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Modified Generalized Linear Exponential Distribution: Properties and Applications

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Abstract In this paper, we propose a new four-parameter lifetime distribution called modified generalized linear exponential distribution. The proposed distribution is a modification of the generalized linear exponential distribution. Several important lifetime distributions in reliability engineering and survival analysis are considered as special sub-models including modified Weibull, Weibull, linear exponential and generalized linear exponential distributions, among others. We study the mathematical and statistical properties of the proposed distribution including moments, moment generating function, modes, and quantile. We then examine hazard rate, mean residual life, and variance residual life functions of the distribution. A significant property of the new distribution is that it can have a bathtub-shaped, which is very flexible for modeling reliability data. The four unknown parameters of the proposed model are estimated by the maximum likelihood. Finally, two practical real data sets are applied to show that the proposed distribution provides a superior fit than the other sub-models and some well-known distributions.

Keywords Modified generalized linear exponential distribution, Reliability analysis, Bathtub hazard rate, parameter estimation

AMS 2010 subject classifications 60E05, 62G05, 62P30, 62F10

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1. Introduction

The quality of procedures utilized in statistical analysis fundamentally relies upon the proposed lifetime distribution. Subsequently, the bigger the available lifetime distributions accessible to the statistician, the simpler it is to select a model. Since the available lifetime distributions are incompatible or fit weakly with data of many important problems in engineering, medicine, and more other topics. Therefore, the generalization of the available distributions become an urgent requirement because of their flexible properties. So, many researchers have proposed various extensions, exponentiated and modified forms of Weibull distribution (WD) and with number of parameters [[1]-[2]]. Several techniques for generalizing the available lifetime distributions are used by several authors . One of these important techniques known as modified distribution by adding one or more parameters to a baseline distribution and was first proposed by [3]. They proposed a new generalization for WD by multiplying its cumulative hazard rate (HR) by exponential function and known as modified Weibull distribution (MWD).

The linear exponential distribution (LED), which was first proposed by [4], has just an increasing HR and notable for fitting lifetime data in reliability analysis. So, many authors were interested in generalizing this distribution to get a flexible model with decreasing, unimodal, and bathtub HR function. The extension of LED, which called as

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Marshall-Olkin linear failure rate distribution (M-OLFRD) was studied by [5]. [6] studied the generalized linear failure rate distribution (ELFRD) and demonstrated that the HR can be increasing, decreasing and bathtub shaped. The transmuted linear failure rate distribution (TLFRD) with increasing, decreasing, unimodal and bathtub HR shaped was introduced by [7]. The Beta linear failure rate distribution (BLFRD), which was introduced by [8], can have a constant, decreasing, increasing and bathtub-shaped HR functions. Another generalization of LED was called the generalized linear exponential distribution (GLED) and was proposed first by [9]. Several authors [[10], [11], [12] and [13]] have considered the generalization for the generalized linear exponential distribution (GLED).

This article attempts to propose a new lifetime distribution with four parameters, called modified generalized linear exponential distribution (MGLED). The proposed distribution is of significance since it includes many sub-models such as GLED, MWD, WD, LED, Rayleigh distribution (RD) and exponential distribution (ED). Also, it provides a new opportunity in modeling the different characteristics of lifetime data sets. Some statistical properties and the properties in terms of reliability analysis for MGLED are studied. Furthermore, while studying paper [14], we found a mistake in Section 4. So, we attempt to discuss this mistake in Section 3 and introduce a numerical study for this purpose. The new distribution gives a reasonable parametric fit for modeling data with bathtub failure rates, which are very useful in modeling reliability analysis. We hope that the proposed distribution will attract many applications in different branches of science, engineering, biology, and others. The remainder of the article is organized into seven sections. Section 2 introduces some statistical functions of MGLED. The correction of the formula of the moments of MWD, which proposed first in [14], is studied in Section 3. Section 4 derives some important statistical properties. Properties of the MGLED in terms of reliability analysis are given in Section 5. Section 6 describes parameters estimation by maximum likelihood estimation (MLE). Two applications to real data are presented in Section 7.

2. Modified Generalized Linear Exponential Distribution

For a non-negative random variable X, the cumulative distribution function (cdf) of GLED is given by

$$F(x;c,b,\xi) = 1-\ e^{-(c\ x+\frac{b}{2}x^2)^{\xi}},\quad c,\ b\geq 0,\ \xi>0,\ x>0. \tag{1}$$

Same to the idea in [3], the cumulative HR of GLED $(c\,y+\frac{b}{2}y^2)^{\xi}$ is multiplied by the term $e^{\varphi(c\ y+\frac{b}{2}y^2)}$ in order to obtain the MGLED. The term takes this form to perform the convergent condition for Lambert W function as shown in Section 4. Then the cdf of the MGLED with parameter vector $\Phi=(c,b,\xi,\varphi)$ is given by

$$F(x;\Phi) = 1-\ e^{-(c\ x+\frac{b}{2}x^2)^{\xi}\ e^{\varphi(c\ x+\frac{b}{2}x^2)}},\qquad c,\ b\geq 0,\ \xi,\ \varphi>0,\ x>0, \tag{2}$$

where the parameters c and b affect the scale of the distribution, the parameter ξ affect the shape of distribution, and the parameter φ is an acceleration parameter. The probability density function (pdf), the survival and HR functions of MGLED are given by:

$$\begin{aligned} f(x;\Phi) &= (c+b\,x)\,(c\,x+\frac{b}{2}x^2)^{\xi-1}\left(\xi+\varphi\,(c\,x+\frac{b}{2}x^2)\right)\\ &\quad\times e^{\varphi(c\ x+\frac{b}{2}x^2)}\ e^{-(c\ x+\frac{b}{2}x^2)^{\xi}\ e^{\varphi(c\ x+\frac{b}{2}x^2)}}, \end{aligned} \tag{3}$$

$$S(t;\Phi) = \ e^{-(c\ t+\frac{b}{2}t^2)^{\xi}\ e^{\varphi(c\ t+\frac{b}{2}t^2)}},\qquad t>0, \tag{4}$$

and

$$h(t;\Phi) = (c+b\,t)\,(c\,t+\frac{b}{2}t^2)^{\xi-1}\left(\xi+\varphi\,(c\,t+\frac{b}{2}t^2)\right)e^{\varphi(c\ t+\frac{b}{2}t^2)}, \tag{5}$$

respectively.

Table 1 indicates a list of distributions that can be derived from MGLED. Some possible shapes for the pdf and the corresponding hazard functions of MGLED are exhibited in Figure 1. Based on different values of parameters, the pdf of MGLED can be decreasing and unimodal while the HR function is decreasing, increasing and bathtub shape.

Table 1. The sub-models from the MGLED

Distribution	c	b	ξ	φ	cdf	Reference
GLED	-	-	-	0	$1-\ e^{-\left(c\ x+\frac{b}{2}x^2\right)^{\xi}}$	[9]
LED	-	-	1	0	$1-\ e^{-\left(c\ x+\frac{b}{2}x^2\right)}$	[4]
MWD	-	0	-	1	$1-\ e^{-(c\ x)^{\xi}\ e^{\varphi(c\ x)}}$	[3]
WD	-	0	-	0	$1-\ e^{-c\ x^{\xi}}$	[15]
RD	0	-	1	0	$1-\ e^{-\frac{b}{2}x^2}$	[4]
ED	-	0	1	0	$1-\ e^{-c\ x}$	[4]
NMLED	-	-	1	-	$1-\ e^{-\left(c\ x+\frac{b}{2}x^2\right)\ e^{\varphi\left(c\ x+\frac{b}{2}x^2\right)}}$	New
NMRD	0	-	1	-	$1-\ e^{-\frac{b}{2}x^2\ e^{\frac{\varphi b}{2}x^2}}$	New
NMED	-	0	1	-	$1-\ e^{-c\ x\ e^{\varphi c\ x}}$	New

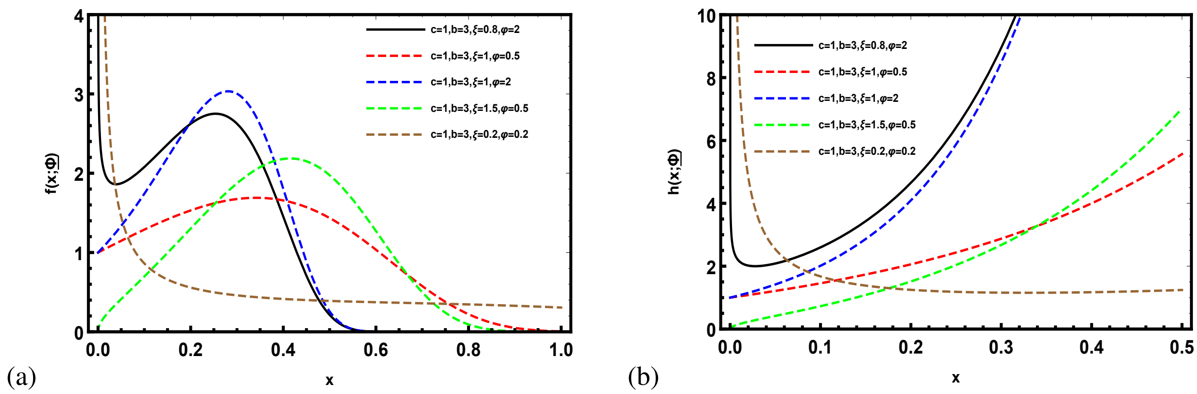


Figure 1. (a) pdfs and (b) corresponding hazard rate functions of MGLED with different values of parameters.

3. Correction of the Formula for the Moments of MWD

In this section, the correction of the formula for the r-th moments of MWD provided in [14] are studied and a numerical study is used to show that our correction formula is precise. For this purpose, three formulae for r-th moments of MWD are presented. All notations and symbols in this section are taken from [14].
Formula I: Numerical integration for original definition of r-th moment of MWD.

$$\mu^{(r)} = \int_0^{\infty} \alpha\, t^r\, t^{\gamma-1}\, e^{\lambda t}\, (\gamma+\lambda\, t) e^{-\alpha\, t^{\gamma}\, e^{\lambda t}}\, dt.$$

Formula II: The correction of the formula given in [14], after taking into consideration the condition of convergent, is given by

$$\mu^{(r)} = \sum_{i_1,\dots,i_r=1}^{\infty} A_{i_1,\dots,i_r} \left(\frac{1}{\alpha}\right)^{\frac{s_r}{\gamma}} \left[\Gamma\left(\frac{s_r}{\gamma}+1\right) - \Gamma\left(\frac{s_r}{\gamma}+1, \alpha\left(\frac{\gamma}{\lambda e}\right)^{\gamma}\right)\right],$$

where $A_{i_1,\dots,i_r}$ and s_r are given on page 453 of [14]. The proof for this formula can be shown in Appendix A. Formula III: The formula of the r-th moments given at the end of Page 454 of [14] which is given by

$$\mu^{(r)} = \sum_{i_1,\dots,i_r=1}^{\infty} A_{i_1,\dots,i_r} \left(\frac{1}{\alpha}\right)^{\frac{s_r}{\gamma}} \Gamma\left(\frac{s_r}{\gamma}+1\right).$$

Table 2 shows that Formula III cannot be given the r-th moments of MWD without any restriction on the parameters which is a contradiction with Lines 10 and 11 Page 454 in [14]. The values of Formula II are more close to the value of Formula I than Formula III. The (-) in Table 2 indicates that the output from Formula III is divergent. From the above, it is clear that Formulae II is more accurate than Formula III which proposed by [14].

Table 2. Comparison for Formulae (I, II and III) of the first two moments of MWD under various values of parameters.

		Formula	$\gamma = 5$	$\gamma = 6$
$r = 1$				
$\lambda = 1$	$\alpha = 1$	I	0.7802	0.8074
		II	0.7802	0.8074
		III	-	-
	$\alpha = 1.5$	I	0.7274	0.7607
		II	0.7274	0.7607
		III	-	-
$r = 2$				
$\lambda = 1$	$\alpha = 1$	I	0.6335	0.6715
		II	0.6335	0.6715
		III	-	-
	$\alpha = 2$	I	0.4985	0.5478
		II	0.4985	0.5478
		III	-	-

4. Statistical Properties

This section deals with the statistical properties of MGLED such as moment, moment generating function, quantiles, and mode.

4.1. Moments

In the following theorem, an explicit forms of the k-th moments of MGLED are given.

Theorem 1

Let X be a non-negative continuous random variable. Then the k-th moments $\mu^{(k)}$ of MGLED; $k = 1, 2, 3, \dots$ are given by

$$\mu^{(k)} = \begin{cases} \sum_{j=0}^{k}\sum_{m=0}^{\infty}\sum_{i=0}^{\infty} A_{j,m}\, d_i(m)\, \gamma\Big(A1_{i,m}, u\Big) + \\ \sum_{j=0}^{k}\sum_{m=0}^{\infty}\sum_{i=0}^{\infty} B_{j,m}\, d_i\big(\frac{k-j-2m}{2}\big) \\ \times \Bigg(\gamma\Big(B1_{i,j,m}, w\Big) - \gamma\Big(B1_{i,j,m}, u\Big)\Bigg) & w > u; \\ \\ \sum_{j=0}^{k}\sum_{m=0}^{\infty}\sum_{i=0}^{\infty} A_{j,m}\, d_i(m)\, \gamma\Big(A1_{i,m}, w\Big)\Bigg) & w < u. \end{cases} \tag{6}$$

where
$A_{j,m} = (-1)^j \binom{k}{j} \binom{\frac{k-j}{2}}{m} (\frac{c}{b})^k (\frac{2\,b}{c^2})^m$, $A1_{i,m} = \frac{\xi+m+i}{\xi}$, $B_{j,m} = (-1)^j \binom{k}{j} \binom{\frac{k-j}{2}}{m} 2^{\frac{k-j}{2}-m}\, c^{j+2\,m}\, b^{-(\frac{k+j+2m}{2})}$
$B1_{i,j,m} = \frac{2\,i+2\,\xi+k-j-2\,m}{2\,\xi}$, $\gamma(.,.)$ is the lower incomplete gamma function, $u = (\frac{c^2}{2\,b})^{\xi}\, e^{\varphi\,(\frac{c^2}{2\,b})}$,
$w = (\frac{\xi}{\varphi\, e})^{\xi}$, $d_0(p) = a_0^p$, $d_i(p) = \frac{1}{i\, d_0} \sum_{l=1}^{i} \Big(l\,(p) - i + l\Big)\, a_l\, d_{i-l}$ and $a_i = \frac{(-1)^{i+2}}{i!}\, (i+1)^{(i-1)}\, (\frac{\varphi}{\xi})^i$.

Proof 1
The k-th moments of MGLED can be written in the form

$$\begin{aligned} \mu^{(k)} &= \int_0^{\infty} x^k\, (c+b\,x)\,(c\,x+\frac{b}{2}x^2)^{\xi-1} \Big(\xi + \varphi\,(c\,x+\frac{b}{2}x^2)\Big) \\ &\quad \times e^{\varphi(c\,x+\frac{b}{2}x^2)}\ e^{-(c\,x+\frac{b}{2}x^2)^{\xi}}\, e^{\varphi(c\,x+\frac{b}{2}x^2)}\, dx. \end{aligned} \tag{7}$$

Upon using the substitution $v = (c\,x+\frac{b}{2}x^2)^{\xi}\, e^{\varphi(c\,x+\frac{b}{2}x^2)}$, it can be shown that $(c\,x+\frac{b}{2}x^2) = \sum_{i=0}^{\infty} a_i\, v^{\frac{i+1}{\xi}}$ (see [14]) and $x = \frac{-c+\sqrt{c^2+2\,b\,\sum_{i=0}^{\infty} a_i\, v^{\frac{i+1}{\xi}}}}{b}$. Here, it is easy to show that the condition of convergence of this sum is $v < (\frac{\xi}{\varphi\, e})^{\xi}$. Then the k-th moments of MGLED are given by

$$\mu^{(k)} = \int_0^{(\frac{\xi}{\varphi\,e})^{\xi}} \Big(\frac{-c+\sqrt{c^2+2\,b\,\sum_{i=0}^{\infty} a_i\, v^{\frac{i+1}{\xi}}}}{b}\Big)^k\, e^{-v}\, dv.$$

Expanding $\Big(\frac{-c+\sqrt{c^2+2\,b\,\sum_{i=0}^{\infty} a_i\, v^{\frac{i+1}{\xi}}}}{b}\Big)^k$ yields,

$$\mu^{(k)} = \int_0^{(\frac{\xi}{\varphi\,e})^{\xi}} \sum_{j=0}^{k} \binom{k}{j} (-c)^j \Big(c^2 + 2\,b \sum_{i=0}^{\infty} a_i\, v^{\frac{i+1}{\xi}}\Big)^{\frac{k-j}{2}}\, e^{-v}\, dv.$$

It is clear that $\left|\frac{2\,b}{c^2}\ \sum_{i=0}^{\infty} a_i\, v^{\frac{i+1}{\xi}}\right| < 1$ if $v < (\frac{c^2}{2\,b})^{\xi}\, e^{\varphi\,(\frac{c^2}{2\,b})}$, and $\left|\frac{c^2}{2\,b\,\sum_{i=0}^{\infty} a_i\, v^{\frac{i+1}{\xi}}}\right| < 1$ if $v > (\frac{c^2}{2\,b})^{\xi}\, e^{\varphi\,(\frac{c^2}{2\,b})}$. Then, by binomial expansion, $\mu^{(k)}$ can be written as

$$\mu^{(k)} = \begin{cases} \sum_{j=0}^{k}\sum_{m=0}^{\infty} A_{j,m} \int_0^u (\sum_{i=0}^{\infty} a_i\, v^{\frac{i+1}{\xi}})^m\, e^{-v}\, dv \\ +\sum_{j=0}^{k}\sum_{m=0}^{\infty} B_{j,m} \int_u^w (\sum_{i=0}^{\infty} a_i\, v^{\frac{i+1}{\xi}})^{\frac{k-j-2\,m}{2}}\, e^{-v}\, dv, & w > u; \\ \\ \sum_{j=0}^{k}\sum_{m=0}^{\infty} A_{j,m} \int_0^w (\sum_{i=0}^{\infty} a_i\, v^{\frac{i+1}{\xi}})^m\, e^{-v}\, dv, & w < u. \end{cases}$$

Since $\left(\sum_{i=0}^{\infty} a_i \quad v^{\frac{i+1}{\xi}} \right)^{\frac{k-j-2\,m}{2}} = \sum_{i=0}^{\infty} d_i(\frac{k-j-2m}{2}) \quad v^{\frac{2\,i+k-j-2\,m}{2\,\xi}}$ and $\left(\sum_{i=0}^{\infty} a_i \quad v^{\frac{i+1}{\xi}} \right)^m = \sum_{i=0}^{\infty} d_i(m)\, v^{\frac{i+m}{\xi}}$ (see [16], page 17), $\mu^{(k)}$ can be presented as

$$\mu^{(k)} = \begin{cases} \sum_{j=0}^{k} \sum_{m=0}^{\infty} A_{j,m}\, d_i(m) \int_0^u v^{\frac{i+m}{\xi}}\, e^{-v}\, dv \\ + \sum_{j=0}^{k} \sum_{m=0}^{\infty} B_{j,m}\, d_i(\frac{k-j-2\,m}{2}) \int_u^w v^{\frac{2\,i+k-j-2\,m}{2\,\xi}} e^{-v}\, dv, & w > u; \\ \\ \sum_{j=0}^{k} \sum_{m=0}^{\infty} A_{j,m}\, d_i(m) \int_0^w v^{\frac{i+m}{\xi}}\, e^{-v}\, dv, & w < u. \end{cases}$$

Then the proof is completed.

Remark 1
When $\varphi = 0$, Equation (6) reduces to the k-moment of GLED (see [17]).

4.2. Moment Generating Function

In this subsection, the explicit form of the moment generating function of MGLED is given. Let X be a non-negative continuous random variable. Then the moment generating function of MGLED $M_X(t); t > 0$ is given by

$$M_X(t) = \begin{cases} \sum_{j=0}^{\infty} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \sum_{i=0}^{\infty} A^*_{j,m,n}\, d_i(n)\, \gamma\Big(A1^*_{i,n}, u \Big) + \\ \sum_{j=0}^{\infty} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \sum_{i=0}^{\infty} B^*_{j,m,n} \\ d_i(\frac{j-m-2n}{2}) \left(\gamma\Big(B1^*_{i,j,m,n}, w \Big) - \gamma\Big(B1^*_{i,j,m,n}, u \Big) \right) & w > u; \\ \\ \sum_{j=0}^{\infty} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \sum_{i=0}^{\infty} A^*_{j,m,n}\, d_i(n)\, \gamma\Big(A1^*_{i,n}, w \Big), & w < u. \end{cases}$$

where
$A^*_{j,m,n} = \binom{j}{m} \binom{\frac{j-m}{2}}{n} (\frac{-c}{b})^j\, \frac{t^j}{j!}\, (\frac{2\,b}{c^2})^n$, $A1^*_{i,n} = \frac{\xi+n+i}{\xi}$, $B^*_{j,m,n} = (-c)^m \binom{j}{m} \binom{\frac{j-m}{2}}{n} (\frac{1}{b})^j\, \frac{t^j}{j!}\, (2\,b)^{\frac{j-m}{2}} (\frac{c^2}{2\,b})^n$,
and $B1^*_{i,j,m,n} = \frac{2\,i+2\,\xi+j-m-2\,n}{2\,\xi}$.

4.3. Mode and Quantile

The mode and the quantile are presented in this section. The first derivative with respet to (w.r.t) x of the pdf for MGLED is given by

$$\frac{d}{dx} f(x;\Phi) = f(x;\Phi)\, p_1(x;\Phi),$$

where

$$\begin{aligned} p_1(x;\Phi) \quad &= \quad \varphi\,(c+b\,x) \left(1 + \frac{1}{\xi + \varphi\,(c\,x + \frac{b}{2}x^2)} \right) + \frac{b}{c+b\,x} + \frac{(\xi-1)(c+b\,x)}{(c\,x+\frac{b}{2}x^2)} \\ & \quad -e^{\varphi(c\,x+\frac{b}{2}x^2)}\, (c\,x + \frac{b}{2}x^2)^{\xi-1}\, (c+b\,x) \left(\xi + \varphi\,(c\,x + \frac{b}{2}x^2) \right). \end{aligned}$$

It is clear that the mode of MGLED is the solution of the non-linear equation $p_1(x;\Phi) = 0$ since $f(x;\Phi) > 0$. The non-linear equation $p_1(x;\Phi) = 0$ can be solved numerically. Moreover, the quantile of MGLED can be given by

$$x_q = \frac{-c + \sqrt{c^2 + 2\,b\, \sum_{i=0}^{\infty} a_i\, (-\ln(1-q))^{\frac{i+1}{\xi}}}}{b}, \quad 0 < q < 1. \tag{8}$$

Then the median of the MGLED is obtained by setting $q = 0.5$ in Equation (8) as

$$x_{0.5} = \frac{-c + \sqrt{c^2 + 2\,b\,\sum_{i=0}^{\infty} a_i\,(\ln 2)^{\frac{i+1}{\xi}}}}{b}. \tag{9}$$

Remark 2
For $\varphi = 0$, Equation (8) reduces to

$$x_q = \frac{-c + \sqrt{c^2 + 2\,b\,(-\ln(1-q))^{\frac{1}{\xi}}}}{b}, \quad 0 < q < 1, \tag{10}$$

which is the quantile of GLED.

Table 3 shows the first four ordinary moments, mode, median, skewness and kurtosis of the MGLED for different values of c, b, ξ and φ.

Table 3. The first four ordinary moments, mode, median, skewness and kurtosis of the MGLED for different parameters.

parameters	Mean	μ_2'	μ_3'	μ_4'	mode	median	skewness	kurtosis
c = 1.5 b =0.1 ξ =2.5 φ = 0.5	0.489	0.272	0.1657	0.1081	0.4918	0.4731	0.0707	2.6136
c = 1.5 b =0.5 ξ =2.5 φ = 0.5	0.4586	0.2367	0.1328	0.0793	0.4712	0.4472	-0.0139	2.5975
c = 1.5 b =1.5 ξ =2.5 φ = 0.5	0.4065	0.1831	0.0887	0.0454	0.4275	0.4004	-0.1317	2.6281
c = 1.5 b =3 ξ =2.5 φ = 0.5	0.3578	1402	0.0586	0.0257	0.3807	0.3547	-0.2189	2.6915
c = 0.1 b =1.5 ξ =2.5 φ = 0.5	0.9137	0.8742	0.8672	0.886	0.9726	0.9159	-0.4576	3.075
c = 0.5 b =1.5 ξ =2.5 φ = 0.5	0.7023	0.5278	0.4169	0.3424	0.7529	0.7021	-0.3705	2.8931
c = 3 b =1.5 ξ =2.5 φ = 0.5	0.2336	0.0616	0.0177	0.0054	0.2388	0.2273	0.0087	2.5987
c = 1.5 b =1.5 ξ =1.5 φ = 0.5	0.3644	0.1643	0.084	0.0469	0.352	0.3743	0.213	2.472
c = 1.5 b =1.5 ξ =3.5 φ = 0.5	0.4313	0.1977	0.0948	0.0472	0.4576	0.4133	-0.3339	2.8846
c = 1.5 b =1.5 ξ =2.5 φ = 0.1	0.4527	0.2311	0.1283	0.076	0.4613	0.4404	0.017	2.62
c = 1.5 b =1.5 ξ =2.5 φ = 1	0.3657	0.1463	0.0624	0.0279	0.3912	0.3631	0.2439	2.695

5. Properties of the MGLED in Terms of Reliability Analysis

In this section, some properties of the MGLED, which is important in reliability analysis, are studied. In particular, the behavior for HR, mean residual life (MRL), and variance residual life (VRL) are discussed.

5.1. Behavior of Hazard Rate Function

The behavior of the HR $h(t;\Phi)$ for MGLED is introduced in the following theorem.

Theorem 2
For $c > 0$ and $b > 0$,

1. MGLED has a bathtub HR if $0 < \xi < 1$, $\begin{cases} 0 < \varphi < \frac{2\,b\,\xi}{c^2} \; and \; t <> t_2 \; (see\; Figure\; 2(a)) or \\ \varphi \geq \frac{2\,b\,\xi}{c^2} \; and \; t <> t_4 \; (see\; Figure\; 2(b)). \end{cases}$
2. MGLED has increasing HR if $\xi \geq 1$, $\varphi > 0$ and $t > 0$ (see Figure 2 (c)),

where t_2 and t_4 given in the proof.

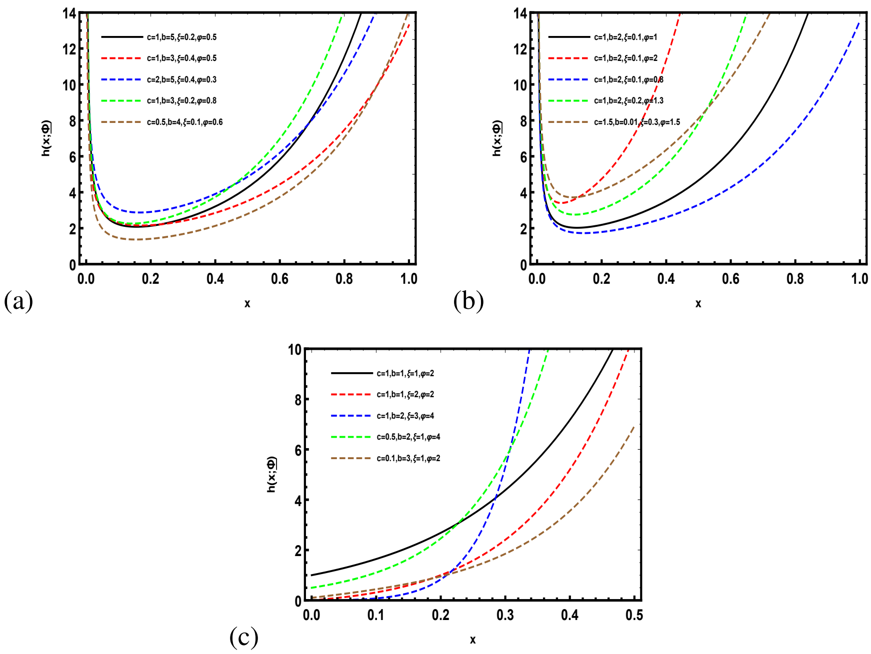


Figure 2. The various shapes of the hazard function of the MGLED.

Proof 2
The first derivative of $h(t;\Phi)$ can be obtained as

$$\frac{d}{dt}h(t;\Phi) = \frac{(c\,t + \frac{b}{2}t^2)^{\xi}}{t^2(2\,c + b\,t)^2}\, e^{\varphi\,(c\,t + \frac{b}{2}t^2)}\; p(t;\Phi),$$

where

$$\begin{aligned} p(t;\Phi) \;\; = \;\; & -2\,\xi\left(b^2\,t^2 + 2\,b\,c\,t + 2\,c^2\right) + \varphi^2\,t^2\,(b\,t + c)^2\,(b\,t + 2\,c)^2 \\ & + 4\,\xi^2\,(b\,t + c)^2 \; + \; \varphi\,t\,(b\,t + 2\,c)\left(4\,\xi\,(b\,t + c)^2 + b\,t(b\,t + 2\,c)\right). \end{aligned}$$

Upon using the package $Reduce$ in $Mathematica 11$ for solving $p(t;\Phi) =<> 0$, where t_2 and t_4 are the second and the fourth roots of the equation of the sixth degree ($p(t;\Phi) = 0$), then the result is satisfied.

5.2. Behavior of Mean Residual Life

Let $\Omega_t = (X - t)|(X \geq t)$ be the residual lifetime which plays an important role in reliability theory. Then the MRL of a non-negative continuous random variable T is defined as

$$m(t,\Phi) = E[\Omega_t;\Phi] = \frac{1}{S(t;\Phi)} \int_t^{\infty} (x - t)\, f(x;\Phi)\, dx, \tag{11}$$

where $E[\Omega_t;\Phi]$ is the expectation of the residual lifetime.

Theorem 3
Using the Equations (3), (4) and (11), the explicit forms for MRL of MGLED are given by:

$$m(t,\Phi) = \begin{cases} \frac{1}{S(t)}\Bigg[-t\,S(t) + \frac{c}{b}\,(e^{-w} - S(t)) + \frac{c}{b}\,\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\binom{\frac{1}{2}}{j}\,(\frac{2\,b}{c^2})^j\,d_i(j) \\ \Big(\gamma(D_{i,j},u) - \gamma(D_{i,j},-\log S(t))\Big) + \frac{\sqrt{2\,b}}{b}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\binom{\frac{1}{2}}{j}\,(\frac{c^2}{2\,b})^j\,d_i(\frac{1}{2}-j) \\ \Big(\gamma(G_{i,j},w) - \gamma(G_{i,j},u)\Big)\Bigg], & -\log S(t) < u < w; \\ \frac{1}{S(t)}\Bigg[-t\,S(t) + \frac{c}{b}\,(e^{-w} - S(t)) + \frac{c}{b}\,\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\binom{\frac{1}{2}}{j}\,(\frac{2\,b}{c^2})^j\,d_i(j) \\ \Big(\gamma(D_{i,j},w) - \gamma(D_{i,j},-\log S(t))\Big)\Bigg], & w < u; \\ \frac{1}{S(t)}\Bigg[-t\,S(t) + \frac{c}{b}\,(e^{-w} - S(t)) + \frac{\sqrt{2\,b}}{b}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\binom{\frac{1}{2}}{j}\,(\frac{c^2}{2\,b})^j\,d_i(\frac{1}{2}-j) \\ \Big(\gamma(G_{i,j},w) - \gamma(G_{i,j},-\log S(t))\Big)\Bigg] & u < -\log S(t), \end{cases}$$

where $D_{i,j} = \frac{i+j+\xi}{\xi}$ and $G_{i,j} = \frac{2i-2j+2\xi+1}{2\xi}$.

Proof 3
To derive the previous explicit form of the MRL for MGLED, the integral $\int_t^{\infty} x\,f(x;\Phi)\,dx$ must be calculated (see Appendix B).

On the other hand, as in [18], Equation (11) can be rewritten as

$$m(t;\Phi) = \int_t^{\infty} e^{-\int_t^{t+x}\ h(t;\Phi)\ dt}\,dx \tag{12}$$

From Equation (12), it is clear that $m(t;\Phi)$ of MGLED is unimodal for the first case in Theorem 2 and decreasing for the second case in the same theorem.

5.3. Behavior of the Variance of Residual Life

In this subsection, the variance of random variable (r.v.) Ω_t and their monotonic and non-monotonic properties are studied. The VRL can be defined as

$$\begin{aligned} Var(\Omega_t;\Phi) &= E((X-t)^2|X \geq t) - [E(X-t|X \geq t)]^2 \\ &= E(X^2|X \geq t) - [E(X|X \geq t)]^2 \\ &= \int_t^{\infty} x^2\,\frac{f(x;\Phi)}{S(t;\Phi)}\,dx - \left(\int_t^{\infty} x\,\frac{f(x;\Phi)}{S(t;\Phi)}\,dx\right)^2. \end{aligned} \tag{13}$$

where $Var(\Omega_t;\Phi)$ is the variance of the residual lifetime.

Theorem 4
Let T be a non-negative continuous r.v., then the explicit forms for VRL of MGLED are given by:

$$
Var(\Omega_t;\Phi) = \begin{cases}
\frac{1}{S(t)}\Bigg[\frac{2\,c^2}{b^2}\,(S(t)-e^{-w}) + \frac{2}{b}\sum_{i=0}^{\infty} a_i\,\Big(\gamma(D_i,w)-\gamma(D_i,-\log S(t))\Big) \\
-\frac{2\,c^2}{b^2}\,\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\binom{\frac{1}{2}}{j}\,(\frac{2\,b}{c^2})^j\,d_i(j)\,\Big(\gamma(D_{i,j},u)-\gamma(D_{i,j},-\log S(t))\Big) \\
-\frac{2\,c\,\sqrt{2\,b}}{b^2}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\binom{\frac{1}{2}}{j}\,(\frac{c^2}{2\,b})^j\,d_i(\frac{1}{2}-j)\,\Big(\gamma(G_{i,j},w)-\gamma(G_{i,j},u)\Big)\Bigg] \\
-(\frac{1}{S(t)})^2\Bigg[\frac{c}{b}\,(e^{-w}-S(t)) + \frac{c}{b}\,\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\binom{\frac{1}{2}}{j}\,(\frac{2\,b}{c^2})^j\,d_i(j)\,\Big(\gamma(D_{i,j},u) \\
-\gamma(D_{i,j},-\log S(t))\Big) + \frac{\sqrt{2\,b}}{b}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\binom{\frac{1}{2}}{j}\,(\frac{c^2}{2\,b})^j\,d_i(\frac{1}{2}-j) \\
\Big(\gamma(G_{i,j},w)-\gamma(G_{i,j},u)\Big)\Bigg]^2 \qquad -\log S(t) < u < w; \\
\\
\frac{1}{S(t)}\Bigg[\frac{2\,c^2}{b^2}\,(S(t)-e^{-w}) + \frac{2}{b}\sum_{i=0}^{\infty} a_i\,\Big(\gamma(D_i,w)-\gamma(D_i,-\log S(t))\Big) \\
-\frac{2\,c^2}{b^2}\,\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\binom{\frac{1}{2}}{j}\,(\frac{2\,b}{c^2})^j\,d_i(j)\,\Big(\gamma(D_{i,j},w)-\gamma(D_{i,j},-\log S(t))\Big)\Bigg] \\
-(\frac{1}{S(t)})^2\Bigg[\frac{c}{b}\,(e^{-w}-S(t)) + \frac{c}{b}\,\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\binom{\frac{1}{2}}{j}\,(\frac{2\,b}{c^2})^j\,d_i(j) \\
\Big(\gamma(D_{i,j},w)-\gamma(D_{i,j},-\log S(t))\Big)\Bigg]^2 \qquad w < u; \\
\\
\frac{1}{S(t)}\Bigg[\frac{2\,c^2}{b^2}\,(S(t)-e^{-w}) + \frac{2}{b}\sum_{i=0}^{\infty} a_i\,\Big(\gamma(D_i,w)-\gamma(D_i,-\log S(t))\Big) \\
-\frac{2\,c\,\sqrt{2\,b}}{b^2}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\binom{\frac{1}{2}}{j}\,(\frac{c^2}{2\,b})^j\,d_i(\frac{1}{2}-j)\,\Big(\gamma(G_{i,j},w)-\gamma(G_{i,j},-\log S(t))\Big)\Bigg] \\
-(\frac{1}{S(t)})^2\Bigg[\frac{c}{b}\,(e^{-w}-S(t)) + \frac{\sqrt{2\,b}}{b}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\binom{\frac{1}{2}}{j}\,(\frac{c^2}{2\,b})^j\,d_i(\frac{1}{2}-j) \\
\Big(\gamma(G_{i,j},w)-\gamma(G_{i,j},-\log S(t))\Big)\Bigg]^2 \qquad u < -\log S(t),
\end{cases}
$$

where $D_{i,j}$ and $G_{i,j}$ are given in the previous theorem and $D_i = \frac{i+\xi+1}{\xi}$.

Proof 4

To derive the previous explicit form of the VRL for MGLED, the integrals $\int_t^{\infty} x\,f(x;\Phi)\,dx$ and $\int_t^{\infty} x^2\,f(x;\Phi)\,dx$ must be calculated (see Appendix B).

To study the behavior of VRL for MGLED, it is important to study the following relations (see [19], [20]):

$$
Var(\Omega_t;\Phi) - m(t,\Phi)^2 = \frac{2}{S(t;\Phi)}\int_t^{\infty} S(x;\Phi)\,\big[m(x,\Phi)-m(t,\Phi)\big]\,dx, \tag{14}
$$

and

$$
\frac{\partial}{\partial t}Var(\Omega_t;\Phi) = h(t;\Phi)\,m(t;\Phi)^2\,\Big[\frac{Var(\Omega_t;\Phi)}{m(t;\Phi)^2}-1\Big]. \tag{15}
$$

Remark 3
Since $x \geq t$, the following properties for VRL are satisfied:

- Upon using the Equation (15), one can show that $Var(\Omega_t;\Phi)$ is increasing if $Var(\Omega_t;\Phi) > m(t;\Phi)^2$, furthermore, it is obvious from Equation (14) that $Var(\Omega_t;\Phi) > m(t;\Phi)^2$ if and only if $m(t;\Phi)$ is increasing.
- Upon using the Equation (15), one can show that $Var(\Omega_t;\Phi)$ is decreasing if $Var(\Omega_t;\Phi) < m(t;\Phi)^2$, furthermore, it is obvious from Equation (14) that $Var(\Omega_t;\Phi) < m(t;\Phi)^2$ if and only if $m(t;\Phi)$ is decreasing.

From Remark 3, it is obvious that the VRL is a unimodal for MGLED given that the MRL for MGLED is a unimodal and is a decreasing for MGLED given that the MRL for MGLED is a decreasing.

6. Maximum Likelihood Estimation

MLE is probably the most widely used method of estimation in statistics. Suppose that $x_1, ..., x_r$ be an independent random sample of size r from MGLED. From Equation (3), the log-likelihood function can be obtained as

$$\begin{aligned} \ell(\Phi) &= -\sum_{i=1}^{r}(\frac{b}{2}\,x_i^2 + c\,x_i)^{\xi}\,e^{\varphi\,(\frac{b}{2}\,x_i^2+c\,x_i)} + \sum_{i=1}^{r}\varphi\,(\frac{b}{2}\,x_i^2 + c\,x_i) \\ &\quad +(\xi-1)\sum_{i=1}^{r}\log\left(\frac{b}{2}\,x_i^2 + c\,x_i\right) + \sum_{i=1}^{r}\log\left(b\,x_i + c\right) \\ &\quad + \sum_{i=1}^{r}\log\left(\xi + \varphi\,(\frac{b}{2}\,x_i^2 + c\,x_i)\right). \end{aligned} \tag{16}$$

By taking the first derivative ($\ell_{\Phi}(\Phi) = \frac{\partial \ell}{\partial \Phi}$) of Equation (16) w.r.t. c, b, ξ and φ we get

$$\begin{aligned} \ell_c(\Phi) &= \sum_{i=1}^{r}\frac{\varphi\,x_i}{\varphi\,\left(\frac{1}{2}b\,x_i^2 + c\,x_i\right)+\xi} - \sum_{i=1}^{r}\left(\xi\,x_i\left(\frac{1}{2}\,b\,x_i^2 + c\,x_i\right)^{\xi-1}\right. \\ &\quad \left. e^{\varphi\,\left(\frac{1}{2}\,b\,x_i^2+c\,x_i\right)} + \varphi\,x_i\left(\frac{1}{2}\,b\,x_i^2 + c\,x_i\right)^{\xi} e^{\varphi\,\left(\frac{1}{2}\,b\,x_i^2+c\,x_i\right)}\right) \\ &\quad + \sum_{i=1}^{r}\frac{(\xi-1)x_i}{\frac{1}{2}\,b\,x_i^2 + c\,x_i} + \sum_{i=1}^{r}\frac{1}{b\,x_i + c} + \sum_{i=1}^{r}\varphi\,x_i, \end{aligned} \tag{17}$$

$$\begin{aligned} \ell_b(\Phi) &= \sum_{i=1}^{r}\frac{\varphi\,x_i^2}{2\left(\varphi\,\left(\frac{1}{2}\,b\,x_i^2 + c\,x_i\right)+\xi\right)} - \sum_{i=1}^{r}\left(\frac{1}{2}\,\xi\,x_i^2\left(\frac{1}{2}\,b\,x_i^2 + c\,x_i\right)^{\xi-1}\right. \\ &\quad \left. e^{\varphi\,\left(\frac{1}{2}\,b\,x_i^2+c\,x_i\right)} + \frac{1}{2}\,\varphi\,x_i^2\left(\frac{1}{2}\,b\,x_i^2 + c\,x_i\right)^{\xi} e^{\varphi\,\left(\frac{1}{2}\,b\,x_i^2+c\,x_i\right)}\right) \\ &\quad + \sum_{i=1}^{r}\frac{(\xi-1)\,x_i^2}{2\left(\frac{1}{2}\,bx_i^2 + c\,x_i\right)} + \sum_{i=1}^{r}\frac{x_i}{b\,x_i + c} + \sum_{i=1}^{r}\frac{1}{2}\,\varphi\,x_i^2, \end{aligned} \tag{18}$$

$$\begin{aligned} \ell_{\xi}(\Phi) &= -\sum_{i=1}^{r}\left(\frac{1}{2}\,b\,x_i^2 + c\,x_i\right)^{\xi} e^{\varphi\,\left(\frac{1}{2}\,b\,x_i^2+c\,x_i\right)}\log\left(\frac{1}{2}\,b\,x_i^2 + c\,x_i\right) + \\ &\quad \sum_{i=1}^{r}\log\left(\frac{1}{2}\,b\,x_i^2 + c\,x_i\right) + \sum_{i=1}^{r}\frac{1}{\varphi\,\left(\frac{1}{2}\,b\,x_i^2 + c\,x_i\right)+\xi}, \end{aligned} \tag{19}$$

and

$$\ell_\varphi(\Phi) = -\sum_{i=1}^{r}\left(\frac{1}{2}\,b\,x_i^2+c\,x_i\right)^{\xi+1} e^{\varphi\,\left(\frac{1}{2}\,b\,x_i^2+c\,x_i\right)} + \sum_{i=1}^{r}\frac{\frac{1}{2}\,b\,x_i^2+c\,x_i}{\varphi\,\left(\frac{1}{2}\,b\,x_i^2+c\,x_i\right)+\xi} + \sum_{i=1}^{r}\left(\frac{1}{2}\,b\,x_i^2+c\,x_i\right). \tag{20}$$

6.1. The Parameters c, b, and φ are Known

The normal equation $\ell_\xi(\Phi)=0$ can be written as

$$\frac{1}{\xi} = \frac{\sum_{i=1}^{r}\left(\frac{b}{2}\,x_i^2+c\,x_i\right)^{\xi} e^{\varphi\left(\frac{b}{2}\,x_i^2+c\,x_i\right)}\log\left(\frac{b}{2}\,x_i^2+c\,x_i\right)-\sum_{i=1}^{r}\log\left(\frac{b}{2}\,x_i^2+c\,x_i\right)}{\sum_{i=1}^{r}\frac{1}{\frac{\varphi\left(\frac{b}{2}\,x_i^2+c\,x_i\right)}{\xi}+1}}. \tag{21}$$

It is clear that the first derivative of the right-side hand ($\Psi_1(\xi)$) of Equation (21) w.r.t. ξ is always positive. This means that the $\Psi_1(\xi)$ is an increasing function. Then by graphical method [21], the MLE of ξ exists and unique.

6.2. The Parameters c, b, and ξ are Known

The normal equation $\ell_\varphi(\Phi)=0$ can be written as

$$\frac{1}{\varphi} = \frac{\sum_{i=1}^{r}\left(\frac{b}{2}\,x_i^2+c\,x_i\right)^{\xi+1} e^{\varphi\left(\frac{b}{2}\,x_i^2+c\,x_i\right)}-\sum_{i=1}^{r}\left(\frac{b}{2}\,x_i^2+c\,x_i\right)}{\sum_{i=1}^{r}\frac{\frac{b}{2}\,x_i^2+c\,x_i}{\left(\frac{b}{2}\,x_i^2+c\,x_i\right)+\frac{\xi}{\varphi}}}. \tag{22}$$

It is clear that the first derivative of the right-side hand ($\Psi_2(\varphi)$) of Equation (22) w.r.t. φ is always positive. This means that the $\Psi_2(\varphi)$ is an increasing function. Then by graphical method [21], the MLE of φ exists and unique.

6.3. The Parameters c, b, ξ, and φ are Unknown

The MLE $\hat{\Phi}$ of Φ is given by solving the four normal equations $\ell_c(\Phi)=0$, $\ell_b(\Phi)=0$, $\ell_\xi(\Phi)=0$, and $\ell_\varphi(\Phi)=0$. These nonlinear equations cannot be solved analytically. So, these four nonlinear equations can be solved numerically by employing $FindRoot$ in software $WolframMathematica11$.

6.4. Fisher Information Matrix

Since the computation of Fisher information matrix (given by taking the expectation of the second derivative of Equation (16)) is very difficult, so, it seems appropriate to approximate these expected values by their MLEs. Then, the asymptotic variance-covariance matrix is given as [see, [22]].
The normal equation $\ell_\varphi(\Phi)=0$ can be written as

$$I^{-1} = \begin{pmatrix} Var(\hat{c}) & Cov(\hat{c},\hat{b}) & Cov(\hat{c},\hat{\xi}) & Cov(\hat{c},\hat{\varphi}) \\ Cov(\hat{b},\hat{c}) & Var(\hat{b}) & Cov(\hat{b},\hat{\xi}) & Cov(\hat{b},\hat{\varphi}) \\ Cov(\hat{\xi},\hat{c}) & Cov(\hat{\xi},\hat{b}) & Var(\hat{\xi}) & Cov(\hat{\varphi},\hat{\xi}) \\ Cov(\hat{\varphi},\hat{c}) & Cov(\hat{\varphi},\hat{b}) & Cov(\hat{\varphi},\hat{\xi}) & Var(\hat{\varphi}) \end{pmatrix} = \begin{pmatrix} -\ell_{cc}(\Phi) & -\ell_{cb}(\Phi) & -\ell_{c\xi}(\Phi) & -\ell_{c\varphi}(\Phi) \\ -\ell_{bc}(\Phi) & -\ell_{bb}(\Phi) & -\ell_{b\xi}(\Phi) & -\ell_{b\varphi}(\Phi) \\ -\ell_{\xi c}(\Phi) & -\ell_{\xi b}(\Phi) & -\ell_{\xi\xi}(\Phi) & -\ell_{\xi\varphi}(\Phi) \\ -\ell_{\varphi c}(\Phi) & -\ell_{\varphi b}(\Phi) & -\ell_{\varphi\xi}(\Phi) & -\ell_{\varphi\varphi}(\Phi) \end{pmatrix}^{-1}_{(\hat{c},\hat{b},\hat{\xi},\hat{\varphi})}, \tag{23}$$

where $\ell_{\Phi_i\Phi_j}(\Phi)=\frac{\partial^2\ell}{\partial\Phi_i\Phi_j}$, $i,j=1,2,3,4$, see Appendix C. Accordingly, the asymptotic confidence intervals (CIs) based on the asymptotic variance-covariance matrix for the parameters c, b, ξ, and φ are respectively given as:

$$\hat{c} \pm z_{\frac{\alpha}{2}} \sqrt{Var(\hat{c})}, \quad \hat{b} \pm z_{\frac{\alpha}{2}} \sqrt{Var(\hat{b})}, \quad \hat{\xi} \pm z_{\frac{\alpha}{2}} \sqrt{Var(\hat{\xi})}, \quad \text{and} \quad \hat{\varphi} \pm z_{\frac{\alpha}{2}} \sqrt{Var(\hat{\varphi})},$$

where $z_{\frac{\alpha}{2}}$ is the percentile of the standard normal distribution with right tail probability $\frac{\alpha}{2}$. Since the parameters c, b, ξ, and φ are positive, the log transformation, which was proposed by [23] is used to obtain the asymptotic CIs. Then, these formulae can be given for the parameters c, b, ξ, and φ as:

$$\hat{c}\, e^{\pm \frac{z_{\frac{\alpha}{2}} \sqrt{Var(\hat{c})}}{\hat{c}}}, \quad \hat{b}\, e^{\pm \frac{z_{\frac{\alpha}{2}} \sqrt{Var(\hat{b})}}{\hat{b}}}, \quad \hat{\xi}\, e^{\pm \frac{z_{\frac{\alpha}{2}} \sqrt{Var(\hat{\xi})}}{\hat{\xi}}}, \quad \text{and} \quad \hat{\varphi}\, e^{\pm \frac{z_{\frac{\alpha}{2}} \sqrt{Var(\hat{\varphi})}}{\hat{\varphi}}},$$

respectively.

7. Applications

In this section, two real data sets are presented for interpretative study. For every data set, we compare MGLED with its sub-models (NMLED, GLED, MWD, WD, and LFRD) and also with the following well-known distributions with bathtub hazard rates:

- The pdf of BLFRD is given by [8]

$$f(x) = \frac{(c+b\,x)}{B[\xi,\varphi]} \left(1 - e^{-(c\,x + \frac{b}{2}x^2)}\right)^{\xi-1} e^{-(c\,\varphi\,x + \frac{b\,\varphi}{2}x^2)}.$$

- The pdf of Marshal-Olkin generalized linear exponential distribution (M-OGLED) is [12]

$$f(x) = \frac{\xi\,\varphi\, e^{-(c\,x+\frac{b}{2}x^2)^{\xi}}\,(c\,x+\frac{b}{2}x^2)^{\xi-1}\,(c+b\,x)}{\left(1-(1-\varphi)\, e^{-(c\,x+\frac{b}{2}x^2)^{\xi}}\right)^2}\,.$$

- The pdf of M-OLFRD is given by [5]

$$f(x) = \frac{\varphi\, e^{-(c\,x+\frac{b}{2}x^2)}\,(c+b\,x)}{\left(1-(1-\varphi)\, e^{-(c\,x+\frac{b}{2}x^2)}\right)^2}\,.$$

- The pdf of Marshal-Olkin Weibull distribution (M-OWD) is given by [24]

$$f(x) = \frac{c\,\xi\,\varphi\, e^{-(c\,x)^{\xi}}\,(c\,x)^{\xi-1}}{\left(1-(1-\varphi)\, e^{-(c\,x)^{\xi}}\right)^2}\,.$$

- The pdf of ELFRD is given by [6]

$$f(x) = \xi\,(c+b\,x)\left(1 - e^{-(c\,x+\frac{b}{2}x^2)}\right)^{\xi-1} e^{-(c\,x+\frac{b}{2}x^2)}\,.$$

- The pdf of TLFRD is given by [7]

$$f(x) = \,(c+b\,x)\left(1-\varphi+2\,\varphi\, e^{-(c\,x+\frac{b}{2}x^2)^{\xi}}\right) e^{-(c\,x+\frac{b}{2}x^2)^{\xi}}\,.$$

For identifying the shapes of hazard rate for given data sets, the scaled TTT transform plot is given as

$$\phi_r(\tfrac{n}{r}) = \frac{\sum_{i=1}^{r} x_{i:r} + (r-n)\,x_{n:r}}{\sum_{i=1}^{r} x_i},$$

where $n = 1, ..., r$ and $x_{i:r}$ is the order statistics of the data (see [25]). Anderson Darling (A^*), Cramér Von-Mises (W^*), and and Kolmogrov-Simnorov $(K-S)$ tests are used for non-parametric test statistic. These tests are defined as:

$$A^* = A^2\left(\frac{9}{4\,n^2} + \frac{3}{4\,n} + 1\right),$$

$$W^{*} = W^{2}\left(\frac{1}{2\,n} + 1\right),$$

and

$$K - S = \max\left(\left|\frac{i}{r} - F(x_i)\right|, \left|F(x_i) - \frac{i-1}{r}\right|\right)$$

respectively, where A^2 and W^2 are given by

$$A^{2} = -r - \sum_{i=1}^{r} \frac{(2\,i-1)}{r}\left(\log\left(F(x_i)\right) + \log\left(1 - F(x_{r-i+1})\right)\right),$$

and

$$W^{2} = \frac{1}{12\,r} + \sum_{i=1}^{r}\left(F(x_i) - \frac{2\,i-1}{2\,r}\right)^{2}.$$

Furthermore, the Akaike information criterion (AIC), which is defined as $AIC = -2\ \ell + 2\ k$ (ℓ is the log-likelihood function and k is the number of parameters), the Bayesian information criterion (BIC), which is defined as $BIC = -2\ \ell + k\ \log[r]$, the corrected Akaike information criterion (AICc), which is defined as $AICc = AIC + \frac{2\,k\,(k+1)}{r-k-1}$ and the Hannan-Quinn information criterion (HQIC), which is defined as $HQIC = -2\,\ell + 2\,k\log[\log(r)]$, are used to compare the candidate distributions for parametric test. All computations were introduced by $Wolfram\ Mathematica 11$.

7.1. Failure Data of Electronic Equipment

Consider the failure times of 107 units for a piece of electronic equipment reported by [26]. The data are:1.0, 1.2, 1.3, 2.0, 2.4, 2.9, 3.0, 3.1, 3.3, 3.8, 4.3, 4.6, 4.7, 4.8, 5.2, 5.4, 5.9, 6.4, 6.8, 6.9, 7.2, 7.9, 8.3, 8.7, 9.2, 9.8, 10.2, 10.4, 11.9, 13.8, 14.4, 15.6, 16.2, 17.0, 17.5, 19.2, 28.1, 28.2, 29.0, 29.9, 30.6, 32.4, 33.0, 35, 35.3, 36.1, 40.1, 42.8, 43.7, 44.5, 50.4, 51.2, 52.0, 53.3, 54.2, 55.6, 56.4, 58.3, 60.2, 63.7, 64.6, 65.3, 66.2, 70.1, 71.0, 75.1, 75.6, 78.4, 79.2, 84.1, 86.0, 87.9, 88.4, 89.9, 90.8, 91.1, 91.5, 92.1, 97.9, 100.8, 102.6, 103.2, 104.0, 104.3, 105.0, 105.8, 106.5, 110.7, 112.6, 113.5, 114.8, 115.1, 117.4, 118.3, 119.7, 120.6, 121.0, 122.9, 123.3, 124.5, 125.8, 126.6, 127.7, 128.4, 129.2, 129.5, 129.9. This data will be used after dividing each failure times by 1000. The TTT plot shows that the hazard rate of this data has first a convex shape and then a concave shape, indicating a bathtub-shaped hazard rate function as shown in Figure 3 (a).

The numerical values of the non-parametric and parametric tests based on MLEs of all comparison distributions are summarized in Table 4 for this data. Based on P-Value associated to K-S test, one can show that the all distributions are in specific form at a level of significance $\alpha = 0.05$ except NMLED, LFRD, M-OLFRD, and TLFRD for the failure data of electronic equipment. It is obvious from Table 4 that the MGLED fits this data better than other distributions because it has the lowest value based on A^*, W^* AIC, $AICc$, BIC, and $HQIC$. The graphical method of ξ and $\Psi_1(\xi)$ based on Equation (21) can be shown in Figure 4 (a) for this data. Also, the graphical method of φ and $\Psi_2(\varphi)$ based on Equation (22) can be shown in Figures 4 (b) for this data. Figure (5) represents the profiles of log-likelihood function for all parameters based on first data.
The asymptotic variance-covariance matrix based on the ML estimates for this data is given by

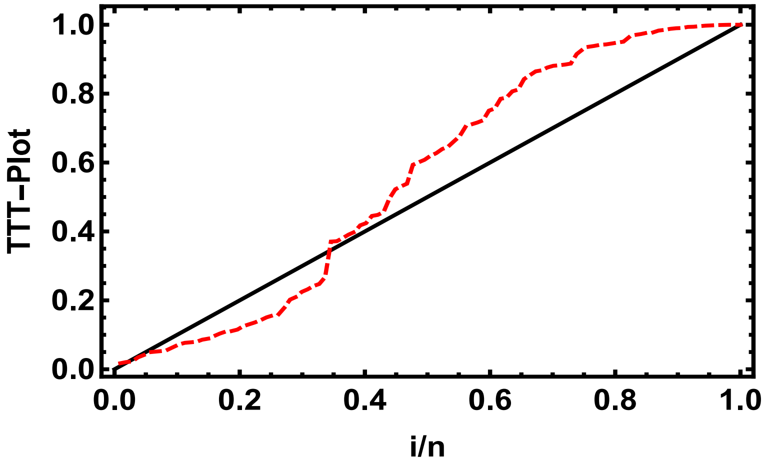
$$I^{-1} = \begin{pmatrix} 3.31761 & 4.9163 & 0.1394 & -0.9091 \\ 4.9163 & 2363.79 & -0.0137 & -37.7665 \\ 0.1394 & -0.0137 & 0.0092 & -0.0351 \\ -0.9091 & -37.7665 & -0.0351 & 0.8274 \end{pmatrix}$$

Thus, the approximate 95% CIs for c, b, ξ and φ are respectively given by [1.12913, 9.73166], [12.784, 290.953], [0.377707, 0.762618], and [0.269678, 5.30927].

Table 4. The ML estimates of unknown parameters, the corresponding SE given in parentheses, (A^*), (W^*), AIC, BIC, $AICc$, and $HQIC$ for the Failure data of electronic equipment

Model	ML estimates	$K-S$	$P-Value$	(A^*)	(W^*)	AIC	BIC	$AICc$	$HQIC$
MGLED	c = 3.3149 (0.18) b =60.9880 (4.7) ξ =0.5367 (0.009) φ = 1.1966 (0.09)	0.1022	0.2137	1.8044	0.2533	-418.011	-407.32	-417.619	-413.677
NMLED	c = 10.3283 (0.19) b =40.1033 (3.6) φ = 0.3037 (0.02)	0.1546	0.0120	4.9051	0.6146	-404.643	-396.624	-404.41	-401.392
GLED	c = 10.2703 (0.28) b = 187.1487 (7.58) ξ = 0.8444 (0.0096)	0.1155	0.1151	3.0456	0.4538	-401.692	-393.674	-401.459	-398.442
MWD	c =3.8575 (0.22) ξ = 0.5839 (0.011) φ = 2.8626 (0.22)	0.1048	0.1905	2.1316	0.3033	-414.127	-406.877	-413.894	-410.877
WD	c = 17.1614 (0.16) ξ = 1.0502 (0.008)	0.1252	0.0698	3.747	0.5741	-394.338	-388.992	-394.223	-392.171
LFRD	c = 12.3776 (0.21) b = 110.7859 (4.09)	0.1356	0.0391	4.0831	0.5564	-401.474	-396.128	-401.358	-399.307
BLFRD	c = 22.0291 b =720.5219 ξ = 0.5829 φ = 0.2627	0.1025	0.2109	2.1632	0.3109	-405.135	-394.443	-404.743	-400.801
M-O-GLED	c = 15.9302 (0.76) b = 260.5389 (15.04) ξ = 0.7643 (0.01) φ = 1.8272 (0.10)	0.1170	0.1068	2.872	0.4082	-400.785	-390.094	-400.393	-396.451
M-O-LFRD	c = 12.3353 (0.43) b = 110.8684 (4.14) φ =0.9952 (0.04)	0.1355	0.0393	7.0789	0.5563	-399.474	-391.455	-399.241	-396.223
M-O-WD	c = 28.1394 (0.99) ξ =0.8894 (0.013) φ =2.4697 (0.14)	0.1228	0.0793	3.4047	0.4772	-395.017	-386.999	-394.784	-391.767
ELFRD	c = 6.0312 (0.29) b = 162.9089 (4.13) ξ =0.6775 (0.01)	0.1114	0.1404	2.5193	0.3787	-405.126	-397.107	-404.893	-401.875
TLFRD	c = 12.3676 (0.28) b = 110.8055 (4.10) ξ =0.0011 (0.02)	0.1356	0.0391	4.0821	0.5564	-399.474	-391.455	-399.241	-396.223

(a)



(b)

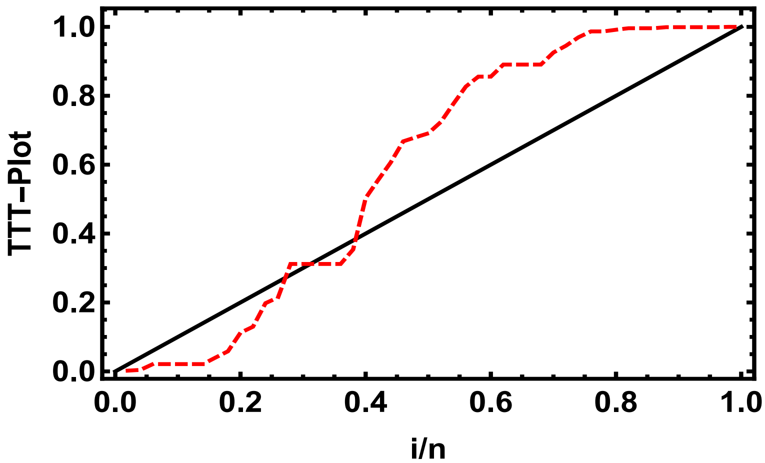
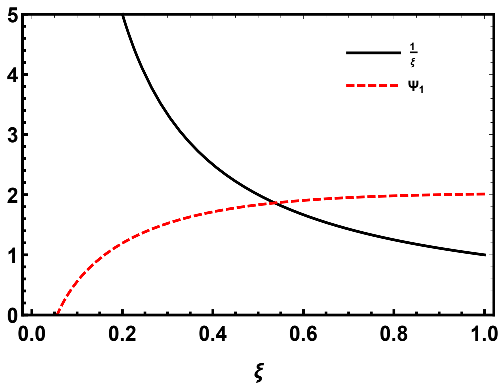


Figure 3. Scaled TTT transform of (a) Failure data of electronic equipment. (b) Aarset data.

(a)



(b)

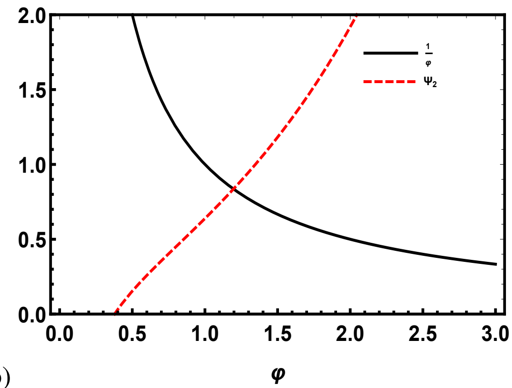


Figure 4. (a) Plot of the $\frac{1}{\xi}$ and $\Psi_1(\xi)$ functions for the power transformer data and (b) Plot of the $\frac{1}{\varphi}$ and $\Psi_2(\varphi)$ functions for failure data of electronic equipment.

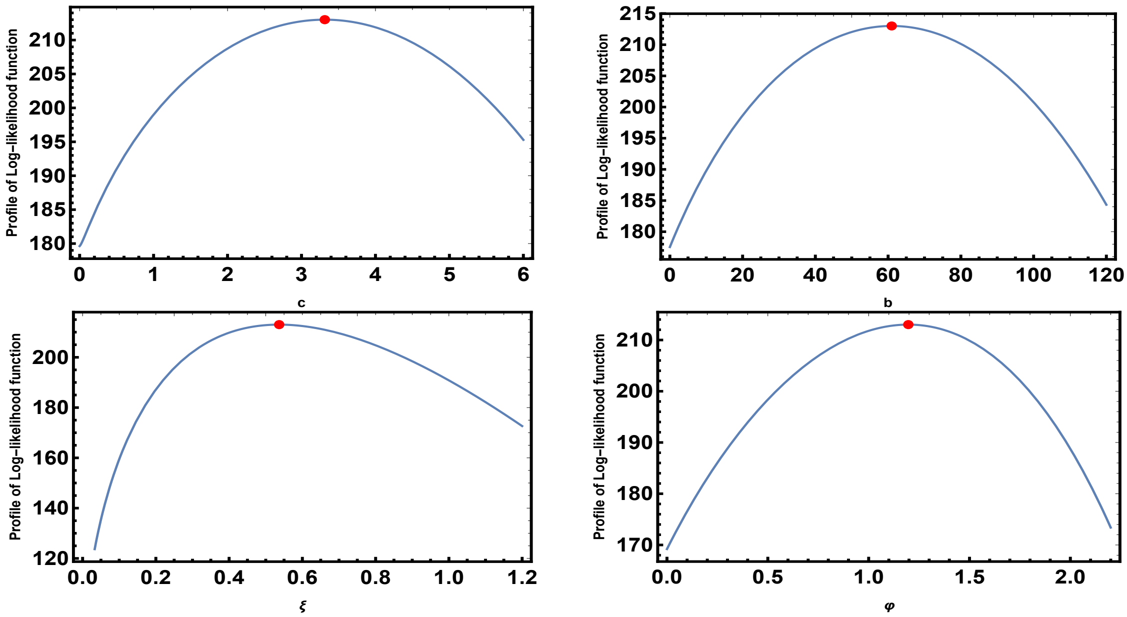


Figure 5. The profile of log-likelihood function of all parameters for the Failure data of electronic equipment .

7.2. *Aarset Data*

Consider the following data set from [25] consisting of 50 observations of failure times for 50 items which are tested to failure. This data has been analysed by many other authors [[1], [10], [27], [28], and [2]]. This data will be used after dividing each failure times by 100. Furthermore, the TTT plot shows that the hazard rate of this data is bathtub as shown in Figure 3 (b).

The numerical values of the non-parametric and parametric tests based on ML estimates of all comparison distributions are summarized in Table 5 for this data. Based on P-Value associated to K-S test, one can show that the all distributions are in specific form at a level of significance $\alpha = 0.05$ except WD for the Aarset data. It is obvious from Table 5 that the MGLED fits this data better than other distributions because it has the lowest value based on A^*, W^* AIC, $AICc$, BIC, and $HQIC$. The graphical method of ξ and $\Psi_1(\xi)$ based on Equation (21) can be shown in Figure 6 (a) for this data. Also, the graphical method of φ and $\Psi_2(\varphi)$ based on Equation (22) can be shown in Figures 6 (b) for this data. Figure (7) represents the profiles of log-likelihood function for all parameters based on this data. The estimated hazard rate function of the failure data of electronic equipment and Aarset data are displayed in Figure (8).

The asymptotic variance-covariance matrix based on the ML estimates for this data is given by

$$I^{-1} = \begin{pmatrix} 0.0052 & 0.0080 & 0.0080 & -0.6794 \\ 0.0080 & 0.1698 & 0.0153 & -7.1688 \\ 0.0080 & 0.0153 & 0.0147 & -1.1291 \\ -0.6794 & -7.1688 & -1.1291 & 328.503 \end{pmatrix}$$

Thus, the approximate 95% CIs for c, b, ξ and φ are respectively given by [-0.116022, 0.167255], [-0.513824, 1.10134], [0.07301, 0.548497], and [-23.9343, 47.1143]. The approximate 95% CIs based on the log transformation method for c, b, ξ and φ are respectively given by [0.000101676, 6.4539], [0.0187952, 4.59122], [0.144596, 0.667845], and [0.540669, 248.447].

Table 5. The ML estimates of unknown parameters, the corresponding SE given in parentheses, (A^*), (W^*), AIC, BIC, $AICc$, and $HQIC$ for the Aarset data

Model	ML estimates	$K-S$	$P-Value$	(A^*)	(W^*)	AIC	BIC	$AICc$	$HQIC$
MGLED	c = 0.0256 (0.01) b =0.2938 (0.06) ξ =0.3108 (0.02) φ = 11.59 (2.56)	0.1341	0.3297	1.4937	0.2069	-5.4331	2.215	-4.5442	-2.5207
NMLED	c = 0.9465 (0.04) b =0.7941 (0.104) φ = 0.6519 (0.07)	0.1734	0.0989	4.88	0.4705	15.8599	21.596	18.0442	16.3816
GLED	c = 0.9621 (0.06) b = 4.5199 (0.26) ξ = 0.7302 (0.016)	0.1799	0.0786	2.9022	0.4349	17.3341	23.0702	17.8559	19.5185
MWD	c =0.0402 (0.011) ξ =0.3548 (0.016) φ = 57.9944 (17.22)	0.1337	0.3332	1.8336	0.2664	-0.2065	5.5295	0.3152	1.9778
WD	c = 2.2266 (0.05) ξ = 0.9490 (0.02)	0.1928	0.0486	3.5445	0.5349	25.4866	29.3107	25.7919	26.9428
LFRD	c = 1.3632 (0.05) b = 2.3997 (0.14)	0.1769	0.08748	4.0988	0.4673	19.6102	23.4343	19.8655	21.0664
BLFRD	c = 1.716 b = 34.7848 ξ = 0.3347 φ = 0.1243	0.1554	0.1786	1.7777	0.2786	8.2400	15.8881	9.1289	11.1525
M-O-GLED	c = 3.592 (0.45) b = 13.8889 (1.81) ξ = 0.5739 (0.02) φ =4.7357 (0.54)	0.1479	0.2241	2.5496	0.3343	15.238	22.8861	16.1269	18.1504
M-O-LFRD	c =1.8106 (0.11) b = 2.4735(0.17) φ = 1.5751 (0.13)	0.1630	0.1403	4.3217	0.4483	21.0621	26.7982	21.5838	23.2464
M-O-WD	c =7.1749 (0.65) ξ = 0.6992 (0.02) φ =6.6973 (0.75)	0.1626	0.1421	3.0031	0.3758	20.9297	26.6658	21.4515	23.1141
ELFRD	c = 0.3821 (0.04) b = 3.0743 (0.11) ξ =0.5327 (0.016)	0.1832	0.06973	2.5343	0.4194	11.7725	17.5085	12.2949	13.9568
TLFRD	c =1.4492 (0.07) b = 2.4185 (0.15) ξ =-0.0944 (0.04)	0.1740	0.0969	4.1388	0.4619	21.4945	27.2306	22.0163	23.6788

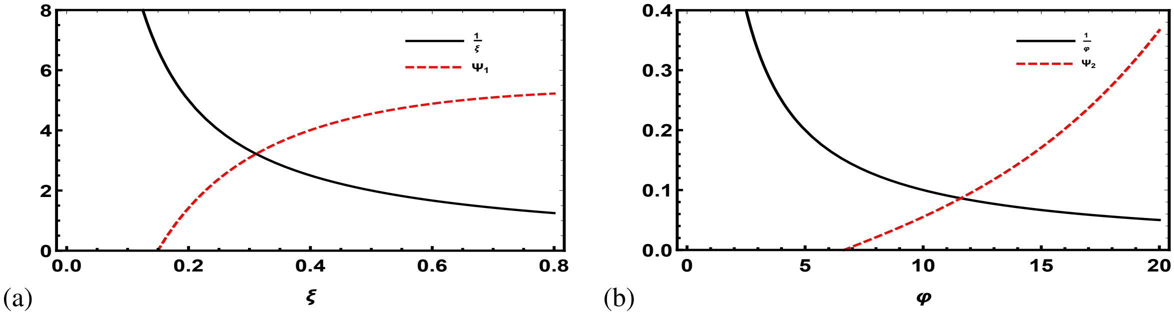


Figure 6. (a) Plot of the $\frac{1}{\xi}$ and $\Psi_1(\xi)$ functions for the failure test data and (b) Plot of the $\frac{1}{\varphi}$ and $\Psi_2(\varphi)$ functions for Aarset data.

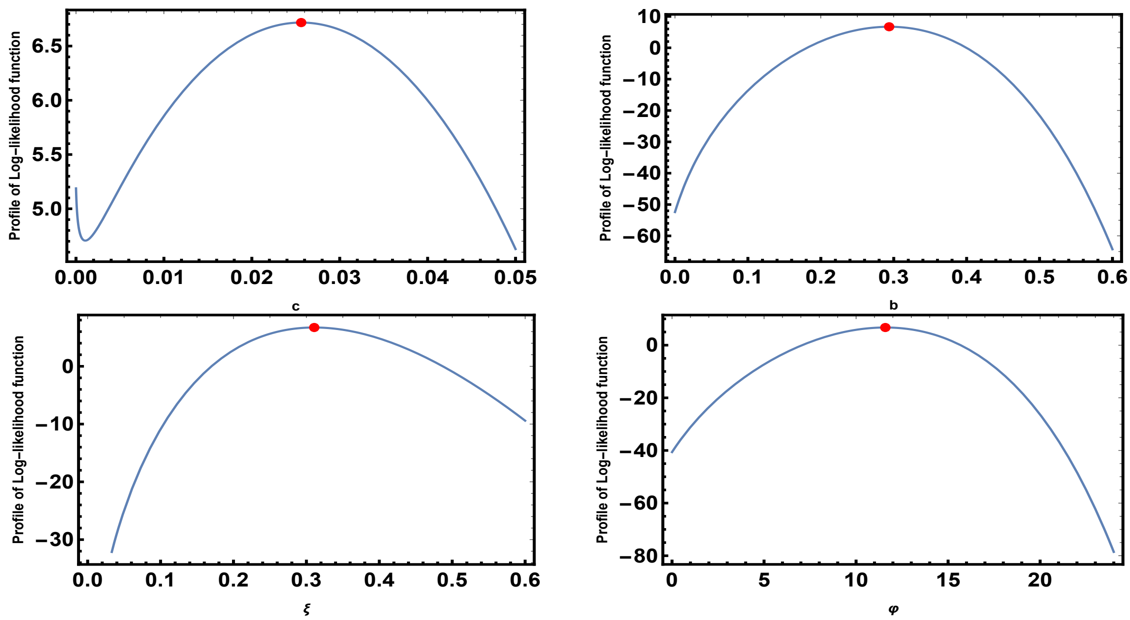


Figure 7. The profile of log-likelihood function of all parameters for the Aarset data .

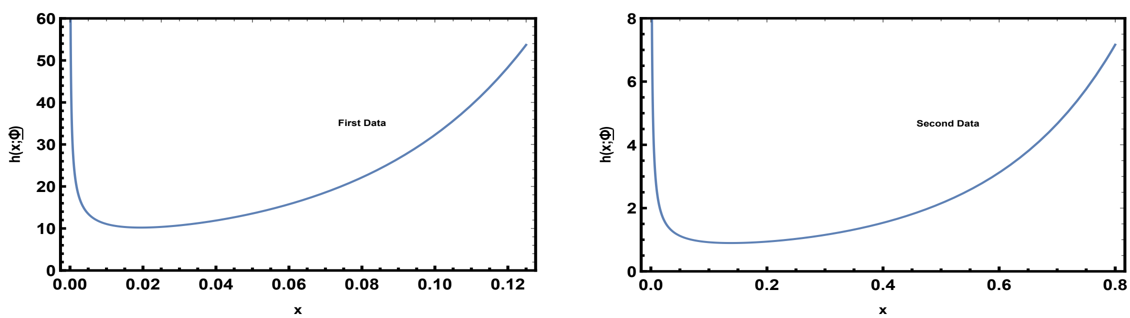


Figure 8. The hazard rate plots of the failure data of electronic equipment and Aarset data.

8. Conclusions

This paper introduced a new lifetime distribution known as MGLED. The new distribution has sub-models like GLED, MWD, LED and WD which are widely used in the lifetime literature. The MGLED has a bathtub and increasing HR function. It was useful to model lifetime with a bathtub-shaped HR function. Some mathematical properties were provided in explicit forms such as moments, moment generating function, and quantiles for MGLED. Furthermore, the explicit forms of MRL and VRL were given and the behavior of their shapes were also studied. The disadvantage of these explicit expressions was that cannot be given for any parameter values and they need modern computer resources to calculate the numeric values of them. The applications of MGLED to two real data sets were given to show that it may engage wider in reliability engineering. Upon using the parametric and non-parametric test, the MGLED has a competitive advantage in modeling lifetime data.

Acknowledgment

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Appendices

A. Correction of the Formula for the Moments of MWD

All notations and symbols in this section are taken from [14]. To get Formula II, it is easy to show that the r-th moments of the MWD can be written as

$$\mu^{(r)} = \int_0^\infty \alpha\, t^r\, t^{\gamma-1}\, e^{\lambda t}\, (\gamma + \lambda\, t)\, e^{-\alpha\, t^\gamma\, e^{\lambda t}}\, dt.$$

Upon using the substitution $x = t^\gamma\, e^{\lambda t}$, it is clear that $t = \sum_{i=1}^{\infty} a_i\, x^{\frac{i}{\gamma}}$ and a_i is given by Equation (7) on page 453 of [14]. It is clear that the r-th moments of the MWD under the convergence condition become

$$\mu^{(r)} = \int_0^{(\frac{\gamma}{\lambda\, e})^\gamma} \alpha\, (\sum_{i=1}^{\infty} a_i\, x^{\frac{i}{\gamma}})^r\, e^{-\alpha x}\, dx,$$

But

$$(\sum_{i=1}^{\infty} a_i\, x^{\frac{i}{\gamma}})^r = \sum_{i_1,\ldots,i_r=1}^{\infty} A_{i_1,\ldots,i_r}\, x^{\frac{s_r}{\gamma}}.$$

as in [14]. So, $\mu^{(r)}$ can be written as

$$\mu^{(r)} = \sum_{i_1,\ldots,i_r=1}^{\infty} A_{i_1,\ldots,i_r} \int_0^{(\frac{\gamma}{\lambda\, e})^\gamma} \alpha\;\, x^{\frac{s_r}{\gamma}}\, e^{-\alpha x}\, dx.$$

Then, the result is satisfied.

B. Proofs

B.1. Constructing the formula for the moment generating function of MGLED

The moment generating function of MGLED can be written as

$$M_X(t) = \int_0^{\infty} e^{t\,x}\, f(x;\Phi)\, dx.$$

Making use of $v = (c\,x + \frac{b}{2}\,x^2)^{\xi}\, e^{\varphi\,(c\,x+\frac{b}{2}\,x^2)}$ and $e^{t\,x} = \sum_{j=0}^{\infty} x^j\, \frac{t^j}{j!}$, yield

$$M_Y(t) = \int_0^{(\frac{\xi}{\varphi\,e})^{\xi}} \sum_{j=0}^{\infty} \frac{t^j}{j!} \Big(\frac{-c + \sqrt{c^2 + 2\,b\,\sum_{i=0}^{\infty} a_i\, v^{\frac{i+1}{\xi}}}}{b}\Big)^j\, e^{-v}\, dv.$$

It is clear that

$$\begin{aligned}
&\Big(\frac{-c + \sqrt{c^2 + 2\,b\,\sum_{i=0}^{\infty} a_i\, v^{\frac{i+1}{\xi}}}}{b}\Big)^j = \\
&b^{-j}\,\Big(c^2 + 2\,b\,\sum_{i=0}^{\infty} a_i\, v^{\frac{i+1}{\xi}}\Big)^{\frac{j}{2}}\,\Big(1 - \frac{c}{\sqrt{c^2 + 2\,b\,\sum_{i=0}^{\infty} a_i\, v^{\frac{i+1}{\xi}}}}\Big)^j = \\
&b^{-j}\,\sum_{m=0}^{\infty} (-1)^m\,(c)^m \binom{j}{m} \Big(c^2 + 2\,b\,\sum_{i=0}^{\infty} a_i\, v^{\frac{i+1}{\xi}}\Big)^{\frac{j-m}{2}}
\end{aligned}$$

Also, it is easy to show that $|\frac{2\,b}{c^2}\,\sum_{i=0}^{\infty} a_i\,v^{\frac{i+1}{\xi}}| < 1$ if $v < (\frac{c^2}{2\,b})^{\xi}\, e^{\varphi\,(\frac{c^2}{2\,b})}$ and $|\frac{c^2}{2\,b\,\sum_{i=0}^{\infty} a_i\,v^{\frac{i+1}{\xi}}}| < 1$ if $v > (\frac{c^2}{2\,b})^{\xi}\, e^{\varphi\,(\frac{c^2}{2\,b})}$. Then, by binomial expansion, $M_X(t)$ can be written as

$$M_X(t) = \begin{cases}
\sum_{j=0}^{\infty}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{i=0}^{\infty} A^*_{j,m,n}\, d_i(n)\, \int_0^u v^{\frac{i+n}{\xi}}\, e^{-v}\, dv \\
+\sum_{j=0}^{\infty}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{i=0}^{\infty} B^*_{j,m,n}\, d_i(\frac{j-m-2n}{2}) \\
\int_u^w v^{\frac{2\,i+j-m-2\,n}{2\,\xi}}\, e^{-v}\, dv, & w > u; \\
\\
\sum_{j=0}^{\infty}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{i=0}^{\infty} A^*_{j,m,n}\, d_i(n)\, \int_0^w v^{\frac{i+n}{\xi}}\, e^{-v}\, dv, & w < u.
\end{cases}$$

Thus the result is satisfied.

B.2. Constructing the formulas for MRL and VRL of MGLED

- To calculate the integral $I_1^* = \int_t^{\infty} x\;f(x;\Phi)\;dx$, the following steps is required. Making use of $v = (c\,x + \frac{b}{2}\,x^2)^{\xi}\, e^{\varphi\,(c\,x+\frac{b}{2}\,x^2)}$, yields

$$\begin{aligned}
I_1^* &= \int_{-\log S(t)}^{w} \Big(\frac{-c + \sqrt{c^2 + 2\,b\,\sum_{i=0}^{\infty} a_i\, v^{\frac{i+1}{\xi}}}}{b}\Big)\, e^{-v}\, dv \\
&= \int_{-\log S(t)}^{w} \Big(\frac{-c}{b}\Big)\, e^{-v}\, dv + \frac{1}{b}\int_{-\log S(t)}^{w} \Big(c^2 + 2\,b\,\sum_{i=0}^{\infty} a_i\, v^{\frac{i+1}{\xi}}\Big)^{\frac{1}{2}}\, e^{-v}\, dv
\end{aligned}$$

Also, it is easy to show that $|\frac{2\,b}{c^2}\ \sum_{i=0}^{\infty} a_i\ v^{\frac{i+1}{\xi}}| < 1$ if $v < (\frac{c^2}{2\,b})^{\xi}\ e^{\varphi\ (\frac{c^2}{2\,b})}$ and $|\frac{c^2}{2\ b\ \sum_{i=0}^{\infty} a_i\ v^{\frac{i+1}{\xi}}}| < 1$ if $v > (\frac{c^2}{2\,b})^{\xi}\ e^{\varphi\ (\frac{c^2}{2\,b})}$. Then, by binomial expansion, I_1^* can be written as

$$I_1^* = \begin{cases} \frac{c}{b}\ (e^{-w} - S(t)) + \frac{c}{b}\ \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \binom{\frac{1}{2}}{j}\ (\frac{2\,b}{c^2})^j\ d_i(j) \\ \Big(\gamma(D_{i,j}, u) - \gamma(D_{i,j}, -\log S(t))\Big) + \frac{\sqrt{2\,b}}{b} \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \\ \binom{\frac{1}{2}}{j}\ (\frac{c^2}{2\,b})^j\ d_i(\frac{1}{2} - j)\ \Big(\gamma(G_{i,j}, w) - \gamma(G_{i,j}, u)\Big), \quad -\log S(t) < u < w; \\ \\ \frac{c}{b}\ (e^{-w} - S(t)) + \frac{c}{b}\ \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \binom{\frac{1}{2}}{j}\ (\frac{2\,b}{c^2})^j\ d_i(j) \\ \Big(\gamma(D_{i,j}, w) - \gamma(D_{i,j}, -\log S(t))\Big), \qquad w < u; \\ \\ \frac{c}{b}\ (e^{-w} - S(t)) + \frac{\sqrt{2\,b}}{b} \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \binom{\frac{1}{2}}{j}\ (\frac{c^2}{2\,b})^j \\ d_i(\frac{1}{2} - j)\ \Big(\gamma(G_{i,j}, w) - \gamma(G_{i,j}, -\log S(t))\Big), \qquad u < -\log S(t). \end{cases}$$

- To calculate the integral $I_2^* = \int_t^{\infty} x^2\ f(x;\Phi)\ dx$, the following steps is required. Making use of $v = (c\,x + \frac{b}{2}\,x^2)^{\xi}\ e^{\varphi\ (c\,x + \frac{b}{2}\,x^2)}$, yields

$$\begin{aligned} I_1^* &= \int_{-\log S(t)}^{w} \Big(\frac{-c + \sqrt{c^2 + 2\,b\ \sum_{i=0}^{\infty} a_i\ v^{\frac{i+1}{\xi}}}}{b}\Big)^2 e^{-v}\ dv \\ &= \int_{-\log S(t)}^{w} \Big(\frac{2\,c^2}{b^2}\Big)\ e^{-v}\ dv + \int_{-\log S(t)}^{w} \Big(\frac{2}{b}\Big) \sum_{i=0}^{\infty} a_i\ v^{\frac{i+1}{\xi}}\ e^{-v}\ dv \\ &- \frac{2\,c}{b^2} \int_{-\log S(t)}^{w} \Big(c^2 + 2\,b \sum_{i=0}^{\infty} a_i\ v^{\frac{i+1}{\xi}}\Big)^{\frac{1}{2}}\ e^{-v}\ dv \end{aligned}$$

Also, it is easy to show that $|\frac{2\,b}{c^2}\ \sum_{i=0}^{\infty} a_i\ v^{\frac{i+1}{\xi}}| < 1$ if $v < (\frac{c^2}{2\,b})^{\xi}\ e^{\varphi\ (\frac{c^2}{2\,b})}$ and $|\frac{c^2}{2\ b\ \sum_{i=0}^{\infty} a_i\ v^{\frac{i+1}{\xi}}}| < 1$ if $v > (\frac{c^2}{2\,b})^{\xi}\ e^{\varphi\ (\frac{c^2}{2\,b})}$. Then, by binomial expansion, I_1^* can be written as

$$I_2^* = \begin{cases} \frac{2\,c^2}{b^2}\ (S(t) - e^{-w}) + \frac{2}{b} \sum_{i=0}^{\infty} a_i\ \Big(\gamma(D_i, w) - \gamma(D_i, -\log S(y))\Big) \\ -\frac{2\,c^2}{b^2}\ \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \binom{\frac{1}{2}}{j}\ (\frac{2\,b}{c^2})^j\ d_i(j)\ \Big(\gamma(D_{i,j}, u) - \gamma(D_{i,j}, -\log S(t))\Big) \\ -\frac{2\,c\,\sqrt{2\,b}}{b^2} \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \binom{\frac{1}{2}}{j}\ (\frac{c^2}{2\,b})^j\ d_i(\frac{1}{2} - j)\ \Big(\gamma(G_{i,j}, w) - \gamma(G_{i,j}, u)\Big), \\ \qquad -\log S(t) < u < w; \\ \\ \frac{2\,c^2}{b^2}\ (S(t) - e^{-w}) + \frac{2}{b} \sum_{i=0}^{\infty} a_i\ \Big(\gamma(D_i, w) - \gamma(D_i, -\log S(t))\Big) \\ -\frac{2\,c^2}{b^2}\ \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \binom{\frac{1}{2}}{j}\ (\frac{2\,b}{c^2})^j\ d_i(j)\ \Big(\gamma(D_{i,j}, w) - \gamma(D_{i,j}, -\log S(t))\Big), \\ \qquad w < u; \\ \\ \frac{2\,c^2}{b^2}\ (S(t) - e^{-w}) + \frac{2}{b} \sum_{i=0}^{\infty} a_i\ \Big(\gamma(D_i, w) - \gamma(D_i, -\log S(t))\Big) \\ -\frac{2\,c\,\sqrt{2\,b}}{b^2} \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \binom{\frac{1}{2}}{j}\ (\frac{c^2}{2\,b})^j\ d_i(\frac{1}{2} - j)\ \Big(\gamma(G_{i,j}, w) - \\ \gamma(G_{i,j}, -\log S(t))\Big), \qquad u < -\log S(t). \end{cases}$$

C. The expressions of the observed information matrix

The second derivatives of (16) can be written as:

$$\begin{aligned}\ell_{cc}(\Phi) &= \sum_{i=1}^{r} -\frac{\varphi^2\, x_i^2}{\left(\varphi\left(\frac{1}{2}\, b\, x_i^2 + c\, x_i\right) + \xi\right)^2} + \sum_{i=1}^{r} -\frac{(\xi-1)x_i^2}{\left(\frac{1}{2}\, b\, x_i^2 + c\, x_i\right)^2} + \sum_{i=1}^{r} -\frac{1}{(b\, x_i + c)^2} - \\ &\sum_{i=1}^{r} e^{\varphi\,\left(\frac{1}{2}\, b x_i^2 + c\, x_i\right)}\left[\varphi^2\, x_i^2 \left(\frac{1}{2}\, b\, x_i^2 + c\, x_i\right)^{\xi} + 2\,\xi\,\varphi\, x_i^2\,\left(\frac{1}{2}\, b\, x_i^2 + c\, x_i\right)^{\xi-1} + \right. \\ &\left. \xi\, x_i\,\left(\xi\, x_i \left(\frac{1}{2}\, b\, x_i^2 + c\, x_i\right)^{\xi-2} - x_i\,\left(\frac{1}{2}\, b\, x_i^2 + c\, x_i\right)^{\xi-2}\right)\right]\end{aligned}$$

$$\begin{aligned}\ell_{bb}(\Phi) &= \sum_{i=1}^{r} -\frac{\varphi^2\, x_i^4}{4\left(\varphi\,\left(\frac{1}{2}\, b\, x_i^2 + c\, x_i\right) + \xi\right)^2} + \sum_{i=1}^{r} -\frac{(\xi-1)\, x_i^4}{4\left(\frac{1}{2}\, b\, x_i^2 + c\, x_i\right)^2} + \sum_{i=1}^{r} -\frac{x_i^2}{(b\, x_i + c)^2} \\ &- \sum_{i=1}^{r} e^{\varphi\,\left(\frac{1}{2}\, b\, x_i^2 + c\, x_i\right)}\left[\frac{1}{4}\,\varphi^2\, x_i^4\,\left(\frac{1}{2}\, b\, x_i^2 + c\, x_i\right)^{\xi} + \frac{1}{2}\,\xi\,\varphi\, x_i^4\,\left(\frac{1}{2}\, b\, x_i^2 + c\, x_i\right)^{\xi-1}\right. \\ &\left. + \frac{1}{2}\,\xi\, x_i^2 \left(\frac{1}{2}\,\xi\, x_i^2 \left(\frac{1}{2}\, b\, x_i^2 + c\, x_i\right)^{\xi-2} - \frac{1}{2}\, x_i^2\,\left(\frac{1}{2}\, b\, x_i^2 + c\, x_i\right)^{\xi-2}\right)\right]\end{aligned}$$

$$\ell_{\xi\xi}(\Phi) = -\sum_{i=1}^{r}\left(\frac{1}{2}\, b\, x_i^2 + c\, x_i\right)^{\xi} e^{\varphi\,\left(\frac{1}{2}\, b\, x_i^2 + c\, x_i\right)} \log^2\left(\frac{1}{2}\, b\, x_i^2 + c\, x_i\right) - \sum_{i=1}^{r}\frac{1}{\left(\varphi\,\left(\frac{1}{2}\, b\, x_i^2 + c\, x_i\right) + \xi\right)^2}$$

$$\begin{aligned}\ell_{\varphi\varphi}(\Phi) &= \sum_{i=1}^{r}\left(\frac{1}{2}\, b\, x_i^2 + c\, x_i\right)\left(-\frac{b\, x_i^2}{2\,\left(\varphi\left(\frac{1}{2}\, b\, x_i^2 + c\, x_i\right) + \xi\right)^2} - \right. \\ &\left.\frac{c\, x_i}{\left(\varphi\,\left(\frac{1}{2}\, b\, x_i^2 + c\, x_i\right) + \xi\right)^2}\right) - \sum_{i=1}^{r}\left(\frac{1}{2}\, b\, x_i^2 + c\, x_i\right)^{\xi+2} e^{\varphi\,\left(\frac{1}{2}\, b\, x_i^2 + c\, x_i\right)}\end{aligned}$$

$$\begin{aligned}\ell_{cb}(\Phi) &= \sum_{i=1}^{r} -\frac{\varphi^2\, x_i^3}{2\left(\varphi\left(\frac{1}{2}\, b\, x_i^2 + c\, x_i\right) + \xi\right)^2} + \sum_{i=1}^{r} -\frac{(\xi-1)\, x_i^3}{2\left(\frac{1}{2}\, b\, x_i^2 + c\, x_i\right)^2} + \sum_{i=1}^{r} -\frac{x_i}{(b\, x_i + c)^2} - \\ &\sum_{i=1}^{r} e^{\varphi\,\left(\frac{1}{2}\, b\, x_i^2 + c\, x_i\right)}\left[\frac{1}{2}\,\varphi^2\, x_i^3 \left(\frac{1}{2}\, b\, x_i^2 + c\, x_i\right)^{\xi} + \xi\,\varphi\, x_i^3 \left(\frac{1}{2}\, b\, x_i^2 + c\, x_i\right)^{\xi-1} + \right. \\ &\left. \xi\, x_i \left(\frac{1}{2}\,\xi\, x_i^2 \left(\frac{1}{2}\, b\, x_i^2 + c\, x_i\right)^{\xi-2} - \frac{1}{2} x_i^2 \left(\frac{1}{2} b x_i^2 + c x_i\right)^{\xi-2}\right)\right]\end{aligned}$$

$$\begin{aligned}\ell_{c\xi}(\Phi) &= -\sum_{i=1}^{r} e^{\varphi\,\left(\frac{1}{2}\, b\, x_i^2 + c\, x_i\right)}\left[x_i \left(\frac{1}{2}\, b\, x_i^2 + c\, x_i\right)^{\xi-1} + \xi\, x_i \left(\frac{1}{2}\, b\, x_i^2 + c\, x_i\right)^{\xi-1}\right. \\ &\left.\log\left(\frac{1}{2}\, b\, x_i^2 + c\, x_i\right) + \varphi\, x_i \left(\frac{1}{2}\, b\, x_i^2 + c\, x_i\right)^{\xi} \log\left(\frac{1}{2}\, b\, x_i^2 + c\, x_i\right)\right] \\ &\sum_{i=1}^{r} -\frac{\varphi\, x_i}{\left(\varphi\,\left(\frac{1}{2}\, b\, x_i^2 + c\, x_i\right) + \xi\right)^2} + \sum_{i=1}^{r}\frac{x_i}{\frac{1}{2}\, b\, x_i^2 + c\, x_i}\end{aligned}$$

$$\ell_{c\varphi}(\Phi) = \sum_{i=1}^{r}\left[\varphi\, x_i\left(-\frac{b\, x_i^2}{2\left(\varphi\ \left(\frac{1}{2}\, b\, x_i^2 + c\, x_i\right)+\xi\right)^2} - \frac{c\, x_i}{\left(\varphi\ \left(\frac{1}{2}\, b\, x_i^2 + c\, x_i\right)+\xi\right)^2}\right) + \frac{x_i}{\varphi\left(\frac{1}{2}\, b\, x_i^2 + c\, x_i\right)+\xi}\right] + \sum_{i=1}^{r} x_i - \sum_{i=1}^{r} e^{\varphi\ \left(\frac{1}{2}\ b\ x_i^2 + c\ x_i\right)}\left[\xi\, x_i\left(\frac{1}{2}\, b\, x_i^2 + c\, x_i\right)^{\xi} + x_i\left(\frac{1}{2}\, b\, x_i^2 + c\, x_i\right)^{\xi} + \varphi\, x_i\ \left(\frac{1}{2}\, b\, x_i^2 + c\, x_i\right)^{\xi+1}\right]$$

$$\ell_{b\xi}(\Phi) = \sum_{i=1}^{r}\frac{x_i^2}{2\left(\frac{1}{2}\, b\, x_i^2 + c\, x_i\right)} + \sum_{i=1}^{r} - \frac{\varphi\, x_i^2}{2\left(\varphi\ \left(\frac{1}{2}\, b\, x_i^2 + c\, x_i\right)+\xi\right)^2} - \sum_{i=1}^{r} e^{\varphi\ \left(\frac{1}{2}\ b\ x_i^2 + c\ x_i\right)} \left[\frac{1}{2}\, x_i^2\left(\frac{1}{2}\, b\, x_i^2 + c\, x_i\right)^{\xi-1} + \frac{1}{2}\,\xi\, x_i^2\left(\frac{1}{2}\, b\, x_i^2 + c\, x_i\right)^{\xi-1} \log\left(\frac{1}{2}\, b\, x_i^2 + c\, x_i\right) + \frac{1}{2}\,\varphi\, x_i^2\left(\frac{1}{2}\, b\, x_i^2 + c\, x_i\right)^{\xi}\log\left(\frac{1}{2}\, b\, x_i^2 + c\, x_i\right)\right]$$

$$\ell_{b\varphi}(\Phi) = \sum_{i=1}^{r}\frac{x_i^2}{2} - \sum_{i=1}^{r} e^{\varphi\ \left(\frac{1}{2}\ b\ x_i^2 + c\ x_i\right)}\left[\frac{1}{2}\,\xi\, x_i^2\left(\frac{1}{2}\, b\, x_i^2 + c\, x_i\right)^{\xi} + \frac{1}{2}\, x_i^2\left(\frac{1}{2}\, b\, x_i^2 + c\, x_i\right)^{\xi} + \frac{1}{2}\,\varphi\, x_i^2\left(\frac{1}{2}\, b\, x_i^2 + c\, x_i\right)^{\xi+1}\right] + \sum_{i=1}^{r}\left[\frac{1}{2}\,\varphi\, x_i^2\left(-\frac{b\, x_i^2}{2\left(\varphi\ \left(\frac{1}{2}\, b\, x_i^2 + c\, x_i\right)+\xi\right)^2} - \frac{c\, x_i}{\left(\varphi\left(\frac{1}{2}\, b\, x_i^2 + c\, x_i\right)+\xi\right)^2}\right) + \frac{x_i^2}{2\left(\varphi\ \left(\frac{1}{2}\, b\, x_i^2 + c\, x_i\right)+\xi\right)}\right]$$

$$\ell_{\xi\varphi}(\Phi) = \sum_{i=1}^{r}\left(-\frac{b\, x_i^2}{2\left(\varphi\ \left(\frac{1}{2}\, b\, x_i^2 + c\, x_i\right)+\xi\right)^2} - \frac{c\, x_i}{\left(\varphi\left(\frac{1}{2}\, b\, x_i^2 + c\, x_i\right)+\xi\right)^2}\right) - \sum_{i=1}^{r} e^{\varphi\ \left(\frac{1}{2}\ b\ x_i^2 + c\ x_i\right)}\left(\frac{1}{2}\, b\, x_i^2 + c\, x_i\right)^{\xi+1}\log\left(\frac{1}{2}\, b\, x_i^2 + c\, x_i\right)$$

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