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Least squares estimation for reflected Ornstein-Uhlenbeck processes with one and two sided barriers

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Abstract Diffusion processes for modelling, among others, dataset for instance, (macro-) econometrics, mathematical finance, biology, queueing, and electrical engineering often involve reflecting one or two barriers. In this paper, we investigate the least squares estimation (LSE) for a one dimensional continuous-time ergodic reflected Ornstein-Uhlenbeck (ROU) processes that returns continuously and immediately to the interior of the state space when it attains one and/or two-sided barriers. Both the estimates based on continuously observed processes and discretely observed processes are considered. So, we derive explicit formulas for the estimates, and then we establish their consistency and asymptotic normality (CAN). We also illustrate the CAN properties of the estimates through a Monte Carlo simulation and comparing with respect to maximum likelihood estimation (LME) as benchmark method showing the performance of the proposed estimators with moderate sample sizes. The method is valid irrespective of the length of the time intervals between consecutive observations.

Keywords Reflected Ornstein-Uhlenbeck processes, Movement Brownian, Least squares estimation, One barrier, Two barriers

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1. Introduction

In recent years, diffusions with one or two reflecting barriers have been widely used to model different phenomena in various fields such as queueing systems, financial engineering, mathematical biology, insurance, and so on. Indeed, from point of view application, among economics and finance applications, we mention the currency exchange rate target-zone models pioneered by Krugman [19] (see also Svensson [27], Bertolla et al. [4], de Jong [10], and Ball et al. [3]), in which the currency exchange rate is allowed to float within a target zone with two barriers enforced by the monetary authority, asset pricing models with price caps and/or price supports (e.g., price supports for agricultural commodities) (see Hanson et al. [15]), interest rate models with targeting by the monetary authority (e.g., Farnsworth et al. [11]), interest rate models with reflection at zero interest rate (e.g., Goldstein et al. [12] and Gorovoi et al. [13]), and stochastic volatility models (e.g., Schobel et al. [26]). References to further applications in economics can be found in Veestraeten [29]. Harrison [14] and Bo et al. [5] model the exchange rates in the European monetary system by an ROU model, Krugman [19] has proposed a ROU process with two reflecting barriers target to model the currency exchange rate dynamics in a target zone. In the field of mathematical biology, the application of the ROU process is discussed by Ricciardi et al. [25] and in ecology by Ricciardi [24]. Bo et al. [7, 6] have presented the ROU process for modelling the so-called regulated financial market. We refer the interested reader to Harrison [14] and Whitt [32] for more details on reflected processes and

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their broad applications. From point of view theoretical, the ROU diffusion occupy a central place in literature due to its flexibility and by the fact that ROU processes behave like the standard OU processes in the interior of its domain, so it becomes an appealing tool for approximating Markovian queue models (that can be characterized as birth-death continuous-time Markov chains) when reneging is present, more precisely, Ward et al. [30] who showed that a queueing system with reneging can be approximated via an appropriate Markovian ROU process. Harrison [14], Abate et al. [1] have shown that the reflected Brownian motion has long played a key role in queueing models. However, the statistical inference for the ROU process has been considered by several authors and some methods were proposed in order to estimate the parameters involved in such models. A shorten rewiew of their estimation is discussed below.

1.1. Literature review

Fundamental properties on the statistical inference of the diffusion processes can be found in Prakasa Rao [22]. In particular, Bo et al. [8] have studied the asymptotic properties of the maximum likelihood estimator (MLE), Lee et al. [20] studied the sequential MLE for the drift parameter based on the continuous time observations. Hu et al. [17] studied the MLE for ROU processes with discrete observations under the assumption that only the state process itself (not the local time process) is observed. Valdivieso et al. [28] investigated the MLE for the OU type processes driven by the Lévy process. More recently, Zhu [35] studied the asymptotic properties of the MLE for the ROU model with two$-$sided barriers. Recently, Yuecaia et al. [33, 34] have proposed a linear and nonlinear LSE for estimation the ROU process.
In the present paper, we focused on the problem of parameter estimation involved in the ROU process using LES for more general cases in which a multiple parameters are present in one and two-sided barriers. So, this paper offers some extensive study of the properties of the ROU process and our body of results expands what is known about the ROU process, both qualitatively and quantitatively. The main scope of our paper is articulated on the following methodology:

1.2. Motivations

In the case of diffusion processes driven by Brownian motions, a popular method is the MLE based on the Girsanov density. It is asymptotically equivalent to LSE. The CAN properties of LSE are widely studied in above references. For the reflected diffusion (in particular ROU processes), we don't know much about its CAN properties. This finding constitute the major motivation of our paper. The parameters estimation problem in ROU with one barrier has gained much attention in recent years due to its increased applications in broad fields, in our paper beside one-side barrier we consider the case of two barriers which constitute the second motivation. The third motivation, raised on the discretization scheme of ROU process with two barriers is also useful to study.

1.3. Contribution

The parameters which characterize the ROU process should be estimated via the data in many real-world applications. As far as we know, that the MLE for the drift parameter is studied in Bo et al. [8]. They studied the CAN properties without any results concerning the asymptotic variance of such estimates. However, there is only limited literature on LSE for the parameters of a ROU process. So, in this paper,we propose two types of LSE's for the parameters involved in ROU process. Specifically, our contributions are as follows:

1. The LSE method is firstly based on continuously observed processes and secondly based on discrete observed processes. The two methods are treated for one and/or two barriers.

2. The CAN properties is studied and more asymptotic results are given.

3. To provide a rigorous proof of the formula for its steady-state distribution; see Theorems 2 and 3.

4. Some new statistical methodology, derived from a discrete approximation procedure, is discussed.

5. The simulation studies is carried out showing the performance of LSE when compared with MLE method.

1.4. Content

The remainder of the paper is organized as follows. In the next section, we give some preliminary results related to the model with one barrier and the asymptotic properties of its LSE estimates for ROU processes. In section 3 we extend LSE to two-barriers case. Section 4 is devoted to highlighting the theoretical results with some Monte Carlo simulations. The last section concludes the paper.

2. Reflected OU with one-sided barrier

In this section, we first recommend the ROU process with one sided barrier briefly. Given a process $X = (X(t))_{t\geq 0}$ be a diffusion process, with infinitesimal variance β^2 and infinitesimal drift $\mu - \alpha x$ (the parameter α carries the physical meaning of customer reneging (or balking) rate from the system). We first deal with ROU with one-sided barrier b^L. According to Ward et al. [31], the process X defined on an usual filtered probability space $\left(\Omega, \Im, (\Im_t)_{t\geq 0}, P\right)$ is called a ROU process if X is the strong solution of the following stochastic differential equation (SDE), almost surely $(a.s)$ for $t \geq 0$,

$$dX(t) = (-\alpha X(t) + \mu)dt + \beta dW(t) + dL(t) \text{ with } X(t) \geq b^L, \text{ and } X(0) = X_0, \tag{1}$$

where μ,$b^L \in \mathbb{R}$, $\alpha, \beta \in (0, \infty)$ and $W = (W(t))_{t\geq 0}$ is a one-dimensional standard Brownian motion defined on the same filtered probability space, $L = (L(t))_{t\geq 0}$ is the minimal continuous increasing processes (which makes $X(t) \geq b^L$ for all $t \geq 0$) and satisfies $L(0) = 0$, $\int\limits_0^\infty 1_{\{X(t)>b^L\}} dL(t) = 0$, $\int\limits_0^t 1_{\{X(s)=b^L\}} dL(s) = L(t)$ and where 1_Δ denotes the indicator function (see Harrison [14], Ata et al. [2] for more discussion). Sometimes L is called the regulator of the point b^L, and has an explicit expression as $L(t) = \max\left\{0, \sup\limits_{u\in[0,t]}\left(-X_0 + \alpha\int\limits_0^u X(s)ds - \mu u - \beta w(u)\right)\right\} = \max\left\{0, \sup_{u\in[0,t]}(L(u) - X(u))\right\}$ and the ROU process (1) can be constructed via a Markovian approximation procedure (see e.g., Bo et al. [9]). Starting from initial position X_0 which is assumed to be not dependent on W, then, a formal Itô solution of (1) is given by

$$X(t) = e^{-\alpha t}\left\{X_0 - \frac{\mu}{\alpha}\left(e^{\alpha t} - 1\right) + \int\limits_0^t e^{\alpha s} dL(s) + \beta\int\limits_0^t e^{\alpha s} dW(s)\right\}. \tag{2}$$

It is easily verified that the process $(X_t)_{t\geq 0}$ as defined by (2) satisfies (1) for any α, μ, β and choice of $X(0)$, it is the unique, strong Markovian solution to (1) ensured by a careful extension of the results of Lions et al. [16] and Ward et al. [31]. The stochastic integral in (2) is well defined and satisfies the properties outlined in Protter [23], for example. Moreover, the process, $M(t) = \int\limits_0^t e^{\alpha s} dW(s)$ is a zero-mean martingale with respect to the natural filtration of $(W(t))_t$.

Remark 1

1. The assumption $\alpha > 0$ and $\beta > 0$, ensure that the equation (1) provide the stationary solution with invariant density is given by Lemma 1 below.
2. When $\alpha = 0$ (balking case), the corresponding ROU process X reduces to the so-called reflected Brownian motion (RBM) process and we refer to Harrison [14] for a rigorous definition of such processes and their properties of interest.
3. In case $b^L = 0$, we refer to Linetsky [16], Ward et al. [31].

In the sequel, we shall note $\phi(u) = \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{u^2}{2}\right\}$ and $\Phi(y) = \int\limits_{-\infty}^{y} \phi(u)du$ is the density and the distribution function associated with $\mathcal{N}(0,1)$. The following lemma is due to Hu et al. [17] which states an ergodic theorem will be used frequently.

Lemma 1 (Hu et al. [17])
Consider the ROU process X defined by (1), suppose that $b^L = 0$, then for any $x \geq 0$.

1. The process X has a unique ergodic and stationary distribution π with associated invariant density $p(x) = \sqrt{\frac{2\alpha}{\beta^2}} \frac{\phi(\overline{x})}{1-\Phi(\overline{0})}$ where $\overline{x} = \sqrt{\frac{2\alpha}{\beta^2}}\left(x - \frac{\mu}{\alpha}\right)$.
2. For any integrable function f the following mean ergodic theorem holds $\lim\limits_{T\to+\infty} \frac{1}{T}\int_0^T f(X(s))\,ds = \int_0^\infty f(x)p(x)dx$ a.s. Moreover, we have

$$E\left\{X^k\right\} = \begin{cases} \frac{\mu}{\alpha} + \frac{\phi(\overline{0})}{1-\Phi(\overline{0})}\sqrt{\frac{\beta^2}{2\alpha}}, k = 1, \\ \frac{\beta^2}{2\alpha} + \frac{\mu^2}{\alpha^2} + \frac{\phi(\overline{0})}{1-\Phi(\overline{0})}\sqrt{\frac{\beta^2}{2\alpha}}\frac{\mu}{\alpha}, k = 2. \end{cases}$$

Proof
See Hu et al. [17]. □

Remark 2
If $b^L \neq 0$, the unique invariant density can be rewritten as $p(x) = \sqrt{\frac{2\alpha}{\beta^2}} \frac{\phi(\overline{x})}{1-\Phi\left(\overline{b}^L\right)}$, $x \in [b^L, \infty)$ and hence the first and the second−order moments are

$$E\left\{X^k\right\} = \begin{cases} \frac{\mu}{\alpha} + \frac{\phi\left(\overline{b}^L\right)}{1-\Phi\left(\overline{b}^L\right)}\sqrt{\frac{\beta^2}{2\alpha}}, k = 1, \\ \frac{\beta^2}{2\alpha} + \frac{\mu^2}{\alpha^2} + 2\frac{\phi\left(\overline{b}^L\right)}{1-\Phi\left(\overline{b}^L\right)}\sqrt{\frac{\beta^2}{2\alpha}}\frac{\mu}{\alpha} + \frac{\phi\left(\overline{b}^L\right)\overline{b}^L}{1-\Phi\left(\overline{b}^L\right)}, k = 2. \end{cases}$$

Moreover, from the above expressions, it can be seen that the parameter β^2 can be expressed in term of α, μ, $E\{X\}$ and the second−order moments as follow

$$\beta^2 = 2\alpha\left(E\left\{X^2\right\} - \frac{\mu^2}{\alpha^2} - \left(b^L + \frac{\mu}{\alpha}\right)\left(E\{X\} - \frac{\mu}{\alpha}\right)\right). \tag{3}$$

2.1. LSE for the ROU processes with one-sided barrier

2.1.1. Continuously observed processes In this subsection, we investigate firstly the estimation of the unknown parameters, α, μ and β of the ROU process (1) from continuous observations, i.e., we suppose that the ROU process $X = (X(t))_{T\geq t\geq 0}$ is observed and $T \to \infty$. For this purpose, we assume that β is known since the process can be observed continuously (see, e.g., Prakasa Rao [22], p. 15). So, we mainly focus on the estimation of the parameters α and μ gathered in vector $\underline{\theta} = (\alpha, \mu)'$ its true value is denoted by $\underline{\theta}_0$. The least squares estimator $\widehat{\underline{\theta}}_T$ of $\underline{\theta}_0$ is defined as any measurable $\widehat{\underline{\theta}}_T$ of $\widehat{\underline{\theta}}_T = Arg\min_{\underline{\theta}} Q_T(\underline{\theta})$ where $Q_T(\underline{\theta}) = \int\limits_0^T \left(X^{(1)}(s) + \alpha X(s) - \mu - L^{(1)}(s)\right)^2 ds$, in which the superscript $^{(j)}$ denotes $j-$fold differentiation with respect to t. Rewriting $Q_T(\underline{\theta})$ as

$$\int_0^T \left(X^{(1)}(s) + \alpha X(s) - \mu - L^{(1)}(s)\right)^2 ds$$
$$= \int_0^T \left(X^{(1)}(s) - L^{(1)}(s)\right)\left(\left(X^{(1)}(s) - L^{(1)}(s)\right) + 2\alpha X(s) - 2\mu\right) ds + \alpha^2 T\widehat{m}_2 - 2\alpha\mu T\widehat{m}_1 + \mu^2 T,$$

where $\widehat{m}_k = \frac{1}{T}\int_0^T X^k(s)ds$. It is easy to see that the minimum is attained when α is given by

$$\widehat{\alpha}_T = \frac{(X(T) - L(T))\,T\widehat{m}_1 + T\int_0^T X(s)dL(s) - T\int_0^T X^{(1)}(s)X(s)ds}{T\widehat{\sigma}_T^2} \tag{4}$$
$$= \frac{(X(T) - L(T))\,T\widehat{m}_1 + Tb^L L(T) - T\int_0^T X(s)dX(s)}{T\widehat{\sigma}_T^2},$$

where $\widehat{\sigma}_T^2 = T\left(\widehat{m}_2 - (\widehat{m}_1)^2\right)$. An estimate of the parameter μ is given by $\widehat{\mu}_T = \frac{1}{T}(X(T) - L(T)) + \widehat{\alpha}_T\widehat{m}_1$. Now, we can use the expression (3) to estimate β^2 as follows

$$\widehat{\beta}_T^2 = 2\widehat{\alpha}_T\left(\widehat{m}_2 - \frac{\widehat{\mu}_T^2}{\widehat{\alpha}_T^2} - \left(b^L + \frac{\widehat{\mu}_T}{\widehat{\alpha}_T}\right)\left(\widehat{m}_1 - \frac{\widehat{\mu}_T}{\widehat{\alpha}_T}\right)\right). \tag{5}$$

We are now in a position to state the main results concerning the asymptotic properties of the estimates $\widehat{\alpha}_T$, $\widehat{\mu}_T$,$\widehat{\beta}_T^2$.

Theorem 1
The estimators $\widehat{\alpha}_T$ and $\widehat{\mu}_T$ are strongly consistent estimators of α and μ i.e., $\lim_{T\to\infty} \widehat{\alpha}_T = \alpha$ and $\lim_{T\to\infty} \widehat{\mu}_T = \mu$. *a.s.*

Proof
To show the least squares estimator $\widehat{\alpha}_T$ converges to α we rewrite the expression (4) as $\widehat{\alpha}_T = \alpha - \frac{\beta}{\widehat{\sigma}_T^2}\int_0^T X(s)dW(s)$. Since $\int_0^T X(s)dW(s)$ is a martingale with bracket $T\widehat{m}_2$ and when $T \to \infty$, $\int_0^T X(s)dW(s)$ is the order of $T\widehat{m}_2$. Moreover, as $T \to \infty$, $\widehat{m}_k$ converges to $E\left\{X^k\right\}$ (this is an immediate consequence of the ergodic theorem in Lemma 1 by setting $f(x) = x^n, n = 1, 2$). Thus $\frac{1}{\widehat{\sigma}_T^2}\int_0^T X(s)dW(s)$ converges to 0 with the order $O(\frac{1}{\sqrt{T}})$. To prove the strong consistent of $\widehat{\mu}_T$, we can see that $\widehat{\mu}_T$ can be rewrite as follows $\widehat{\mu}_T - \mu = (\widehat{\alpha}_T - \alpha)\,\widehat{m}_1 + \frac{\beta}{T}W(T)$. The convergence of $\widehat{\mu}_T$ to μ follows immediately from the above expression and the fact that $\widehat{\alpha}_T \to \alpha$ and $\frac{1}{T}\int_0^T X(s)ds \to E\{X\}$ by the Lemma 1 . □

Theorem 2
The estimator $\widehat{\alpha}_T$ of α is asymptotically normal i.e., $\widehat{\sigma}_T^2\left(\frac{\alpha - \widehat{\alpha}_T}{\beta}\right)(T\widehat{m}_2)^{-1/2} \rightsquigarrow \mathcal{N}(0, 1)$ as $T \to \infty$.

Proof
It is no difficult to see that $\widehat{\sigma}_T^2\left(\frac{\alpha - \widehat{\alpha}_T}{\beta}\right) = \int_0^T X(s)dW(s)$, and the fact that $E\left\{\int_0^T X(s)dW(s)\right\} = 0$,

$E\left\{\left(\int\limits_0^T X(s)dW(s)\right)^2\right\} = TE\left\{\widehat{m}_1\right\}$. Then applying the Central Limit Theorem (Theorem B.10, p. 313 of Prakasa Rao [22]), the result follows. □

The following theorem shows the strong consistency of the estimator $\widehat{\beta}_T^2$.

Theorem 3
The estimator $\widehat{\beta}_T^2$ of β^2 is strong consistency i.e., $a.s.$ $\lim\limits_{T\to\infty}\widehat{\beta}_T^2 = \beta^2$ as $T \to \infty$.

Proof
The strong consistency of the estimator $\widehat{\beta}_T^2$ follows immediately from the expression (3) and the Theorem 1 where we prove that both estimators $\widehat{\alpha}_T$ and $\widehat{\mu}_T$ are strong consistency for α and μ respectively and the strong convergence of $\widehat{m}_j$ for $j = 1, 2$ by the ergodic theorem (see Lemma 1). □

2.1.2. Discretized observed processes In this subsection, we investigate secondly the estimation of the unknown parameters, α, μ and β of the ROU process (1) from discrete observations , i.e., when the processes is observed at the discrete time instants $\{t_k = kh, k = 0; ..., n\}$, $h \to 0$ and $nh \to +\infty$. Some elementary computations yield the following expressions for the stationary moments of the invariant measure with $\mu \geq 0$.

$$E\left\{X^k\right\} = \begin{cases} \dfrac{\mu}{\alpha} + \dfrac{\beta}{2\alpha}\dfrac{\phi\left(\frac{\sqrt{2}\mu}{\beta}\right)}{1-\Phi\left(-\frac{\sqrt{2}\mu}{\beta}\right)}, k = 1, \\ \dfrac{\beta^2}{2\alpha} + \dfrac{\mu^2}{\alpha^2} + \dfrac{\mu}{\alpha}\frac{\beta}{\sqrt{2\alpha}}\dfrac{\phi\left(\frac{\sqrt{2}\mu}{\beta}\right)}{1-\Phi\left(-\frac{\sqrt{2}\mu}{\beta}\right)}, k = 2. \end{cases}$$

The discrete-type LSE is motivated by minimizing the following contrast function $\sum_{k=0}^n (X(t_{k+1}) - X(t_k) - (\mu - \alpha X(_k))\, h - \Delta_k L)^2$ where $\Delta_k L = L(t_{k+1}) - L(t_k)$. The minimum is achieved when

$$\widehat{\underline{\theta}}_n = \left(\frac{1}{nh}\sum_{t=0}^{n-1}\underline{X}(t_k)\,\underline{X}(t_k)'\right)^{-1}\frac{1}{nh}\sum_{t=0}^{n-1} y(t_k)\,\underline{X}(t_k), \tag{6}$$

where $\underline{X}(t) = (1, X(t))'$, $y(t_k) = \Delta_k X - \Delta_k L$. According to independent increments of $(W(t))_{t\geq 0}$, then $\Delta_k W \backsimeq \sqrt{h}e_{k+1}$ where $e_k \rightsquigarrow \mathcal{N}(0,1)$, and β^2 will be treated as nuisance parameters, otherwise, a possible consistent estimator of β^2 is given by (5) . Thus we shall focus on the parameter $\underline{\theta} = (\mu, -\alpha)$. The following theorem proves the CAN properties of the discrete version of LSE.

Theorem 4
Suppose that $0 < \alpha < 1$, The estimator $\widehat{\underline{\theta}}_n$ of $\underline{\theta}$ admits the asymptotic properties, i.e.,

1. $\widehat{\underline{\theta}}_n \to \underline{\theta}$ $a.s.$, as $n \to +\infty$.
2. $\sqrt{hn}\left(\widehat{\underline{\theta}}_n\text{-}\underline{\theta}\right) \rightsquigarrow N(0, \beta^2\Sigma_{(1)}^{-1}(\underline{\theta}_0))$, as $n \to +\infty$ where $\Sigma_{(1)}(\underline{\theta}_0) = E\left\{\underline{X}(t)\,\underline{X}'(t)\right\}$.

Proof

1. The first assertion holds true upon the observation that by (6) we have an alternative expressions of $\widehat{\underline{\theta}}_n$

$$\widehat{\underline{\theta}}_n - \underline{\theta} = h^{-1}S_n^{-1}\frac{1}{nh}\sum_{k=0}^{n-1}\underline{X}(t_k)\left(\int\limits_{t_k}^{t_{k+1}}\underline{\theta}\left(\underline{X}(t) - \underline{X}(t_k)\right)dt + \beta\Delta_k W\right), \tag{7}$$

where $S_n = \frac{1}{nh}\sum_{t=0}^{n-1} \underline{X}(t_k)\,\underline{X}(t_k)'$. We first consider the estimate of $\sup_{t_k\le t\le t_{k+1}} \|\underline{X}(t) - \underline{X}(t_k)\|$ for any norm $\|.\|$. Indeed, since $\underline{X}(t) - \underline{X}(t_k) = (0, X(t) - X(t_k))'$, then we have for any $t \in \Delta(k) = [t_k t_{k+1}]$,

$$|X(t) - X(t_k)| = \left|\mu(t-t_k) - \alpha\int_{t_k}^{t}(X(s)-X(t_k))ds - \alpha X(t_k)(t-t_k) + (L(t)-L(t_k)) + \beta(W(t)-W(t_k))\right|$$

$$\le |\mu|\,h + \alpha\int_{t_k}^{t}|(X(s)-X(t_k))|\,ds + \alpha|X(t_k)|\,h + \sup_t(|(L(t)-L(t_k))| + |\beta(W(t)-W(t_k))|).$$

By Gronwall$'$s inequality,

$$|X(t) - X(t_k)| \le |\mu|\,h + \left(\alpha|X(t_k)|\,h + \sup_t(|(L(t)-L(t_k))| + |\beta(W(t)-W(t_k))|)\right)e^{\alpha(t-t_k)}.$$

So, $\sup_{t\in\Delta(k)}|X(t) - X(t_k)| \le |\mu|\,h + \left(\alpha|X(t_k)|\,h + \sup_{t\in\Delta(k)}(|(L(t)-L(t_k))| + |\beta(W(t)-W(t_k))|)\right)e^{\alpha h}$. From the properties of the process $(L(t))_t$ and $\gamma-$Hölder continuity (see Yuecaia et al. [33]), we obtain $\sup_{t\in\Delta(k)}|X(t) - X(t_k)| \le |\mu|\,h + Ch^{\gamma}e^{\alpha h} = O(h^{\gamma})$, where $0<\gamma<1/2$ and C is some positive constant. Then

$$\frac{1}{nh}\sum_{k=0}^{n-1}\left\|\underline{X}(t_k)\int_{t_k}^{t_{k+1}}\underline{\theta}(\underline{X}(t)-\underline{X}(t_k))\,dt\right\| \le \frac{\alpha}{nh}\sum_{k=0}^{n-1}\|\underline{X}(t_k)\|\sup_t(|X(t)-X(t_k)|)\,h = O(h^{\gamma}),$$

which tends to 0 as $h\to 0$. Next, let $\underline{\phi}_k(t) = \underline{X}(t_k)\,I_{\Delta(k)}(t)$, then $\lim_{n\to\infty}\sum_{k=0}^{n-1}\underline{X}(t_k)\Delta_k W = \lim_{n\to\infty}\sum_{k=0}^{n-1}\int_0^{nh}\underline{\phi}_k(t)dW(t)$, so as above we obtain $\lim_{n\to\infty}\frac{1}{nh}\sum_{k=0}^{n-1}\underline{X}(t_k)\beta\Delta_k W = 0$.

2. From (7), we have

$$\sqrt{nh}\left(\widehat{\underline{\theta}}_n - \underline{\theta}\right) = h^{-1}S_n^{-1}\frac{1}{\sqrt{nh}}\sum_{k=0}^{n-1}\underline{X}(t_k)\left(\int_{t_k}^{t_{k+1}}\underline{\theta}(\underline{X}(t)-\underline{X}(t_k))\,dt + \beta\Delta_k W\right),$$

and $\frac{1}{\sqrt{nh}}\sum_{k=0}^{n-1}\underline{X}(t_k)\int_{t_k}^{t_{k+1}}\underline{\theta}(\underline{X}(t)-\underline{X}(t_k))\,dt \le O(\sqrt{nh^{1+2\gamma}}$ which goes to 0 an $n\to+\infty$. So by standard central limit theorem and Slutsky's theorem, the results follows.

□

Remark 3
It is worth noting that the matrix $\Sigma(\underline{\theta}_0)$ may be estimate by replacing its entries by their corresponding sample approximations i.e., $E\{X\}$ by $\frac{1}{n}\sum_{k=0}^{n-1}X(t_k)$ and $E\{X^2\}$ by $\frac{1}{n}\sum_{k=0}^{n-1}X^2(t_k)$.

3. Reflected OU processes with two-sided barriers

Following the motivations by Ward and Glynn [31] for one-sided barrier *ROU* processes and from the point of view of queuing system, it is natural to suggest a flexible model with finite buffer capacity. This leads us to consider a

ROU process $(X(t))_{t\geq 0}$ with two-sided barriers b^L and b^U defined as

$$dX(t) = (-\alpha X(t) + \mu)dt + \beta dW(t) + dL(t) - dU(t),\ X(t) \in \left[b^L, b^U\right],\ \text{for all } t \geq 0 \text{ and } X(0) = X_0. \quad (8)$$

By the standard definition in Harrison [14] or Ata et al. [2], the processes $L = (L(t))_{t\geq 0}$ is uniquely determined and associated with the lower barrier b^L and the upper barrier b^U, respectively. Both processes L and U are minimal continuous increasing processes which makes $X(t) \in \left[b^L, b^U\right]$ for all $t \geq 0$ with $L_0 = U_0 = 0$ and satisfies

$$\int_0^\infty 1_{\{X(t)>b^L\}} dL(t) = \int_0^\infty 1_{\{X(t)<b^U\}} dU(t) = 0. \quad (9)$$

Actually, the ROU process with two-sided barriers can be constructed via a Markovian approximation procedure (see,e.g., Bo et al. [9], Ward et al. [30, 31] for details). We can also refer to Linetsky [16] to see that the invariant density of X is given by $p(x) = \sqrt{\frac{2\alpha}{\beta^2}} \frac{\phi(\overline{x})}{\Phi\left(\overline{b}^U\right) - \Phi\left(\overline{b}^L\right)}$, $x \in \left[b^L, b^U\right]$ and for any $t \in \mathbb{R}$, the generator function has the form

$$E\left\{e^{tX}\right\} = \frac{\exp\left\{\frac{\mu}{\alpha}t + \frac{\beta^2}{4\alpha}t^2\right\}\left(\Phi\left(\overline{b}^U - t\sqrt{\frac{\beta^2}{2\alpha}}\right) - \Phi\left(\overline{b}^L - t\sqrt{\frac{\beta^2}{2\alpha}}\right)\right)}{\Phi\left(\overline{b}^U\right) - \Phi\left(\overline{b}^L\right)}, \quad (10)$$

where we have used the identity $\int\limits_b^\infty \phi(z) e^{az} dz = e^{\frac{a^2}{2}}\left[1 - \Phi(b-a)\right]$. Set $g(t) = \Phi\left(\overline{b}^U - t\sqrt{\frac{\beta^2}{2\alpha}}\right) - \Phi\left(\overline{b}^L - t\sqrt{\frac{\beta^2}{2\alpha}}\right)$ and $f(t) = exp\left\{\frac{\mu}{\alpha}t + \frac{\beta^2}{4\alpha}t^2\right\}$, then we can compute the following quantities which are used to obtain the moments of the third $-$order.

$$f(0) = 1,\ g(0) = \Phi\left(\overline{b}^U\right) - \Phi\left(\overline{b}^L\right),\ f'(t) = \left(\frac{\mu}{\alpha} + \frac{\beta^2}{2\alpha}t\right) f(t) \text{ and } f'(0) = \frac{\mu}{\alpha},$$

$$g'(t) = -\sqrt{\frac{\beta^2}{2\alpha}}\left(\phi\left(\overline{b}^U - t\sqrt{\frac{\beta^2}{2\alpha}}\right) - \phi\left(\overline{b}^L - t\sqrt{\frac{\beta^2}{2\alpha}}\right)\right) \text{ and } g'(0) = -\sqrt{\frac{\beta^2}{2\alpha}}\left(\phi\left(\overline{b}^U\right) - \phi\left(\overline{b}^L\right)\right),$$

$$f''(t) = \left(\frac{\beta^2}{2\alpha} + \left(\frac{\mu}{\alpha} + \frac{\beta^2}{2\alpha}t\right)^2\right) f(t) \text{ and } f''(0) = \frac{\beta^2}{2\alpha} + \frac{\mu^2}{\alpha^2},$$

$$g''(t) = -\frac{\beta^2}{2\alpha}\left(\overline{b}^U - t\sqrt{\frac{\beta^2}{2\alpha}}\right)\phi\left(\overline{b}^U - t\sqrt{\frac{\beta^2}{2\alpha}}\right) + \frac{\beta^2}{2\alpha}\left(\overline{b}^L - t\sqrt{\frac{\beta^2}{2\alpha}}\right)\phi\left(\overline{b}^L - t\sqrt{\frac{\beta^2}{2\alpha}}\right),$$

$$g''(0) = \sqrt{\frac{\beta^2}{2\alpha}}\left(-\left(b^U - \frac{\mu}{\alpha}\right)\phi\left(\overline{b}^U\right) + \left(b^L - \frac{\mu}{\alpha}\right)\phi\left(\overline{b}^L\right)\right),$$

$$f'''(t) = 2\frac{\beta^2}{2\alpha}f'(t) + \left(\frac{\mu}{\alpha} + \frac{\beta^2}{2\alpha}t\right) f''(t) \text{ and } f'''(0) = \frac{3\beta^2\mu}{2\alpha^2} + \frac{\mu^3}{\alpha^3},$$

$$g'''(t) = \frac{\beta^2}{2\alpha}\sqrt{\frac{\beta^2}{2\alpha}}\phi\left(\overline{b}^U - t\sqrt{\frac{\beta^2}{2\alpha}}\right)\left(1 - \left(\overline{b}^U - t\sqrt{\frac{\beta^2}{2\alpha}}\right)^2\right) - \frac{\beta^2}{2\alpha}\sqrt{\frac{\beta^2}{2\alpha}}\phi\left(\overline{b}^L - t\sqrt{\frac{\beta^2}{2\alpha}}\right)\left(1 - \left(\overline{b}^L - t\sqrt{\frac{\beta^2}{2\alpha}}\right)^2\right),$$

$$g'''(0) = \frac{\beta^2}{2\alpha}\sqrt{\frac{\beta^2}{2\alpha}}\left(\phi\left(\overline{b}^U\right)\left(1 - \overline{b}^{U2}\right) - \phi\left(\overline{b}^L\right)\left(1 - \overline{b}^{L2}\right)\right).$$

Using the relationship between the generator function and the moments we can obtain the third order moments as follows

$$E\{X\}=\frac{\mu}{\alpha}-\frac{1}{g(0)}\left(\phi\left(\bar{b}^U\right)-\phi\left(\bar{b}^L\right)\right)\sqrt{\frac{\beta^2}{2\alpha}}, \tag{11}$$

$$\begin{aligned}E\left\{X^2\right\}&=\frac{\beta^2}{2\alpha}+\frac{\mu^2}{\alpha^2}-\frac{2}{g(0)}\left(\phi\left(\bar{b}^U\right)-\phi\left(\bar{b}^L\right)\right)\sqrt{\frac{\beta^2}{2\alpha}}\frac{\mu}{\alpha}\\&-\frac{1}{g(0)}\left(\left(b^U-\frac{\mu}{\alpha}\right)\phi\left(\bar{b}^U\right)-\left(b^L-\frac{\mu}{\alpha}\right)\phi\left(\bar{b}^L\right)\right)\sqrt{\frac{\beta^2}{2\alpha}}.\end{aligned} \tag{12}$$

$$\begin{aligned}E\left\{X^3\right\}&=\frac{3\beta^2\mu}{2\alpha^2}+\frac{\mu^3}{\alpha^3}-\frac{3}{g(0)}\left(\phi\left(\bar{b}^U\right)-\phi\left(\bar{b}^L\right)\right)\sqrt{\frac{\beta^2}{2\alpha}}\left(\frac{\beta^2}{2\alpha}+\frac{\mu^2}{\alpha^2}\right)\\&-\frac{3}{g(0)}\left(\left(b^U-\frac{\mu}{\alpha}\right)\phi\left(\bar{b}^U\right)-\left(b^L-\frac{\mu}{\alpha}\right)\phi\left(\bar{b}^L\right)\right)\sqrt{\frac{\beta^2}{2\alpha}}\frac{\mu}{\alpha}\\&+\frac{1}{g(0)}\left(\phi\left(\bar{b}^U\right)\left(1-\bar{b}^{U2}\right)-\phi\left(\bar{b}^L\right)\left(1-\bar{b}^{L2}\right)\right)\sqrt{\frac{\beta^2}{2\alpha}}\frac{\beta^2}{2\alpha}.\end{aligned} \tag{13}$$

From the above expressions, the parameter β^2 can be written as

$$\beta^2=\frac{2\alpha\left\{M1\left(\alpha,\mu\right)-M2\left(\alpha,\mu\right)\right\}}{M3\left(\alpha,\mu\right)}, \tag{14}$$

where

$$\begin{aligned}M1\left(\alpha,\mu\right)&=\left(E\left\{X^2\right\}-\frac{\mu^2}{\alpha^2}-\left(E\left\{X\right\}-\frac{\mu}{\alpha}\right)\left(b^L+\frac{\mu}{\alpha}\right)\right)\left(b^L+b^U+\frac{\mu}{\alpha}\right),\\M2\left(\alpha,\mu\right)&=E\left\{X^3\right\}-\frac{\mu^3}{\alpha^3}-\left(E\left\{X\right\}-\frac{\mu}{\alpha}\right)\left(\left(b^L+\frac{\mu}{\alpha}\right)^2-b^L\frac{\mu}{\alpha}\right),\\M3\left(\alpha,\mu\right)&=b^L+b^U+\frac{\mu}{\alpha}-\frac{\mu}{2\alpha^2}-\frac{E\left\{X\right\}}{\alpha},\end{aligned}$$

which depends on α, μ, b^U, b^L and on $E\left\{X^k\right\}$, $k=1,2,3$.

3.1. LSE for the ROU processes with two-sided barriers

3.1.1. Continuously observed processes In this subsection, we intend to estimate the parameters of the ROU process (8) from continuous observations,i.e., we suppose that the ROU process $(X(t))_{T\geq t\geq 0}$ is observed and $T\to\infty$. For this purpose, we assume that β is known since the process can be observed continuously (see, e.g., Prakasa Rao [22], p. 15). So we mainly focus on the estimation of the parameters α and μ. The LSE of α,μ is obtained by minimizing the quadratic function

$$\begin{aligned}&\int_0^T\left(X^{(1)}(s)+\alpha X(s)-\mu-L^{(1)}(s)+U^{(1)}(s)\right)^2ds=\int_0^T\left(X^{(1)}-L^{(1)}(s)+U^{(1)}(s)\right)^2ds\\&+2\alpha\int_0^T X(s)\left(X^{(1)}(s)-L^{(1)}(s)+U^{(1)}(s)\right)ds+\alpha^2\int_0^T X^2(s)ds\\&-2\mu\int_0^T\left(X^{(1)}(s)-L^{(1)}(s)+U^{(1)}(s)\right)ds-2\alpha\mu\int_0^T X(s)ds+\mu^2T.\end{aligned} \tag{15}$$

It is easy to see that the minimum is attained when α is given by

$$\widehat{\alpha}_T = \frac{(X(T) - L(T) + U(T)) \int_0^T X(s)ds + T\int_0^T X(s)dL(s) - T\int_0^T X(s)dU(s) - T\int_0^T X^{(1)}(s)X(s)ds}{T\widehat{\sigma}_T^2}$$
$$= \frac{(X(T) - L(T) + U(T))\, T\widehat{m}_1 + Tb^L L(t) - Tb^U U(t) - T\int_0^T X(s)dX(s)}{T\widehat{\sigma}_T^2}, \tag{16}$$

and hence the parameter μ is estimate by $\widehat{\mu}_T = \frac{1}{T}(X(T) - L(T) + U(T)) + \widehat{\alpha}_T \widehat{m}_1$ where the equality (16) follows from (9). Now, we can use the expression (14) to estimate the parameter β^2 as follow

$$\widehat{\beta}_T^2 = \frac{2\widehat{\alpha}_T \{M1(\widehat{\alpha}_T, \widehat{\mu}_T) - M2(\widehat{\alpha}_T, \widehat{\mu}_T)\}}{M3(\widehat{\alpha}_T, \widehat{\mu}_T)}. \tag{17}$$

We now state the main results concerning the consistency and the asymptotic normality of the LSE of $\widehat{\mu}_T$ and $\widehat{\alpha}_T$.

Theorem 5
The estimator $(\widehat{\alpha}_T, \widehat{\mu}_T)'$ of $(\alpha, \mu)'$ is strong consistency, i.e., $\lim_{T\to\infty} \widehat{\alpha}_T = \alpha$, $a.s$ and $\lim_{T\to\infty} \widehat{\mu}_T = \mu$, $a.s.$

Proof
The proof follows essentially the same arguments as in the proof of theorem 1. □

Theorem 6
The estimators $\widehat{\alpha}_T$ of α is asymptotically normal i.e., $\sqrt{T}\left(\frac{\widehat{\alpha}_T - \alpha}{\beta}\right) \rightsquigarrow \mathcal{N}(0, \sigma_\alpha)$, as $T \to \infty$, where $\sigma_\alpha = \frac{E\{X^2\}}{\left((E\{X\})^2 - E\{X^2\}\right)^2}$.

Proof
It is not difficult to see that $\left(\frac{\alpha - \widehat{\alpha}_T}{\beta}\right) T^{-1}\widehat{\sigma}_T^2 = \int_0^T X(s)dW(s)$, and the fact that $E\left\{\int_0^T X(s)dW(s)\right\} = 0$, $E\left\{\left(\int_0^T X(s)dW(s)\right)^2\right\} = TE\{\widehat{m}_1\}$. Then applying the Central Limit Theorem (Theorem B.10, p. 313 of Prakasa Rao [22]), we obtain $\sqrt{T}\left(\frac{\alpha - \widehat{\alpha}_T}{\beta}\right)\frac{T^{-1}\widehat{\sigma}_T^2}{\sqrt{\widehat{m}_2}} \rightsquigarrow \mathcal{N}(0,1)$ as $T \to \infty$. Now, we use the continuous-time ergodic theorem (Theorem 9.8, page 161, Kallenberg [18]) to see that $\lim_{T\to\infty} \frac{T^{-1}\widehat{\sigma}_T^2}{\sqrt{\widehat{m}_2}} = \frac{E\{X^2\} - (E\{X\})^2}{\sqrt{E\{X^2\}}}$ a.s where $E\{X\}$ and $E\{X^2\}$ are given respectively by the expressions (11) and (12), so, it follows that $\sqrt{T}\left(\frac{\widehat{\alpha}_T - \alpha}{\beta}\right) \rightsquigarrow \mathcal{N}(0, \sigma_\alpha)$, as $T \to \infty$, with $\sigma_\alpha = \frac{E\{X^2\}}{\left[(E\{X\})^2 - E\{X^2\}\right]^2}$. □

The following theorem shows the strong consistency of the estimator $\widehat{\beta}_T^2$ of β^2.

Theorem 7
The estimator $\widehat{\beta}_T^2$ is strong consistency, i.e., almost surely $\lim_{T\to\infty} \widehat{\beta}_T^2 = \beta^2$.

Proof
The strong consistency of the estimator $\widehat{\beta}_T^2$ follows immediately from the expression (17) and the Theorem 5 where we have proved that both estimators $\widehat{\alpha}_T$, $\widehat{\mu}_T$ are strongly consistent to α, μ respectively and the strong convergence of $\widehat{m}_j$ for $j = 1, 2, 3$ by the ergodic theorem (Theorem 9.8, page 161, Kallenberg [18]). □

3.1.2. Discretized observed processes When the processes is observed at the discrete time instants $\{t_k = kh, k = 0; ..., n\}$, the discrete-type LSE for ROU with two barriers is motivated by minimizing the following contrast function $\sum_{k=0}^{n}(X\left(t_{k+1}\right) - X\left(t_k\right) - \left(\mu - \alpha X\left(_k\right)\right)h - \Delta_k L + \Delta_k U)^2$. As in subsection 2.1.2, the minimum is achieved when

$$\underline{\widehat{\theta}}_n = \left(\frac{1}{nh}\sum_{t=0}^{n-1} \underline{X}\left(t_k\right)\underline{X}\left(t_k\right)'\right)^{-1} \frac{1}{nh}\sum_{t=0}^{n-1} z\left(t_k\right)\underline{X}\left(t_k\right), \tag{18}$$

where $z\left(t_k\right) = \Delta_k X - \Delta_k L + \Delta_k U$. The following theorem proves the CAN properties of the discrete LSE for ROU with two-sided barriers.

Theorem 8
The estimator $\underline{\widehat{\theta}}_n$ of $\underline{\theta}$ admits the asymptotic properties, i.e.,

1. $\underline{\widehat{\theta}}_n \to \underline{\theta}$ $a.s.$, as $n \to +\infty$.
2. $\sqrt{nh}\left(\underline{\widehat{\theta}}_n\text{-}\underline{\theta}\right) \rightsquigarrow N(0, \beta^2\Sigma_{(2)}^{-1}\left(\underline{\theta}_0\right))$, as $n \to +\infty$ where the entries of the matrix $\Sigma_{(2)}\left(\underline{\theta}_0\right)$ are given by (11) and (12).

Proof
The CAN properties follow the same arguments as the proof of Theorem 4 using the statistics $\underline{\widehat{\theta}}_n$ given by (18). □

4. Numerical results

As already pointed out in the introduction, the class of birth-death continuous-time Markov chains can be approximated (in the distribution sense) by an ROU process. For an illustration, consider a birth-death process $\left(Z\left(t\right)\right)_{t\geq 0}$ with birth rate $\lambda_n = \lambda\ (n \geq 0)$ and death rate $\mu_n = \mu + (n-1)\gamma\ (n \geq 1)$ where λ, μ and γ are positive parameters. The process is described through the following equations

$$\begin{cases} \dfrac{dp_n\left(t\right)}{dt} = -\left(\lambda_n + \mu_n\right)p_n(t) + \lambda_{n-1}p_{n-1}(t) + \mu_{n+1}p_{n+1}(t), \text{ if } n \geq 1, \\ \dfrac{dp_1\left(t\right)}{dt} = -\lambda_0 p_0(t) + \mu_1 p_1(t), n = 1 \text{ with } \sum\limits_{n\geq 0} p_n(t) = 1, t \geq 0, \end{cases}$$

where $p_n\left(t\right) = P(Z(t) = n)$ is the probability that the population must be in state n at time t. The approximation for particular case $\alpha = 0.3$, $\mu = 5.5$ and $\beta = 1.5$ is showed in Figure 1 and their descriptive statistics is summarized in next Table 1.

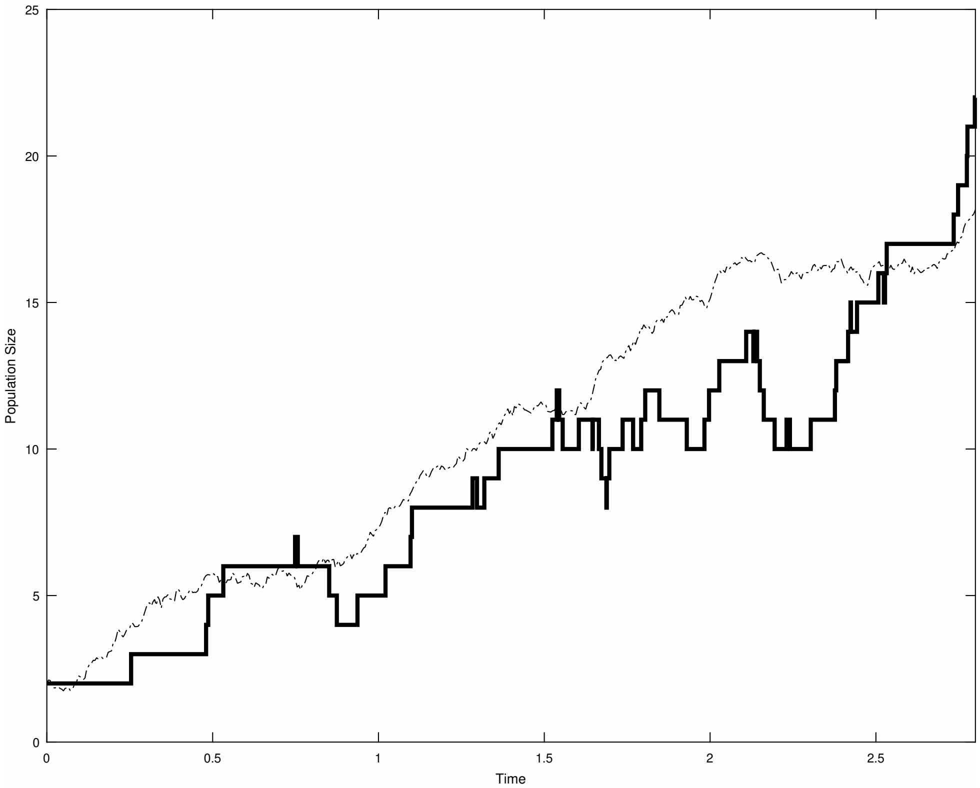

Figure 1. Approximation of birth-death process by ROU with one-sided barrier.

The processes	min	max	mean	var	Skew	Kurt
$10^3 \times$Birth-Death	0.0010	0.1420	0.0747	1.5711	-0.0003	0.0021
$10^3 \times$ROU(one sided barrier)	0.0017	0.1415	0.0861	1.5415	-0.0005	0.0019

Table 1. Some descriptive statistics of ROU compared with Birth-Death.

In order to investigate the finite sample properties of the LSE method developed in the previous section, we shall examine and compare the performance of our proposed LSE compared with MLE as a benchmark method. So, we simulate 500 independent trajectories according to a ROU processes with length $n \in \{1000, 2000, 3000\}$. Since, in practice, the observations are recorded at discrete times, then we suppose that the data are observed at times $0 = t_1 < t_2 < ... < t_n = T$ and let $\Delta_i = t_{i+1} - t_i$ and $h = \sup_t \Delta_i$. So, for small h, it seems reasonable to estimate $E\{X\}$, $E\{X^2\}$ and $E\{X^3\}$ by the numerical integrals respectively $\widetilde{m}_1 = \frac{1}{T}\sum_{i=0}^{n-1} x(t_i)\Delta_i$, $\widetilde{m}_2 = \frac{1}{T}\sum_{i=0}^{n-1} x^2(t_i)\Delta_i$ and $\widetilde{m}_3 = \frac{1}{T}\sum_{i=0}^{n-1} x^3(t_i)\Delta_i$. The results of simulation for estimating the vector $\underline{\theta}' = (\alpha, \mu, \beta)$ are reported in the tables below in which we have indicated in the columns Mean corresponds to the average of the parameters estimates over 500 simulations. In order to show the performance of such a method, we have reported (results between brackets) the root-mean-square errors $(RMSE)$ of each estimate. Hence, we define a discrete form of the LSE by replacing the approximation of the integrals and the estimators of the first and second moments in the above expressions $(4) - (5)$ for the case of one-sided barrier and in the expressions $(16) - (17)$ for the case of two-sided barriers, it follows that the estimators of the parameters α, μ and β^2 in discrete form are given by

1. Case of one-sided barrier

$$\widetilde{\alpha}_T = \frac{(X(t_n) - L(t_n))\,\widetilde{m}_1 + b^L L(t_n) - \sum_{i=0}^{n-1} x(t_i)\,(x(t_i) - x(t_{i-1}))}{(\widetilde{m}_2 - \widetilde{m}_1^2)},$$
$$\widetilde{\mu}_T = \frac{1}{T}\,(X(t_n) - L(t_n)) + \widetilde{\alpha}_T \widetilde{m}_1,$$
$$\widetilde{\beta}_T^2 = 2\widetilde{\alpha}_T\left(\widetilde{m}_2 - \frac{\widetilde{\mu}_T^2}{\widetilde{\alpha}_T^2} - \left(b^L + \frac{\widetilde{\mu}_T}{\widetilde{\alpha}_T}\right)\left(\widetilde{m}_1 - \frac{\widetilde{\mu}_T}{\widetilde{\alpha}_T}\right)\right).$$

2. Case of two-sided barriers

$$\widetilde{\alpha}_T = \frac{(X(t_n) - L(t_n) + U(t_n))\,\widetilde{m}_1 + b^L L(t_n) - b^U U(t_n) - \sum_{i=0}^{n-1} x(t_i)\,(x(t_i) - x(t_{i-1}))}{(\widetilde{m}_2 - \widetilde{m}_1^2)},$$
$$\widetilde{\mu}_T = \frac{1}{T}\,(X(t_n) - L(t_n) + U(t_n)) + \widetilde{\alpha}_T \widetilde{m}_1,$$
$$\widetilde{\beta}_T^2 = \frac{2\widetilde{\alpha}_T\,\{M1\,(\widetilde{\alpha}_T, \widetilde{\mu}_T) - M2\,(\widetilde{\alpha}_T, \widetilde{\mu}_T)\}}{M3\,(\widetilde{\alpha}_T, \widetilde{\mu}_T)}.$$

4.1. ROU with one-sided barrier

The results of simulation by the LSE and MLE methods for ROU with one-sided barrier are reported in the Table 2 below.

n	1000		2000		3000		1000		2000		3000	
$\underline{\widehat{\theta_T}}$	$\underline{Mean}$		$\underline{Mean}$		$\underline{Mean}$		$\underline{Mean}$		$\underline{Mean}$		$\underline{Mean}$	
	LSE	MLE	LSE	MLE	LSE	MLE	LSE	MLE	LSE	MLE	LSE	MLE
$\widehat{\alpha}_T$	1.9935 (0.1727)	1.9983 (0.1715)	2.0044 (0.1273)	2.0109 (0.1135)	1.9846 (0.1170)	2.0000 (0.1223)	1.0041 (0.1303)	1.0090 (0.1328)	0.9980 (0.0644)	1.0001 (0.0793)	0.9997 (0.0709)	1.0026 (0.0671)
$\widehat{\mu}_T$	0.9968 (0.1277)	1.0001 (0.1258)	1.0029 (0.0823)	1.0072 (0.0723)	0.9898 (0.0729)	1.0003 (0.0730)	1.5061 (0.1523)	1.5116 (0.1580)	1.4979 (0.0824)	1.5005 (0.1023)	1.5002 (0.0934)	1.5037 (0.0910)
$\widehat{\beta}_T$	0.0199 (0.0004)	0.0199 (0.0004)	0.0199 (0.0002)	0.0199 (0.0002)	0.0199 (0.0002)	0.0199 (0.0002)	0.0099 (0.0002)	0.0099 (0.0002)	0.0099 (0.0001)	0.0099 (0.0001)	0.0099 (0.0001)	0.0099 (0.0001)
	Design(1): $\underline{\theta} = (2, 1, 0.02)'$						Design(2): $\underline{\theta} = (1, 1.5, 0.01)'$					

Table 2. Mean, $RMSE$ of LSE and MLE estimators with $b^L = 0.5$.

The asymptotic distribution of the density of the actual errors $\sqrt{T}\,(\mu - \widehat{\mu}_T)$, $\sqrt{T}\,(\alpha - \widehat{\alpha}_T)$ and $\sqrt{T}\left(\beta - \widehat{\beta}_T\right)$ associated with LSE compared with those associated with MLE are plotted in three figures in Figure 2 respectively by using the built-in function "pltdens" in Matlab.

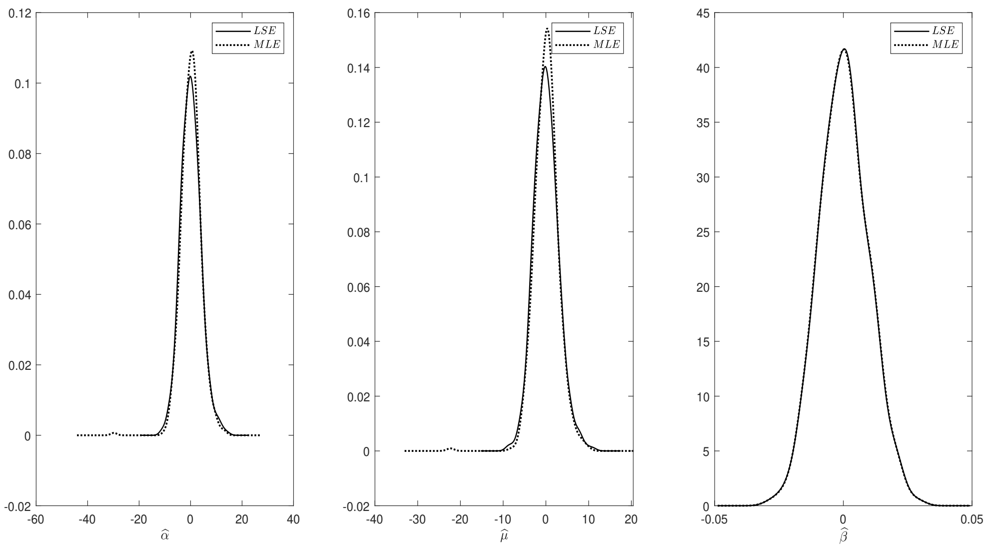

Figure 2. The superposition of the asymptotic densities associated to LSE and MLE for ROU with one-sided barrier according to design(1).

The box plots summary of the estimates $\widehat{\alpha}_T, \widehat{\mu}_T$ and of $\widehat{\beta}_T$ are showed in Figure 3.

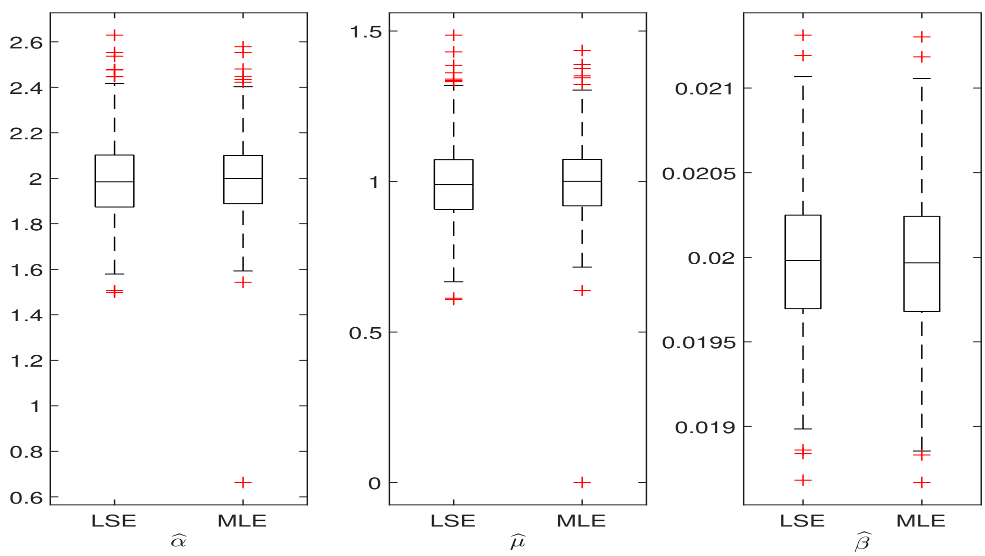

Figure 3. The box plot of LSE (left) and of MLE (right) for ROU with one-sided barrier according to design(1).

Now, we examine the sensitivity of the LSE estimates $\widehat{\alpha}_n$ with respect to the changes in μ which we remark that the percentage changes in the Mean of the estimates is less than those for μ. Table 3 contains some results. The variance decrease as μ increases. If $\mu \in [0, 10]$, the empirical $RMSE$ (root mean squared error) of the estimator is minimized at $\mu = 10$ as shown in Figure 4.

μ	0	1	2	3	4	5	6	7	8	9	10
$Mean(\widehat{\alpha}_T)$	1.9956	1.9974	1.9980	1.9984	1.9987	1.9989	1.9990	1.9991	1.9992	1.9993	1.9994
$RMSE(\widehat{\alpha}_T)$	0.0452	0.0282	0.0209	0.0165	0.0136	0.0116	0.0100	0.0089	0.0079	0.0072	0.0065

Table 3. $Mean\ (\widehat{\alpha})$ and $RMSE\ (\widehat{\alpha}\ )$ against μ for the ROU process with one-sided barrier,$\alpha = 2.0$, $\beta = 0.02$, and $b^L = 0.5$.

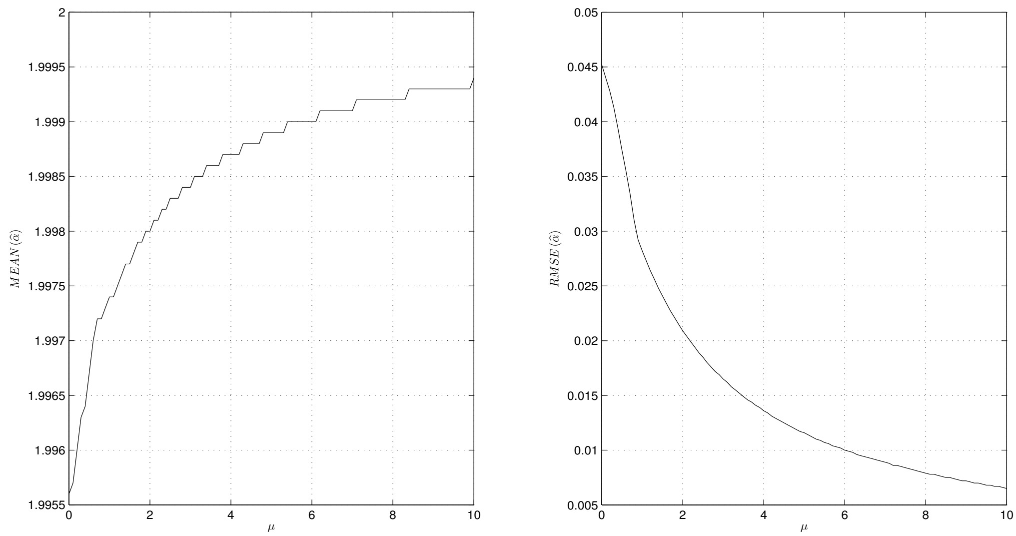

Figure 4. $Mean\ (\widehat{\alpha}_T)$ and $RMSE\ (\widehat{\alpha}_T)$ against μ for the ROU process with one-sided barrier, $\alpha = 2.0$, $\beta = 0.02$, $b^L = 0.5$.

4.1.1. Comments The results reported in Table 2 are in accordance with the asymptotic theory. Indeed, for LSE (resp. MLE) method's the $RMSE$ (results in bracket) for the parameter β involved in Designs(i) $i = 1, 2$, are with order $O(10^{-4})$ even for the moderate sample size suggests that $\widehat{\beta}_n$ is consistent. For the others parameters, it can be seen that there are no significant difference between their parameters estimates values in LSE (resp. MLE) despite an order $O\left(10^{-1}\right)$ for the $RMSE$. The order $O\left(10^{-3}\right)$ for the bias of $\widehat{\mu}_n$ and $\widehat{\alpha}_n$ indicates the consistency of such estimates. Regarding now the Figure 2 which compare the asymptotic distribution of the methods LSE (displayed in solid line) and MLE (displayed in dashed line). It is clearly observed that these methods are competitive and there are no substantial difference between them, which confirms the results of Table 2. Moreover, it shows that $\widehat{\mu}_n$ and $\widehat{\alpha}_n$ exhibit a sharp distribution with little variation in values equal to or close to their means. In contrast with the distribution of $\widehat{\beta}_n$ which is more flatted than the others with significant variations values far from its mean. The box plot summary in Figure 3 for the parameters estimates shows a closed similarity between the LSE and MLE. The outlier values are moderate and approximately the same for LSE and for MLE methods showing their robustness.

4.2. ROU with two-sided barriers

The results of simulation by the least square estimator for ROU with two-sided barriers are reported in Table 4 below.

n	1000		2000		3000		1000		2000		3000	
$\underline{\widehat{\theta_T}}$	$\underline{Mean}$		$\underline{Mean}$		$\underline{Mean}$		$\underline{Mean}$		$\underline{Mean}$		$\underline{Mean}$	
	LSE	MLE	LSE	MLE	LSE	MLE	LSE	MLE	LSE	MLE	LSE	MLE
$\widehat{\alpha}_T$	1.9935 (0.1727)	1.9983 (0.1715)	2.0079 (0.1253)	2.0184 (0.1088)	2.0116 (0.1175)	2.0341 (0.0916)	1.0041 (0.1303)	1.0090 (0.1328)	0.9955 (0.0850)	1.0001 (0.0793)	0.9996 (0.0738)	1.0026 (0.0671)
$\widehat{\mu}_T$	0.9968 (0.1277)	1.0001 (0.1258)	1.0055 (0.0805)	1.0130 (0.0677)	1.0085 (0.0726)	1.0256 (0.0510)	1.5061 (0.1523)	1.5116 (0.1580)	1.4950 (0.1061)	1.5005 (0.1023)	1.5002 (0.0934)	1.5037 (0.0910)
$\widehat{\beta}_T$	0.0199 (0.0004)	0.0199 (0.0004)	0.0199 (0.0002)	0.0199 (0.0002)	0.0198 (0.0002)	0.0198 (0.0002)	0.0099 (0.0002)	0.0099 (0.0002)	0.0099 (0.0001)	0.0099 (0.0001)	0.0099 (0.0001)	0.0099 (0.0001)
	Design(1): $\underline{\theta} = (2, 1, 0.02)'$						Design(2): $\underline{\theta} = (1, 1.5, 0.1)'$					

Table 4. Mean, $RMSE$ of LSE and MLE estimators with $b^L = 0.5, b^U = 1.5$.

The asymptotic distribution of the actual errors $\sqrt{T}\,(\mu-\widehat{\mu}_T)$, $\sqrt{T}\,(\alpha-\widehat{\alpha}_T)$ and $\sqrt{T}\left(\beta\widehat{\beta}_T\right)$ associated with LSE compared with those associated with MLE are plotted in three in Figure 5.

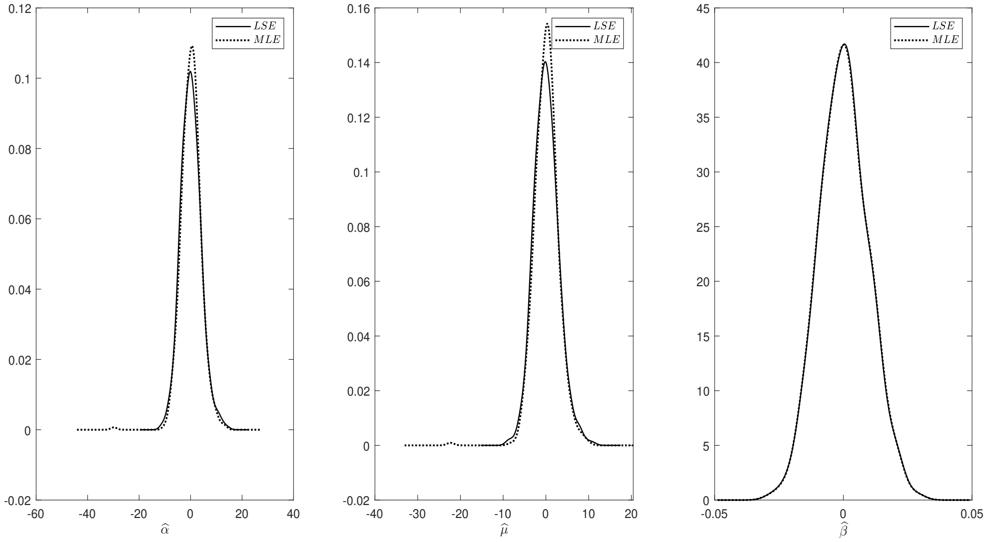

Figure 5. The superposition of the asymptotic densities associated to of LSE and MLE for ROU with two-sided barriers according to design(1).

The box plot of such estimates are shown in Figure 6 bellow

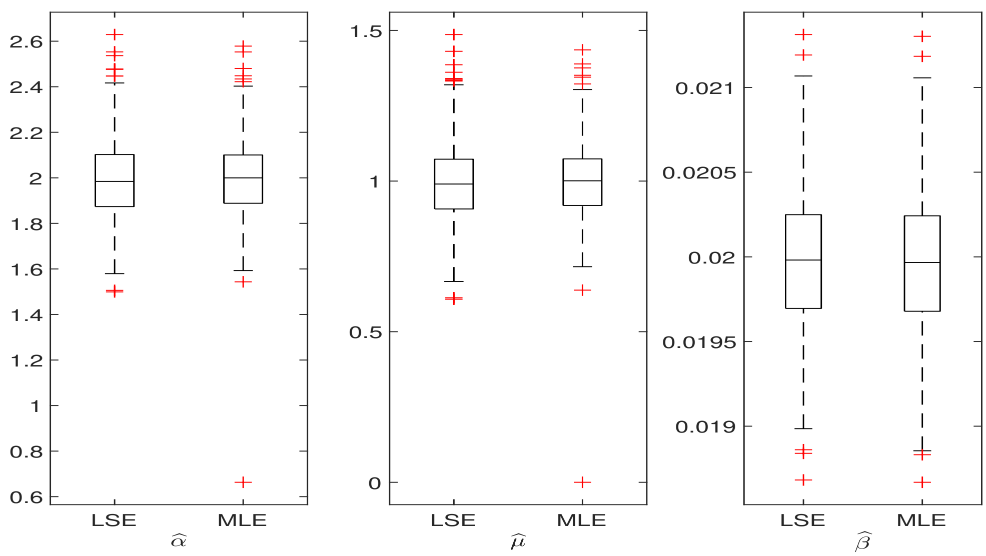

Figure 6. The box plot of LSE (left) and of MLE (right) for ROU with two-sided barriers according to design(1).

As in first model, the sensitivity of the estimator $\widehat{\alpha}$ for α with respect to the changes in μ exists, where we remark that the influence of the change of μ on the values of the Mean of $\widehat{\alpha}$, some results are reported in Table 5. For example in the interval $[0, 10]$, if $\mu \in [0, 1.78)$, the empirical $RMSE$ (root mean squared error) of the estimator decreases as μ increases in that case $RMSE$ is minimized at $\mu \in (1.77, 1.78)$ but if $\mu > 1.78$, the variance tends to increase as μ increases which is maximized at 10 as shown in Figure 7.

μ	0	1	1.77	1.78	2	3	4	5	6	7	8	9	10
$Mean(\widehat{\alpha}_T)$	0.9984	0.9989	0.9991	0.9991	0.9989	0.9982	0.9980	0.9979	0.9976	0.9971	0.9968	0.9964	0.9959
$RMSE(\widehat{\alpha}_T)$	0.0163	0.0105	0.008478	0.008485	0.0102	0.0159	0.0203	0.0244	0.0282	0.0318	0.0349	0.0376	0.0398

Table 5. $Mean\ (\widehat{\alpha}_T)$ and $RMSE\ (\widehat{\alpha}_T)$ against μ for the ROU process with two-sided barriers, $\alpha = 1$, $\beta = 0.01$, $b^L = 0.5$, $b^U = 1.5$.

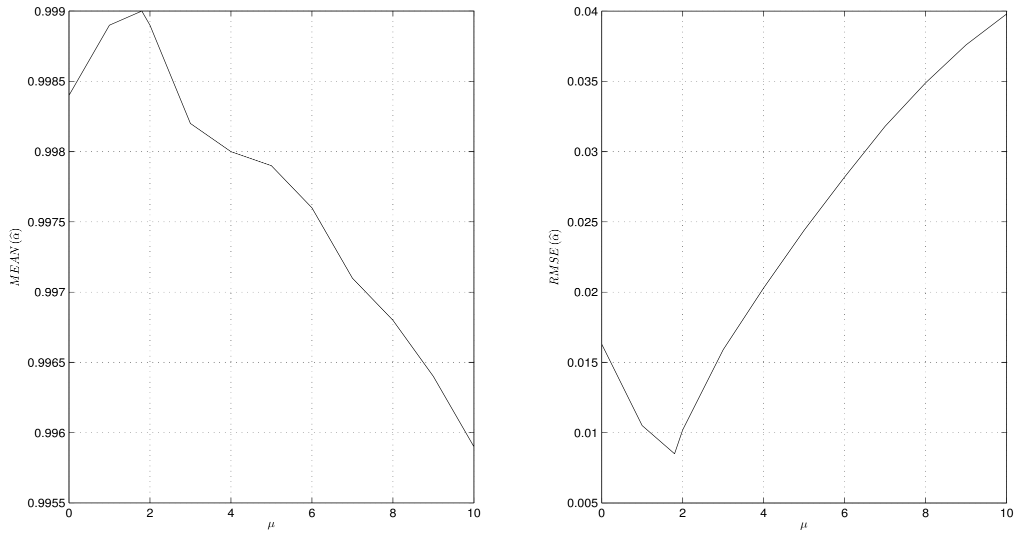

Figure 7. $Mean\ (\widehat{\alpha})$ and $RMSE\ (\widehat{\alpha})$ against μ for the ROU process with two-sided barriers, $\alpha = 1$, $\beta = 0.01$, $b^L = 0.5$ and $b^U = 1.5$.

Remark 4
Notice that, in the above approach, we cannot estimate $\alpha > 0$ and β simultaneously. This is because the invariant density $p(x)$ is a function of α/β^2. We can use $p(x)$ to determine only α/β^2.

4.2.1. Comments Beside the comments reported in 4.1.1 which rest in general valid here for describing the results of Table 4. Figure 6 compare the asymptotic distribution of the least squares estimator and its competitor (MLE) in the case of two-barriers. The distributions of $\widehat{\mu}_n$ and $\widehat{\alpha}_T$ are more accurate in the MLE case than in the LSE one in the sense that the variance of MLE estimates for $\widehat{\alpha}_n$ and $\widehat{\mu}_n$ are loss than of LSE. Moreover, it is clear these estimates represents more outliers values compared with one-sided barrier. In end regarding the Figures 4, 7 which shows the influence of the values of μ on the estimates values $\widehat{\alpha}_T$. It is clear that an observed dissimilarity is present. Indeed, in one-sided barrier $\widehat{\alpha}_T$ increase to the true values α proportionally with μ and its $RMSE$ decrease inversely to 0. This finding is completely the inverse in the case of two-sided barriers.

5. Conclusion

In this paper, we have considered the ROU processes in both one and two-sided barriers with multiple parameters case. We conclude that the LSE for the ROU processes based on the continuous and discrete observations provide to us a new tentative for the estimation of the parameters involved in the ROU processes. More precisely, the explicit expressions of the moments of the ROU processes are used to obtain an exact formulae of the estimates. So, the strong consistency and the asymptotic normality (CAN) of the LSE are sufficiently discussed in continuous and discrete observations. A simulation study on parameter estimation was carried-out for both one and two-sided barriers and compared with the MLE method. As a conclusion of the comparison shows that there are no significant difference between the LSE and MLE methods which confirm our proposed method. However, a reflected non linear diffusion processes as a Pearson processes is interesting to be studied with some adequate method of estimation. The moments method of estimation the reflected general diffusion processes is also of interest to be considered. This workstream constitutes our future research.

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