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A Note on a Strong Persistence of Stochastic Predator-Prey Model with Jumps

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Abstract We study the non-autonomous stochastic predator-prey model with a modified version of Leslie-Gower term and Holling-type II functional response driven by the system of stochastic differential equations with white noise, centered and non-centered Poisson noises. The sufficient conditions of strong persistence in the mean of the solution to the considered system are obtained.

Keywords Non-autonomous, Stochastic Predator-Prey Model, Strong Persistence in the Mean

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1. Introduction

Predator-prey interaction is one of a basic mechanism for two species dynamics. Let us consider a two-species food chain model describing a prey population x which serves as food for a predator population y. Let r_1 be the growth rate of prey, b_1 measures the strength of competition among individuals of species x, $\phi(x)$ be the functional response of the predator to the prey density refers to the change in the density of prey attaches per unit time per predator as the prey density changes. Then the equation for prey population dynamics has a form

$$\frac{dx(t)}{dt} = r_1 x(t) - b_1 x^2(t) - \phi(x(t))y(t).$$

Various types of functional response are used. For example, $\frac{ax}{b+x}$ (Holling-type II), $\frac{ax^2}{b+x^2}$ (Holling-type III) (see [1]); $\frac{ax}{b+cx+x^2}$ (Holling-type IV) (see [2]).

For the predator population dynamics in [3], [4] it is introduced and discussed a predator-prey model where the carrying capacity of the predator's environment is proportional to the number of prey. The dynamics of predator population is described by the equation

$$\frac{dy(t)}{dt} = r_2 y(t)\left(1 - \frac{y(t)}{\alpha x(t)}\right),$$

where r_2 is the growth rate of predator, α is the conversion factor of prey into predators. The term $y/\alpha x$ of this equation is called the Leslie-Gower term. It measures the loss in the predator population due to rarity (per capita

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y/x) of its favorite food. In the case of severe scarcity, predator y can switch over to other populations but its grows will be limited by the fact that its most favorite food x is not available in abundance. So in [5] the authors suggested using a modified Leslie-Gower term in the form $y/(\alpha x+d)$. The deterministic predator-prey model with modified version of Leslie-Gower and Holling-type II functional response (see [5]) is driven by the system of differential equations

$$\begin{aligned} dx(t) &= x(t)\left(a - bx(t) - \frac{cy(t)}{m_1 + x(t)}\right)dt, \\ dy(t) &= y(t)\left(r - \frac{fy(t)}{m_2 + x(t)}\right)dt, \end{aligned} \tag{1}$$

where $x(t)$ and $y(t)$ are the prey and predator population densities at time t, respectively. Positive constants a, b, c, r, f, m_1, m_2 defined as follows: a is the growth rate of prey x; b measures the strength of competition among individuals of species x; c is the maximum value of the per capita reduction rate of x due to y; m_1 and m_2 measure the extent to which the environment provides protection to prey x and to the predator y, respectively; r is the growth rate of predator y, and f has a similar meaning to c. In [5] the authors study boundedness and global stability of the positive equilibrium of the model (1).

The system (1) may, for example, be considered as a representation of an insect pest – spider food chain and other population systems in nature (see [5]).

In the papers [6], [7], [8] it is considered the stochastic version of model (1) in the following form

$$\begin{aligned} dx(t) &= x(t)\left(a - bx(t) - \frac{cy(t)}{m_1 + x(t)}\right)dt + \alpha x(t)dw_1(t), \\ dy &= y\left(r - \frac{fy}{m_2 + x}\right)dt + \beta y(t)dw_2(t), \end{aligned} \tag{2}$$

where $w_1(t)$ and $w_2(t)$ are mutually independent Wiener processes in [6], [7], and processes $w_1(t), w_2(t)$ are correlated in [8]. In [6] the authors proved that there is a unique positive solution to the system (2), obtained the sufficient conditions for extinction and persistence in the mean of predator and prey. In [7] it is shown, that under appropriate conditions there is a stationary distribution of the solution to the system (2) which is ergodic. In [8] the authors prove that the densities of the distributions of the solution to the system (2) can converges in L^1 to an invariant density or can converge weakly to a singular measure under appropriate conditions.

Population systems may suffer abrupt environmental perturbations, such as epidemics, fires, earthquakes, etc. So it is natural to introduce Poisson noises into the population model for describing such discontinuous systems. It is worth noting that the impact of centered and non-centered Poisson noises to the stochastic non-autonomous logistic model and to the stochastic two-species mutualism model is studied in the papers [9] – [12].

In the paper [13] the authors consider the non-autonomous stochastic predator-prey model with modified version of Leslie-Gower and Holling-type II functional response, disturbed by white noise and jumps generated by centered and non-centered Poisson measures. This model is driven by the system of stochastic differential equations

$$\begin{aligned} dx_i(t) &= x_i(t)\left[a_i(t) - b_i(t)x_i(t) - \frac{c_i(t)x_2(t)}{m(t) + x_1(t)}\right]dt + \sigma_i(t)x_i(t)dw_i(t) \\ &+ \int_{\mathbb{R}} \gamma_i(t,z)x_i(t)\tilde{\nu}_1(dt,dz) + \int_{\mathbb{R}} \delta_i(t,z)x_i(t)\nu_2(dt,dz),\ x_i(0) = x_{i0} > 0,\ i = 1,2. \end{aligned} \tag{3}$$

where $x_1(t)$ and $x_2(t)$ are the prey and predator population densities at time t, respectively, $b_2(t) \equiv 0$, $w_i(t), i = 1,2$ are independent standard one-dimensional Wiener processes, $\nu_i(t,A), i = 1,2$ are independent Poisson measures, which are independent on $w_i(t), i = 1,2$, $\tilde{\nu}_1(t,A) = \nu_1(t,A) - t\Pi_1(A)$, $E[\nu_i(t,A)] = t\Pi_i(A), i = 1,2$, $\Pi_i(A), i = 1,2$ are a finite measures on the Borel sets A in $\mathbb{R}$.

The authors proved that system (3) has a unique, positive, global (no explosion in a finite time) solution for any positive initial value, and that this solution is stochastically ultimate bounded. The sufficient conditions for stochastic permanence, non-persistence in the mean, weak persistence in the mean, and extinction of solution are derived.

In this paper, we derive the sufficient conditions for the strong persistence in the mean of predator and prey population densities, driven by system (3).

In the following we will use the notations $X(t) = (x_1(t), x_2(t))$, $X_0 = (x_{10}, x_{20})$, $|X(t)| = \sqrt{x_1^2(t) + x_2^2(t)}$, $\mathbb{R}^2_+ = \{X \in \mathbb{R}^2 :\ x_1 > 0, x_2 > 0\}$,

$$\beta_i(t) = \sigma_i^2(t)/2 + \int\limits_{\mathbb{R}} [\gamma_i(t,z) - \ln(1+\gamma_i(t,z))]\Pi_1(dz) - \int\limits_{\mathbb{R}} \ln(1+\delta_i(t,z))]\Pi_2(dz), i = 1,2.$$

$$\bar{f}(t) = \frac{1}{t}\int_0^t f(s)\,ds,\ f_* = \liminf_{t\to\infty} f(t),\ f^* = \limsup_{t\to\infty} f(t).$$

For the bounded, continuous functions $f_i(t), t \in [0,+\infty), i = 1,2$, let us denote

$$f_{i\,\sup} = \sup_{t\geq 0} f_i(t), f_{i\,\inf} = \inf_{t\geq 0} f_i(t), i = 1,2.$$

2. Strong persistence

Let $(\Omega, \mathcal{F}, \mathrm{P})$ be a probability space, $w_i(t), i = 1,2, t \geq 0$ are independent standard one-dimensional Wiener processes on $(\Omega, \mathcal{F}, \mathrm{P})$, and $\nu_i(t,A), i = 1,2$ are independent Poisson measures defined on $(\Omega, \mathcal{F}, \mathrm{P})$ independent on $w_i(t), i = 1,2$. Here $\mathrm{E}[\nu_i(t,A)] = t\Pi_i(A), i = 1,2$, $\tilde{\nu}_i(t,A) = \nu_i(t,A) - t\Pi_i(A), i = 1,2$, $\Pi_i(\cdot), i = 1,2$ are finite measures on the Borel sets in $\mathbb{R}$. On the probability space $(\Omega, \mathcal{F}, \mathrm{P})$ we consider an increasing, right continuous family of complete sub-σ-algebras $\{\mathcal{F}_t\}_{t\geq 0}$, where $\mathcal{F}_t = \sigma\{w_i(s), \nu_i(s,A), s \leq t, i = 1,2\}$.

We need the following assumption.

Assumption 1
It is assumed, that $a_i(t), b_1(t)$, $c_i(t), \sigma_i(t), \gamma_i(t,z), \delta_i(t,z), i = 1,2$, $m(t)$ are bounded, continuous on t functions, $a_i(t) > 0, i = 1,2$, $b_{1\,\inf} > 0$, $c_{i\,\inf} > 0, i = 1,2$, $m_{\inf} > 0$, and $\ln(1+\gamma_i(t,z)), \ln(1+\delta_i(t,z)), i = 1,2$ are bounded, $\Pi_i(\mathbb{R}) < \infty, i = 1,2$.

In what follows we will assume that Assumption 1 holds.

Definition 1
The population density $x(t), t \geq 0$ is said to be strongly persistence in the mean if for every initial data $x(0) > 0$, we have $\bar{x}_* > 0$ a.s.

Theorem 1
If $\bar{p}_{2*} = \liminf_{t\to\infty} \frac{1}{t}\int_0^t p_2(s)\,ds > 0$, where $p_2(t) = a_2(t) - \beta_2(t)$, then for every initial data $x_{20} > 0$ we have

$$\bar{x}_{2*} = \liminf_{t\to\infty} \frac{1}{t}\int_0^t x_2(s)ds \geq \frac{m_{\inf}}{c_{2\,\sup}}\bar{p}_{2*}, \quad a.s. \tag{4}$$

Therefore predator population density $x_2(t)$ will be strongly persistence in the mean.

Proof. Under Assumption 1 there exists a unique global solution $X(t)$ to the system (3) for any initial value $X(0) = X_0 > 0$, and $\mathrm{P}\{X(t) \in \mathbb{R}^2_+\} = 1$ (Theorem 1, [13]). Applying the Itô's formula to the process $\ln x_2(t)$, we obtain

$$\begin{aligned} \ln x_2(t) &= \ln x_{20} + \int_0^t \left[a_2(s) - \frac{c_2(s)x_2(s)}{m(s)+x_1(s)} - \beta_2(s)\right] ds + M_2(t) \\ &\geq \ln x_{20} + \int_0^t p_2(s)\,ds - \frac{c_{2\,\sup}}{m_{\inf}}\int_0^t x_2(s)\,ds + M_2(t), \end{aligned} \tag{5}$$

where the martingale

$$M_2(t) = \int\limits_0^t \sigma_2(s)dw_2(s) + \int\limits_0^t\int\limits_{\mathbb{R}} \ln(1+\gamma_2(s,z))\tilde{\nu}_1(ds,dz) + \int\limits_0^t\int\limits_{\mathbb{R}} \ln(1+\delta_2(s,z))\tilde{\nu}_2(ds,dz),$$

has a quadratic characteristic (Meyer's angle bracket process)

$$\langle M_2, M_2\rangle(t) = \int\limits_0^t \sigma_2^2(s)ds + \int\limits_0^t\int\limits_{\mathbb{R}} \ln^2(1+\gamma_2(s,z))\Pi_1(dz)ds + \int\limits_0^t\int\limits_{\mathbb{R}} \ln^2(1+\delta_2(s,z))\Pi_2(dz)ds \le Kt,$$

for some constant $K > 0$. Then the strong law of large numbers for local martingales ([14]) yields $\lim_{t\to\infty} M_2(t)/t = 0$ a.s.

From definition of $\bar{p}_{2*}$, strong law of large numbers for $M_2(t)$ it follows that $\forall\varepsilon > 0$, $\exists t_0 \ge 0$, $\exists\Omega_\varepsilon \subset \Omega$, such that $\mathrm{P}(\Omega_\varepsilon) \ge 1-\varepsilon$,

$$\frac{1}{t}\int_0^t p_2(s)\,ds \ge \bar{p}_{2*} - \frac{\varepsilon}{2}, \quad \frac{M_2(t)}{t} \ge -\frac{\varepsilon}{2}, \quad \forall t \ge t_0, \forall\omega\in\Omega_\varepsilon.$$

So from (5) we have

$$\ln x_2(t) \ge \ln x_{20} + t(\bar{p}_{2*} - \varepsilon) - \hat{c}\int_0^t x_2(s)\,ds \quad \forall t \ge t_0, \forall\omega\in\Omega_\varepsilon,$$

where $\hat{c} = \frac{c_{2\,\sup}}{m_{\inf}}$. Therefore for the process $y_2(t) = \int_0^t x_2(s)\,ds$ we have inequality

$$\ln\left(\frac{dy_2(t)}{dt}\right) \ge (\bar{p}_{2*} - \varepsilon)t - \hat{c}y_2(t) + \ln x_{20}, \quad \forall t \ge t_0, \forall\omega\in\Omega_\varepsilon.$$

Hence

$$e^{\hat{c}y_2(t)}\frac{dy_2(t)}{dt} \ge x_{20}e^{(\bar{p}_{2*}-\varepsilon)t}, \quad \forall t \ge t_0, \forall\omega\in\Omega_\varepsilon.$$

Integrating the last inequality from t_0 to t and using obvious calculations, yields

$$\frac{1}{t}\int_0^t x_2(s)ds \ge \frac{1}{\hat{c}t}\ln\left[e^{\hat{c}y_2(t_0)} + \frac{\hat{c}x_{20}}{\bar{p}_{2*}-\varepsilon}\left(e^{(\bar{p}_{2*}-\varepsilon)t} - e^{(\bar{p}_{2*}-\varepsilon)t_0}\right)\right] \quad \forall t \ge t_0, \forall\omega\in\Omega_\varepsilon.$$

So

$$\bar{x}_{2*} = \liminf_{t\to\infty}\frac{1}{t}\int_0^t x_2(s)\,ds \ge \frac{\bar{p}_{2*}-\varepsilon}{\hat{c}} = \frac{m_{\inf}}{c_{2\,\sup}}(\bar{p}_{2*}-\varepsilon), \quad \forall\omega\in\Omega_\varepsilon.$$

Using the arbitrariness of $\varepsilon > 0$, we get (4).

Lemma 1
If $p_{2\,\inf} > 0$, then

$$\lim_{t\to\infty}\frac{\ln x_2(t)}{t} = 0, \quad a.s.$$

Proof. The density of predator population $x_2(t)$ has the property (Lemma 2, [13]):

$$\limsup_{t\to\infty}\frac{\ln x_2(t)}{t} \le 0, \quad a.s.$$

So it suffices to show

$$\liminf_{t\to\infty}\frac{\ln x_2(t)}{t} \ge 0, \quad a.s.$$

For the process $U(t) = 1/x_2(t)$ by the Itô's formula we have

$$U(t) = U(0) + \int\limits_0^t U(s)\left[\frac{c_2(s)x_2(s)}{m(s)+x_1(s)} - a_2(s) + \sigma_2^2(s) + \int\limits_{\mathbb{R}}\frac{\gamma_2^2(s,z)}{1+\gamma_2(s,z)}\Pi_1(dz)\right]ds$$
$$-\int\limits_0^t U(s)\sigma_2(s)dw_2(s) - \int\limits_0^t\int\limits_{\mathbb{R}} U(s)\frac{\gamma_2(s,z)}{1+\gamma_2(s,z)}\tilde{\nu}_1(ds,dz) - \int\limits_0^t\int\limits_{\mathbb{R}} U(s)\frac{\delta_2(s,z)}{1+\delta_2(s,z)}\nu_2(ds,dz).$$

Then by the Itô formula we derive for $0<\theta<1$

$$
\begin{aligned}
(1+U(t))^\theta = (1+U(0))^\theta + \int_0^t \theta(1+U(s))^{\theta-2}\Bigg\{&(1+U(s))U(s)\left[\frac{c_2(s)x_2(s)}{m(s)+x_1(s)}-a_2(s)\right.\\
&\left.+\sigma_2^2(s)+\int_{\mathbb{R}}\frac{\gamma_2^2(s,z)}{1+\gamma_2(s,z)}\Pi_1(dz)\right]+\frac{\theta-1}{2}U^2(s)\sigma_2^2(s)\\
+\frac{1}{\theta}\int_{\mathbb{R}}\left[(1+U(s))^2\left(\left(\frac{1+U(s)+\gamma_2(s,z)}{(1+\gamma_2(s,z))(1+U(s))}\right)^\theta-1\right)+\theta(1+U(s))\frac{U(s)\gamma_2(s,z)}{1+\gamma_2(s,z)}\right]&\Pi_1(dz)\\
+\frac{1}{\theta}\int_{\mathbb{R}}(1+U(s))^2\left[\left(\frac{1+U(s)+\delta_2(s,z)}{(1+\delta_2(s,z))(1+U(s))}\right)^\theta-1\right]\Pi_2(dz)\Bigg\}&ds\\
-\int_0^t\theta(1+U(s))^{\theta-1}U(s)\sigma_2(s)dw_2(s)+\int_0^t\int_{\mathbb{R}}\left[\left(1+\frac{U(s)}{1+\gamma_2(s,z)}\right)^\theta-(1+U(s))^\theta\right]&\tilde{\nu}_1(ds,dz)\\
+\int_0^t\int_{\mathbb{R}}\left[\left(1+\frac{U(s)}{1+\delta_2(s,z)}\right)^\theta-(1+U(s))^\theta\right]&\tilde{\nu}_2(ds,dz)\\
=(1+U(0))^\theta+\int_0^t\theta(1+U(s))^{\theta-2}J(s)ds-I_{1,stoch}(t)+I_{2,stoch}(t)+I_{3,stoch}(t),& \qquad (6)
\end{aligned}
$$

where $I_{j,stoch}(t), j=\overline{1,3}$ are corresponding stochastic integrals in (6). Under the Assumption 1 there exists constants $|K_1(\theta)|<\infty$, $|K_2(\theta)|<\infty$ such, that for the process $J(t)$ we have the estimate

$$
\begin{aligned}
J(t)\le(1+U(t))U(t)&\left[-a_2(t)+\frac{c_{2\,\sup}U^{-1}(t)}{m_{inf}}+\sigma_2^2(t)\quad+\int_{\mathbb{R}}\frac{\gamma_2^2(s,z)}{1+\gamma_2(s,z)}\Pi_1(dz)\right]+\frac{\theta-1}{2}U^2(s)\sigma_2^2(s)\\
+\frac{1}{\theta}\int_{\mathbb{R}}&\left[(1+U(s))^2\left(\left(\frac{1}{1+\gamma_2(s,z)}+\frac{1}{1+U(s)}\right)^\theta-1\right)+\theta(1+U(s))\frac{U(s)\gamma_2(s,z)}{1+\gamma_2(s,z)}\right]\Pi_1(dz)\\
&+\frac{1}{\theta}\int_{\mathbb{R}}(1+U(s))^2\left[\left(\frac{1}{1+\delta_2(s,z)}+\frac{1}{1+U(s)}\right)^\theta-1\right]\Pi_2(dz)\\
\le U^2(t)&\left[-a_2(t)+\frac{\sigma_2^2(t)}{2}+\int_{\mathbb{R}}\gamma_2(t,z)\Pi_1(dz)+\frac{\theta}{2}\sigma_2^2(t)+\frac{1}{\theta}\int_{\mathbb{R}}[(1+\gamma_2(t,z))^{-\theta}-1]\Pi_1(dz)\right.\\
&\left.+\frac{1}{\theta}\int_{\mathbb{R}}[(1+\delta_2(t,z))^{-\theta}-1]\Pi_2(dz)\right]+K_1(\theta)U(t)+K_2(\theta)=-K_0(t,\theta)U^2(t)+K_1(\theta)U(t)+K_2(\theta).
\end{aligned}
$$

Here we used the inequality $(x+y)^\theta\le x^\theta+\theta x^{\theta-1}y$, $0<\theta<1$, $x,y>0$. Due to

$$
\begin{aligned}
\lim_{\theta\to0+}&\left[\frac{\theta}{2}\sigma_2^2(t)+\frac{1}{\theta}\int_{\mathbb{R}}\left[(1+\gamma_2(t,z))^{-\theta}-1\right]\Pi_1(dz)+\frac{1}{\theta}\int_{\mathbb{R}}\left[(1+\delta_2(t,z))^{-\theta}-1\right]\Pi_2(dz)\right.\\
&\left.+\int_{\mathbb{R}}\ln(1+\gamma_2(t,z))\Pi_1(dz)+\int_{\mathbb{R}}\ln(1+\delta_2(t,z))\Pi_2(dz)\right]=\lim_{\theta\to0+}\Delta(\theta)=0,
\end{aligned}
$$

and condition $p_{2\,\inf} > 0$ we can choose a sufficiently small $0 < \theta < 1$ to satisfy

$$K_0(\theta) = \inf_{t\geq 0} K_0(t,\theta) = \inf_{t\geq 0}[p_2(t) - \Delta(\theta)] = p_{2\,\inf} - \Delta(\theta) > 0.$$

So from (6) and the estimate for $J(t)$ we derive

$$\begin{aligned} d\left[(1+U(t))^{\theta}\right] &\leq \theta(1+U(t))^{\theta-2}[-K_0(\theta)U^2(t) + K_1(\theta)U(t) + K_2(\theta)]dt \\ - \theta(1+U(t))^{\theta-1}U(t)\sigma_2(t)dw_2(t) &+ \int\limits_{\mathbb{R}} \left[\left(1+\frac{U(t)}{1+\gamma_2(t,z)}\right)^{\theta} - (1+U(t))^{\theta}\right]\tilde{\nu}_1(dt,dz) \\ &+ \int\limits_{\mathbb{R}} \left[\left(1+\frac{U(t)}{1+\delta_2(t,z)}\right)^{\theta} - (1+U(t))^{\theta}\right]\tilde{\nu}_2(dt,dz). \end{aligned} \tag{7}$$

It is easy to see that

$$\theta(1+U(t))^{\theta-2}[-K_0(\theta)U^2(t) + K_1(\theta)U(t) + K_2(\theta)] \leq K$$

on $U(t) > 0$ for some constant $K > 0$. So from (7) we observed that

$$\begin{aligned} \mathrm{E}\left[\sup_{t\leq r\leq t+1}(1+U(r))^{\theta}\right] &\leq \mathrm{E}\left[(1+U(t))^{\theta}\right] + K + \mathrm{E}\left[\sup_{t\leq r\leq t+1}\left|\int_t^r \theta(1+U(s))^{\theta-1}U(s)\sigma_2(s)dw_2(s)\right|\right] \\ &+\mathrm{E}\left[\sup_{t\leq r\leq t+1}\left|\int_t^r\int\limits_{\mathbb{R}}\left[\left(1+\frac{U(s)}{1+\gamma_2(s,z)}\right)^{\theta} - (1+U(s))^{\theta}\right]\tilde{\nu}_1(ds,dz)\right|\right] \\ &+\mathrm{E}\left[\sup_{t\leq r\leq t+1}\left|\int_t^r\int\limits_{\mathbb{R}}\left[\left(1+\frac{U(s)}{1+\delta_2(s,z)}\right)^{\theta} - (1+U(s))^{\theta}\right]\tilde{\nu}_2(ds,dz)\right|\right] \end{aligned} \tag{8}$$

By the inequalities for the moments of local square integrable martingales ([15]) and the Hölder inequality we have

$$\begin{aligned} \mathrm{E}\left[\sup_{t\leq r\leq t+1}\left|\int_t^r \theta(1+U(s))^{\theta-1}U(s)\sigma_2(s)dw_2(s)\right|\right] &\leq 3\left(\mathrm{E}\left[\int_t^{t+1}\theta^2(1+U(s))^{2(\theta-1)}U^2(s)\sigma_2^2(s)\,ds\right]\right)^{1/2} \\ &\leq K_1\left(\int_t^{t+1}\mathrm{E}\left[(1+U(s))^{2\theta}\right]\,ds\right)^{1/2}, \end{aligned}$$

$$\mathrm{E}\left[\sup_{t\leq r\leq t+1}\left|\int_t^r\int\limits_{\mathbb{R}}\left[\left(1+\frac{U(s)}{1+\gamma_2(s,z)}\right)^{\theta} - (1+U(s))^{\theta}\right]\tilde{\nu}_1(ds,dz)\right|\right]$$

$$\leq 3\mathrm{E}\left[\left|\int_t^r\int\limits_{\mathbb{R}}\left[\left(1+\frac{U(s)}{1+\gamma_2(s,z)}\right)^{\theta} - (1+U(s))^{\theta}\right]^2\Pi_1(dz)ds\right|^{1/2}\right] \leq K_2\left(\int_t^{t+1}\mathrm{E}\left[(1+U(s))^{2\theta}\right]\,ds\right)^{1/2},$$

$$\mathrm{E}\left[\sup_{t\leq r\leq t+1}\left|\int_t^r\int\limits_{\mathbb{R}}\left[\left(1+\frac{U(s)}{1+\delta_2(s,z)}\right)^{\theta} - (1+U(s))^{\theta}\right]\tilde{\nu}_2(ds,dz)\right|\right]$$

$$\leq 3\mathrm{E}\left[\left|\int_t^r\int\limits_{\mathbb{R}}\left[\left(1+\frac{U(s)}{1+\delta_2(s,z)}\right)^{\theta} - (1+U(s))^{\theta}\right]^2\Pi_2(dz)ds\right|^{1/2}\right] \leq K_3\left(\int_t^{t+1}\mathrm{E}\left[(1+U(s))^{2\theta}\right]\,ds\right)^{1/2},$$

for some constants $K_i > 0, i = \overline{1,3}$ due to the Assumption 1. Substituting this into (8) gives

$$\mathrm{E}\left[\sup_{t\le r\le t+1}(1+U(r))^{\theta}\right] \le \mathrm{E}\left[(1+U(t))^{\theta}\right] + K + K_{\max}\left(\int_t^{t+1}\mathrm{E}\left[(1+U(s))^{2\theta}\right]ds\right)^{1/2}, \tag{9}$$

where $K_{\max} = \max\{K_i, i = \overline{1,3}\}$. If $p_{2\,\inf} > 0$, then for any initial value $x_{20} > 0$, the predator population density $x_2(t)$ has the property that

$$\limsup_{t\to\infty}\mathrm{E}\left[\left(\frac{1}{x_2(t)}\right)^q\right] \le \limsup_{t\to\infty}\mathrm{E}\left[(1+U(t))^q\right] \le K(q), \tag{10}$$

for arbitrary sufficiently small $0 < q < 1$ (Lemma 4, [13]). Letting $t \to +\infty$ in (9) and using (10) we obtain that

$$\limsup_{t\to\infty}\mathrm{E}\left[\sup_{t\le r\le t+1}(1+U(r))^{\theta}\right] \le \bar{K}(\theta),$$

for some $\bar{K}(\theta) > 0$ and for all sufficiently small $0 < \theta < 1$. Therefore

$$\limsup_{t\to\infty}\mathrm{E}\left[\sup_{t\le r\le t+1}\frac{1}{x_2^{\theta}(r)}\right] \le \limsup_{t\to\infty}\mathrm{E}\left[\sup_{t\le r\le t+1}(1+U(r))^{\theta}\right] \le \bar{K}(\theta).$$

So there is a $k_0 \in \mathbb{N}$ such that $\forall k \ge k_0$ by the well-known Chebyshev inequality, we have

$$\mathrm{P}\left\{\sup_{k\le t\le k+1}\frac{1}{x_2^{\theta}(t)} > k^{1+\varepsilon}\right\} \le \frac{\bar{K}(\theta)}{k^{1+\varepsilon}},\ \forall\varepsilon > 0.$$

Applying the Borel-Cantelli lemma, we obtain that for almost all $\omega \in \Omega$

$$\sup_{k\le t\le k+1}\frac{1}{x_2^{\theta}(t)} \le k^{1+\varepsilon},\ \forall k \ge k_0.$$

Hence

$$\frac{\ln x_2(t)}{\ln t} \ge -\frac{1+\varepsilon}{\theta},\ \forall k \ge k_0,\ k \le t \le k+1,\ \forall\varepsilon > 0 \quad a.s.$$

Using the arbitrariness of $\varepsilon > 0$, we get

$$\liminf_{t\to\infty}\frac{\ln x_2(t)}{\ln t} \ge -\frac{1}{\theta} \quad a.s.$$

for sufficiently small $0 < \theta < 1$. Therefore $\liminf_{t\to\infty}\frac{\ln x_2(t)}{t} \ge 0$, a.s. and we complete the proof.

Theorem 2
Let us denote

$$\bar{p}_{1*} = \liminf_{t\to\infty}\frac{1}{t}\int_0^t p_1(s)\,ds,\ \bar{p}_2^* = \limsup_{t\to\infty}\frac{1}{t}\int_0^t p_2(s)\,ds,\ p_i(t) = a_i(t) - \beta_i(t), i = 1, 2.$$

If $\bar{p}_{1*} > \frac{c_{1\,\sup}}{c_{2\,\inf}}\bar{p}_2^*$ and $p_{2\,\inf} > 0$, then

$$\bar{x}_{1*} = \liminf_{t\to\infty}\frac{1}{t}\int_0^t x_1(s)\,ds \ge \frac{1}{b_{1\,\sup}}\left(\bar{p}_{1*} - \frac{c_{1\,\sup}}{c_{2\,\inf}}\bar{p}_2^*\right) \quad a.s. \tag{11}$$

Hence the prey population density $x_1(t)$ will be strongly persistence in the mean.

Proof. By Itô formula we obtain

$$\ln x_i(t) = \ln x_{i0} + \int_0^t p_i(s)\,ds - \int_0^t b_i(s)x_i(s)\,ds - \int_0^t \frac{c_i(s)x_2(s)}{m(s)+x_1(s)}\,ds + M_i(t),\ i=1,2 \tag{12}$$

where the martingale

$$M_i(t) = \int\limits_0^t \sigma_i(s)dw_i(s) + \int\limits_0^t\int\limits_{\mathbb{R}} \ln(1+\gamma_i(s,z))\tilde{\nu}_1(ds,dz) + \int\limits_0^t\int\limits_{\mathbb{R}} \ln(1+\delta_i(s,z))\tilde{\nu}_2(ds,dz),\ i=1,2$$

has a quadratic characteristic (Meyer's angle bracket process)

$$\langle M_i, M_i\rangle(t) = \int\limits_0^t \sigma_i^2(s)ds + \int\limits_0^t\int\limits_{\mathbb{R}} \ln^2(1+\gamma_i(s,z))\Pi_1(dz)ds + \int\limits_0^t\int\limits_{\mathbb{R}} \ln^2(1+\delta_i(s,z))\Pi_2(dz)ds \le Kt,\ i=1,2$$

for some constant $K>0$. Then the strong law of large numbers for local martingales ([14]) yields $\lim_{t\to\infty} M_i(t)/t = 0, i=1,2$ a.s.

From (12) we have

$$\ln x_1(t) \ge \ln x_{10} + \int_0^t p_1(s)\,ds - b_{1\,\sup}\int_0^t x_1(s)\,ds - c_{1\,\sup}\int_0^t \frac{x_2(s)}{m(s)+x_1(s)}\,ds + M_1(t) \tag{13}$$

and

$$\frac{c_{2\,\inf}}{t}\int_0^t \frac{x_2(s)}{m(s)+x_1(s)}\,ds \le -\frac{\ln x_2(t)-\ln x_{20}}{t} + \frac{1}{t}\int_0^t p_2(s)\,ds + \frac{M_2(t)}{t}. \tag{14}$$

From definition of $\bar{p}_2^*$ and $\bar{p}_{1*}$, strong law of large numbers for $M_i(t), i=1,2$ and Lemma 1 it follows that $\forall\varepsilon>0$, $\exists t_0\ge 0$, $\exists\Omega_\varepsilon\subset\Omega$, such that $\mathrm{P}(\Omega_\varepsilon)\ge 1-\varepsilon$,

$$\frac{1}{t}\int_0^t p_2(s)\,ds \le \bar{p}_2^* + \frac{\varepsilon c_{2\,\inf}}{6c_{1\,\sup}},\ \frac{M_2(t)}{t} \le \frac{\varepsilon c_{2\,\inf}}{6c_{1\,\sup}},\ -\frac{\ln x_2(t)-\ln x_{20}}{t} \le \frac{\varepsilon c_{2\,\inf}}{6c_{1\,\sup}}$$
$$\frac{1}{t}\int_0^t p_1(s)\,ds \ge \bar{p}_{1*} - \frac{\varepsilon}{4},\ \frac{M_1(t)}{t} \ge -\frac{\varepsilon}{4} \quad \forall t\ge t_0, \forall\omega\in\Omega_\varepsilon.$$

Therefore from (13), using (14), we obtain

$$\ln x_1(t) \ge \ln x_{10} + \left(\bar{p}_{1*} - \frac{c_{1\,\sup}}{c_{2\,\inf}}\bar{p}_2^* - \varepsilon\right)t - b_{1\,\sup}\int_0^t x_1(s)\,ds, \quad \forall t\ge t_0, \forall\omega\in\Omega_\varepsilon.$$

Using arguments similar to those in the proof of the Theorem 1 we derive

$$\liminf_{t\to\infty}\frac{1}{t}\int_0^t x_1(s)\,ds \ge \frac{1}{b_{1\,\sup}}\left(\bar{p}_{1*} - \frac{c_{1\,\sup}}{c_{2\,\inf}}\bar{p}_2^* - \varepsilon\right), \quad \forall\omega\in\Omega_\varepsilon.$$

Using the arbitrariness of $\varepsilon>0$, we get (11).

3. Conclusion

In this paper we derive the sufficient conditions for the strong persistence in the mean of predator and prey populations in the predator-prey model with a modified version of Leslie-Gower term and Holling-type II functional response driven by the system of stochastic differential equations with white noise, centered and non-centered Poisson noises. In [13] it is obtained sufficient conditions for the weak persistence in the mean for predator and prey populations in the considered predator-prey model. Under the weak persistence in the mean $\liminf_{t\to\infty} \bar{x}_i(t) = 0, i = 1, 2$ is allowed but is not allowed under the strong persistence in the mean, which means that the survival in Theorem 1 and in Theorem 2 is stronger than in [13] (Theorem 8 and Theorem 9).

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