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The Marshall-Olkin Topp-Leone Half-Logistic-G Family of Distributions with Applications

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Abstract A new family of distributions called the Marshall-Olkin Topp-Leone Half-Logistic-G (MO-TLHL-G) family of distributions is proposed and studied. Structural properties of the new family of distributions including moments, incomplete moments, distribution of the order statistics and Rényi entropy are derived. Maximum likelihood estimation technique is used to estimate the model parameters. A simulation study to examine the bias and mean square error of the maximum likelihood estimators and applications to real data sets to illustrates the usefulness of the generalized distribution are given.

Keywords Marshall-Olkin Distribution, Topp-Leone Distribution, Half-Logistic Distribution, Maximum Likelihood Estimation

Mathematics Subject Classifications 62E99; 60E05

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1. Introduction

The need for more flexible models that can explain and provide better fit to real life data sets has motivated many researchers to develop new generalized distributions by extending the classical ones. Marshall and Olkin [23] introduced a new method of adding a parameter into a family of distributions. Addition of parameters to an existing baseline distribution has proven to be an effective technique for improving the flexibility of new families of distributions (Barreto-Souza et al. [8]). The Marshall-Olkin technique has been applied to build several well known distributions which include papers by Barreto-Souza et al. [8], Cordeiro et al. [16], Santos-Neo et al. [27], Chakraborty et al. [10], Lazhar et al. [21], Kumar et al. [20], Lepetu et al. [22] and Ghitany et al. [19]).

In this paper, we employ the Marshall-Olkin transformation to come up with a new family of distributions called the Marshall-Olkin Topp-Leone Half-Logistic-G (MO-TLHL-G) family of distributions. The desirable properties and flexibility obtained from this new extended family of distributions in terms of the shapes of the density and hazard rate function which makes the model useful in lifetime analysis has motivated us to develop this model.

The results in this note are organized in the following manner. Section 2 contain the MO-TLHL-G family of distributions and its sub-models, hazard function and the quantile function, stochastic ordering, series expansion of the density function and some special cases of the MO-TLHL-G family of distributions. In Section 3, moments, generating function and incomplete moments are given. Section 4 contain results on the distribution of order statistics and Rényi entropy. Presented in Section 5 is the estimation of the parameters of the MO-TLHL-G family of distributions via the method of maximum likelihood, followed by a Monte Carlo simulation study to examine the bias and mean square error of the maximum likelihood estimators in Section 6. Some applications to real data sets are given in section 7. Concluding remarks are given in Section 8.

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2. The Model, Properties and Special Cases

The derivation of some of the statistical properties of the Marshall-Olkin Topp-Leone Half Logistic-G (MO-TLHL-G) family of distributions including special cases, expansion of the density, stochastic ordering, hazard function and quantile function are presented in this section.

2.1. The Model

Marshall and Olkin [23] developed the Marshall-Olkin (MO) extended family of distributions. For any baseline distribution with probability density function (pdf) $f(x)$ and cumulative distribution function (cdf) $F(x)$, Marshall and Olkin [23] defined a new survival function $\overline{F}(x)$ by introducing an additional parameter $\alpha > 0$. The survival function is defined by

$$\overline{F}_{MO}(x) = \frac{\alpha\overline{F}(x)}{1-\overline{\alpha}\overline{F}(x)}, \tag{1}$$

where $-\infty < x < \infty, \alpha > 0, \overline{\alpha} = 1-\alpha, \alpha$ is known as the tilt parameter since the hazard rate function of the new family is shifted either above or below the hazard rate function of the baseline distribution for $0 < \alpha < 1$ and $\alpha > 1$, respectively. The cdf and the pdf of the MO-extended distribution are given by

$$F_{MO}(x) = \frac{F(x)}{1-\overline{\alpha}\overline{F}(x)}, \tag{2}$$

and

$$f_{MO}(x) = \frac{\alpha f(x)}{[1-\overline{\alpha}\overline{F}(x)]^2}, \tag{3}$$

respectively, where $f(x)$ and $F(x)$ are any respective baseline pdf and cdf.

Following the Marshall-Olkin extension, the cdf and pdf of the Mashall-Olkin Topp-Leone Half-Logistic-G family of distributions are given by

$$F(x;b,\delta,\varphi) = \frac{\frac{[1-\overline{G}^2(x;\varphi)]^b}{(1+(1-[1-\overline{G}^2(x;\varphi)]^b)}}{1-\overline{\delta}\left(1-\left[\frac{[1-\overline{G}^2(x;\varphi)]^b}{1+(1-[1-\overline{G}^2(x;\varphi)]^b)}\right]\right)} \tag{4}$$

and

$$f(x;b,\delta,\varphi) = \frac{4b\delta g(x;\varphi)\overline{G}(x;\varphi)(1-\overline{G}^2(x;\varphi))^{b-1}}{(1+(1-[1-\overline{G}^2(x;\varphi)]^b))^2\left\{1-\overline{\delta}\left(1-\left[\frac{[1-\overline{G}^2(x;\varphi)]^b}{1+(1-[1-\overline{G}^2(x;\varphi)]^b)}\right]\right)\right\}^2}, \tag{5}$$

for $\delta, b > 0$, $\overline{\delta} = 1-\delta$ and φ is a parameter vector for the baseline distribution $G(.)$.

2.2. Hazard and Quantile Functions

In this section, we present the hazard and quantile functions of the MO-TLHL-G family of distributions. The hazard rate function (hrf) of the MO-TLHL-G family of distributions is given by

$$h(x) = \frac{2bg(x;\varphi)\overline{G}(x;\varphi)(1-\overline{G}^2(x;\varphi))^{b-1}(1-(1-\overline{G}^2(x;\varphi))^b)^{-1}}{(1+(1-[1-\overline{G}^2(x;\varphi)]^b))\left\{1-\overline{\delta}\left(1-\left[\frac{[1-\overline{G}^2(x;\varphi)]^b}{1+(1-[1-\overline{G}^2(x;\varphi)]^b)}\right]\right)\right\}}. \tag{6}$$

The quantile function of the MO-TLHL-G family of distributions is given by

$$Q(u) = G^{-1}\left[1-\left(1-\left[\frac{2\delta u}{1+u-2u\overline{\delta}}\right]^{\frac{1}{b}}\right)^{\frac{1}{2}}\right]. \tag{7}$$

2.3. Stochastic Order

Stochastic ordering is an important tool in the comparison of probability models in areas such as reliability, survival analysis, finance, risks and economics. In this section, we present a stochastic ordering result from the MO-TLHL-G family of distributions.

Suppose X and Y be two random variables with cumulative distribution functions (cdfs) $F_X(t)$ and $F_Y(t)$, respectively. Then X is stochastically smaller than Y if $\overline{F}_X(t) \leq \overline{F}_Y(t)$ for all t, where $\overline{F}_X(t) = 1 - F(t)$ is the survival function. Stochastic ordering of X and Y is denoted by $X <_{st} Y$. A stronger ordering is the likelihood ratio order denoted by $X <_{lr} Y$, and given by $\frac{f_X(t)}{f_Y(t)}$ decreasing in t. Hazard rate order is stronger than stochastic order but weaker than likelihood ratio order. Hazard rate order is given by $h_X(t) \geq h_Y(t)$ for all t, and denoted by $X <_{hr} Y$. Thus, $X <_{lr} Y \Rightarrow X <_{hr} Y \Rightarrow X <_{st} Y$ (see Shaked and Shanthikumar [28]).

Theorem 2.1. Suppose $X_1 \sim MO-TLHL-G(\delta_1, b, \varphi)$ and $X_2 \sim MO-TLHL-G(\delta_2, b, \varphi)$. If $\delta_1 < \delta_2$, then $\frac{f(x;\delta_1,b,\varphi)}{g(x;\delta_2,b,\varphi)}$ is decreasing in x.

Proof:
After some simplification, we have

$$\frac{f(x;\delta_1,b,\varphi)}{g(x;\delta_2,b,\varphi)} = \frac{\delta_1}{\delta_2}\left[\frac{1-\overline{\delta}_2\left(1-\left[\frac{[1-\overline{G}^2(x;\varphi)]^b}{1+(1-[1-\overline{G}^2(x;\varphi)]^b)}\right]\right)}{1-\overline{\delta}_1\left(1-\left[\frac{[1-\overline{G}^2(x;\varphi)]^b}{1+(1-[1-\overline{G}^2(x;\varphi)]^b)}\right]\right)}\right]^2, \tag{8}$$

where $\overline{\delta}_1 = 1-\delta_1$, and $\overline{\delta}_2 = 1-\delta_2$.
Differentiating equation (8) with respect to x, we have

$$\frac{\partial}{\partial x}\left(\frac{f(x;\delta_1,b,\varphi)}{g(x;\delta_2,b,\varphi)}\right) = \frac{2\delta_1}{\delta_2}(\overline{\delta_2}-\overline{\delta_1})\frac{(1-\overline{\delta}_2(1-W))}{(1-\overline{\delta}_1(1-W))^3}W',$$

where $W = \frac{[1-\overline{G}^2(x;\varphi)]^b}{1+(1-[1-\overline{G}^2(x;\varphi)]^b)}$ and $W' = \frac{dW}{dx}$.

If $\delta_1 < \delta_2$, then $\frac{\partial}{\partial x}\left(\frac{f(x;\delta_1,b,\varphi)}{g(x;\delta_2,b,\varphi)}\right) < 0$ which implies that the likelihood ratio order exist among X_1 and X_2, that is $X_2 <_{lr} X_1$. Consequently, X_1 and X_2 are stochastically ordered, that is $X_1 <_{st} X_2$.

2.4. Linear Representation

In this section, a useful linear representation for the MO-TLHL-G pdf is presented. The pdf in equation (5) can be expressed as

$$\begin{aligned} f(x) &= 4b\delta g(x;\varphi)\overline{G}(x;\varphi)(1-\overline{G}^2(x;\varphi))^{b-1}(1+(1-[1-\overline{G}^2(x;\varphi)]^b))^{-2} \\ &\times \underbrace{\left\{1-\overline{\delta}\left(1-\left[\frac{[1-\overline{G}^2(x;\varphi)]^b}{1+(1-[1-\overline{G}^2(x;\varphi)]^b)}\right]\right)\right\}^{-2}}_{A}. \end{aligned} \tag{9}$$

Using the generalized binomial series expansion, we have

$$A = \sum_{l,k=0}^{\infty}(-1)^{l+k}\binom{-2}{l}\binom{l}{k}\overline{\delta}^l\frac{[1-\overline{G}^2(x;\varphi)]^{bk}}{[1+(1-[1-\overline{G}^2(x;\varphi)]^b)]^k}.$$

Subtituting A in equation (9), we have

$$\begin{aligned} f(x) &= 4b\delta g(x;\varphi)\overline{G}(x;\varphi)\sum_{l,k=0}^{\infty}(-1)^{l+k}\binom{-2}{l}\binom{l}{k}\overline{\delta}^{l}(1-\overline{G}^{2}(x;\varphi))^{b(k+1)-1} \\ &\times \underbrace{(1+(1-[1-\overline{G}^{2}(x;\varphi)]^{b}))^{-(k+2)}}_{B}. \end{aligned} \tag{10}$$

Applying the generalized binomial series expansion on B, we obtain

$$B = \sum_{i,j=0}^{\infty}(-1)^{i}\binom{-(k+2)}{j}\binom{j}{i}[1-\overline{G}^{2}(x;\varphi)]^{bi}.$$

Substituting B in equation (10), we have

$$\begin{aligned} f(x) &= 4b\delta g(x;\varphi)\overline{G}(x;\varphi)\sum_{l,k,i,j=0}^{\infty}(-1)^{l+k+i}\binom{-2}{l}\binom{-(k+2)}{j}\binom{j}{i}\binom{l}{k}\overline{\delta}^{l} \\ &\times \underbrace{(1-\overline{G}^{2}(x;\varphi))^{b(k+i+1)-1}}_{C}. \end{aligned} \tag{11}$$

Using the binomial series expansion, we have

$$C = \sum_{q=0}^{\infty}(-1)^{q}\binom{b(k+i+1)-1}{q}\overline{G}^{2q}(x;\varphi).$$

Substituting C in equation (11) we obtain

$$\begin{aligned} f(x) &= 4b\delta g(x;\varphi)\sum_{l,k,i,j,q=0}^{\infty}(-1)^{l+k+i+q}\binom{-2}{l}\binom{-(k+2)}{j}\binom{j}{i}\binom{l}{k} \\ &\times \binom{b(k+i+1)-1}{q}\overline{\delta}^{l}\overline{G}^{2q+1}(x;\varphi). \end{aligned} \tag{12}$$

Using the binomial series expansion, we have

$$(1-G(x;\varphi))^{2q+1} = \sum_{p=0}^{\infty}(-1)^{p}\binom{2q+1}{p}G^{p}(x;\varphi).$$

Thus, the pdf of the MO-TLHL-G family of distributions can be expressed as

$$\begin{aligned} f(x) &= 4b\delta g(x;\varphi)\sum_{l,k,i,j,q,p=0}^{\infty}(-1)^{l+k+i+q+p}\binom{-2}{l}\binom{-(k+2)}{j}\binom{j}{i}\binom{l}{k} \\ &\times \binom{b(k+i+1)-1}{q}\binom{2q+1}{p}\overline{\delta}^{l}G^{p}(x;\varphi). \end{aligned} \tag{13}$$

Or equivalently, we can write

$$f(x) = \sum_{p=0}^{\infty}c_{p+1}h_{p+1}(x;\varphi), \tag{14}$$

where

$$c_{p+1} = 4b\delta \sum_{l,k,i,j,q=0}^{\infty} \frac{(-1)^{l+k+i+q+p}}{p+1} \binom{-2}{l} \binom{-(k+2)}{j} \binom{j}{i} \binom{l}{k} \times \binom{b(k+i+1)-1}{q} \binom{2q+1}{p} \bar{\delta}^{l}, \quad (15)$$

and $h_{p+1}(x) = (p+1)g(x;\varphi)G(x;\varphi)^p$ is the exp-G density with power parameter $(p+1)$. Thus, the MO-TLHL-G family of distributions can be written as an infinite linear combination of exponentiated-G densities. The mathematical and statistical properties of the MO-TLHL-G family of distributions follow directly from those of the exponentiated-G distribution.

2.4.1. Sub-Families

- If $\delta = 1$, we obtain a new family of distributions called the Topp-Leone Half-Logistic-G (TLHL-G) distribution with pdf given by

$$f(x;b,\varphi) = 2bg(x;\varphi)(1-\overline{G}^2(x;\varphi))^{b-1} \exp\Big[-\Big(\frac{[1-\overline{G}^2(x;\varphi)]^b}{1-[1-\overline{G}^2(x;\varphi)]^b}\Big)\Big].$$

- If $b = 1$, we obtain a new family of distributions with pdf given by

$$f(x;\delta,\varphi) = \frac{2\delta g(x;\varphi)\exp\Big[-\Big(\frac{[1-\overline{G}^2(x;\varphi)]}{\overline{G}^2(x;\varphi)}\Big)\Big]}{\Big\{1-\bar{\delta}\exp\Big[-\Big(\frac{1-\overline{G}^2(x;\varphi)}{\overline{G}^2(x;\varphi)}\Big)\Big]\Big\}^2}.$$

- If $b = \delta = 1$, we obtain a new family of distributions with pdf given by

$$f(x;\varphi) = 2g(x;\varphi)\exp\Big[-\Big(\frac{[1-\overline{G}^2(x;\varphi)]}{\overline{G}^2(x;\varphi)}\Big)\Big].$$

2.5. Special Cases

In this section, we consider some special cases of the MO-TLHL-G family of distributions, specifically when the baseline distribution function $G(x;\varphi)$ are Burr XII, Weibull, Kumaraswamy, Burr III and Uniform distributions, respectively.

2.5.1. Marshall-Olkin Topp-Leone Half-Logistic-Burr XII distribution Suppose the cdf and pdf of the baseline distribution are given by $G(x;\theta,\gamma) = 1-(1+x^\theta)^{-\gamma}$ and $g(x;\theta,\gamma) = \theta\gamma x^{\theta-1}(1+x^\theta)^{-\gamma-1}$ for $\theta,\gamma > 0$ and $x > 0$. Then, the MO-TLHL-Burr XII (MO-TLHL-BXII) distribution has cdf and pdf given by

$$F(x;b,\delta,\theta,\gamma) = \frac{\frac{[1-(1+x^\theta)^{-2\gamma}]^b}{(1+(1-[1-(1+x^\theta)^{-2\gamma}]^b)}}{1-\bar{\delta}\Big(1-\Big[\frac{[1-(1+x^\theta)^{-2\gamma}]^b}{1+(1-[1-(1+x^\theta)^{-2\gamma}]^b)}\Big]\Big)}$$

and

$$f(x;b,\delta,\theta,\gamma) = \frac{4b\delta\theta\gamma x^{\theta-1}(1+x^\theta)^{-2\gamma-1}(1-(1+x^\theta)^{-2\gamma})^{b-1}}{(1+(1-[1-(1+x^\theta)^{-2\gamma}]^b))^2\Big\{1-\bar{\delta}\Big(1-\Big[\frac{[1-(1+x^\theta)^{-2\gamma}]^b}{1+(1-[1-(1+x^\theta)^{-2\gamma}]^b)}\Big]\Big)\Big\}^2},$$

respectively for $b,\delta,\theta,\gamma > 0$. Figure 1 shows the plots of the MO-TLHL-BXII distribution for different parameter values. The pdf can take various shapes including the reverse J, uni-modal, almost symmetric, left skewed and right skewed.

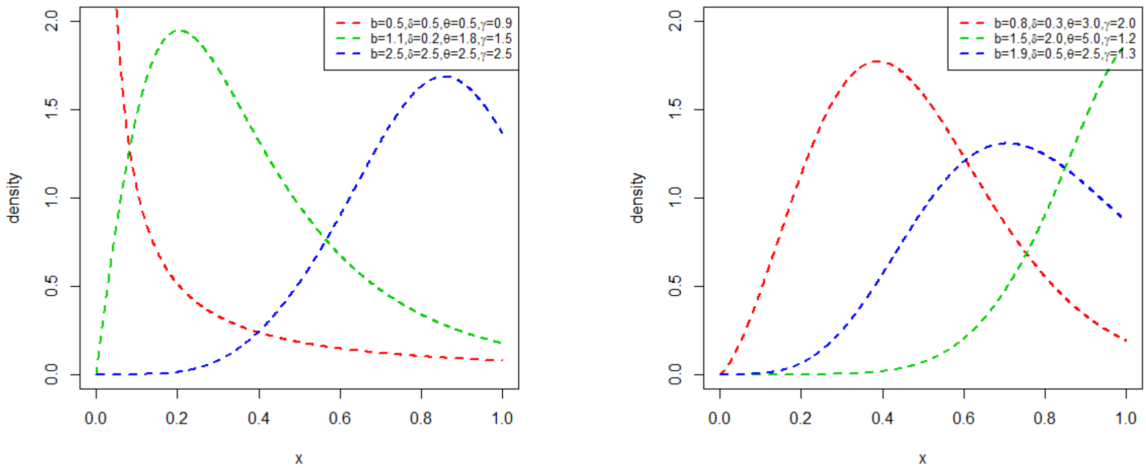

Figure 1. Plots of MO-TLHL-BXII density function

Figure 2 shows the plots of the hrf, the graphs exhibits increasing, decreasing, upside-down bathtub and bathtub followed by upside-down bathtub shapes.

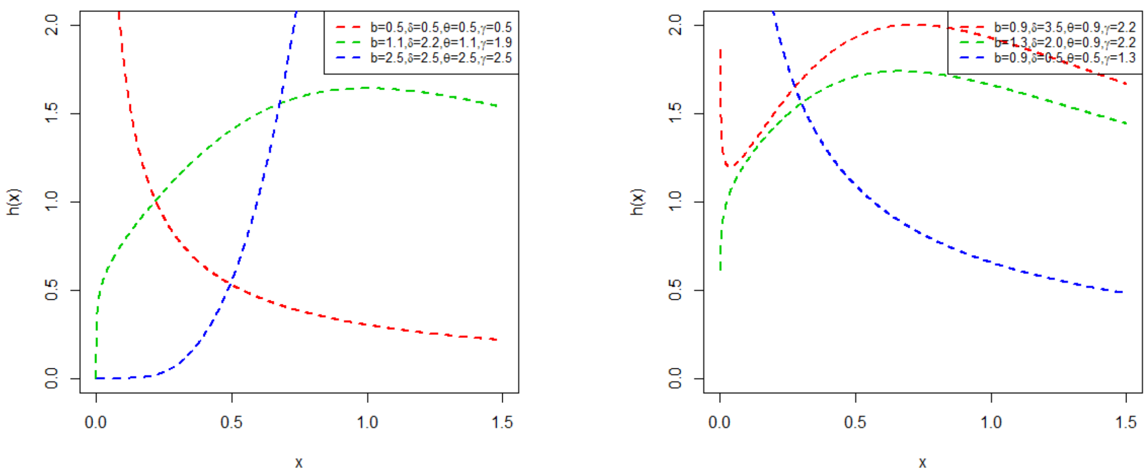

Figure 2. Plots of MO-TLHL-BXII hazard rate function

2.5.2. Marshall-Olkin Topp-Leone Half-Logistic-Weibull distribution Suppose the cdf and pdf of the baseline distribution are Weibull and are given by $G(x;\theta,\gamma) = 1 - \exp(-\theta x^{\gamma})$ and $g(x;\theta,\gamma) = \gamma\theta x^{\gamma-1}\exp(-\theta x^{\gamma})$ for $\theta, \gamma > 0$ and $x > 0$. Then, the MO-TLHL-Weibull (MO-TLHL-W) distribution has cdf and pdf given by

$$F(x;b,\delta,\theta,\gamma) = \frac{\frac{[1-\exp(-2\theta x^{\gamma})]^{b}}{(1+(1-[1-\exp(-2\theta x^{\gamma})]^{b})}}{1-\overline{\delta}\left(1-\left[\frac{[1-\exp(-2\theta x^{\gamma})]^{b}}{1+(1-[1-\exp(-2\theta x^{\gamma})]^{b})}\right]\right)} \tag{16}$$

and

$$f(x;b,\delta,\theta,\gamma)=\frac{4b\delta\gamma\theta x^{\gamma-1}\exp(-2\theta x^{\gamma})(1-\exp(-2\theta x^{\gamma}))^{b-1}}{(1+(1-[1-\exp(-2\theta x^{\gamma})]^{b}))^{2}\left\{1-\overline{\delta}\left(1-\left[\frac{[1-\exp(-2\theta x^{\gamma})]^{b}}{1+(1-[1-\exp(-2\theta x^{\gamma})]^{b})}\right]\right)\right\}^{2}}. \tag{17}$$

Figure 3 shows the plots of the pdf of the MO-TLHL-W distribution for different parameter values. The pdf can take various shapes including the reverse J, uni-modal and right skewed.

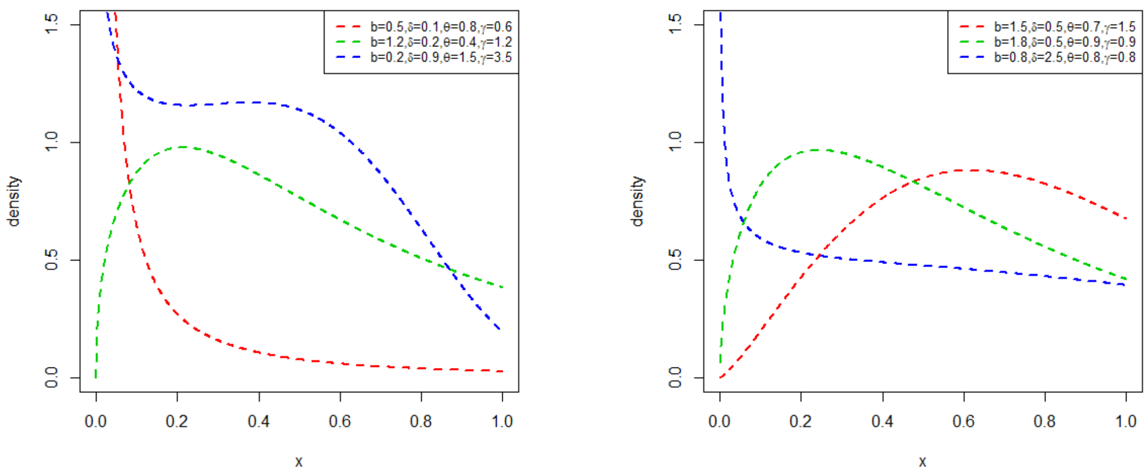

Figure 3. Plots of MO-TLHL-W density function

Figure 4 shows the plots of the hrf and the graphs show increasing, decreasing, bathtub and upside-down bathtub shapes.

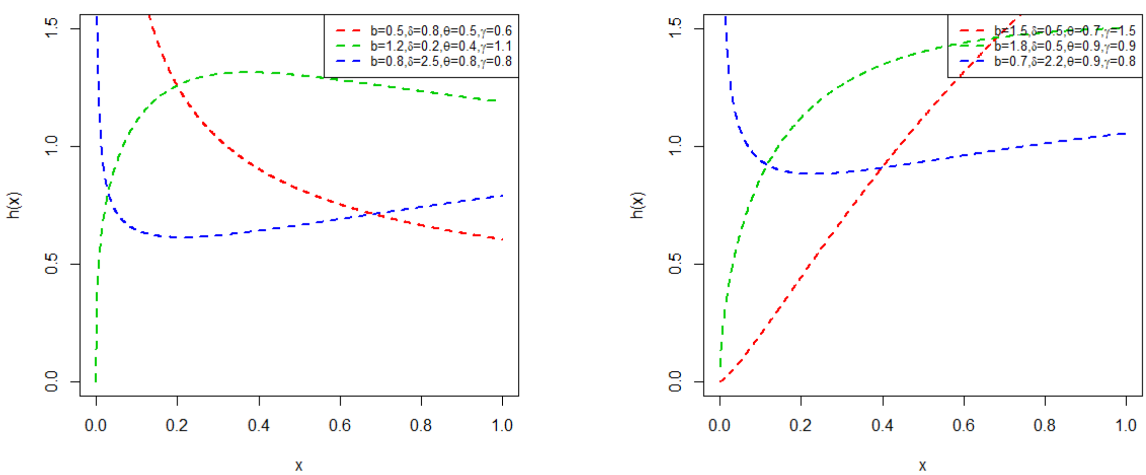

Figure 4. Plots of MO-TLHL-W hazard rate function

2.5.3. Marshall-Olkin Topp-Leone Half-Logistic-Kumaraswamy distribution Suppose the cdf and pdf of the baseline distribution are given by $G(x;\theta,\gamma)=1-(1-x^{\theta})^{\gamma}$ and $g(x;\theta,\gamma)=\theta\gamma x^{\theta-1}(1-x^{\theta})^{\gamma-1}$ for $\theta,\gamma>0$ and $x>0$. Then, the MO-TLHL-Kumaraswamy (MO-TLHL-Kum) distribution has cdf and pdf given by

$$F(x;b,\delta,\theta,\gamma)=\frac{\frac{[1-(1-x^{\theta})^{2\gamma}]^{b}}{(1+(1-[1-(1-x^{\theta})^{2\gamma}]^{b})}}{1-\bar{\delta}\Big(1-\Big[\frac{[1-(1-x^{\theta})^{2\gamma}]^{b}}{1+(1-[1-(1-x^{\theta})^{2\gamma}]^{b})}\Big]\Big)}$$

and

$$f(x;b,\delta,\theta,\gamma)=\frac{4b\delta\theta\gamma x^{\theta-1}(1-x^{\theta})^{2\gamma-1}(1-(1-x^{\theta})^{2\gamma})^{b-1}}{(1+(1-[1-(1-x^{\theta})^{2\gamma}]^{b}))^{2}\Big\{1-\bar{\delta}\Big(1-\Big[\frac{[1-(1-x^{\theta})^{2\gamma}]^{b}}{1+(1-[1-(1-x^{\theta})^{2\gamma}]^{b})}\Big]\Big)\Big\}^{2}}.$$

Figure 5 shows the plots of the MO-TLHL-Kum distribution for different parameter values. The pdf can take various shapes including the reverse-J, uni-modal, almost symmetric, left skewed and right skewed.

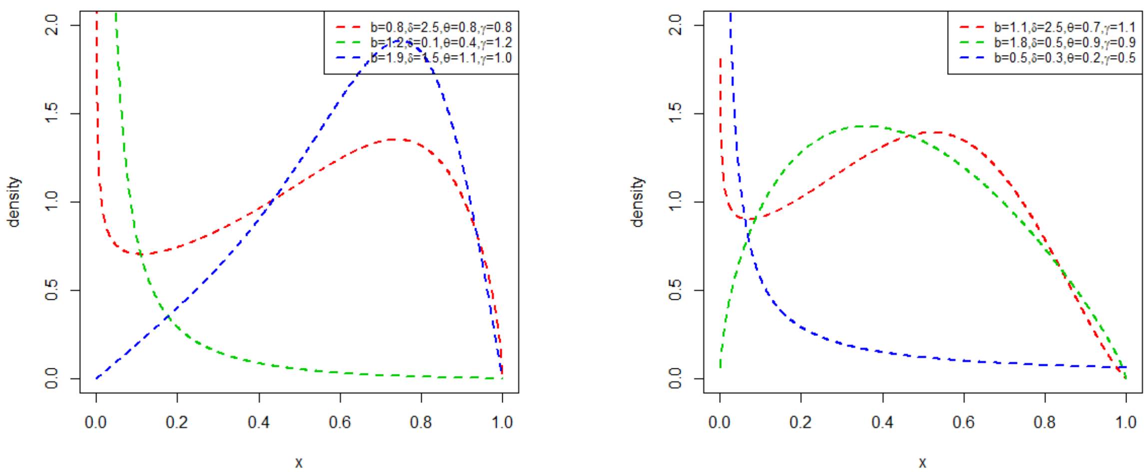

Figure 5. Plots of MO-TLHL-Kum density function

Figure 6 shows the plots of the hrf. We have increasing and bathtub shapes.

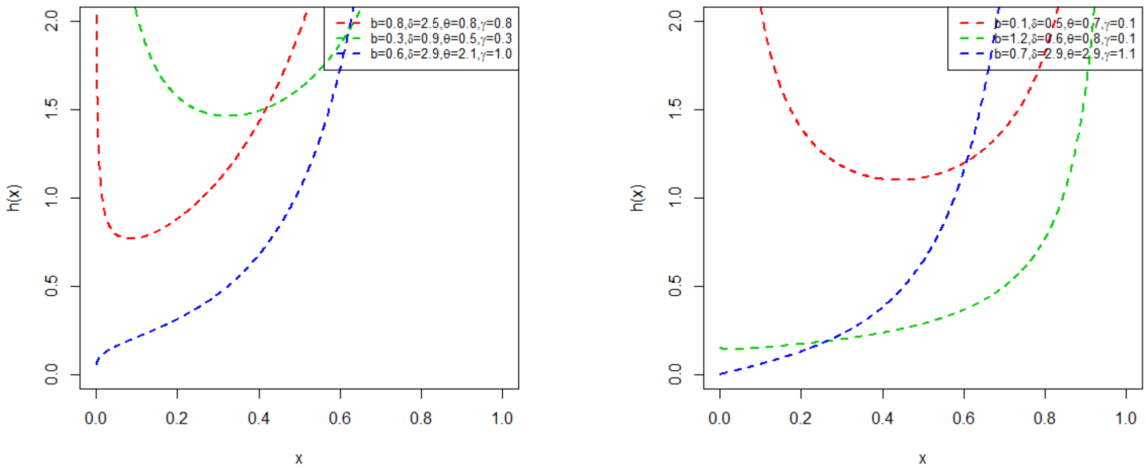

Figure 6. Plots of MO-TLHL-Kum hazard rate function

2.5.4. Marshall-Olkin Topp-Leone Half-Logistic-Burr III distribution Suppose the cdf and pdf of the baseline distribution are given by $G(x;\theta,\gamma) = (1+x^{-\theta})^{-\gamma}$ and $g(x;\theta,\gamma) = \theta\gamma x^{-\theta-1}(1+x^{-\theta})^{-\gamma-1}$ for $\theta, \gamma > 0$ and $x > 0$. Then, the MO-TLHL-Burr III (MO-TLHL-BIII) distribution has cdf and pdf given by

$$F(x;b,\delta,\theta,\gamma) = \frac{\frac{[1-(1-(1+x^{-\theta})^{-\gamma})^2]^b}{(1+(1-[1-(1-(1+x^{-\theta})^{-\gamma})^2]^b)}}{1-\bar{\delta}\left(1-\left[\frac{[1-(1-(1+x^{-\theta})^{-\gamma})^2]^b}{1+(1-[1-(1-(1+x^{-\theta})^{-\gamma})^2]^b)}\right]\right)}$$

and

$$f(x;b,\delta,\theta,\gamma) = \frac{4b\delta\theta\gamma x^{-\theta-1}(1+x^{-\theta})^{-\gamma-1}(1-(1+x^{-\theta})^{-\gamma})(1-(1-(1+x^{-\theta})^{-\gamma})^2)^{b-1}}{(1+(1-[1-(1-(1+x^{-\theta})^{-\gamma})^2]^b))^2\left\{1-\bar{\delta}\left(1-\left[\frac{[1-(1-(1+x^{-\theta})^{-\gamma})^2]^b}{1+(1-[1-(1-(1+x^{-\theta})^{-\gamma})^2]^b)}\right]\right)\right\}^2}.$$

Figure 7 shows the plots of the MO-TLHL-BIII distribution for different parameter values. The pdf can take various shapes including the reverse-J, uni-modal, left skewed and right skewed.

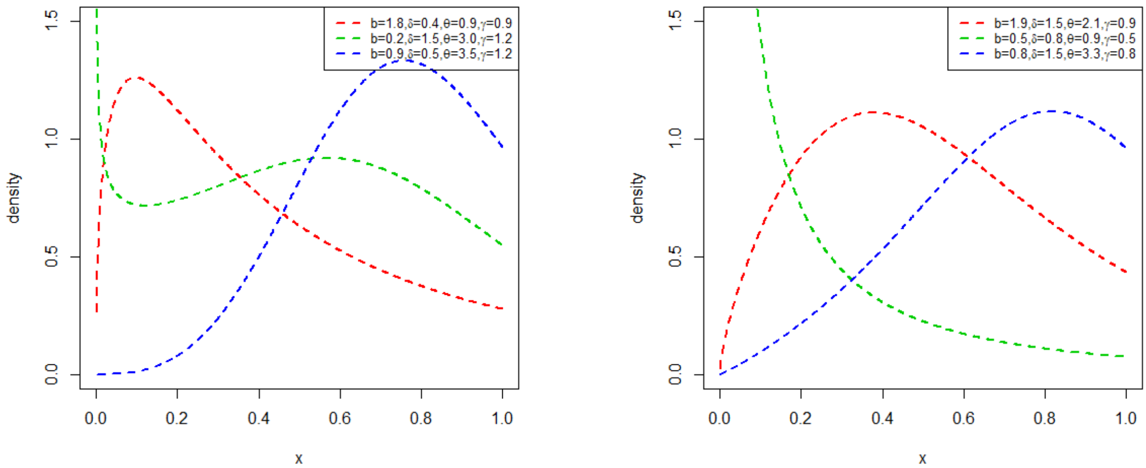

Figure 7. Plots of MO-TLHL-BIII density function

Figure 8 shows the plots of the hrf of the MO-TLHL-BIII distribution. Graphs exhibits decreasing, reverse-J, upside-down bathtub and bathtub followed by upside-down shapes.

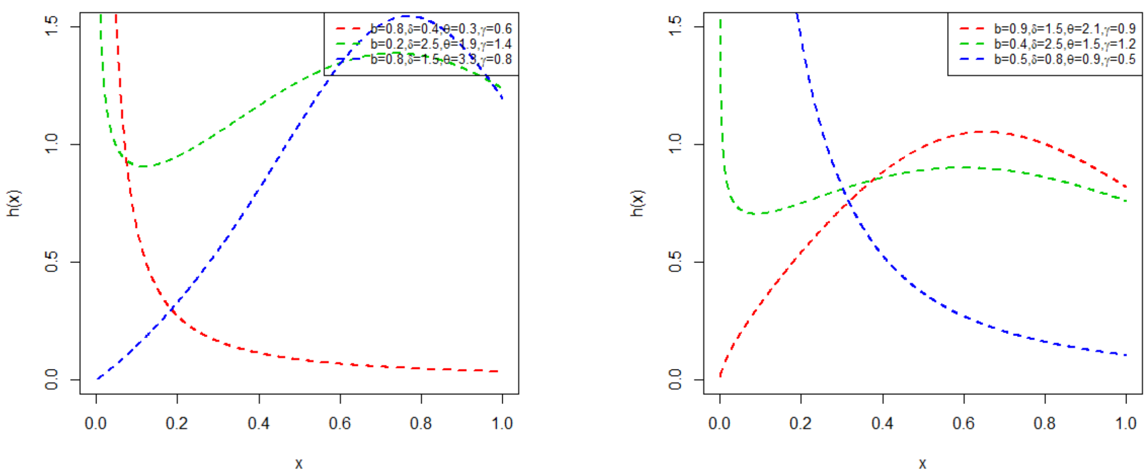

Figure 8. Plots of MO-TLHL-BIII hazard rate function

2.5.5. Marshall-Olkin Topp-Leone Half-Logistic-Uniform Distribution Suppose the cdf and pdf of the baseline distribution are given by $G(x;\theta) = \frac{x}{\theta}$ and $g(x;\theta) = \frac{1}{\theta}$ for $0 < x < \theta < \infty$. Then, the MO-TLHL-Uniform (MO-TLHL-U) distribution has cdf and pdf given by

$$F(x;b,\delta,\theta) = \frac{\frac{[1-(1-\frac{x}{\theta})^2]^b}{(1+(1-[1-(1-\frac{x}{\theta})^2]^b)}}{1-\bar{\delta}\left(1-\left[\frac{[1-(1-\frac{x}{\theta})^2]^b}{1+(1-[1-(1-\frac{x}{\theta})^2]^b)}\right]\right)}$$

and

$$f(x;b,\delta,\theta)=\frac{4b\delta\frac{1}{\theta}(1-\frac{x}{\theta})(1-(1-\frac{x}{\theta})^2)^{b-1}}{(1+(1-[1-(1-\frac{x}{\theta})^2]^b))^2\left\{1-\bar{\delta}\left(1-\left[\frac{[1-(1-\frac{x}{\theta})^2]^b}{1+(1-[1-(1-\frac{x}{\theta})^2]^b)}\right]\right)\right\}^2}.$$

Figure 9 shows the plots of the MO-TLHL-U distribution for different parameter values. The pdf can take various shapes including the reverse-J, uni-modal, almost symmetric and left skewed.

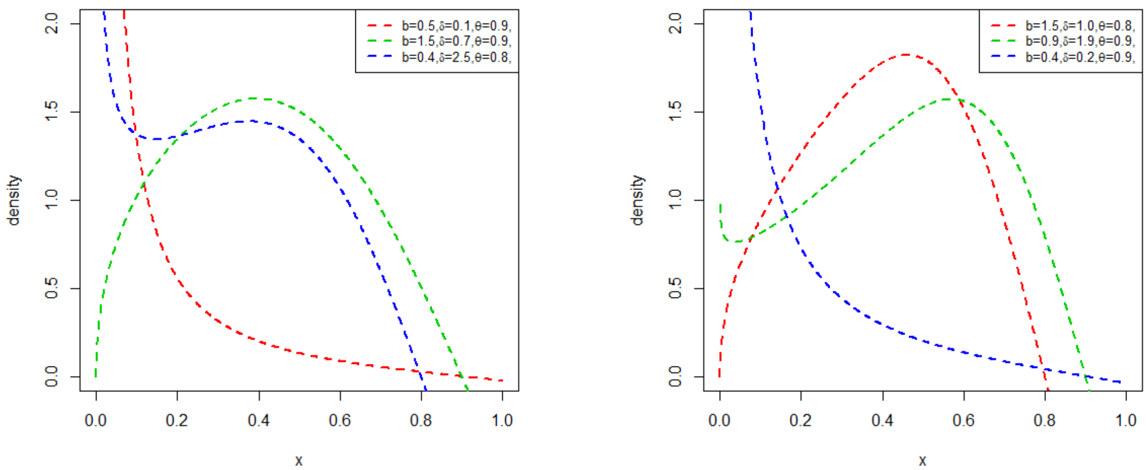

Figure 9. Plots of MO-TLHL-U density function

Figure 10 shows the plots of the hrf of the MO-TLHL-U distribution. The graphs exhibits increasing and bathtub shapes.

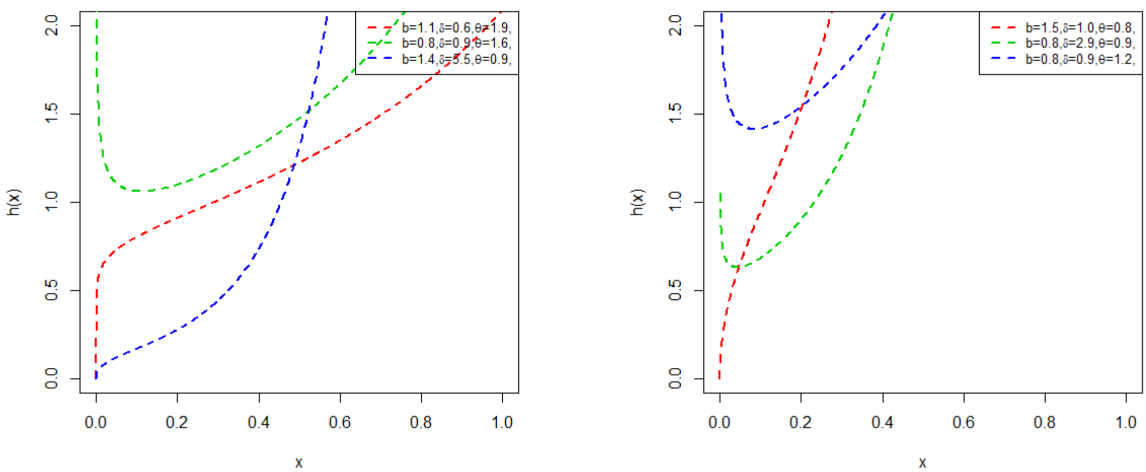

Figure 10. Plots of MO-TLHL-U hazard rate function

Table 1 shows some random numbers that are generated from the MO-TLHL-W distribution for selected parameter values.

Table 1. Table of Quantiles

	$(b, \delta, \theta, \gamma)$				
u	(0.9, 0.9, 0.9, 0.9)	(2.0, 1.5, 0.9, 0.5)	(0.5, 0.9, 1.5, 1.5)	(0.9, 0.8, 1.8, 0.5)	(0.8, 0.5, 0.6, 0.4)
0.1	0.06175	0.14829	0.04451	0.00131	0.00052
0.2	0.14579	0.34892	0.10445	0.00631	0.00496
0.3	0.24409	0.59315	0.17000	0.01625	0.02000
0.4	0.35749	0.88728	0.24021	0.03292	0.05750
0.5	0.48982	1.24708	0.31611	0.05905	0.13932
0.6	0.64882	1.70300	0.40048	0.09966	0.30978
0.7	0.84978	2.31676	0.49872	0.16490	0.67010
0.8	1.12786	3.23657	0.62263	0.27974	1.50459
0.9	1.59681	4.97860	0.80860	0.53480	4.01451

3. Moments and Incomplete Moments

In this section, moments, moment generating function and incomplete moments for the MO-TLHL-G family of distributions are presented. These measures (moments, moment generating function and incomplete moments) can be readily obtained for the sub-families and sub-models given in section 2.

3.1. Moments and Generating Function

Let T_{p+1}, denote the Exp-G random variable with power parameter $p+1$. Then the r^{th} raw moment of X, say μ'_r follows from equation (5) as

$$\mu'_r = E(X^r) \quad = \quad \sum_{p=0}^{\infty} c_{p+1} E(T_{p+1}^r), \tag{18}$$

where c_{p+1} is as given in equation (15).

The moment generating function (mgf) $M_X(t) = E(e^{tX})$ of X can be derived from equation (5) and is given by

$$M_X(t) = \sum_{p=0}^{\infty} c_{p+1} M_{p+1}(t),$$

where $M_{p+1}(t)$ is the mgf of T_{p+1}. Hence, $M_X(t)$ can be determined from the Exp-G generating function.

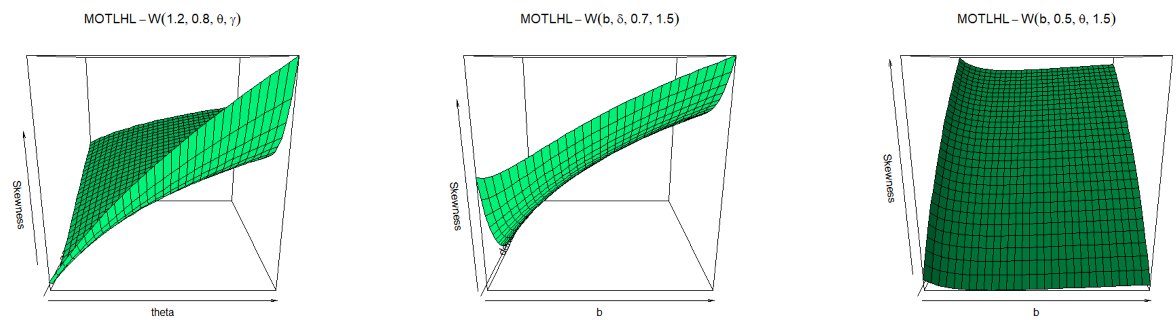

Figure 11. Plots of the skewness of the MO-TLHL-W distribution.

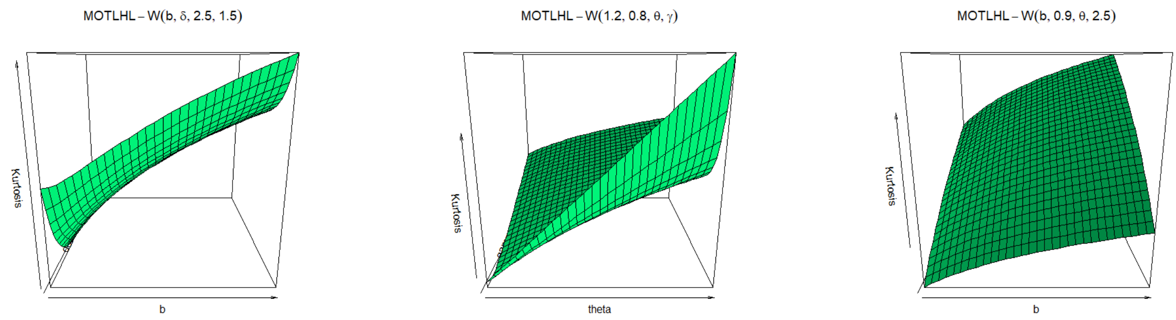

Figure 12. Plots of the kurtosis of the MO-TLHL-W distribution.

Figure 11 and 12 shows the skewness and kurtosis plots of the MO-TLHL-W distribution, it can be seen that the members of the MO-TLHL-G family are capable of modelling various data sets with different levels of skewness and kurtosis.

3.2. Incomplete Moments

Conditional and incomplete moments are particularly useful for the analysis of lifetime models, reliability and in measuring inequality. The r^{th} incomplete moment of the MO-TLHL-G family of distributions is given by

$$\begin{aligned} m_r(t) &= \int_{-\infty}^{t} x^r f(x;b,\delta,\varphi)dx \\ &= \sum_{p=0}^{\infty} c_{p+1} \int_{-\infty}^{t} x^r h_{p+1}(x;\varphi)dx. \end{aligned} \tag{19}$$

Note that $m_1(t)$ can be used in the construction of Bonferroni and Lorenz curves. These curves are of great importance in insurance, demography, reliability, medicine and economics.

A general equation for $m_1(t)$ can be obtained from equation (19) as

$$m_1(t) = \sum_{p=0}^{\infty} c_{p+1} K_{p+1}(t),$$

where $K_{p+1}(t) = \int_{-\infty}^{t} x h_{p+1}(x;\varphi)dx$ is the first incomplete moment of the Exp-G distribution.

4. Order Statistics and Rényi Entropy

Order statistics and entropy play important roles in probability and statistics, particularly in reliability, lifetime data analysis and information theory. In this section, we present the distribution of the i^{th} order statistics and Rényi entropy for the MO-TLHL-G family of distributions.

4.1. Order Statistics

Let $X_1, X_2, ..., X_n$ be independent and identically distributed MO-TLHL-G random variables. The pdf of $X_{i:n}$ can be written as

$$f_{i:n}(x) = \frac{f(x)}{B(i, n-i+1)} \sum_{j=0}^{n-i} (-1)^j \binom{n-i}{j} F(x)^{j+i-1}, \tag{20}$$

where $B(.,.)$ is the beta function. Note that

$$\begin{aligned} f(x)F(x)^{j+i-1} &= 4b\delta g(x;\varphi)(1-\overline{G}^2(x;\varphi))^{b(j+i)-1}(1+(1-(1-\overline{G}^2(x;\varphi))))^{-(j+i+1)} \\ &\times \overline{G}(x;\xi)\Big\{1-\overline{\delta}\Big(1-\Big[\frac{[1-\overline{G}^2(x;\varphi)]^b}{1+(1-[1-\overline{G}^2(x;\varphi)]^b)}\Big]\Big)\Big\}^{-(j+i+1)}. \end{aligned}$$

Using the generalized binomial series expansion, we obtain

$$\begin{aligned} f(x)F(x)^{j+i-1} &= 4b\delta g(x;\varphi)\overline{G}(x;\varphi)\sum_{k=0}^{\infty}(-1)^k\binom{-(j+i+1)}{k}\overline{\delta}^k \\ &\times (1-\overline{G}^2(x;\varphi))^{b(j+i)-1}(1+(1-(1-\overline{G}^2(x;\varphi))))^{-(j+i+1)} \\ &\times \Big(1-\Big[\frac{[1-\overline{G}^2(x;\varphi)]^b}{1+(1-[1-\overline{G}^2(x;\varphi)]^b)}\Big]\Big)^k \\ &= 4b\delta g(x;\varphi)\overline{G}(x;\varphi)\sum_{k,q=0}^{\infty}(-1)^{k+q}\binom{-(j+i+1)}{k}\binom{k}{q}\overline{\delta}^k \\ &\times (1-\overline{G}^2(x;\varphi))^{b(j+i+q)-1}(1+(1-(1-\overline{G}^2(x;\varphi))))^{-(j+i+q+1)} \\ &= 4b\delta g(x;\varphi)\overline{G}(x;\varphi)\sum_{k,q,i,p=0}^{\infty}(-1)^{k+q+p}\binom{-(j+i+1)}{k}\binom{k}{q}\binom{i}{p} \\ &\times \binom{-(j+i+q+1)}{i}\overline{\delta}^k(1-\overline{G}^2(x;\varphi))^{b(j+i+q+p)-1} \\ &= 4b\delta g(x;\varphi)\sum_{k,q,i,p,w,s=0}^{\infty}(-1)^{k+q+p+w+s}\binom{-(j+i+1)}{k}\binom{k}{q} \\ &\times \binom{i}{p}\binom{-(j+i+q+1)}{i}\binom{b(j+i+q+p)-1}{w}\binom{2w+1}{s} \\ &\times \overline{\delta}^k G(x;\varphi)^s. \end{aligned} \quad (21)$$

Substituting equation (21) in equation (20), the pdf of $X_{i:n}$ can be expressed as

$$f_{i:n}(x)=\sum_{s=0}^{\infty}c_{s+1}h_{s+1}(x;\varphi), \quad (22)$$

where $h_{s+1}(x;\varphi)$ is the Exp-G density function with power parameter $(s+1)$ and

$$\begin{aligned} c_{s+1} &= 4b\delta\sum_{k,q,i,p,w=0}^{\infty}\sum_{j=0}^{n-i}\frac{(-1)^{k+q+p+w+s+j}}{B(i,n-i+1)(s+1)}\binom{n-i}{j}\binom{-(j+i+1)}{k}\binom{k}{q} \\ &\times \binom{i}{p}\binom{-(j+i+q+1)}{i}\binom{b(j+i+q+p)-1}{w}\binom{2w+1}{s}\overline{\delta}^k. \end{aligned}$$

It follows that the pdf of the i^{th} order statistics can be expressed as an infinite linear combination of exponentiated-G densities.

4.2. Rényi Entropy

Rényi entropy (Rényi [26]) is an extension of Shannon entropy. Rényi entropy is defined to be

$$I_R(v)=\frac{1}{1-v}\log\left(\int_0^{\infty}[f(x;b,\delta,\varphi)]^v dx\right), v\neq 1, v>0. \quad (23)$$

Rényi entropy tends to Shannon entropy as $v \to 1$. Set $[f(x;b,\delta,\varphi)]^v = [f(x)]^v$. From equation (5),

$$
\begin{aligned}
[f(x)]^v &= (4b\delta g(x;\varphi)\overline{G}(x;\varphi))^v(1-\overline{G}^2(x;\varphi))^{v(b-1)}(1+(1-[1-\overline{G}^2(x;\varphi)]^b))^{-2v} \\
&\times \Big\{1-\overline{\delta}\Big(1-\Big[\frac{[1-\overline{G}^2(x;\varphi)]^b}{1+(1-[1-\overline{G}^2(x;\varphi)]^b)}\Big]\Big)\Big\}^{-2v}.
\end{aligned}
$$

Using the generalized binomial series expansion, we have

$$
\begin{aligned}
[f(x)]^v &= (4b\delta g(x;\varphi)\overline{G}(x;\varphi))^v \sum_{l,k=0}^{\infty}(-1)^{l+k}\binom{-2v}{l}\binom{l}{k}\overline{\delta}^l(1-\overline{G}^2(x;\varphi))^{b(v+k)-v} \\
&\times (1+(1-[1-\overline{G}^2(x;\varphi)]^b))^{-2v-k} \\
&= (4b\delta g(x;\varphi)\overline{G}(x;\varphi))^v \sum_{l,k,i,j=0}^{\infty}(-1)^{l+k+i}\binom{-2v}{l}\binom{l}{k}\binom{-(2v+k)}{j}\binom{j}{i} \\
&\times \overline{\delta}^l(1-\overline{G}^2(x;\varphi))^{b(v+k+i)-v} \\
&= (4b\delta)^v \sum_{l,k,i,j,q,p=0}^{\infty}(-1)^{l+k+i+q+p}\binom{-2v}{l}\binom{l}{k}\binom{-(2v+k)}{j} \\
&\times \binom{j}{i}\binom{b(k+i+v)-v}{q}\overline{\delta}^l\overline{G}^{2q+v}(x;\varphi) \\
&= \sum_{l,k,i,j,q,p=0}^{\infty}(-1)^{l+k+i+q+p}(4b\delta)^v\binom{-2v}{l}\binom{l}{k}\binom{-(2v+k)}{j} \\
&\times \binom{j}{i}\binom{b(k+i+v)-v}{q}\binom{2q+v}{p}\overline{\delta}^l[g(x;\varphi)]^v G(x;\varphi)^p.
\end{aligned}
$$

Thus, the Rényi entropy of the MO-TLHL-G family of distributions can be expressed as

$$
I_R(v) = \frac{1}{1-v}\log\Big[\sum_{p=0}^{\infty} b_p\left(\int_0^{\infty}[g(x;\varphi)]^v G(x,\varphi)^p dx\right)\Big], \tag{24}
$$

where

$$
\begin{aligned}
b_p &= \sum_{l,k,i,j,q=0}^{\infty}(-1)^{l+k+i+q+p}(4b\delta)^v\binom{-2v}{l}\binom{l}{k}\binom{-(2v+k)}{j} \\
&\times \binom{j}{i}\binom{b(k+i+v)-v}{q}\binom{2q+v}{p}\overline{\delta}^l.
\end{aligned}
$$

Note that, $\int_0^{\infty}[g(x;\varphi)]^v G(x,\varphi)^p dx$ can be obtained numerically. The Rényi entropy of the MO-TLHL-G family of distributions can be obtained directly from that of the exponentiated-G distribution as follows

$$
I_R(v) = \frac{1}{1-v}\log\left[\sum_{p=0}^{\infty}\Phi_p e^{(1-v)I_{REG}}\right], \tag{25}
$$

where

$$
\begin{aligned}
\Phi_p &= \sum_{l,k,i,j,q=0}^{\infty}(-1)^{l+k+i+q+p}(4b\delta)^v\binom{-2v}{l}\binom{l}{k}\binom{-(2v+k)}{j} \\
&\times \binom{j}{i}\binom{b(k+i+v)-v}{q}\binom{2q+v}{p}\Big(\frac{p}{v}+1\Big)\overline{\delta}^l
\end{aligned}
$$

and $I_{REG} = \int_0^\infty [(\frac{p}{v}+1)g(x;\varphi)G(x;\varphi)^{\frac{p}{v}}]^v dx$ is the Rényi entropy of the exponentiated-G distribution with power parameter $\frac{p}{v}$.

5. Maximum Likelihood Estimation

In this section, we present the maximum likelihood method of estimation for estimating the parameters of the MO-TLHL-G family of distributions. Let $X \sim MO-TLHL-G(b,\delta,\varphi)$ and $\mathbf{\Delta} = (b,\delta,\varphi)^T$ be the vector of model parameters. The log-likelihood function $\ell_n = \ell_n(\mathbf{\Delta})$ based on a random sample of size n from the MO-TLHL-G distribution is given by

$$\begin{aligned} \ell_n(\mathbf{\Delta}) &= nln(4b) + nln(\delta) + \sum_{i=1}^n ln(g(x_i;\varphi)) + \sum_{i=1}^n ln(\overline{G}(x_i;\varphi)) + (b-1) \\ &\times \sum_{i=1}^n ln(1-\overline{G}^2(x_i;\varphi)) - 2\sum_{i=1}^n ln(1+(1-(1-\overline{G}^2(x_i;\varphi))^b)) \\ &- 2\sum_{i=1}^n ln\Big(1-\overline{\delta}\Big(1-\Big[\frac{[1-\overline{G}^2(x;\varphi)]^b}{1+(1-[1-\overline{G}^2(x;\varphi)]^b)}\Big]\Big)\Big). \end{aligned} \quad (26)$$

The first derivative of the log-likelihood function with respect to each of component of the parameters $\mathbf{\Delta} = (b,\delta,\xi)^T$ are given by

$$\begin{aligned} \frac{\partial \ell_n(\mathbf{\Delta})}{\partial b} &= \frac{n}{b} + \sum_{i=1}^n ln(1-\overline{G}^2(x_i;\varphi)) + 2\sum_{i=1}^n \frac{(1-\overline{G}^2(x_i;\varphi))^b ln(1-\overline{G}^2(x_i;\varphi))}{1+(1-[1-\overline{G}^2(x;\varphi)]^b)} \\ &+ 4\overline{\delta}\sum_{i=1}^n \frac{(1-\overline{G}^2(x_i;\varphi))^b ln(1-\overline{G}^2(x_i;\varphi))}{(1+(1-[1-\overline{G}^2(x;\varphi)]^b))^2(1-\overline{\delta}\Big(1-\Big[\frac{[1-\overline{G}^2(x;\varphi)]^b}{1+(1-[1-\overline{G}^2(x;\varphi)]^b)}\Big]\Big))}, \\ \frac{\partial \ell_n(\mathbf{\Delta})}{\partial \delta} &= \frac{n}{\delta} - 2\sum_{i=1}^n \frac{\Big(1-\Big[\frac{[1-\overline{G}^2(x;\varphi)]^b}{1+(1-[1-\overline{G}^2(x;\varphi)]^b)}\Big]\Big)}{1-\overline{\delta}\Big(1-\Big[\frac{[1-\overline{G}^2(x;\varphi)]^b}{1+(1-[1-\overline{G}^2(x;\varphi)]^b)}\Big]\Big)} \end{aligned}$$

and

$$\begin{aligned} \frac{\partial \ell_n(\mathbf{\Delta})}{\partial \varphi_k} &= \sum_{i=1}^n \frac{g'(x_i;\varphi)}{g(x_i;\varphi)} + \sum_{i=1}^n \frac{\overline{G}'(x_i;\varphi)}{\overline{G}(x_i;\varphi)} + 2(b-1)\sum_{i=1}^n \frac{\overline{G}(x_i;\varphi)\overline{G}'(x_i;\varphi)}{1-\overline{G}^2(x_i;\varphi)} \\ &+ 2b\sum_{i=1}^n \frac{\overline{G}(x_i;\varphi)(1-\overline{G}^2(x_i;\varphi))^{b-1}\overline{G}'(x_i;\varphi)}{1+(1-[1-\overline{G}^2(x;\varphi)]^b)} \\ &- 8b\overline{\delta}\sum_{i=1}^n \frac{\overline{G}(x_i;\varphi)(1-\overline{G}^2(x_i;\varphi))^{b-1}\overline{G}'(x_i;\varphi)}{(1+(1-[1-\overline{G}^2(x;\varphi)]^b))^2\Big(1-\overline{\delta}\Big(1-\Big[\frac{[1-\overline{G}^2(x;\varphi)]^b}{1+(1-[1-\overline{G}^2(x;\varphi)]^b)}\Big]\Big)\Big)}, \end{aligned}$$

where $g'(x_i;\varphi) = \frac{\partial g(x_i;\varphi)}{\partial \varphi_k}$ and $\overline{G}'(x_i;\varphi) = \frac{\partial \overline{G}(x_i;\varphi)}{\partial \varphi_k}$.

The maximum likelihood estimates of the parameters, denoted by $\hat{\mathbf{\Delta}}$ is obtained by solving the non-linear equation $(\frac{\partial \ell_n}{\partial b}, \frac{\partial \ell_n}{\partial \delta}, \frac{\partial \ell_n}{\partial \varphi_k})^T = \mathbf{0}$, using a numerical method such as Newton-Raphson procedure. The multivariate normal distribution $N_{p+2}(\underline{0}, J(\hat{\mathbf{\Delta}})^{-1})$, where the mean vector $\underline{\mathbf{0}} = (0,0,\underline{0})^T$ and $J(\hat{\mathbf{\Delta}})^{-1}$ is the observed Fisher information matrix evaluated at $\hat{\mathbf{\Delta}}$, can be used to construct confidence intervals and confidence regions for the individual model parameters.

6. Simulation Study

The performance of the MO-TLHL-W distribution is examined by conducting various simulations for different sizes (n=35, 70, 140, 280, 560, 1120) via the R package. We simulate $N = 1000$ samples for the true parameters values given in Table 2 and 3. The table lists the mean MLEs of the model parameters along with the respective bias and root mean squared errors (RMSEs). The bias and RMSE for the estimated parameter, say, $\hat{\theta}$, say, are given by:

$$Bias(\hat{\theta}) = \frac{\sum_{i=1}^{N} \hat{\theta}_i}{N} - \theta, \quad \text{and} \quad RMSE(\hat{\theta}) = \sqrt{\frac{\sum_{i=1}^{N} (\hat{\theta}_i - \theta)^2}{N}},$$

respectively.

The results in Table 2 and 3 show that the mean estimates of the parameters are approaching the true parameter values as the sample size n increases. Also, the RMSEs and the bias are decaying toward zero with increasing sample size, which suggest strong consistency of the MLE of the model parameters.

Table 2. Simulation Results

			$(1.0, 0.02, 0.02, 2.5)$			$(1.1, 0.02, 0.7, 2.5)$	
parameter	Sample Size	Mean	RMSE	Bias	Mean	RMSE	Bias
b	35	1.6053	3.4577	0.5053	3.1300	9.3935	2.0300
	70	1.5432	2.7637	0.4432	1.9672	6.3920	0.8672
	140	1.3546	1.2978	0.2546	1.4503	1.6396	0.3503
	280	1.2463	1.4873	0.1463	1.2514	1.0963	0.1514
	560	1.1752	0.7361	0.0752	1.1512	0.4241	0.0512
	1120	1.1158	0.1775	0.0158	1.1308	0.3042	0.0308
δ	35	1.2490	8.4814	1.2290	0.3040	1.6737	0.2840
	70	0.4390	3.2529	0.4190	0.2851	2.7249	0.2651
	140	0.2196	1.5353	0.1996	0.1184	0.6326	0.0984
	280	0.0528	0.2760	0.0328	0.0536	0.2470	0.0336
	560	0.0359	0.1469	0.0159	0.0284	0.0368	0.0084
	1120	0.0248	0.0146	0.0048	0.0253	0.0233	0.0053
θ	35	0.3165	0.7680	0.2965	3.2153	13.7223	2.5153
	70	0.1306	0.4106	0.1106	1.5605	5.2816	0.8605
	140	0.0997	0.3245	0.0797	1.0748	1.1403	0.3748
	280	0.0516	0.1806	0.0316	0.8964	0.5911	0.1964
	560	0.0390	0.1278	0.0190	0.8177	0.3515	0.1177
	1120	0.0265	0.0370	0.0065	0.7888	0.2640	0.0888
γ	35	2.4130	0.9596	-0.0870	3.0604	2.5888	0.5604
	70	2.4991	0.6519	-0.0009	2.8395	1.8071	0.3395
	140	2.4208	0.4515	-0.0792	2.5923	1.1554	0.0923
	280	2.4532	0.2922	-0.0468	2.5751	0.9044	0.0751
	560	2.4592	0.2291	-0.0408	2.5024	0.4382	0.0024
	1120	2.4807	0.1365	-0.0193	2.4900	0.2929	-0.0100

Table 3. Simulation Results

			$(0.6, 0.03, 0.03, 2.3)$			$(0.6, 0.6, 1.2, 2.3)$	
parameter	Sample Size	Mean	RMSE	Bias	Mean	RMSE	Bias
b	35	0.7733	0.8171	0.1733	1.1633	6.6970	0.5633
	70	0.7947	0.9815	0.1947	0.6500	0.6908	0.0500
	140	0.8149	1.0174	0.2149	0.5904	0.3583	-0.0096
	280	0.7350	0.7175	0.1350	0.5826	0.2273	-0.0174
	560	0.7128	0.5948	0.1128	0.5915	0.1577	-0.0085
	1120	0.6438	0.2526	0.0438	0.5874	0.1092	-0.0126
δ	35	0.6950	3.6399	0.6650	4.1928	12.1237	3.5928
	70	0.3995	2.0308	0.3695	2.5914	7.4818	1.9914
	140	0.2219	0.9081	0.1919	1.4276	6.6402	0.8276
	280	0.0958	0.3359	0.0658	0.8436	1.1266	0.2436
	560	0.0583	0.1451	0.0283	0.6975	0.5683	0.0975
	1120	0.0397	0.0507	0.0097	0.6410	0.3183	0.0410
θ	35	0.2768	0.7176	0.2468	1.5077	1.0855	0.3077
	70	0.2013	0.5964	0.1713	1.3462	0.8986	0.1462
	140	0.1845	0.5469	0.1545	1.2007	0.6686	0.0007
	280	0.1217	0.3775	0.0917	1.1652	0.5233	-0.0348
	560	0.0961	0.2717	0.0661	1.1777	0.3828	-0.0223
	1120	0.0595	0.1397	0.0295	1.1793	0.2780	-0.0207
γ	35	2.4825	1.0870	0.1825	3.7828	3.3829	1.4828
	70	2.4440	0.9155	0.1440	3.3699	2.6276	1.0699
	140	2.3637	0.7804	0.0637	3.0936	1.9246	0.7936
	280	2.3118	0.6016	0.0118	2.7996	1.3650	0.4996
	560	2.2344	0.4810	-0.0656	2.5424	0.8203	0.2424
	1120	2.2589	0.3606	-0.0411	2.4501	0.5733	0.1501

7. Applications

In this section, we present examples to illustrate the flexibility and usefulness of the MO-TLHL-W distribution and its sub-models for data modeling.The MO-TLHL-W distribution is fitted to data sets and these fits are compared to the fits of the Odd Log-Logistic Exponentiated Weibull (OLLE-W) distribution (Afify et al. [1]), Kumaraswamy Weibull (KumW) dustribution (Cordeiro et al. [14]), Weibull Lomax (WL) distribution (Tahir et al. [29]), the Beta generalized Lindley (BGL) distribution (Oluyede and Yang [25]) and the Generalized Weibull Log-Logistic (GWLL) distribution (Cordeiro et al. [13]. The pdfs of the listed distributions are given below:

$$f_{OLLE-W}(x) = \frac{\theta\beta\gamma x^{\beta-1}e^{-(\frac{x}{\alpha})^{\beta}}[1-e^{-(\frac{x}{\alpha})^{\beta}}]^{\gamma\theta-1}\{1-[1-e^{-(\frac{x}{\alpha})^{\beta}}]^{\alpha}\}^{\theta-1}}{\alpha^{\beta}\{[1-e^{-(\frac{x}{\alpha})^{\beta}}]^{\gamma\theta}+\{1-[1-e^{-(\frac{x}{\alpha})^{\beta}}]^{\gamma}\}^{\theta}\}^{2}}, \tag{27}$$

for $\alpha, \beta, \theta, \gamma > 0$ and $x > 0$,

$$f_{Kum-W}(x) = ab\beta\alpha^{\beta}x^{\beta-1}e^{-(\alpha x)^{\beta}}(1-e^{-(\alpha x)^{\beta}})^{a-1}\{1-[1-e^{-(\alpha x)^{\beta}}]^{a}\}^{b-1}, \tag{28}$$

for $\alpha, \beta, a, b > 0$ and $x > 0$,

$$\begin{aligned} f_{WL}(x) &= \frac{ab\alpha}{\beta}\Big[1+\Big(\frac{x}{\beta}\Big)\Big]^{b\alpha-1}\Big\{1-\Big[1+\Big(\frac{x}{\beta}\Big)\Big]^{-\alpha}\Big\}^{b-1} \\ &\times \exp\Big\{-a\Big\{\Big[1+\Big(\frac{x}{\beta}\Big)\Big]^{\alpha}-1\Big\}\Big\}, \end{aligned} \tag{29}$$

for $\alpha, \beta, a, b > 0$ and $x > 0$,

$$\begin{aligned} f_{BGL}(x) &= \frac{\alpha\theta^{2}(1+x)e^{-\theta x}}{B(a,b)(1+\theta)}\Big[1-\frac{1+\theta+\theta x}{1+\theta}e^{-\theta x}\Big]^{a\alpha-1} \\ &\times \Big[1-\Big(1-\frac{1+\theta+\theta x}{1+\theta}e^{-\theta x}\Big)^{a}\Big]^{b-1}, \end{aligned} \tag{30}$$

for $\alpha, \theta, a, b > 0$ and $x > 0$, and

$$\begin{aligned} f_{GWLL}(x) &= \frac{\alpha\beta\gamma x^{\gamma-1}}{a^{\theta}}(1+\Big(\frac{x}{a}\Big)^{\gamma})^{-1}[\log(1+\Big(\frac{x}{a}\Big)^{\gamma})]^{\beta-1} \\ &\times \exp\{-\alpha[\log(1+\Big(\frac{x}{a}\Big)^{\gamma})]^{\beta}\}, \end{aligned} \tag{31}$$

for $\alpha, \beta, a, \gamma > 0$ and $x > 0$.

Plots of the fitted densities, the histogram of the data and probability plots (Chambers et al. [11]) are given in Figure 13 and Figure 15. For the probability plot, we plotted $F(x_{(j)}; \hat{b}, \hat{\delta}, \hat{\varphi})$ against $\frac{j-0.375}{n+0.25}$, $j = 1, 2, \cdots, n$, where $x_{(j)}$ are the ordered values of the observed data. The measure of closeness is given by the sum of squares

$$SS = \sum_{j=1}^{n}\Big[F(x_{(j)}; \hat{b}, \hat{\delta}, \hat{\varphi}) - \Big(\frac{j-0.375}{n+0.25}\Big)\Big]^{2}.$$

The goodness-of-fit statistics W^* and A^*, described by Chen and Balakrishnan [12] are also presented in the tables. These statistics can be used to verify which distribution fits better to the data. In general, the smaller the values of W^* and A^*, the better the fit.

The MLEs of the parameters with standard errors in parenthesis and the values of the goodness-of-fit statistics (-2log(L), AIC, AICC, BIC, W^*, A^*, K-S, p-value of the K-S statistic) and the SS are given in Tables 4 and 5.

7.1. Failure times of 50 components

The data set on the failure times of 50 components (per 1000 hours) were taken from Murthy et al. [24]. The data are: 0.036 0.148 0.590 3.076 6.816 0.058 0.183 0.618 3.147 7.896 0.061 0.192 0.645 3.625 7.904 0.074 0.254 0.961 3.704 8.022 0.078 0.262 1.228 3.931 9.337 0.086 0.379 1.600 4.073 10.940 0.102 0.381 2.006 4.393 11.020 0.103 0.538 2.054 4.534 13.880 0.114 0.570 2.804 4.893 14.73 0.116 0.574 3.058 6.274 15.08. The estimated parameters (standard error of estimates in parenthesis) and the goodness-of-fit statistics are presented in Table 4.

Table 4. Estimation of the MO-TLHL-W model for the failure times data set

			Estimates										
Distribution	b	δ	θ	γ	-2logL	AIC	CAIC	BIC	W^*	A^*	K−S	P−value	SS
MOTLHL-W	0.1731	0.1405	1.8835×10^{-05}	3.8215	200.2998	208.2998	209.1887	215.9479	0.1148	0.7461	0.1162	0.4739	0.1166
	(0.0164)	(0.0616)	(1.1390×10^{-05})	(0.0007)									
	α	θ	a	b									
BGL	0.1186	1.6897×10^{-07}	0.3015	10.1000	361.3658	369.5053	370.3942	377.1534	0.1592	0.9857	0.4576	4.3100×10^{-10}	3.2251
	(0.0077)	(2.5943×10^{-06})	(0.0007)	(2.2357×10^{-05})									
	α	β	θ	γ									
OLLE-W	44.9931	3.4902	2.2405	0.0575	205.8275	213.8275	214.7164	218.0930	0.1444	0.8940	0.1169	0.4662	0.1393
	(0.1415)	(4.4068)	(0.2619)	(0.0728)									
	a	b	c	λ									
Kum-W	3.8120	0.1245	0.5299	29.7109	202.4449	210.4449	211.3338	218.0930	0.1444	0.8940	0.1169	0.4662	0.1895
	(3.1860)	(0.0916)	(0.0981)	(28.9529)									
	a	b	α	β									
WL	0.0944	1.5271	0.3356	0.0129	202.6376	210.6376	211.5265	218.2857	0.1461	0.9038	0.1182	0.4523	0.1606
	(0.1692)	(1.1335)	(0.3456)	(0.0316)									
	α	β	a	γ									
GWLL	0.0064	9.9661	4.1023×10^{-05}	0.1320	204.4580	212.4579	213.3468	220.1060	0.1538	0.9582	0.1166	0.4701	0.1578
	(0.0364)	(0.0006)	(0.0002)	(0.0469)									

The histogram and fitted densities as well as the probability plots are given in Figure 13. From the values of the goodness-of-fit statistics, p-value of the K-S statistic and the plots in Figure 13, we can conclude that the MO-TLHL-W distribution provide a better fit compared to the other models in Table 4.

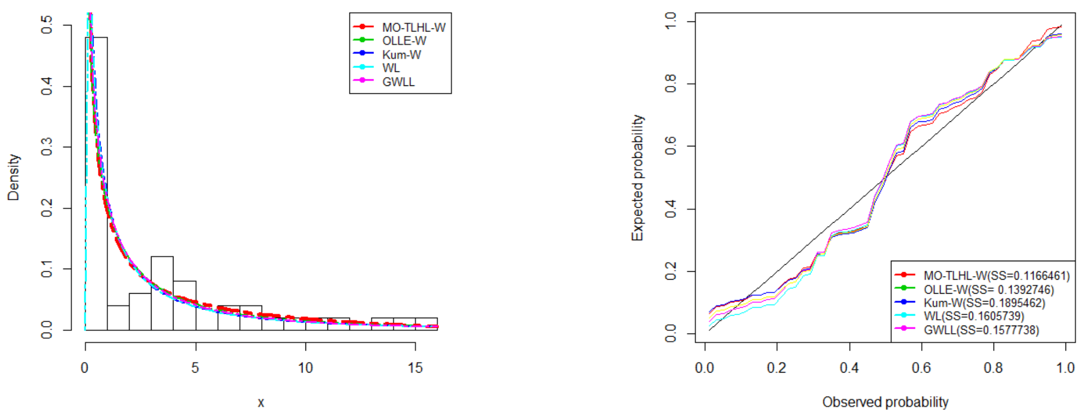

Figure 13. Fitted densities and probability plots for the failure times data set.

The estimated variance-covariance matrix for the failure times of 50 components data is given by,

$$\begin{bmatrix} 0.00027 & 0.00010 & 7.6689\times10^{-08} & 0.00001 \\ 0.00010 & 0.00004 & 2.8741\times10^{-08} & 4.7879\times10^{-06} \\ 7.6689\times10^{-08} & 2.8741\times10^{-08} & 1.2973\times10^{-10} & 3.6216\times10^{-09} \\ 0.00001 & 4.7879\times10^{-06} & 3.6216\times10^{-09} & 6.0331\times10^{-07} \end{bmatrix}$$

and the 95% asymptotic confidence intervals for the parameters b, δ, θ and γ are: 0.1731 $\pm$0.32144, 0.1405 $\pm$ 0.12074, $1.8835\times10^{-05} \pm 0.00002$ and 3.8215 ± 0.00137, respectively.

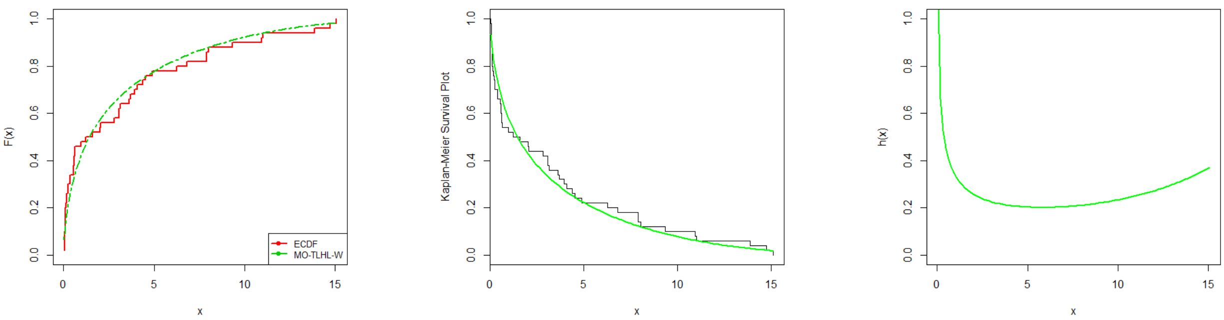

Figure 14. Estimated cdf, Kaplan-Meier survival plots and estimated hazard rate plot of the MO-TLHL-W distribution for the failure times data set.

7.2. Kevlar 49/Epoxy Strands Failure at 90% Stress Level

The data set represent the stress-rupture life of kevlar 49/epoxy strands which were subjected to constant sustained pressure at the 90% stress level until all had failed, so that we have complete data with exact times of failure, (see Andrews and Herzberg [6]; Barlow et al. [7] for additional details). The data are: 0.01 0.01 0.02 0.02 0.02 0.03 0.03 0.04 0.05 0.06 0.07 0.07 0.08 0.09 0.09 0.10 0.10 0.11 0.11 0.12 0.13 0.18 0.19 0.20 0.23 0.24 0.24 0.29 0.34 0.35 0.36 0.38 0.40 0.42 0.43 0.52 0.54 0.56 0.60 0.60 0.63 0.65 0.67 0.68 0.72 0.72 0.72 0.73 0.79 0.79 0.80 0.80 0.83 0.85 0.90 0.92 0.95 0.99 1.00 1.01 1.02 1.03 1.05 1.10 1.10 1.11 1.15 1.18 1.20 1.29 1.31 1.33 1.34 1.40 1.43 1.45 1.50 1.51 1.52 1.53 1.54 1.54 1.55 1.58 1.60 1.63 1.64 1.80 1.80 1.81 2.02 2.05 2.14 2.17 2.33 3.03 3.03 3.34 4.20 4.69 7.89.

Table 5. Estimation of the MO-TLHL-W model for the Kevlar 49/Epoxy Strands Failure at 90% Stress Level data set

			Estimates										
Distribution	b	δ	θ	γ	-2logL	AIC	CAIC	BIC	W^*	A^*	K−S	P−value	SS
MOTLHL-W	1.0922 (0.6325)	2.0489 (2.0543)	1.0613 (0.5921)	0.6613 (0.2560)	203.66	211.6600	212.0767	222.1205	0.1260	0.7663	0.0724	0.6657	0.1189
BGL	α 0.1233 (0.00559)	θ 1.6897×10^{-06} (2.8310×10^{-06})	a 0.3020 (0.0005)	b 10.100 (1.7373×10^{-05})	596.9422	604.9524	605.3691	615.4129	0.4136	2.2013	0.4450	2.2000×10^{-16}	7.0925
OLLE-W	α 1.6254 (2.6056)	β 1.1122 (0.4411)	θ 1.1688 (0.9425)	γ 0.6079 (0.8972)	205.5335	213.5335	213.9501	223.9939	0.1616	0.9432	0.0798	0.5407	0.1548
Kum-W	a 0.7546 (0.2978)	b 0.3338 (0.7513)	c 1.0338 (0.1645)	λ 2.5736 (6.0408)	205.2936	213.2936	213.7103	223.7541	0.1489	0.8846	0.0792	0.5505	0.1463
WL	a 0.2506 (0.4173)	b 0.7860 (0.1804)	α 1.3581 (0.4580)	β 0.3303 (0.6282)	205.1976	213.1976	213.6143	223.6581	0.1440	0.8627	0.0787	0.5587	0.1426
GWLL	α 9.8528×10^{06} (7.4924×10^{-07})	β 8.1611 (2.6690)	a 6.3927×10^{06} (9.5495×10^{-07})	γ 0.1213 (0.4427)	206.0739	214.0739	214.4906	224.5344	0.2059	1.1450	0.0918	0.3622	0.2019

The histogram and fitted densities as well as the probability plots are given in Figure 15. From the values of the goodness-of-fit statistics, p-value of the K-S statistic and the plots in Figure 15, we can conclude that the MO-TLHL-W distribution provide a better fit compared to the other models in Table 5.

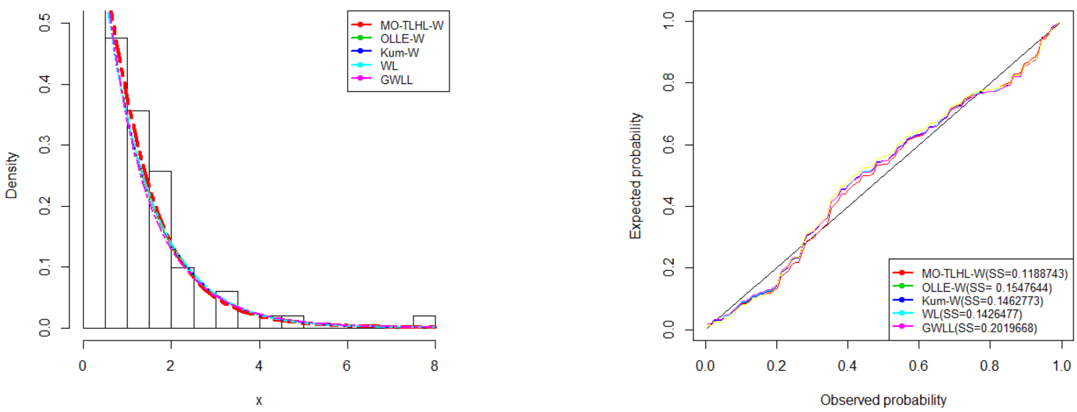

Figure 15. Fitted densities and probability plots for the Kevlar 49/Epoxy Strands Failure at 90% Stress Level data set.

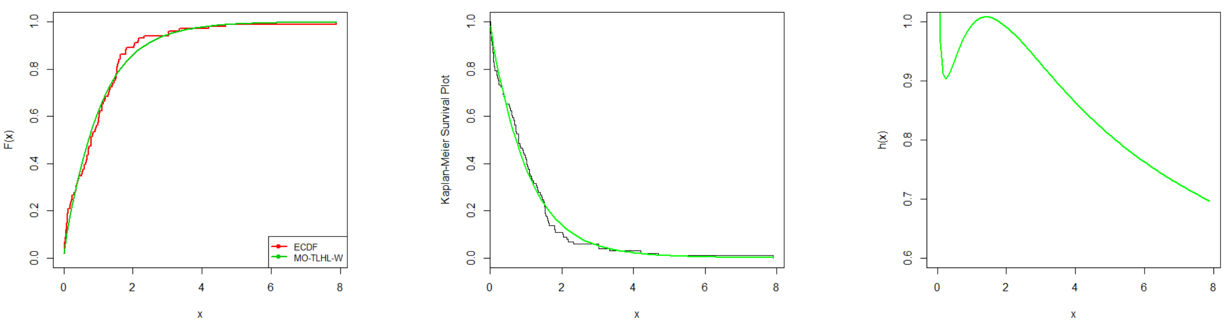

Figure 16. Estimated cdf, Kaplan-Meier survival plots and estimated hazard rate plot of the MO-TLHL-W distribution for the Kevlar 49/ Epoxy strands failure data set.

The estimated variance-covariance matrix for the Kevlar 49/ Epoxy strands failure at 90% stress level data is given by,

$$\begin{bmatrix} 0.40008 & 0.41146 & 0.27047 & -0.13001 \\ 0.41146 & 4.22002 & 1.06270 & -0.40941 \\ 0.27047 & 1.06270 & 0.35061 & -0.14638 \\ -0.13001 & -0.40941 & -0.14638 & 0.06555 \end{bmatrix}$$

and the 95% confidence intervals for the parameters b, δ, θ and γ are: 1.0922 ± 1.2397, 2.0489 ± 4.02643, 1.0613 ± 1.16052 and 0.6613 ± 0.50176, respectively.

8. Concluding Remarks

A new generalized family of distributions called the Marshall-Olkin Topp-Leone Half Logistic-G (MO-TLHL-G) distribution was developed and presented. The MO-TLHL-G distribution possesses hazard rate function with flexible behavior. We also obtain closed form expressions for the moments, incomplete moments, distribution of order statistics and Rényi entropy. The method of maximum likelihood estimation (MLE) was used to estimate the model parameters. The performance of the special case of the MO-TLHL-W distribution was examined by conducting various simulations for different sizes. Finally, the special case of the MO-TLHL-W distribution was fitted to real data sets to illustrate the applicability and usefulness of the proposed family of distributions.

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A. R Code: Define Functions

```
##Mashall Olkin Topp Leone Half Logistic-Weibull Distribution
#MONTE-CARLO SIMULATION STUDY
rm(list=ls())
library(stats4)
library(bbmle)
library(stats)
library(numDeriv)
library(rootSolve)
#define pdf MOTLHLW
MOTLHLW_pdf<-function(b,delta,theta,gamma,x)
{
4*b*delta*theta*gamma*x^(gamma -1)*exp(-2*theta*x^gamma)
*(1-exp(-2*theta*x^gamma))^(b-1)*(1+(1-(1-exp(-2*theta*x^gamma))^b))^(-2)
*(1-(1-delta)*(1-((1-exp(-2*theta*x^gamma))^b
/(1+(1-(1-exp(-2*theta*x^gamma))^b)))))^(-2)
}
##define cdf MOTLHLW
MOTLHLW_cdf=function(b,delta,theta,gamma,x){
y= ((1-exp(-2*theta*x^gamma))^b)*(1+(1-(1-exp(-2*theta*x^gamma))^b))
^(-1)*(1-(1-delta)*(1-((1-exp(-2*theta*x^gamma))^b/
(1+(1-(1-exp(-2*theta*x^gamma))^b)))))^(-1)

}

MOTLHLW_neglogl=function(b,delta,theta,gamma,x){
  -sum(log(MOTLHLW_pdf(b,delta,theta,gamma,x)))}

MOTLHLW_hazard=function(b,delta,theta,gamma,x){
  f=MOTLHLW_pdf(b,delta,theta,gamma,x)
  F=MOTLHLW_cdf(b,delta,theta,gamma,x)
  y=f/(1-F)
  return(y)}

## define MOTLHLW quantile
MOTLHLW_quantile =function(parameter,u){
  b=parameter[1]
  delta=parameter[2]
  theta=parameter[3]
  gamma=parameter[4]

  f=function(x){
    MOTLHLW_cdf(b,delta,theta,gamma,x)-u
  }
  x=min(uniroot.all(f,lower=0,upper=10,tol=1000))
  return(x)
}

MOTLHLW_QuantileTable=function(parameter_matrix){
```

```
  u=seq(0.1,0.9,0.1)
  size=dim(parameter_matrix)[1]
  Table_Quantile=matrix(NA, nrow=length(u), ncol=size)
  row.names(Table_Quantile)=u
  colnames(Table_Quantile)=apply(parameter_matrix,1,function(x)
  {paste0('(',paste0(x,collapse=','),")")})
  Table_Quantile
  for(iter in 1:size){
    parameter=parameter_matrix[iter,]
    for(i in 1:length(u)){
    Table_Quantile[i,iter]=MOTLHLW_quantile(parameter,u[i])
    }
  }
  return(Table_Quantile)
}
## table of quantile
parameter_matrix=as.matrix(rbind(
  par1=c(0.9,0.9,0.9,0.9),
  par2=c(2.0,1.5,0.9,0.5),
  par3=c(0.5,0.9,1.5,1.5),
  par4=c(0.9,0.8,1.8,0.5),
  par5=c(0.8,0.5,0.6,0.4)

))

MOTLHLW_QuantileTable(parameter_matrix)
print(parameter_matrix)
print(MOTLHLW_QuantileTable(parameter_matrix))

##define pdf MOTLHLW
MOTLHLW_pdf=function(b,delta,theta,gamma,x){
  y= 4*b*delta*theta*gamma*x^(gamma -1)*exp(-2*theta*x^gamma)
  *(1-exp(-2*theta*x^gamma))^(b-1)*(1+(1-(1-exp(-2*theta*x^gamma))^b))
  ^(-2)*(1-(1-delta)*(1-((1-exp(-2*theta*x^gamma))^b
  /(1+(1-(1-exp(-2*theta*x^gamma))^b)))))^(-2)
}

MOTLHLW_moments=function(b,delta,theta,gamma,n){
  f=function(b,delta,theta,gamma,n,x){
    (x^n)*(MOTLHLW_pdf(b,delta,theta,gamma,x))
  }
  y=integrate(f,lower=0,upper=1,subdivisions=10,b=b,delta=delta,
  theta=theta,gamma=gamma,n=n)
  return(y$value)
}

# Table of moments
parameter_matrix_Moments=matrix(c(0.9,0.9,0.9,0.9,2.0,1.5,
0.9,0.5,0.5,0.9,1.5,1.5,0.9,0.8,1.8,0.5,0.8,0.5,0.6,0.4)
,ncol=4,byrow=T) parameter_matrix_Moments
```

```
list_Moments=c('EX','EX2','EX3','EX4','EX5','EX6','SD','CV','CS','CK')
Table_Moments=matrix(0,nrow=length(list_Moments)
,ncol=dim(parameter_matrix_Moments)[1])
row.names(Table_Moments)=list_Moments
Table_Moments

for(iter in 1:dim(parameter_matrix_Moments)[1]){
  parameter=parameter_matrix_Moments[iter,]
  for(i in 1:10){
    Table_Moments[i,iter]=MOTLHLW_moments(parameter[1]
    ,parameter[2],parameter[3],parameter[4],i)
  }
}
print(Table_Moments)
###moments end

MOTLHLW_simulation=function(size=c(35, 70, 140, 280, 560, 1120),samp,par)
  {
  set.seed(2000)
  K=length(par)
  Mean=vector()
  RMSE=vector()
  Bias=vector()
  samplesize=as.vector(t(mapply(rep,size,K)))
  for(iter_size in 1:length(size)){
    coef1=matrix(NA,samp,K)
    colnames(coef1)=c('b','delta','theta','gamma')
    for(nsamp in 1:samp){
      tryCatch(
        {
  x1_MOTLHLW=NULL
  q=runif(size[iter_size],0,1)
  x1=sapply(q,MOTLHLW_quantile,parameter=par)
# MLE
x1_MOTLHLW=mle2(MOTLHLW_neglogl,start=list(b=par[1]
,delta=par[2],theta=par[3],gamma=par[4]),method="L-BFGS-B",
data=list(x=x1),lower=c(b=0,delta=0,theta=0,gamma=0),
upper=c(b=Inf,delta=Inf,theta=Inf,gamma=Inf),use.ginv=TRUE)
coef1[nsamp,]=coef(x1_MOTLHLW)
        },error=function(e){}
      )
    }
Mean[length(size)*(0:(K-1))+iter_size]=apply(coef1,2,mean,na.rm=TRUE)
RMSE[length(size)*(0:(K-1))+iter_size]=apply((coef1-matrix(rep(par,nsamp)
,ncol=K,byrow=T))^2,2,function(x){sqrt(mean(x,na.rm=TRUE))})
  }
  Bias=as.vector(sapply(1:K,function(x){Bias[(length(size)*(x-1)+1)
:(length(size)*x)]=Mean[(length(size)*(x-1)+1):(length(size)*x)]-par[x]}))
  return(cbind(samplesize,Mean,RMSE,Bias))
}
```

```
par=c(1.1, 0.02, 0.7, 2.5)
par_k=c(2.5)
for(i in 1:length(par)){
for(j in 1:length(par)){
for(l in 1:length(par)){
for(p in 1:length(par_k)){
print(c(par[i],par[j],par[l],par_k[p]))
print(MOTLHLW_simulation(samp=1000,par=c(par[i],par[j],par[l],par_k[p])))
      }
    }
  }
}
```