigma}_{js})^*$ for 10 zones case.



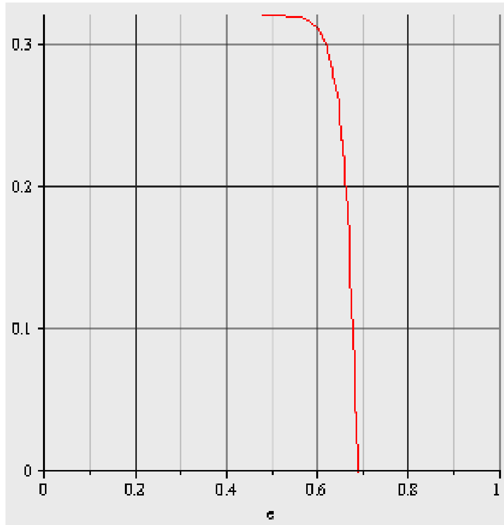
(a) $N = 30$ **(b)** $N = 40$

(c) $N = 50$ **(d)** $N = 60$

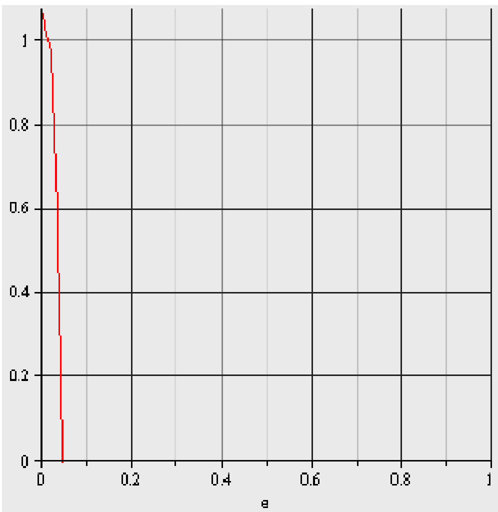
Figure 4. Plotting $3D$ which presents the solution of (22) in 10 zones and different values of N.



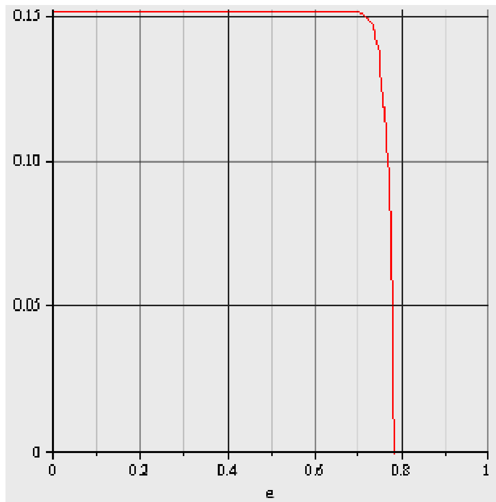
(a) $N = 30$



(b) $N = 40$



(c) $N = 50$



(d) $N = 60$

Figure 5. The optimal values of $(\delta_{js}^{\sigma})^*$ and $(W_{\sigma js}^{(\hat{S})})^*$ in 10 zones and different values of N.

6. Concluding Remarks and the Future Work

A novel probabilistic search model has been presented here to find the target with maximum probability and minimum search effort. The target move randomly on a known region, which divided into a finite number of zones. Each zone is divided also into a finite number of cells (not necessary identical). Also, each zone contains one sensor. The sensor aims to find the target with minimum effort and maximum probability. The effort is bounded by a known probability distribution.

We solve a difficult discrete stochastic optimization problem by using the discounted effort reward function. The solution of this problem can be obtained from the constructed algorithm which provided here. The effectiveness of this algorithm appeared in the numerical application. Moreover, this algorithm is more flexible where it is provided the optimal division of the region and then determined the optimal number of used sensors.

In the future work, we can extend our model to use multiple sensors in each zone. Also, we can solve this difficult discrete stochastic optimization problem under the fuzzy concept for the discounted effort reward.

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