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Partial Bayes Estimation in Two Parameter Gamma Distribution under Non-Informative Prior

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Abstract In Bayesian analysis, empirical and hierarchical methods are two main approaches for the estimation of the parameter(s) involved in the prior distribution of one parameter. But in the multi-parameter model, e.g., $Gamma(\alpha, p)$, where both the parameters are unknown, idea of the 'Partial Bayes (PB) Estimation' is introduced. When we do not have proper belief regarding the joint parameters of the distribution of the variable and when we are estimating one parameter in presence of others, such method may be used. Partial Bayes estimation of the scale parameter p is done by putting the estimate of the another parameter α obtained by some other classical method in case of two parameter Gamma distribution. Using non-informative prior and computing the risk, it is found that the Partial Bayes estimator has less risk than the Bayes estimator. For this, simulation studies for some choices of shape parameter values have been done. In case of the shape parameter, posterior mean and posterior variance are evaluated through simulations to obtain the risk values for estimator of α with known scale parameter. Finally after fitting this distribution, two real datasets are illustrated to see the performance of the Partial Bayes estimator.

Keywords Partial Bayes Estimation, Non-Informative Prior, Jeffreys Prior, Digamma Function

AMS 2010 subject classifications 62C10, 62F15

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1. Introduction

In Classical and Bayesian inference Gamma distribution has been quite extensively used for modeling positive data. It offers better fit in several applied fields, like reliability, life testing experiments and climatology, e.g., [1, 2, 3] and many others in real life situations.

A random variable $\boldsymbol{X}$ is said to follow Gamma distribution and written as $G(\alpha, p)$, if it has the following probability density function:

$$f(x|\alpha, p) = \frac{p^{\alpha}}{\Gamma(\alpha)} x^{\alpha-1} e^{-px}; \;\; x > 0 \tag{1}$$

with the shape and scale parameters $\alpha > 0$ and $p > 0$ respectively.

In literature, there are many papers on Bayesian inference of Gamma parameters. Under vague priors, Son and Oh [4] computed the Bayes estimates of the unknown parameters using Gibbs sampling procedure and the results were compared with the maximum likelihood estimators (MLE's) and modified moment estimators. For generalized Gamma distribution, Tsionas [5] obtained the Bayes estimates for a specific non-informative prior using Gibbs sampling techniques. In 2009, Apolloni and Bassis [6] developed an estimation procedure for two parameter Gamma distribution based on the algorithmic inference approach, without assuming any prior for these parameters involved in joint probability distribution. In the cases where the prior information regarding the

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parameters are available, estimation in Bayesian context for transition probability matrix has been done by Seal and Hossain [20, 21]. Pradhan and Kundu [7] assumed Gamma prior for the scale parameter and log-concave prior for the shape parameter and assumed independence between them. Using these priors they computed the Bayes estimates of the unknown Gamma parameter(s) and also constructed highest posterior density credible intervals. Moala et al. [8] assumed different non-informative prior to derive the Bayesian estimates and credible intervals for the two parameters of Gamma distribution.

Apart from these type of various works, a new approach is addressed in this paper for estimating the parameter and it is named as 'Partial Bayes (PB) Estimation'. Suppose we have a distribution which has two unknown parameters and our interest is to estimate one of this only. Then according to the present proposed procedure, we choose the prior of that parameter of interest and find the Bayes estimate, which is not fully known due to the presence of another unknown parameter in that estimate and then we estimate that unknown parameter by some other classical method e.g., by method of maximum likelihood, method of moments etc and plug that for unknown part. As we estimate our target parameter partially by mixing Bayesian idea and other classical ideas it may be called as 'Partial Bayes Estimation' and this is clearly different from Empirical Bayes method.

In this paper, the Partial Bayes procedure is described for Gamma distribution where p is the parameter of interest in presence of the nuisance parameter α. Prior for the scale parameter p is assumed and the Bayes estimator involves the unknown shape parameter α. Then we put $\hat{\alpha}$ obtained by the method of MLE. Of course, some other estimates may be attempted. Thus, the Partial Bayes estimator of p is obtained. We investigate risk for both the estimators i.e. Bayes and Partial Bayes estimators of p through simulations and then compare them. To derive the risk for the shape parameter, posterior mean and posterior variance of α are obtained by extensive simulation works. To demonstrate the performance of the proposed estimate in real-life situation, two datasets namely rainfall data [8] and bladder cancer patient's data [18] are considered also. For these two sets of data, two parameter Gamma family fits well compared to different competitors. Thus this method is employed for two parameter Gamma distribution to these datasets in good manner.

The materials of the article are arranged in following sequence. In Section 2, we derive estimates using the likelihood equation after being approximated. Jeffrey's non-informative prior for Gamma family is described in Section 3. Bayes and Partial Bayes estimators for scale parameter p under non-informative prior are obtained in Section 4. For comparison, numerical results of Bayes and Partial Bayes risk are shown in Section 5. In Section 6, posterior mean and posterior variance for shape parameter are calculated along with the risk values through extensive simulation works when scale parameter is known. Two real datasets have been used in Section 7, to understand clearly the behavior of the estimator in real situations. Finally, the paper is accomplished in Section 8, with the expectation of better performance of Partial Bayes estimator.

2. Maximum likelihood estimation from approximated likelihood equation

For estimating the parameters in classical paradigm, maximum likelihood method is most frequently used as it has several desirable properties like consistency, efficiency, asymptotic normality and invariance. Let $x_1, x_2, ..., x_n$ be a random sample from Gamma distribution. Then the likelihood function becomes

$$L(x|\alpha, p) = \left(\frac{p^{\alpha}}{\Gamma(\alpha)}\right)^n \prod_{i=1}^{n} x_i^{\alpha-1} e^{-p\sum_{i=1}^{n} x_i} \tag{2}$$

for $\alpha > 0$ and $p > 0$.

Taking logarithm on both sides of the equation (2), we get

$$ln\ L = n\ \alpha\ ln(p) - n\ ln\Gamma(\alpha) + (\alpha - 1)\sum_{i=1}^{n} ln(x_i) - p\sum_{i=1}^{n} x_i. \tag{3}$$

Now differentiating (3) w.r.t. the parameters p and α respectively, estimate of p becomes

$$\hat{p}_{MLE} = \frac{\alpha}{\bar{x}} \quad \text{if } \alpha \text{ is known.} \tag{4}$$

Due to the complexity in likelihood function, closed form solution for estimating α is difficult to find. So it is approximated for large α, as in the following.

After differentiating we get

$$nlnp - n\frac{\partial}{\partial\alpha}ln\Gamma(\alpha) + \sum_{i=1}^{n} lnx_i = 0,$$

and using di-gamma approximation from Beal [19] i.e. $\frac{\partial}{\partial\alpha}(ln\Gamma(\alpha)) \approx ln\alpha - \frac{1}{2\alpha}$, it is enough to solve the following approximate equation as,

$$nln\hat{p} - n(ln\alpha - \frac{1}{2\alpha}) + \sum_{i=1}^{n} ln\, x_i = 0.$$

After some algebraic manipulations and using (4), we get approximate MLE of α given by

$$\hat{\alpha}_{MLE} = \frac{1}{2\left[ln\bar{x} - \overline{lnx}\right]}. \quad (5)$$

3. Jeffrey's prior for Gamma family

The proper selection of prior(s) for the parameters is an important step in Bayesian estimation. From Bayesian perspective, Arnold et al. [9] pointed out that there is no clearly mentioned way to conclude the superiority of one prior over other. It is widely known that the conjugate prior do not exist when both the parameters in the baseline distribution are unknown Singh et al. [10]. However, if sufficient information about the parameter(s) are available, use of informative prior is reasonable otherwise it is better to use non-informative prior.

In this study, we prefer to use a well known non-informative prior proposed by Jeffreys [11]. Although there is a controversy regarding the use of Jeffrey's prior for models with multidimensional parameters, it is widely used in Bayesian analysis due to its invariance property under one-to-one transformations of parameters, e.g., Link and Barker [12]. The prior $\Pi(p)$ for scale parameter p is derived by constructing the Fisher information matrix as in the following.

$$\Pi(p) \propto \sqrt{detI(\alpha, p)},$$

where the Fisher information matrix is given by, $I(\alpha, p) = \begin{bmatrix} \frac{\alpha}{p^2} & -\frac{1}{p} \\ -\frac{1}{p} & \frac{1+2\alpha}{2\alpha^2} \end{bmatrix}$,

and determinant of the information matrix is ,

$$det[I(\alpha, p)] = \frac{1}{2\alpha p^2}.$$

Therefore, the Jeffrey's prior for the Gamma parameters is given by,

$$\Pi(p) = \sqrt{\frac{1}{2\alpha p^2}} = \frac{1}{p\sqrt{2\alpha}}. \quad (6)$$

4. Bayes and Partial Bayes estimates of the scale parameter

In this section, we discuss the derivation of the Bayes estimate (BE) and the Partial Bayes estimate (PBE) under squared error loss function (SELF). In Bayesian inference the posterior distribution, which is useful for future inferences and prediction, contains all the information from sample and prior knowledge about the unknown parameter. For this prior selection, the posterior distribution becomes,

$$\begin{aligned} h(p|\underline{\mathrm{x}}) &= \frac{f(\underline{\mathrm{x}}|p)\Pi(p)}{\int_0^\infty f(\underline{\mathrm{x}}|p)\Pi(p)\,dp} \\ &= \frac{\left(\frac{p^\alpha}{\Gamma(\alpha)}\right)^n e^{-p\sum_{i=1}^n x_i}\frac{1}{p\sqrt{2\alpha}}}{\int_0^\infty \left(\frac{p^\alpha}{\Gamma(\alpha)}\right)^n e^{-p\sum_{i=1}^n x_i}\frac{1}{p\sqrt{2\alpha}}\,dp}. \end{aligned}$$

By looking at the form of above, the posterior density becomes proportional to Gamma and to make this as density we get

$$h(p|\underline{\mathrm{x}}) = \frac{\left(\sum_{i=1}^n x_i\right)^{\alpha n}}{\Gamma(\alpha n)} e^{-p\ \sum_{i=1}^n x_i} p^{\alpha n-1}. \tag{7}$$

Therefore, $p|\underline{\mathrm{x}} \sim Gamma\left(\alpha n, \sum_{i=1}^n x_i\right)$.

Hence, under the squared error loss function, Bayes estimator of the parameter p is the posterior mean i.e.

$$p_{Bayes} = \frac{\alpha n}{\sum_{i=1}^n x_i} = \frac{\alpha}{\bar{x}}. \tag{8}$$

If in the expression (8), α is unknown then we put the maximum likelihood estimator of α derived in (5) and finally the ‘Partial Bayes Estimator’ of scale parameter p is obtained as

$$p_{PB} = \frac{1}{2\left[ln\ \bar{x} - \overline{ln\ x}\right]\ \bar{x}}. \tag{9}$$

5. Computation of risk functions of the Bayes and Partial Bayes estimators using simulation

In decision theory, quality of an estimator is quantified in its risk function. The estimators developed in Section 4, are studied on the basis of their risks obtained under SELF. Being an estimator of θ, $\delta(\boldsymbol{X})$ attains the following risk

$$R(\theta, \delta) = E_\theta L(\theta, \delta(\boldsymbol{X})).$$

Now to study the performance of the two estimators i.e. Bayes in (8) and PB estimate in (9), their risk functions are calculated as in the following.

$$\text{Risk of Bayes estimator} = E\left[\frac{\alpha}{\bar{x}} - p\right]^2 \quad \text{and} \tag{10}$$

$$\text{Risk of Partial Bayes estimator} = E\left[\frac{1}{2\left[ln\ \bar{x} - \overline{ln\ x}\right]\ \bar{x}} - p\right]^2. \tag{11}$$

In the above equations (10) and (11), we have the risk functions for Bayes and PB estimator respectively. But due to complexity in mathematical calculation, we prefer to proceed further calculation numerically.

The simulation study is performed for different choices of the shape and scale parameters of Gamma distribution and we want to obtain the risks for the Bayes and Partial Bayes estimators of scale parameter p. We take different samples of sizes $n = 100, 500, 800, 1000$ from Gamma distribution with the parameters choice $\alpha = (0.05, 1, 5, 10, 15, 20, 25, 30)$ and $p = (0.05, 0.10, 0.25, 0.50, 0.75)$. We are to take moderately large α as we use the di-gamma approximation to calculate the PB estimate in (9). As it is the beginning, we start with the small values of the scale parameter and comparison of both the risk functions for all the combination of (α, p) is performed by using the following steps:

- *Step 1 :* Consider (α_1, p_1) as the first choice of parameter.

Table 1. Comparison of risks for Bayes and Partial Bayes (PB) estimators for different choices of parametric values when sample size n = 100.

p↓	$\alpha \longrightarrow$	0.50	1	5	10	15	20	25	30
0.05	Bayes Risk	423.446	409.954	400.222	399.127	398.925	398.502	398.371	398.421
	PB Risk	281.634	332.226	407.313	417.909	423.202	426.023	427.064	428.005
0.10	Bayes Risk	104.356	100.952	98.579	98.339	98.235	98.161	98.125	98.111
	PB Risk	69.209	81.606	100.024	103.294	104.519	104.760	105.069	105.588
0.25	Bayes Risk	15.015	14.547	14.165	14.106	14.093	14.085	14.078	14.078
	PB Risk	9.717	11.636	14.424	14.886	15.040	15.090	15.217	15.142
0.50	Bayes Risk	2.461	2.350	2.272	2.260	2.256	2.255	2.256	2.254
	PB Risk	1.434	1.787	2.340	2.431	2.469	2.485	2.486	2.498
0.75	Bayes Risk	0.414	0.374	0.346	0.344	0.343	0.342	0.342	0.342
	PB Risk	0.172	0.244	0.381	0.404	0.410	0.416	0.417	0.424

Table 2. Comparison of risks for Bayes and Partial Bayes (PB) estimators for different choices of parametric values when sample size n = 500.

p↓	$\alpha \longrightarrow$	0.50	1	5	10	15	20	25	30
0.05	Bayes Risk	402.051	400.408	398.520	398.256	398.078	398.197	398.120	398.130
	PB Risk	252.335	304.965	379.465	391.816	395.291	397.908	399.077	399.862
0.10	Bayes Risk	99.230	98.537	98.139	98.071	98.032	98.039	98.032	98.013
	PB Risk	61.999	74.916	93.372	96.534	97.351	98.032	98.212	98.602
0.25	Bayes Risk	14.256	14.143	14.085	14.069	14.067	14.067	14.064	14.066
	PB Risk	8.653	10.565	13.360	13.819	13.978	14.068	14.126	14.147
0.50	Bayes Risk	2.294	2.270	2.253	2.251	2.252	2.250	2.251	2.251
	PB Risk	1.205	1.568	2.114	2.204	2.240	2.253	2.264	2.269
0.75	Bayes Risk	0.353	0.346	0.341	0.341	0.341	0.341	0.340	0.340
	PB Risk	0.104	0.178	0.307	0.332	0.341	0.344	0.347	0.349

- *Step 2 :* Generate sample of sizes $n = 100, 500, 800, 1000$ from Gamma distribution.
- *Step 3 :* Calculate the Bayes Risk and Partial Bayes Risk using (10) and (11) simultaneously.
- *Step 4 :* Repeat above *Step 2* and *Step 3* for $K = 30000$ times.
- *Step 5 :* Compute the average of the above Bayes and Partial Bayes risk to get the overall risks for p of both the estimators.
- *Step 6 :* Repeat *Step 1 – Step 5* for the remaining combinations of (α, p).

In Table 1–4, a clear comparison is done for the different combination of the shape and scale parameters. Tables are constructed for small, moderate and large sample values. When the sample sizes are small the Bayes risk is relatively small compared to Partial Bayes risk for the large combinations of parameter values. But as the sample size increases the PB risk is comparatively small for the maximum number of parameter combinations. For large sample sizes, when both the parameters are small, the risk for the proposed estimator, i.e. PB estimate is better than the Bayes estimate with respect to the risk value. Whereas, for large values of the shape parameter α and relatively

Table 3. Comparison of risks for Bayes and Partial Bayes (PB) estimators for different choices of parametric values when sample size n = 800.

p↓	$\alpha \longrightarrow$	0.50	1	5	10	15	20	25	30
0.05	Bayes Risk	400.689	399.439	398.251	398.183	398.126	398.017	398.025	398.038
	PB Risk	250.169	302.347	376.483	388.952	393.009	395.329	396.683	398.023
0.10	Bayes Risk	98.753	98.384	98.092	98.065	98.022	98.021	98.023	98.020
	PB Risk	61.360	74.349	92.663	95.804	96.780	97.419	97.800	97.821
0.25	Bayes Risk	14.176	14.117	14.071	14.069	14.067	14.063	14.064	14.064
	PB Risk	8.556	10.498	13.271	13.732	13.870	13.949	14.028	14.052
0.50	Bayes Risk	2.273	2.261	2.253	2.251	2.250	2.250	2.251	2.250
	PB Risk	1.185	1.547	2.092	2.187	2.214	2.234	2.242	2.247
0.75	Bayes Risk	0.349	0.345	0.341	0.341	0.341	0.341	0.341	0.340
	PB Risk	0.099	0.174	0.303	0.326	0.333	0.338	0.340	0.342

Table 4. Comparison of risks for Bayes and Partial Bayes (PB) estimators for different choices of parametric values when sample size n = 1000.

p↓	$\alpha \longrightarrow$	0.50	1	5	10	15	20	25	30
0.05	Bayes Risk	400.238	399.185	398.251	398.121	398.022	398.068	398.031	398.005
	PB Risk	249.327	301.468	375.891	388.041	392.188	394.982	395.883	396.747
0.10	Bayes Risk	98.660	98.327	98.055	98.046	98.032	98.012	98.018	98.016
	PB Risk	61.203	74.139	92.546	95.606	96.608	97.157	97.554	97.728
0.25	Bayes Risk	14.152	14.107	14.069	14.069	14.065	14.065	14.063	14.063
	PB Risk	8.523	10.454	13.242	13.712	13.857	13.933	13.976	14.024
0.50	Bayes Risk	2.268	2.259	2.252	2.251	2.250	2.251	2.251	2.250
	PB Risk	1.179	1.542	2.087	2.180	2.211	2.227	2.236	2.242
0.75	Bayes Risk	0.347	0.344	0.341	0.341	0.341	0.340	0.340	0.340
	PB Risk	0.097	0.172	0.300	0.323	0.331	0.335	0.338	0.339

large value of p, risks of both the estimator are almost equal. Although, for small values of α, the Partial Bayes risk is always small irrespective of the sample sizes.

6. Evaluation of posterior mean and variance of the estimator for shape parameter

For the Gamma distribution, there is a problem in finding Bayes estimator for shape parameter α. In the past years, Damsleth [13], Miller [14] attempted this problem and developed conjugate distributions for the model but it is difficult to handle these analytically. In recent time, some works have been done numerically to find the Bayes estimate for the shape parameter (e.g., Son and Oh [4], Apolloni and Bassis [6], Pradhan and Kundu [7] etc). Importing two extra parameters, namely location and another shape parameter in Gamma density and setting these to 0 and 1 respectively, is also an idea applied by Tsionas [5] for Bayes estimation using Gibbs sampling. Miller [15] has shown that the full conditional distribution of the Gamma shape parameter is well approximated by a Gamma distribution, even for small sample sizes, when the prior on the shape parameter is also a Gamma distribution.

But in this paper, we consider Jeffrey's non-informative prior as a prior information for the shape parameter α and squared error loss function is chosen to obtain the Bayes estimate from posterior distribution. Our goal is to compute the risk values for shape parameter which is quite difficult to achieve analytically. So instead of doing integration over the prior distribution, here we obtain the posterior mean and posterior variance of the shape parameter. With the help of these, we find out the Bayes estimate as well as the risk functions for this Bayes estimator of shape parameter of the Gamma distribution through some extensive numerical integration.

Under the Jeffrey's prior (6), posterior density becomes,

$$h(\alpha|\underline{x}) = \frac{\left(\frac{p^\alpha}{\Gamma(\alpha)}\right)^n \prod_{i=1}^n x_i^{\alpha-1} \frac{1}{p\sqrt{2\alpha}}}{\int_0^\infty \left(\frac{p^\alpha}{\Gamma(\alpha)}\right)^n \prod_{i=1}^n x_i^{\alpha-1} \frac{1}{p\sqrt{2\alpha}}\, d\alpha}. \tag{12}$$

Now, we first derive the denominator part of (12)

$$\begin{aligned}
&\int_0^\infty \left(\frac{p^\alpha}{\Gamma(\alpha)}\right)^n \prod_{i=1}^n x_i^{\alpha-1} \frac{1}{p\sqrt{2\alpha}}\, d\alpha \\
&= \frac{1}{p\sqrt{2}(\prod_{i=1}^n x_i)} \int_0^\infty \frac{(p^n \prod_{i=1}^n x_i)^\alpha}{\sqrt{\alpha}(\Gamma(\alpha))^n}\, d\alpha \\
&= \frac{1}{p\sqrt{2}(\prod_{i=1}^n x_i)} \int_0^\infty \frac{u^\alpha}{\sqrt{\alpha}(\Gamma(\alpha))^n}\, d\alpha \qquad \text{where, } u = \left(p^n \prod_{i=1}^n x_i\right) \\
&= \frac{p^n}{u\, p\sqrt{2}} \int_0^\infty \frac{u^\alpha}{(\Gamma(\alpha))^n \sqrt{\alpha}}\, d\alpha \\
&= \frac{p^{n-1}}{u\sqrt{2}}\, I(u).
\end{aligned}$$

Hence, after putting the value of the integral in denominator, (12) becomes

$$h(\alpha|\underline{x}) = \frac{\left(\frac{p^\alpha}{\Gamma(\alpha)}\right)^n \prod_{i=1}^n x_i^{\alpha-1} \frac{1}{p\sqrt{2\alpha}}}{\frac{p^{n-1}}{u\sqrt{2}}\, I(u)}.$$

Under SELF, the posterior mean is nothing but the Bayes estimator of α. Therefore the posterior mean becomes,

$$\begin{aligned}
E(\alpha|\underline{x}) &= \frac{\int_0^\infty \alpha \left(\frac{p^\alpha}{\Gamma(\alpha)}\right)^n \prod_{i=1}^n x_i^{\alpha-1} \frac{1}{p\sqrt{2\alpha}}\, d\alpha}{\frac{p^{n-1}}{u\sqrt{2}}\, I(u)} \\
&= \frac{\frac{p^{n-1}}{\sqrt{2}}\, I'(u)}{\frac{p^{n-1}}{u\sqrt{2}}\, I(u)} \\
&= \frac{u\, I'(u)}{I(u)} = \text{Bayes estimate of } \alpha.
\end{aligned} \tag{13}$$

We calculate the posterior variance using the following expression,

$$V(\alpha|\underline{x}) = E\left(\alpha^2|\underline{x}\right) - \left(E(\alpha|\underline{x})\right)^2,$$

$$\text{where,} \quad E(\alpha^2|\underline{x}) = \int_0^\infty \frac{\alpha^2 \left(\frac{p^\alpha}{\Gamma(\alpha)}\right)^n \prod_{i=1}^n x_i^{\alpha-1} \frac{1}{p\sqrt{2\alpha}}\, d\alpha}{\frac{p^{n-1}}{u\sqrt{2}}\, I(u)}. \tag{14}$$

In (14), denominator part is constant, so the numerator part is to be simplified.

$$\text{Now,} \quad \int_0^\infty \alpha^2 \left(\frac{p^\alpha}{\Gamma(\alpha)}\right)^n \prod_{i=1}^n x_i^{\alpha-1} \frac{1}{p\sqrt{2\alpha}}\, d\alpha$$
$$= \frac{p^{n-1}}{u\sqrt{2}} \int_0^\infty \frac{\alpha^2 u^\alpha}{(\Gamma(\alpha))^n \sqrt{\alpha}}\, d\alpha. \tag{15}$$

$$\text{Putting,} \quad I(u) = \int_0^\infty \frac{u^\alpha}{(\Gamma(\alpha))^n \sqrt{\alpha}}\, d\alpha, \quad \text{it follows that}$$
$$I'(u) = \int_0^\infty \frac{\alpha\, u^{\alpha-1}}{(\Gamma(\alpha))^n \sqrt{\alpha}}\, d\alpha \quad \text{and}$$
$$I''(u) = \int_0^\infty \frac{\alpha(\alpha-1)\, u^{\alpha-2}}{(\Gamma(\alpha))^n \sqrt{\alpha}}\, d\alpha.$$

$$\text{Thus} \quad I''(u) = \int_0^\infty \frac{\alpha^2 u^{\alpha-2}}{(\Gamma(\alpha))^n \sqrt{\alpha}}\, d\alpha - \int_0^\infty \frac{\alpha\, u^{\alpha-2}}{(\Gamma(\alpha))^n \sqrt{\alpha}}\, d\alpha.$$
$$\implies u^2 I''(u) = \int_0^\infty \frac{\alpha^2\, u^\alpha}{(\Gamma(\alpha))^n \sqrt{\alpha}}\, d\alpha - \int_0^\infty \frac{\alpha\, u^\alpha}{(\Gamma(\alpha))^n \sqrt{\alpha}}\, d\alpha$$
$$\implies u^2 I''(u) + u \int_0^\infty \frac{\alpha\, u^{\alpha-1}}{(\Gamma(\alpha))^n \sqrt{\alpha}}\, d\alpha = \int_0^\infty \frac{\alpha^2\, u^\alpha}{(\Gamma(\alpha))^n \sqrt{\alpha}}\, d\alpha$$
$$\implies \left[u^2 I''(u) + u I'(u)\right] = \int_0^\infty \frac{\alpha^2\, u^\alpha}{(\Gamma(\alpha))^n \sqrt{\alpha}}\, d\alpha. \tag{16}$$

Now using the expression (16) in (15), the value of the integral becomes

$$\frac{p^{n-1}}{u\sqrt{2}} \left[u^2 I''(u) + u\, I'(u)\right]$$
$$= \frac{p^{n-1}}{\sqrt{2}} \left[u\, I''(u) + I'(u)\right]. \tag{17}$$

Again, substituting the expression (17) in (14), we get

$$E(\alpha^2|\underline{x}) = \frac{\frac{p^{n-1}}{\sqrt{2}} \left[u\, I''(u) + I'(u)\right]}{\frac{p^{n-1}}{u\, \sqrt{2}} I(u)}$$
$$= \frac{\left[u^2 I''(u) + u I'(u)\right]}{I(u)}. \tag{18}$$

Finally, the posterior variance becomes,

$$Var(\alpha|\underline{x}) = \left[\frac{u^2 I''(u) + u I'(u)}{I(u)}\right] - \left[\frac{u\, I'(u)}{I(u)}\right]^2. \tag{19}$$

Now it will be difficult to obtain the risk of α analytically. So we proceed the further calculation numerically using the following steps.

- *Step 1 :* Initialize the parameters $\alpha = 5.5, 6.5, 7, 7.5, 8$ and $p = 0.2, 0.3$.

Table 5. Average Bayes estimate and corresponding Bayes risk (in parenthesis) for the shape parameter.

p	Sample sizes (n)	$\alpha = 5.5$	$\alpha = 6.5$	$\alpha = 7$	$\alpha = 7.5$	$\alpha = 8$
p = 0.2	25	0.587074 (0.010433)	0.638052 (0.011993)	0.662800 (0.012758)	0.687120 (0.01352)	0.711188 (0.014285)
	30	0.585804 (0.00866)	0.636770 (0.009984)	0.661598 (0.010623)	0.685952 (0.01126)	0.710086 (0.011901)
	50	0.584160 (0.005383)	0.635868 (0.005803)	0.660472 (0.006378)	0.684672 (0.006886)	0.708787 (0.007244)
	75	0.583393 (0.003264)	0.636296 (0.002687)	0.657151 (0.005488)	0.680799 (0.006175)	0.70698 (0.005046)
	80	0.583952 (0.002661)	0.636719 (0.002151)	0.655930 (0.005958)	0.679960 (0.00645)	0.707072 (0.004651)
	100	0.588964 (-0.001282)	0.637604 (0.00059)	0.648305 (0.009769)	0.674608 (0.008793)	0.708693 (0.002184)
p = 0.3	25	0.883195 (0.019841)	0.981651 (0.023442)	1.030223 (0.025163)	1.078385 (0.026873)	1.126429 (0.028594)
	30	0.882036 (0.01603)	0.980087 (0.019519)	1.028815 (0.020979)	1.077052 (0.022419)	1.125278 (0.023814)
	50	0.881307 (0.009081)	0.979623 (0.011735)	1.028686 (0.012487)	1.077008 (0.013384)	1.124631 (0.014799)
	75	0.879196 (0.006551)	0.977713 (0.00799)	1.026446 (0.008684)	1.075502 (0.008254)	1.124583 (0.007876)
	80	0.879287 (0.006047)	0.977760 (0.007497)	1.026217 (0.00834)	1.075540 (0.00779)	1.125248 (0.006599)
	100	0.879777 (0.004201)	0.977597 (0.005826)	1.025489 (0.007051)	1.073526 (0.007656)	1.126718 (0.002292)

- *Step 2 :* For each combination of (α, p) generate sample of sizes $n = 25, 30, 50, 75, 80, 100$ from Gamma distribution and then calculate Bayes estimate for each.
- *Step 3 :* Repeat *Step 2* for $K = 1000$ times. For e.g., if we use n = 25 samples and calculate posterior variance and then such average of 1000 posterior risks will give Bayes risk by strong law of large number i.e. that will be equivalent to finding integration over sample space.
- *Step 4 :* Calculate $u = (p^n \prod_{i=1}^{n} x_i)$, K times considering each of the sample size.
- *Step 5 :* Posterior mean and posterior variance are obtained from (13) and (19) respectively.
- *Step 6 :* Take the average of the posterior variance to obtain the risk values.

At present, the idea behind the simulation study is to produce low value of posterior variance so that we can obtain the minimum risk value of α. To attain this target, it is preferable to multiply small quantity of u in (19), so that the variance can not be large. As u incorporates the scale parameter and the generated sample in a multiplicative form, both should be small in quantity. With the choices of shape parameter as mentioned in *step 1*, we are able to generate small value of $X_i's$ as much as possible so that the values of u becomes small for most of the samples. The sample sizes are chosen in an increasing order to represent small, moderate and large sample performances. All the numerical results are tabulated in Table 5 from R Core Team software [16].

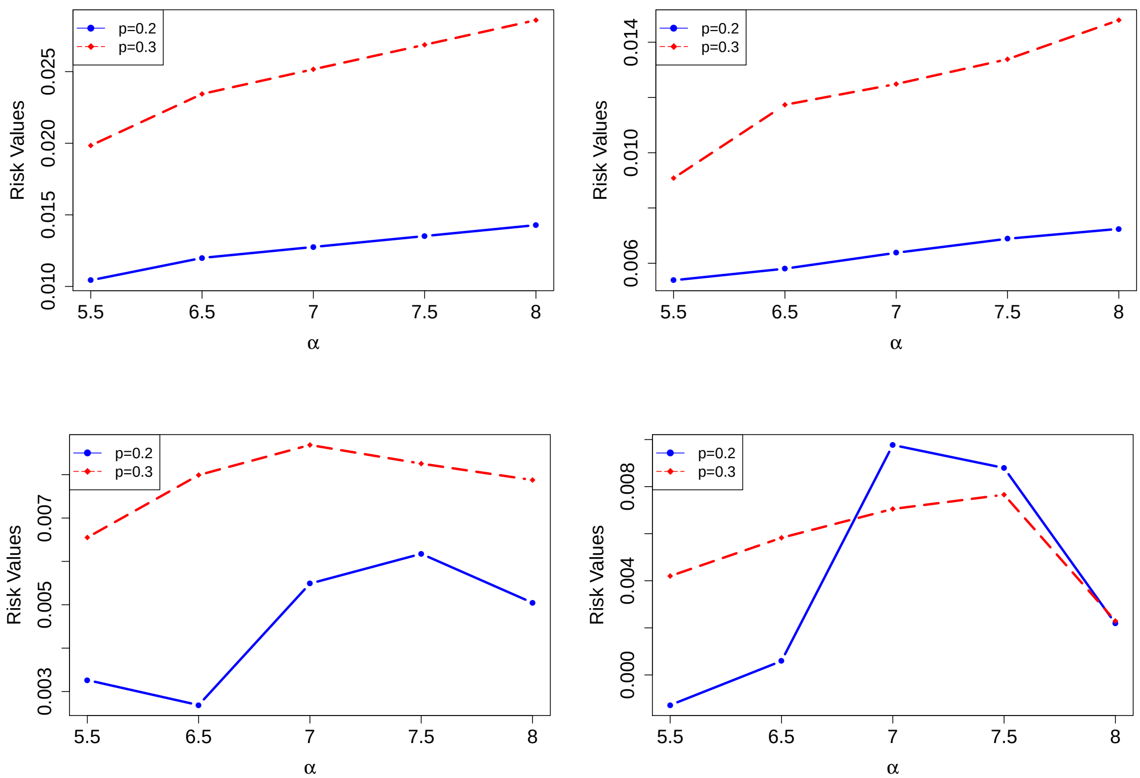

Figure 1. Pictorial representations of above tabular calculation. From upper-left to lower-right: When sample size n = 25, When sample size n = 50, When sample size n = 75 and When sample size n = 100.

To study the risk functions graphically, the risk values are plotted vertically while the initial values of the shape parameter are in the horizontal axis. In Figure 1 it has been observed that, for the small size i.e., when n = 25, the risk values are gradually increasing with the increment of shape parameter α. For large size of sample (n = 100), the risk of $\hat{\alpha}$ corresponding to p = 0.2 and p = 0.3 converge to the almost same point 0.002 . All the risk values are very small in quantity. So it is expected that risk function values are small. Thus we get an algorithm for calculating Bayes estimate for the shape parameter.

7. Real data demonstrations

In this section, we analyze two datasets to illustrate the methodology mentioned in this article. In first example we consider average monthly rainfall data while in the second scenario the remission times (in months) of bladder cancer patient's data are used. One natural concern about these datasets is whether or not they fit the Gamma distribution. Goodness-of-fit testing is a crucial part of any statistical studies, since it reveals the gap between a considered statistical model and the available data Ali et al. [24]. There are several approaches available in literature for determining a model's goodness of fit to a specific dataset. Among them, Pearson's χ^2 and Kolmogorov–Smirnov tests are widely used. Here we only focus on Kolmogorov–Smirnov (K-S) distances between empirical and fitted distributions as it is well known that for limited sample observations Pearson's χ^2 test may not perform well. The K-S distance values are derived for the Rainfall dataset and Bladder cancer dataset and it has been found that they are 0.0978 and 0.0683 respectively. The fitting of these two real datasets are also verified through some diagnostics plots. We compare the fits of the Gamma distribution (G) with some other well-known two parameter models like Weibull (W), Logistic (L), Inverse Gamma (InvG), Exponential power (EP) [22], Marshall-Olkin Exponential (MOE) [23], the Burr X Exponential (BrXE) [33] and Burr-Hatke Logarithmic BXII (BH-BXII) [31] distributions. In order to compare the fits of the distributions, we consider Cramèr-Von-Mises (CVM) and Anderson-Darling (AD) goodness of fit statistics. These two statistics are essentially modifications of the K-S test statistic and are usually thought to be more powerful than the original K-S test, which are also favoured by many researchers (for example Yousof et al. [32], Ibrahim et al. [30], Khalil et al. [26], Lak et al. [27], Elgohari [29], Mohamed et al. [28]). Moreover for more accuracy, we consider another five goodness of fit measures based on the log-likelihood functions namely, -2 $\times$ log-likelihood (-2logL), Akaike Information Criterion (AIC), Consistent Akaike Information Criterion (CAIC), Hannan-Quinn Information Criterion (HQIC), Bayesian Information Criterion (BIC). The smaller values of these statistics are indication of the better fitting. Table 7 and 10, have come with a summary of the fitted information criteria and estimated MLE's for both the datasets with different models. Thus it is appropriate to use two parameter Gamma distributions.

7.1. Rainfall data application

Table 6. Historical rainfall averages over last 56 years in State of São Paulo.

0.2	3.5	2.8	3.7	8.7	6.9	7.4	0.8	4.8	2.5	2.9	3.1	4.0	5.0	3.8	3.5	5.4	3.3
2.9	1.7	7.3	2.9	4.6	1.1	1.4	3.9	6.2	4.1	10.8	3.8	7.3	1.8	6.7	3.5	3.2	5.2
2.8	5.2	5.4	2.2	9.9	2.1	4.7	5.5	2.6	4.1	5.4	5.5	2.1	1.9	8.8	1.3	24.1	5.4
6.2	2.9																

Table 7. Parameter estimate and Information criteria for Rainfall data.

Model	**MLE**	**-2logL**	**AIC**	**CAIC**	**HQIC**	**BIC**	**CVM**	**AD**
G (α, p)	$\hat{\alpha}$=2.3962 $\hat{p}$=0.5185	264.159	268.1579	268.3843	269.7284	272.2086	0.06992	0.50617
W (θ, λ)	$\hat{\theta}$=1.5028 $\hat{\lambda}$=0.1941	268.6427	272.6427	272.8691	274.2131	276.6934	0.15167	1.0343
InvG (α, p)	$\hat{\alpha}$=1.4834 $\hat{p}$=3.7758	295.5672	299.5672	299.7936	301.1376	303.6179	0.56990	3.23680
EP (α, β)	$\hat{\alpha}$=8.5345 $\hat{\beta}$=0.8683	285.8483	289.8483	290.0747	291.4187	293.899	0.50788	3.14460
L (μ, s)	$\hat{\mu}$=4.1907 $\hat{s}$=1.5171	275.8025	279.8025	280.0289	281.3730	283.8532	0.08714	0.83050
MOE (α, λ)	$\hat{\alpha}$=7.6916 $\hat{\lambda}$=0.5285	266.1421	270.1421	270.3685	271.7125	274.1928	0.07459	0.59099
BrXE (θ, λ)	$\hat{\theta}$=0.3588 $\hat{\lambda}$=0.0612	305.8106	309.8106	310.0370	311.3810	313.8613	1.30572	6.67065
BH-BXII (a_1, a_2)	$\hat{a_1}$=3.3113 $\hat{a_2}$=0.1199	317.6960	321.6960	321.9224	323.2664	325.7467	1.62932	8.11363

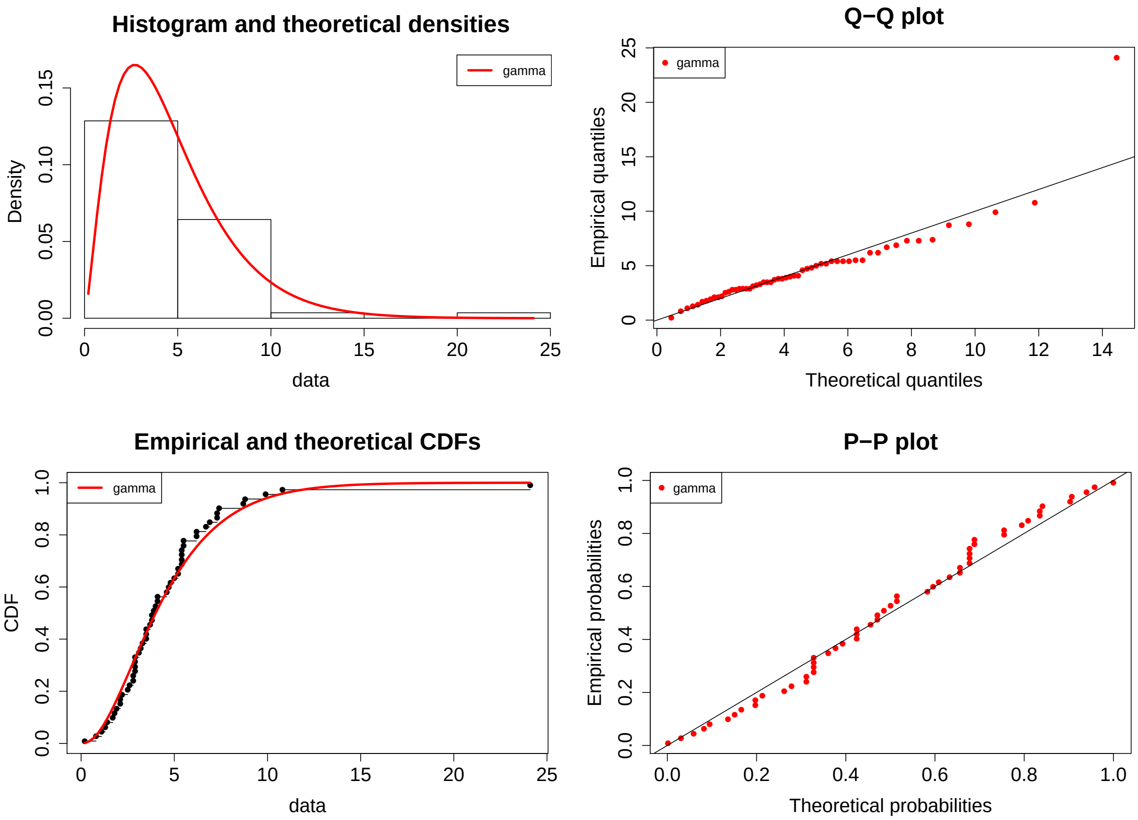

Figure 2. Diagnostic plots of fitted Gamma distribution for Rainfall data. From upper-left to lower-right: Histogram, Q-Q plot, Empirical VS Theoretical CDF and P-P plot.

The dataset in Table 6 represents the average monthly rainfall obtained from the Information System for Management of Water Resources of the State of São Paulo, including a period of 56 years from 1947 to 2003, by considering the month of November reported by Moala et al. [8]. The sample mean and sample variance for the rainfall data are 4.621 and 12.217 respectively. The maximum likelihood estimate of the parameters α and p obtained as $\hat{\alpha}_{MLE}$ = 2.396 and $\hat{p}_{MLE}$ = 0.518 for the dataset. We further calculate the PB estimate and Bayes estimate from (9) and (8) for the complete sample and splitting the original datasets in two groups.

Table 8. Partial Bayes and Bayes estimates for the Rainfall data.

Group	Partial Bayes estimate (PB)	Bayes estimate (BE) when $\alpha = 2.396$
Total	0.4853	0.5185
Group 1	0.6344	0.6189
Group 2	0.4317	0.4461

From Table 7, we conclude that the Gamma distribution provides better fit for Rainfall data compared to other comparative models with $CVM_{statistic} = 0.06992$ and $AD_{statistic} = 0.50617$. Also it has lowest value regarding -2logL = 264.1579, AIC =268.1579, CAIC = 268.3843, HQIC = 269.7284 and BIC = 272.2086.

In Table 8, we divide the rainfall data equally in two groups consisting of sample sizes n = 28 for both the groups and for each of these groups, the Bayes estimate and the PB estimate are given. The idea is to study how the proposed Partial Bayes estimate performs in real situations. As the Bayes estimate involves the shape parameter α, we use MLE $\hat{\alpha}$ to calculate the Bayes estimate of p. From Table 8, it is observed that PB estimate of the total dataset is 0.4853 which does not differ too much from $\hat{p}_{MLE}$ as well as the Bayes estimate of p. Also after splitting the dataset into two groups, the PB estimators for group 1 and group 2 are very close to original Bayes estimators. So, we can say that the performance of the PB estimate is as stable as the Bayes estimate. Moreover, it is to be noted that the Bayes estimate depends on the shape parameter while the PB estimate is fully dependent on the data. Therefore, we can calculate the Partial Bayes estimate directly from the data without any prior knowledge of parameters. So, our proposed estimate behaves logically sound and not too much different from Bayes estimate.

7.2. Bladder cancer data application

Table 9. Remission times (in months) of a random sample of 128 bladder cancer patients.

0.08	2.09	3.48	4.87	6.94	8.66	13.11	23.63	0.20	2.23	3.52	4.98	6.97	9.02
13.29	0.40	2.26	3.57	5.06	7.09	9.22	13.80	25.74	0.50	2.46	3.64	5.09	7.26
9.47	14.24	25.82	0.51	2.54	3.70	5.17	7.28	9.74	14.76	26.31	0.81	2.62	3.82
5.32	7.32	10.06	14.77	32.15	2.64	3.88	5.32	7.39	10.34	14.83	34.26	0.90	2.69
4.18	5.34	7.59	10.66	15.96	36.66	1.05	2.69	4.23	5.41	7.62	10.75	16.62	43.01
1.19	2.75	4.26	5.41	7.63	17.12	46.12	1.26	2.83	4.33	5.49	7.66	11.25	17.14
79.05	1.35	2.87	5.62	7.87	11.64	17.36	1.40	3.02	4.34	5.71	7.93	1.46	18.10
11.79	4.40	5.85	8.26	11.98	19.13	1.76	3.25	4.50	6.25	8.37	12.02	2.02	13.31
4.51	6.54	8.53	12.03	20.28	2.02	3.36	12.07	6.76	21.73	2.07	3.36	6.93	8.65
12.63	22.69												

Table 10. Parameter estimate and Information criteria for Bladder cancer patient's data.

Model	MLE	-2logL	AIC	CAIC	HQIC	BIC	CVM	AD
G (α, p)	$\hat{\alpha}$=1.1781 $\hat{p}$=0.1247	828.7471	832.7471	832.8431	835.0646	838.4511	0.11784	0.68492
W (θ, λ)	$\hat{\theta}$=1.0528 $\hat{\lambda}$=0.1035	830.1968	834.1968	834.2928	836.5144	839.9009	0.13804	0.87435
InvG (α, p)	$\hat{\alpha}$=0.7098 $\hat{p}$=1.7715	913.8213	917.8213	917.9173	920.1389	923.5254	1.5969	8.5313
EP (α, β)	$\hat{\alpha}$=18.8723 $\hat{\beta}$=0.6599	854.9746	858.9746	859.0706	861.2922	864.6787	0.57446	3.5541
L (μ, s)	$\hat{\mu}$=7.6863 $\hat{s}$=4.4958	913.6324	917.6324	917.7284	919.9500	923.3364	0.5841	4.7896
MOE (α, λ)	$\hat{\alpha}$=1.0924 $\hat{\lambda}$=0.1109	830.7273	834.7273	834.8233	837.0049	840.4314	0.15363	1.02399
BrXE (θ, λ)	$\hat{\theta}$=0.2389 $\hat{\lambda}$=0.0198	911.5886	915.5886	915.6846	917.9061	921.2926	2.64599	13.19588
BH-BXII(a_1, a_2)	$\hat{a_1}$=2.3622 $\hat{a_2}$=0.1247	927.1633	931.1633	931.2593	933.4809	936.8674	2.86226	14.32391

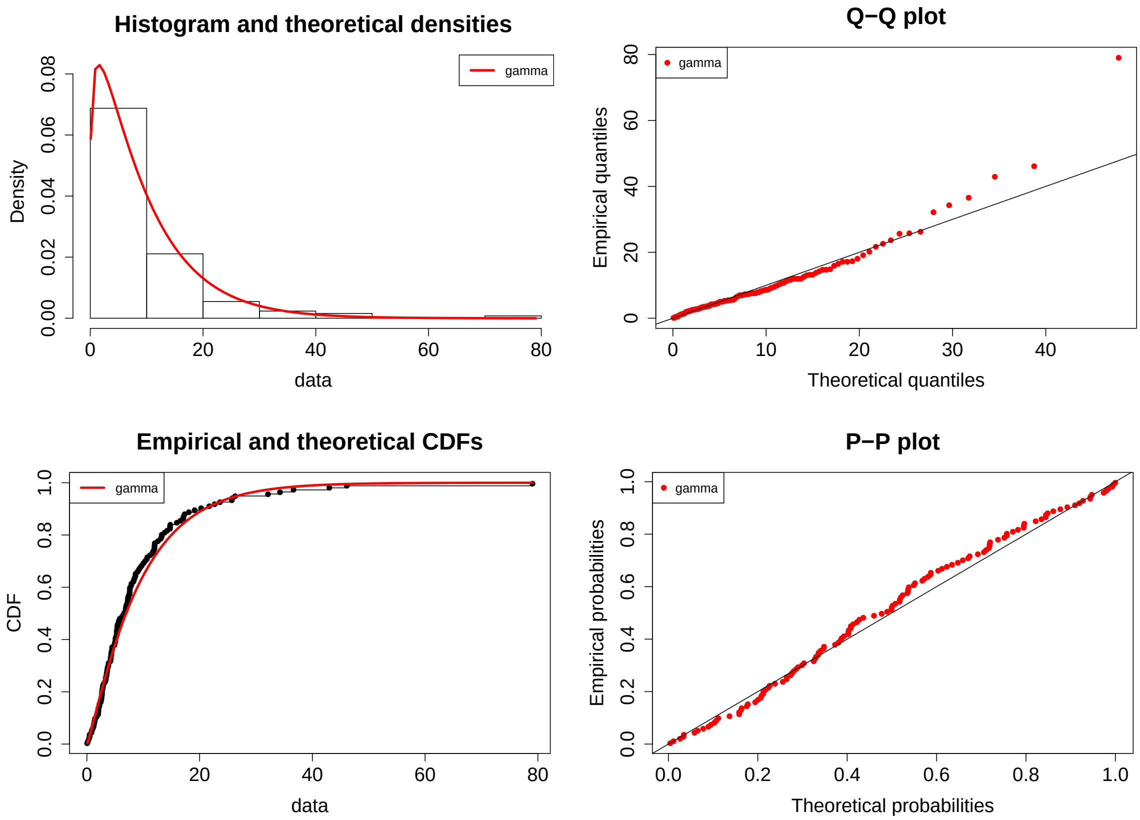

Figure 3. Diagnostic plots of fitted Gamma distribution for Rainfall data. From upper-left to lower-right: Histogram, Q-Q plot, Empirical VS Theoretical CDF and P-P plot.

Now, let us consider a survival dataset taken from Elgohari and Yousof [17]. This dataset is originally reported by Lee and Wang [18] and represents the remission times (in months) of 128 bladder cancer patients. Respective sample mean and variance for the considered dataset are 9.444 and 110.253. The maximum likelihood estimates of the parameters computed using the dataset given as $\hat{\alpha}_{MLE} = 1.178$ and $\hat{p}_{MLE} = 0.125$.

Similarly, as mentioned in previous Subsection 7.1, here also from Table 10 we see that Gamma distribution provides much better fit than the other comparative model for Bladder cancer patient's data with $CVM_{statistic} = 0.11784$ and $AD_{statistic} = 0.68492$. Also by considering other Information Criteria, i.e., -2logL = 828.7471, AIC = 832.7471 , CAIC = 832.8431, HQIC = 835.0646 and BIC = 838.4511 the Gamma model is selected better alternative for modeling this data. For this dataset, we also compute the PB estimate and Bayes estimate for

Table 11. Partial Bayes and Bayes estimates for the Bladder cancer patient's data.

Group	Partial Bayes estimate (PB)	Bayes estimate (BE) when $\alpha = 1.178$
Total	0.1101	0.1247
Group 1	0.1088	0.1366
Group 2	0.1156	0.1148

complete sample and consequently splitting this dataset into two groups. The results are tabulated in Table 11 when the size of the each group is n = 64. Here, we also try to understand the consistency of the PB estimate compared to the MLE and the Bayes estimate for the large as well as for small sample groups. It has been clearly verified from Table 11 that the proposed PB estimate does not differ too much from the $\hat{p}_{MLE}$ as well as Bayes estimate of the scale parameter p. However, it may also be noticeable that the Partial Bayes estimate is directly computed from the available sample whereas the Bayes estimate depends on the another parameter of the baseline distribution.

8. Conclusion

In this article, a new variant of Bayes estimate is introduced and we call it as Partial Bayes (PB) estimate. The main idea behind the PB estimation is that, in multi-parameter model when we do not have any knowledge about the joint prior but we know it partially. This means that one of the parameter of the model is unknown, then that unknown parameter is replaced by the estimated value obtained from any of the classical method (i.e. maximum likelihood estimator, method of moments estimator etc). But it is clear that it is neither empirical nor hierarchical method. Here, in the Gamma model, the parameter of interest is the scale parameter p only which is estimated by using Partial Bayes approach by putting the MLE of the another parameter α involved in the Bayes estimate of p. We compare the newly obtained PB estimate with the conventional Bayes estimate of the scale parameter in terms of risks for different choices of the parameter values. It is seen that the risk values are smaller for the PB estimator than the Bayes estimator for maximum combinations of the parameter choices. Also, it is observed that for higher choice of parameter values risks are quite same for both the estimators but for small values of parameter PB estimator performs very well.

In case of the shape parameter, Bayes estimation of α is performed through numerical evaluation with the help of the posterior mean and the posterior variance. Also from Table 5, risks for the shape parameter becomes small.

To associate the theoretical concepts with the real life situation, we have considered two real data sets and have demonstrated the performance of PB estimate along with the traditional Bayes estimate. From these, we can proceed with the PB estimation method to estimate our parameter of interest in spite of the absence of prior knowledge of the other parameter in a multi-parameter model. There are enough scopes for doing this type of works for some other distributions also.

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