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Prediction of Censored Weibull Lifetimes in a Simple Step-Stress Plan With Khamis-Higgins Model

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Abstract In this paper, we discuss the prediction of the lifetimes to failure of censored units from Weibull distribution for a simple step-stress plan under Khamis-Higgins model. Different methods of prediction are considered including maximum likelihood predictor, modified maximum likelihood predictor, conditional median predictor, and best unbiased predictor. Another aspect of prediction is constructing prediction limits for future lifetimes of the censored units. The pivotal quantity, highest conditional density, and shortest-length based methods are discussed in this paper. Monte Carlo simulations are performed to compare all the prediction methods developed here and one real data set is analyzed for illustrative purposes.

Keywords Best unbiased predictor; Conditional median predictor; Highest conditional density; Khamis-Higgins model; Maximum likelihood predictor; Mean square prediction error; Prediction interval; Step-stress accelerated life tests.

AMS 2010 subject classifications 62G32, 62M20, 62N01

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1. Introduction

Accelerated Life Test (ALT) is commonly used to evaluate the lifetime of highly reliable products or components within a reasonable testing time. In ALT, the products or components are run at higher than usual levels of stress (including temperature, voltage, pressure, etc.) to obtain failures quickly. The data obtained from such an accelerated test are then transformed to estimate the distribution of failures under specified conditions. If a constant stress level is used and some selected stress levels are very low, there are many non-failed products or components during the testing time, which reduces the effectiveness of accelerated tests.

In this case, the step stress accelerated life test (SSALT) is used to overcome such problems. For further details, one may refer to Lawless [15] and Nelson [18]. In the SSALT, the stress-level in the model will be changed in steps at stages of experiment. Specifically, a test unit is subjected to a specified level of stress for a prefixed period of time. If it does not fail during that period of time, then the stress level is increased for future prefixed time. This process continues until the test units fail or some termination conditions are met. Simple SSALT contains only two levels of stress. To analyze the data under SSALT, there is more than one model that relates the lifetime's distribution under different stress levels to the lifetimes under the step stress test. The most popular one is the cumulative exposure model (CEM), which was proposed first by Seydyakin [20], and later by Nelson [17]. The model assumes that the remaining lifetime of the experiment units depends only on the cumulative exposure the units have experienced, with no memory on how this exposure was accumulated. Inferential aspects of step-stress model under Type-I and Type-II censoring schemes are addressed by Bai and Kim [3] and Kateri and Balakrishnan [13], respectively.

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Though CEM is the most popular model for exponential distribution, it is not the case for Weibull distribution, the reason is that Weibull CEM does not transform to exponential CEM under power transformation. An alternative model is Khamis and Higgins (K-H) model, which was proposed by Khamis and Higgins [14]. This model is based on a time-transformation of the exponential model. The K-H model is analytically more tractable than the CEM without sacrificing flexibility for fitting data. For this reason, the prediction problem is addressed in this article under K-H model. The Weibull K-H model has been discussed by many authors, Alhadeed and Yang [1] obtained optimal simple step-stress plan for K-H model. Ganguly et al. [9] presented Bayesian inference for Weibull K-H model under restricted and unrestricted priors. For further inferences on SSALT under censoring data, one may refer to Ismail [10, 11].

However, it may worth mentioning that no attention has been paid to the problem of prediction of future lifetimes of Weibull distribution under K-H model. Generally, the prediction problem has not been discussed extensively for step-stress model in the literature. Basak [5], Basak and Balakrishnan [6] and Basak and Balakrishnan [7] considered the problem of predicting the failure times of censored items for a simple step-stress model from exponential distribution with progressive Type-I censoring, progressive Type-II censoring and Type-II censoring, respectively. Most recently, Amleh and Raqab [2] discussed the prediction problem under SSALT for Lomax distribution when the data are Type-II censored.

In this paper, we consider the simple SSALT for the Weibull distribution based on Type-II censoring data, in which the experiment is terminated as soon as the r-th failure takes place. It is assumed that failures occur according to K-H model. Mainly, the paper is aimed at predicting future order statistics based on Type-II censored observations under simple step-stress with K-H model via point prediction as well as prediction intervals.

The rest of the paper is organized as follows. The K-H model, basic model assumptions, and maximum likelihood estimation of the unknown parameters based on the observed data are presented in Section 2. The Maximum likelihood predictor, modified maximum likelihood predictor, best unbiased predictor, and conditional median predictor are discussed in Section 3. In Section 4, we propose different methods for constructing prediction intervals of the censored lifetimes. Numerical simulation study has been performed to assess the effectiveness of the prediction procedures and a real data set is analyzed for illustration in Section 5. Finally, we conclude the paper in Section 6.

2. Model Assumption and Estimation Problem

In the simple step-stress model under Type-II censoring, the test is conducted as follows. All n units are initially put on the lower stress $S_1$and run until time τ. Then, the stress is changed to high level S_2, and the test continues until a pre-specified number of failures $r \leq n$ are observed. Let n_1 denotes the random number of failures before the stress change time τ, and $n_2 = r - n_1$denotes the number of failures after τ. If $n_1 = r$, then the test is terminated at the first step. Otherwise, the stress level is increased to the next step, and the test continues until required r failures.

It is further assumed that the lifetimes of the items being tested have a Weibull distribution

$$f(t, \alpha, \lambda) = \alpha\lambda t^{\alpha-1} e^{-\lambda t^{\alpha}}, t > 0, \alpha > 0, \lambda > 0, \tag{2.1}$$

and its respective cumulative distribution function (CDF):

$$F(t, \alpha, \lambda) = 1 - e^{-\lambda t^{\alpha}}, t > 0, \alpha > 0, \lambda > 0. \tag{2.2}$$

Here, $\alpha > 0, \lambda > 0$ are the shape and scale parameters, respectively. The Weibull distribution with the shape and scale parameters as α and λ will be denoted by $WE(\alpha, \lambda)$. The Weibull distribution is one of the most widely used lifetime distributions in reliability engineering that aims to plan maintenance, determine the life-cycle cost and prediction failures to determine warranty periods of products.

Basic Assumptions:

1. Units are tested at two stress levels $S_1 < S_2$;
2. The failure times of the units for any stress level follow Weibull distribution;

3. The scale parameters for the life distribution are $\lambda_i, i = 1, 2$, corresponding to stress level S_i,$i = 1, 2$;
4. α is independent of the stress level S_i,$i = 1, 2$;
5. Failures follow the K-H model that was described above.

Let us denote the ordered observed lifetimes by $\mathbf{t} = (t_{1:n}, \ldots, t_{n_1:n}, t_{n_1+1:n}, \ldots, t_{r:n})$ with the following cases:

$$\begin{cases} CaseI : \tau < t_{1:n} < \ldots < t_{r:n}; \\ CaseII : t_{1:n} < \ldots < t_{n_1:n} < \tau \leq t_{n_1+1:n} < t_{r:n}; \\ CaseIII : t_{1:n} < \ldots < t_{r:n} < \tau. \end{cases} \tag{2.3}$$

Here, the data vector $\mathbf{t}$ represents the observed values of the variable

$$T = (T_{1:n}, \ldots, T_{n_1:n}, T_{n_1+1:n}, \ldots, T_{r:n}),$$

which denotes the Type-II censored order statistics. Suppose the lifetimes CDFs at stress levels $S_1$and S_2 are $F_1$and F_2, respectively. The stress level is changed from $S_1$to S_2 at a prefixed time τ. Different models are available in the literature to relate the distributions of lifetimes under different stress levels. The most popular one is the CEM which assumes that the remaining lifetime of a unit depends only on the cumulative exposure accumulated at the current stress level, regardless of the previous accumulated exposure. By denoting the CDF of the lifetime under the step-stress pattern by $G(.)$, then $G(t) = F_1(t)$, for $0 \leq t < \tau$ and $G(t) = F_2(t - h)$, for $\tau \leq t < \infty$, where h is the solution of the equation

$$F_2(\tau - h) = F_1(\tau).$$

This model has been widely discussed in the literature, especially for exponential lifetimes; see for example, a review article by Balakrishnan [4]. Actually, the Weibull CEM does not transform to the exponential CEM under a power transformation. For this reason, an alternative proposed model to the Weibull CEM is the K-H model. The K-H model in step-stress accelerated life testing is based on a power time-transformation of the exponential model. Therefore, the distribution of the lifetimes has the CDF

$$G(t, \alpha, \lambda_1, \lambda_2) = \begin{cases} G_1(t) = 1 - e^{-\lambda_1 t^\alpha}, & 0 \leq t < \tau, \\ G_2(t) = 1 - e^{-\lambda_2 (t^\alpha - \tau^\alpha) - \lambda_1 \tau^\alpha}, & \tau \leq t < \infty, \end{cases} \tag{2.4}$$

and its corresponding PDF

$$g(t, \alpha, \lambda_1, \lambda_2) = \begin{cases} g_1(t) = \alpha \lambda_1 t^{\alpha-1} e^{-\lambda_1 t^\alpha}, & 0 \leq t < \tau, \\ g_2(t) = \alpha \lambda_2 t^{\alpha-1} e^{-\lambda_2 (t^\alpha - \tau^\alpha) - \lambda_1 \tau^\alpha}, & \tau \leq t < \infty. \end{cases} \tag{2.5}$$

For convenience, let us denote the parameters vector $\theta = (\alpha, \lambda_1, \lambda_2)$. It is clear to note that Case I and Case III in (2.3) are included in Case II by setting ($n_1 = 0$, $n_2 = r$) and ($n_1 = r$, $n_2 = 0$), respectively. Now, for Case II, where $0 < n_1 < r$, the likelihood function of the lifetimes can be written as:

$$L(\theta|data) = \alpha^r \lambda_1^{n_1} \lambda_2^{n_2} e^{-(\lambda_1 A_1 + \lambda_2 A_2)} \prod_{i=1}^{r} t_{i:n}^{\alpha-1}, \tag{2.6}$$

where

$$A_1 = A_1(\alpha, n_1, \tau) = \sum_{i=1}^{n_1} t_{i:n}^\alpha + (n - n_1)\tau^\alpha,$$

and

$$A_2 = A_2(\alpha, r, \tau) = \sum_{i=n_1+1}^{r} (t_{i:n}^\alpha - \tau^\alpha) + (n - r)(t_{r:n}^\alpha - \tau^\alpha),$$

or equivalently,

$$A_2 = \sum_{i=n_1+1}^{r} t_{i:n}^{\alpha} + (n-r)t_{r:n}^{\alpha} - (n-n_1)\tau^{\alpha}.$$

So, the corresponding log-likelihood function (l) can be written as

$$l(\theta|data) = r\ln\alpha + n_1\ln\lambda_1 + n_2\ln\lambda_2 - (\lambda_1 A_1 + \lambda_2 A_2) + (\alpha-1)\sum_{i=1}^{r}\ln t_{i:n}. \tag{2.7}$$

By differentiating (2.7) with respect to λ_1 and λ_2, we immediately obtain the maximum likelihood estimators (MLEs) of λ_1 and λ_2, respectively, as

$$\widehat{\lambda}_1 = \frac{n_1}{A_1}, \tag{2.8}$$

and

$$\widehat{\lambda}_2 = \frac{n_2}{A_2}. \tag{2.9}$$

Now, the MLE of $\alpha, \widehat{\alpha}$, can be obtained as a solution of the following equation:

$$\begin{aligned}\frac{\partial l}{\partial\alpha} = \varphi(\alpha) = \frac{r}{\alpha} + \sum_{i=1}^{r}\ln t_{i:n} - \lambda_1[\sum_{i=1}^{n_1} t_{i:n}^{\alpha}\ln t_{i:n} + (n-n_1)\tau^{\alpha}\ln\tau] \\ -\lambda_2[\sum_{i=n_1+1}^{r} t_{i:n}^{\alpha}\ln t_{i:n} + (n-r)t_{r:n}^{\alpha}\ln t_{r:n} - (n-n_1)\tau^{\alpha}\ln\tau] = 0.\end{aligned} \tag{2.10}$$

By plugging $\widehat{\lambda}_1$ and $\widehat{\lambda}_2$ into (2.10), $\widehat{\alpha}$ is obtained numerically by solving the following simplified equation:

$$\varphi(\alpha) = \frac{r}{\alpha} + \sum_{i=1}^{r}\ln t_{i:n} - n_1\frac{B_1}{A_1} - n_2\frac{B_2}{A_2} = 0, \tag{2.11}$$

where

$$B_1 = B_1(\alpha, n_1, \tau) = \sum_{i=1}^{n_1} t_{i:n}^{\alpha}\ln t_{i:n} + (n-n_1)\tau^{\alpha}\ln\tau,$$

and

$$B_2 = B_2(\alpha, r, \tau) = \sum_{i=n_1+1}^{r} t_{i:n}^{\alpha}\ln t_{i:n} + (n-r)t_{r:n}^{\alpha}\ln t_{r:n} - (n-n_1)\tau^{\alpha}\ln\tau.$$

In fact, the MLE $\widehat{\alpha}$ of α can be obtained as a fixed point solution of the following equation:

$$\alpha = h(\alpha), \tag{2.12}$$

where

$$h(\alpha) = \frac{r}{n_1\frac{B_1}{A_1} + n_2\frac{B_2}{A_2} - \sum\limits_{i=1}^{r}\ln t_{i:n}}.$$

The simple iterative technique $\alpha^{(j)} = h(\alpha)^{(j)}$, can be used to find a numerical solution of (2.12), where $\alpha^{(j)}$ is the value computed in the j-th iteration. The following theorem shows the existence and uniqueness of the MLEs of $\alpha, \lambda_1$and λ_2.

Theorem 1
Let $\{t_{i:n} : 1 \leq i \leq r\}$ be observed Weibull lifetimes to failure under simple SSALT with K-H model. Then, the MLEs of $\alpha, \lambda_1$and λ_2 are unique real values.

Proof
It suffices to show that the non-linear equation (2.12) has a unique positive real root. Once the MLE of α is shown to be a unique positive root, then the MLEs of $\lambda_1$and λ_2 should be unique since they can be obtained uniquely based on (2.8) and (2.9). Firstly, we show there is only one real root to the equation, $\varphi(\alpha) = 0$, where

$$\varphi(\alpha) = \frac{r}{\alpha} + \sum_{i=1}^{r} \ln t_{i:n} - n_1 C_1 - n_2 C_2 = 0,$$

with

$$C_1 = C_1(\alpha, n_1, \tau) = \frac{\sum\limits_{i=1}^{n_1} t_{i:n}^{\alpha} \ln t_{i:n} + (n - n_1)\tau^{\alpha} \ln \tau}{\sum\limits_{i=1}^{n_1} t_{i:n}^{\alpha} + (n - n_1)\tau^{\alpha}},$$

and

$$C_2 = C_2(\alpha, r, \tau) = \frac{\sum\limits_{i=n_1+1}^{r} t_{i:n}^{\alpha} \ln t_{i:n} + (n - r)t_{r:n}^{\alpha} \ln t_{r:n} - (n - n_1)\tau^{\alpha} \ln \tau}{\sum\limits_{i=n_1+1}^{r} t_{i:n}^{\alpha} + (n - r)t_{r:n}^{\alpha} - (n - n_1)\tau^{\alpha}}.$$

It is clear that C_1 tends to be a finite number and C_2 tends to ∞ as α approaches 0. By applying l'Hospital's rule, we readily have

$$\begin{aligned}\lim_{\alpha \to 0}\left(\tfrac{r}{\alpha} - n_2 C_2\right) &= \lim_{\alpha \to 0}\left(\tfrac{rA_2 - n_2 \alpha B_2}{\alpha A_2}\right)\\ &= \infty,\end{aligned}$$

which implies that $\varphi(\alpha)$ moves to ∞ when α gets 0. On using the facts $\tau \geq t_{i:n}, i = 1, 2, ..., n_1$ and that $t_{r:n} \geq t_{i:n} > \tau, i = n_1 + 1, ..., r$, it follows that:

$$\lim_{\alpha \to \infty} C_1(\alpha, n_1, \tau) = \ln \tau, \text{ and } \lim_{\alpha \to \infty} C_2(\alpha, n_1, \tau) = \ln t_{r:n}.$$

This turns out that $\varphi(\alpha)$ reaches to a negative real value as α moves away to ∞. That is,

$$\lim_{\alpha \to \infty} \varphi(\alpha) = \sum_{i=1}^{r} \ln t_{i:n} - n_1 \ln \tau - n_2 \ln t_{r:n} < 0.$$

Since

$$\begin{aligned}\varphi'(\alpha) = &-\frac{r}{\alpha^2} - \lambda_1 \Big[\sum_{i=1}^{n_1} t_{i:n}^{\alpha} (\ln t_{i:n})^2 + (n - n_1)\tau^{\alpha} (\ln \tau)^2\Big] \\ &- \lambda_2 \Big[\sum_{i=n_1+1}^{r} t_{i:n}^{\alpha} (\ln t_{i:n})^2 + (n - r)t_{r:n}^{\alpha} (\ln t_{r:n})^2 - (n - n_1)\tau^{\alpha} (\ln \tau)^2\Big] < 0,\end{aligned}$$

we conclude that $\varphi(\alpha)$ is a continuous function on $(0, \infty)$ and it is decreasing monotonically from ∞ to negative values. The required result then follows. □

3. Prediction for Simple Step-Stress Model

Suppose a sample of n experimental units is placed on a simple step-stress life test. As described in the previous section, the test continues until required r failures. Let us consider Case II in (2.3) where $0 < n_1 < r$. Our purpose is to discuss how to predict the unobserved value of $Y = T_{s:n}, s = r+1, ..., n$, of all $(n-r)$ censored units based on the observed data $\mathbf{t} = (t_{1:n}, \dots, t_{n_1:n}, t_{n_1+1:n}, \dots, t_{r:n})$.
Using the Markovian property of censored-order statistics, it is well-known that the conditional distribution of $Y = T_{s:n}$ given $\mathbf{T} = \mathbf{t}$ is just the distribution of $Y = T_{s:n}$ given $T_{r:n} = t_{r:n}$. This implies that the density of Y given $\mathbf{T} = \mathbf{t}$ is the same as the density of the $(s-r)-th$ order statistic out of $(n-r)$ units from the population with density $\frac{g(y)}{1-G(t_{r:n})}$ (left truncated density at $t_{r:n}$, where $G(y)$ is given in Eq. (2.4). Precisely,

$$\begin{aligned} g_{T_{s:n}|\mathbf{T}}(y|\theta, data) = &\frac{(n-r)!}{(s-r-1)!(n-s)!}\alpha\lambda_2 y^{\alpha-1}(1-e^{-\lambda_2(y^\alpha - t^\alpha_{r:n})})^{s-r-1} \\ &\times e^{-\lambda_2(n-s+1)(y^\alpha - t^\alpha_{r:n})}, y > t_{r:n}. \end{aligned} \tag{3.1}$$

3.1. Maximum Likelihood Predictor

The maximum likelihood predictor (MLP) was proposed by Kaminsky and Rhodin [12]. This method involves prediction of future order statistics and also estimation of the parameters in the model. The predictive likelihood function (PLF) of $Y = T_{s:n}$ is given by

$$\begin{aligned} L(y, \theta|\mathbf{t}) = L &= g_{T_{s:n}|\mathbf{T}}(y|\mathbf{t}, \theta).g_{\mathbf{T}}(\mathbf{t}, \theta) \\ &= g_{T_{s:n}|T_{r:n}}(y|t_{r:n}, \theta).g_{\mathbf{T}}(\mathbf{t}, \theta), \end{aligned} \tag{3.2}$$

where $g_{T_{s:n}|T_{r:n}}(y|t_{r:n}, \theta)$ is the conditional density of $Y = T_{s:n}$ given the observed value of $\mathbf{T} = \mathbf{t}$, as in Eq. (3.1), and $g_{\mathbf{T}}(\mathbf{t}, \theta)$ is the density of $\mathbf{T}$. Therefore, Eq. (3.2) can be written as

$$\begin{aligned} L \propto &\prod_{i=1}^{n_1} g_1(t_{i:n}) \prod_{i=n_1+1}^{r} g_2(t_{i:n})[G_2(y) - G_2(t_{r:n})]^{s-r-1} g_2(y)[1-G_2(y)]^{n-s}, \\ &0 \le n_1 \le r, r+1 \le s \le n. \end{aligned} \tag{3.3}$$

Here, if we take $n_1 = 0$, we get Case I in (2.3), and if we take $n_1 = r$, we get Case III, so considering Case II in (2.3), i.e. $1 \le n_1 < r \le n$, we obtain

$$\begin{aligned} L \propto &\lambda_2 \alpha y^{\alpha-1} e^{-\lambda_2(y^\alpha - \tau^\alpha) - \lambda_1 \tau^\alpha}.e^{-(n-s)[\lambda_2(y^\alpha-\tau^\alpha)+\lambda_1\tau^\alpha]} \prod_{i=1}^{n_1} \alpha\lambda_1 t_{i:n}^{\alpha-1} e^{-\lambda_1 t^\alpha_{i:n}} \\ \times &\prod_{i=n_1+1}^{r} \alpha\lambda_2 t_{i:n}^{\alpha-1} e^{-\lambda_2(t^\alpha_{i:n}-\tau^\alpha)-\lambda_1\tau^\alpha} [e^{-\lambda_2(t^\alpha_{i:n}-\tau^\alpha)-\lambda_1\tau^\alpha} - e^{-\lambda_2(y^\alpha-\tau^\alpha)-\lambda_1\tau^\alpha}]^{s-r-1}, \end{aligned} \tag{3.4}$$

which can be simplified to:

$$\begin{aligned} L \propto &(y \prod_{i=1}^{r} t_{i:n})^{\alpha-1} \lambda_1^{n_1} \lambda_2^{n_2+1} \alpha^{r+1} e^{-\lambda_1[\sum_{i=1}^{n_1} t^\alpha_{i:n} + (n-n_1)\tau^\alpha]}.[e^{-\lambda_2(t^\alpha_{i:n}-\tau^\alpha)} - e^{-\lambda_2(y^\alpha-\tau^\alpha)}]^{s-r-1} \\ \times &e^{-\lambda_2[\sum_{i=n_1+1}^{r}(t^\alpha_{i:n}-\tau^\alpha)+(n-s+1)(y^\alpha-\tau^\alpha)]}. \end{aligned} \tag{3.5}$$

Consequently, the log PLF is expressed as

$$\begin{aligned}\ln L \propto &(r+1)\ln\alpha + n_1\ln\lambda_1 + (n_2+1)\ln\lambda_2 + (\alpha-1)(\ln y + \sum_{i=1}^{r}\ln t_{i:n}) \\ &- \lambda_1[\sum_{i=1}^{n_1} t_{i:n}^{\alpha} + (n-n_1)\tau^{\alpha}] - \lambda_2[\sum_{i=n_1+1}^{r}(t_{i:n}^{\alpha}-\tau^{\alpha}) + (n-s+1)(y^{\alpha}-\tau^{\alpha})] \\ &+ (s-r-1)\ln[e^{-\lambda_2(t_{r:n}^{\alpha}-\tau^{\alpha})} - e^{-\lambda_2(y^{\alpha}-\tau^{\alpha})}].\end{aligned} \tag{3.6}$$

By (3.6), the predictive likelihood equations (PLEs) for y, λ_1, λ_2 and α, are obtained and presented as follows:

$$\frac{\partial \ln L}{\partial y} = \frac{\alpha-1}{y} - \lambda_2(n-s+1)\alpha y^{\alpha-1} + \frac{(s-r-1)\lambda_2\alpha y^{\alpha-1}e^{-\lambda_2(y^{\alpha}-\tau^{\alpha})}}{e^{-\lambda_2(t_{r:n}^{\alpha}-\tau^{\alpha})} - e^{-\lambda_2(y^{\alpha}-\tau^{\alpha})}} = 0, \tag{3.7}$$

$$\frac{\partial \ln L}{\partial \lambda_1} = \frac{n_1}{\lambda_1} - [\sum_{i=1}^{n_1} t_{i:n}^{\alpha} + (n-n_1)\tau^{\alpha}] = 0, \tag{3.8}$$

$$\begin{aligned}\frac{\partial \ln L}{\partial \lambda_2} = &\frac{n_2+1}{\lambda_2} - [\sum_{i=n_1+1}^{r}(t_{i:n}^{\alpha}-\tau^{\alpha}) + (n-s+1)(y^{\alpha}-\tau^{\alpha})] \\ &+ (s-r-1)\frac{[(y^{\alpha}-\tau^{\alpha})e^{-\lambda_2(y^{\alpha}-\tau^{\alpha})} - (t_{r:n}^{\alpha}-\tau^{\alpha})e^{-\lambda_2(t_{r:n}^{\alpha}-\tau^{\alpha})}]}{e^{-\lambda_2(t_{r:n}^{\alpha}-\tau^{\alpha})} - e^{-\lambda_2(y^{\alpha}-\tau^{\alpha})}} = 0,\end{aligned} \tag{3.9}$$

and

$$\begin{aligned}\frac{\partial \ln L}{\partial \alpha} = &\frac{r+1}{\alpha} + \ln y + \sum_{i=1}^{r}\ln t_{i:n} - \lambda_1\left[\sum_{i=1}^{n_1} t_{i:n}^{\alpha}\ln t_{i:n} + (n-n_1)\tau^{\alpha}\ln\tau\right] \\ &- \lambda_2\left[\sum_{i=n_1+1}^{r}(t_{i:n}^{\alpha}\ln t_{i:n} - \tau^{\alpha}\ln\tau) + (n-s+1)(y^{\alpha}\ln y - \tau^{\alpha}\ln\tau)\right] \\ &+ (s-r-1)\lambda_2\frac{e^{-\lambda_2(y^{\alpha}-\tau^{\alpha})}(y^{\alpha}\ln y - \tau^{\alpha}\ln\tau) - e^{-\lambda_2(t_{r:n}^{\alpha}-\tau^{\alpha})}(t_{r:n}^{\alpha}\ln t_{r:n} - \tau^{\alpha}\ln\tau)}{e^{-\lambda_2(t_{r:n}^{\alpha}-\tau^{\alpha})} - e^{-\lambda_2(y^{\alpha}-\tau^{\alpha})}} = 0.\end{aligned} \tag{3.10}$$

The predictive maximum likelihood estimator (PMLE) of λ_1, $\widetilde{\lambda}_1$, is obtained immediately from Eq. (3.8), and it is given by

$$\widetilde{\lambda}_1 = \frac{n_1}{\sum_{i=1}^{n_1} t_{i:n}^{\alpha} + (n-n_1)\tau^{\alpha}}. \tag{3.11}$$

Eq. (3.9) can be rewritten as follows:

$$\begin{aligned}\frac{n_2+1}{\lambda_2} - &\left[\sum_{i=n_1+1}^{r}(t_{i:n}^{\alpha}-\tau^{\alpha}) + (n-s+1)(y^{\alpha}-\tau^{\alpha})\right] \\ &+ (s-r-1)\left[\frac{(y^{\alpha}-\tau^{\alpha})}{e^{-\lambda_2(t_{r:n}^{\alpha}-y^{\alpha})}-1} - \frac{(t_{r:n}^{\alpha}-\tau^{\alpha})}{1-e^{-\lambda_2(y^{\alpha}-t_{r:n}^{\alpha})}}\right] = 0.\end{aligned} \tag{3.12}$$

Now, it follows from Eq. (3.7), we have

$$e^{-\lambda_2(t_{r:n}^{\alpha}-y^{\alpha})} - 1 = \frac{1}{\frac{n-s+1}{s-r-1} - \frac{\alpha-1}{\alpha\lambda_2(s-r-1)y^{\alpha}}}, \tag{3.13}$$

and

$$1-e^{-\lambda_2(y^{\alpha}-t^{\alpha}_{r:n})}=\frac{1}{\frac{n-s+1}{s-r-1}-\frac{\alpha-1}{\alpha\lambda_2(s-r-1)y^{\alpha}}+1}. \tag{3.14}$$

The PMLE of λ_2, $\widetilde{\lambda}_2$, can be obtained by substituting Eq.'s (3.13) and (3.14) into (3.12). That is,

$$\widetilde{\lambda}_2=\frac{n_2+1-(1-\frac{1}{\alpha})\left[1-\left(\frac{t_{r:n}}{y}\right)^{\alpha}\right]}{\sum\limits_{i=n_1+1}^{r}(t^{\alpha}_{i:n}-\tau^{\alpha})+(n-r)(t^{\alpha}_{r:n}-\tau^{\alpha})}. \tag{3.15}$$

Using $\widetilde{\lambda}_1$ and $\widetilde{\lambda}_2$, Eq.'s (3.7) and (3.10) can be solved simultaneously with respect to y and α. Consequently, we obtain the MLP of Y, $\hat{Y}_M$, and the PMLE of α, $\widetilde{\alpha}$. Numerical methods can be used to solve these simultaneous equations.

3.2. Modified MLP

The modified maximum likelihood predictor (MMLP) can be obtained based on 2-stage procedure. First, we obtain the MLEs of $\alpha, \lambda_1$and λ_2 based on the observed data discussed in Section 2. In the second stage, we substitute these MLEs, $\widehat{\alpha}, \widehat{\lambda}_2$ of α and λ_2 into (3.7). As a result of that the MMLP of $Y=T_{s:n}$ can be obtained by solving Eq. (3.7) for Y, which can be written as the following simplified equation:

$$e^{-\widehat{\lambda}_2(y^{\widehat{\alpha}}-t^{\widehat{\alpha}}_{r:n})}=\frac{(n-r)\widehat{\lambda}_2y^{\widehat{\alpha}}-(1-\frac{1}{\widehat{\alpha}})}{(n-s+1)\widehat{\lambda}_2y^{\widehat{\alpha}}-(1-\frac{1}{\widehat{\alpha}})}, y>t_{r:n}. \tag{3.16}$$

Since Eq. (3.16) can't be solved analytically, a numerical method is needed to compute the MMLP of Y, say, $\hat{Y}_{MML}$. For the special case when $s=r+1$, it can be easily checked that $\hat{Y}_{MML}=t_{r:n}$. The existence and uniqueness of the solution of (3.16) are shown in Theorem 2 below.

Theorem 2
Let $\{t_{i:n}:1\le i\le r\}$ be observed Weibull lifetimes to failure under simple SSALT with K-H model. Then, the MMLP of $Y=T_{s:n}$ is unique real value.

Proof
To show the result, we need to check the variabilities of the functions $h_1(y)$ and $h_2(y)$, where $h_1(y)$ and $h_2(y)$ are the left hand and right hand sides of Eq. (3.16), respectively. That is,

$$h_1(y)=\frac{ay^{\widehat{\alpha}}-b}{cy^{\widehat{\alpha}}-b},$$

and

$$h_2(y)=e^{\lambda_2(y^{\widehat{\alpha}}-t^{\widehat{\alpha}}_{r:n})}, \tag{3.17}$$

where $a=(n-r)\ \widehat{\lambda}_2$,$b=1-\frac{1}{\widehat{\alpha}}$ and $c=(n-s+1)\ \widehat{\lambda}_2$, with $a>c$. Thus, we can obtain the derivative of $h_1(y)$ as follows:

$$h_1'(y)=\frac{\widehat{\alpha}b(c-a)y^{\widehat{\alpha}-1}}{(cy^{\widehat{\alpha}}-b)^2}. \tag{3.18}$$

It follows from Eq. (3.18) that $h_1(y)$ is increasing, decreasing and constant for $\widehat{\alpha}<1, \widehat{\alpha}>1, \widehat{\alpha}=1$, respectively. It can be easily noticed that $h_1(y)$ starts from $h_1(t_{r:n})=\frac{at^{\widehat{\alpha}}_{r:n}-b}{ct^{\widehat{\alpha}}_{r:n}-b}>1$ and reaches $\frac{a}{c}>1$, when y moves away to ∞. Note that if $\widehat{\alpha} < 1$ $(b < 0)$, $1 < h_1(t_{r:n}) < \frac{a}{c}$ and if $\widehat{\alpha} > 1$ $(b > 0)$, $1 < \frac{a}{c} < h_1(t_{r:n})$. For the case when $\widehat{\alpha} = 1$ $(b = 0)$, $h_1(t_{r:n}) = \frac{a}{c}$. It is evident that $h_2(y)$ is non-decreasing function and going faster than $h_1(y)$ as y approaches ∞, with $h_2(t_{r:n}) = 1$ and $h_2(\infty) = \infty$. This shows that the difference between the two functions, $h_1(y) - h_2(y)$, intersects the original horizontal line $y = 0$, in exactly one point. Therefore, the required result follows. □

3.3. Best Unbiased Predictor

A predictor $\hat{Y}$ of $Y = T_{s:n}$ is called a best unbiased predictor (BUP) of Y, if the prediction error $\hat{Y} - Y$ has a mean zero and its prediction error variance, $Var(\hat{Y} - Y)$ is less than or equal to that of any other unbiased predictor of Y. Based on the conditional density of Y given $\mathbf{T} = \mathbf{t}$, as in Eq. (3.1), the BUP of Y is given by

$$\hat{Y}_{BUP} = E(Y|\mathbf{T}) = \int_{t_{r:n}}^{\infty} y g_{T_{s:n}|\mathbf{T}}(y|\alpha, \lambda_1, \lambda_2, data)dy. \tag{3.19}$$

Using the binomial expansion:

$$\left[1 - e^{-\lambda_2(y^\alpha - t_{r:n}^\alpha)}\right]^{s-r-1} = \sum_{k=0}^{s-r-1} \binom{s-r-1}{k} (-1)^{s-r-1-k} \, e^{-(s-r-k-1)\lambda_2(y^\alpha - t_{r:n}^\alpha)},$$

we obtain

$$\begin{aligned}\hat{Y}_{BUP} &= (s-r)\binom{n-r}{s-r}\lambda_2^{-\frac{1}{\alpha}} \\ &\times \sum_{k=0}^{s-r-1} \frac{\binom{s-r-1}{k}(-1)^{s-r-1-k} . e^{-(n-r-k-1)\lambda_2 t_{r:n}^\alpha} . \gamma(\frac{1}{\alpha}+1; (n-r-k)\lambda_2 t_{r:n}^\alpha)}{(n-r-k)^{\frac{1}{\alpha}+1}},\end{aligned} \tag{3.20}$$

where

$$\gamma(a;t) = \int_t^\infty u^{a-1}e^{-u}du,$$

is the incomplete gamma function for $a > 0$. Since the parameters α and λ_2 are unknown, the BUP of Y can be approximated by replacing α and λ_2 by their corresponding MLEs.

3.4. Conditional Median Predictor

The conditional median predictor (CMP) was first suggested by Raqab and Nagaraja [19]. A predictor $\hat{Y}$ is called the CMP of Y, if it is the median of the conditional distribution of Y given $\mathbf{T} = \mathbf{t}$, that is

$$P_\theta(Y \le \hat{Y}|\mathbf{T} = \mathbf{t}) = P_\theta(Y \ge \hat{Y}|\mathbf{T} = \mathbf{t}). \tag{3.21}$$

Based on the conditional distribution of Y given $\mathbf{T} = \mathbf{t}$, we can obtain

$$P_\theta(Y \le \hat{Y}|\mathbf{T} = \mathbf{t}) = P_\theta(1 - e^{-\lambda_2(y^\alpha - t_{r:n}^\alpha)} \ge 1 - e^{-\lambda_2(\hat{Y}^\alpha - t_{r:n}^\alpha)}|\mathbf{T} = \mathbf{t}). \tag{3.22}$$

It can be shown that, given $\mathbf{T} = \mathbf{t}$, the distribution of $1 - e^{-\lambda_2(y^\alpha - t_{r:n}^\alpha)}$ is a Beta distribution with parameters $s - r$ and $n - s + 1$, denoted by $Beta(s - r, n - s + 1)$. So, we can obtain the CMP of Y as

$$\hat{Y}_{CMP} = \left[t_{r:n}^\alpha - \frac{1}{\lambda_2}\ln(1 - Med_B)\right]^{\frac{1}{\alpha}}, \tag{3.23}$$

where B is a random $Beta(s - r, n - s + 1)$, and Med_B represents the median of B. We compute an approximate CMP of Y by replacing α and λ_2 by their corresponding MLEs.

4. Prediction Intervals

Another aspect of prediction problem is to predict the future censored lifetimes by establishing prediction intervals (PIs) for $Y = T_{s:n}$,$s = r + 1, \ldots, n$ based on the Type II censored sample $\mathbf{T} = (T_{1:n}, T_{2:n}, \ldots, T_{r:n})$.

4.1. Pivotal Method

Let us consider the random variable

$$Z = 1 - e^{-\lambda_2(y^\alpha - t^\alpha_{r:n})}, Y > t_{r:n}. \tag{4.1}$$

The distribution of Z given $\mathbf{T} = \mathbf{t}$, is a Beta distribution with parameters $s - r$ and $n - s + 1$. So, Z can be considered as a pivotal quantity to obtain the PI of Y. Taking $(1 - \gamma)$ as prediction coefficient and using (4.1), we obtain

$$P(B_{\frac{\gamma}{2}} < Z < B_{1-\frac{1}{\gamma}}) = 1 - \gamma,$$

where B_γ is the $100\gamma - th$ percentile of the distribution $Beta(s - r, n - s + 1)$. Therefore, a $(1 - \gamma)100\%$ PI of Y is $(L_1(\mathbf{T}), U_1(\mathbf{T}))$, where

$$L_1(\mathbf{T}) = \left[t^\alpha_{r:n} - \frac{1}{\lambda_2}\ln\left(1 - B_{\frac{\gamma}{2}}\right)\right]^{\frac{1}{\alpha}},$$

$$U_1(\mathbf{T}) = \left[t^\alpha_{r:n} - \frac{1}{\lambda_2}\ln\left(1 - B_{1-\frac{\gamma}{2}}\right)\right]^{\frac{1}{\alpha}}. \tag{4.2}$$

Since α and λ_2 are unknown, the MLEs of the parameters can be used to obtain approximation of the prediction limits, $L_1(\mathbf{T})$ and $U_1(\mathbf{T})$.

4.2. Highest Conditional Density Method

Here we consider the conditional distribution of $Z = 1 - e^{-\lambda_2(y^\alpha - t^\alpha_{r:n})}$ given $\mathbf{T} = \mathbf{t}$. Its density takes the form

$$g(z|\mathbf{t}, \theta) = \frac{(n-r)!}{(s-r-1)!(n-s)!} z^{s-r-1}(1-z)^{n-s}, 0 < z < 1. \tag{4.3}$$

The density in (4.3) is unimodal function. An interval (d_1, d_2) is called highest conditional density (HCD) PI of content $1 - \gamma$ if $(d_1, d_2) = \{d : d \in [0, 1], f(d) \geq k\} \subseteq [0, 1]$, where

$$\int_{d_1}^{d_2} f(u)du = 1 - \gamma,$$

for some $k > 0$. Now, if $r + 1 < s < n$, then $g(z|\mathbf{t}, \theta)$ is a unimodal function in z, and it attains its maximum value at $\delta = \frac{(s-r-1)}{n-r-1} \in (0, 1)$. So, the HCD PI can be obtained by finding two points $d_1 = 100(\frac{\gamma}{2}) - th$ percentile, and $d_1 = 100(1 - \frac{\gamma}{2}) - th$ percentile, with $d_1 \leq \delta \leq d_2$, satisfying

$$\int_{d_1}^{d_2} g(z|\mathbf{t}, \theta)dz = 1 - \gamma, \tag{4.4}$$

and

$$g(d_1|\mathbf{t}, \theta) = g(d_2|\mathbf{t}, \theta), \tag{4.5}$$

(see Casella and Berger [8], 441- 442). Eq.'s (4.4) and (4.5) can be simplified as

$$B_{d_2}(s-r,n-s+1) - B_{d_1}(s-r,n-s+1) = 1-\gamma, \tag{4.6}$$

and

$$\left(\frac{1-d_2}{1-d_1}\right)^{n-s} = \left(\frac{d_1}{d_2}\right)^{s-r-1}, \tag{4.7}$$

where, $B_\nu(a,b)$ is the incomplete beta function

$$B_\nu(a,b) = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \int_0^\nu u^{a-1}(1-u)^{b-1}du.$$

Consequently, a $(1-\gamma)100\%$ HCD PI of Y is given by $(L_2(\mathbf{T}), U_2(\mathbf{T}))$, with

$$L_2(\mathbf{T}) = \left[t_{r:n}^\alpha - \frac{1}{\lambda_2}\ln(1-d_1)\right]^{\frac{1}{\alpha}},$$
$$U_2(\mathbf{T}) = \left[t_{r:n}^\alpha - \frac{1}{\lambda_2}\ln(1-d_2)\right]^{\frac{1}{\alpha}}. \tag{4.8}$$

For the special case when $s=n$ and $s>r+1$, the density $g(z|\mathbf{t},\theta)$ is increasing function with $g(0|\mathbf{t},\theta)=0$ and $g(1|\mathbf{t},\theta)=(n-r)$. Therefore, we choose the PI for Y of the form $(d_1,1)$ such that

$$\int_{d_1}^1 g(z|\mathbf{t},\theta)dz = 1-\gamma,$$

which implies that

$$d_1 = \gamma^{\frac{1}{n-r}}.$$

So, a $(1-\gamma)100\%$ HCD PI of Y is given by

$$L_2(\mathbf{T}) = \left[t_{r:n}^\alpha - \frac{1}{\lambda_2}\ln\left(1-\gamma^{\frac{1}{n-r}}\right)\right]^{\frac{1}{\alpha}}, U_2(\mathbf{T}) = \infty.$$

When $s=r+1$ and $s<n, g(z|\mathbf{t},\theta)$ is decreasing function starting from $(n-r)$ at $z=0$ to 0 at $z=1$. In this case, the PI for Y is of the form $(0,d_2)$ such that $d_2 = 1-\gamma^{\frac{1}{n-r}}$. This in turns implies that

$$L_2(\mathbf{T}) = t_{r:n}, U_2(\mathbf{T}) = \left[t_{r:n}^\alpha - \frac{1}{\lambda_2(n-r)}\ln\gamma\right]^{\frac{1}{\alpha}}.$$

Finally, for $s=r+1$ and $s=n$, $g(z|\mathbf{t},\theta)$ is uniform $U(0,1)$. Here d_1 and d_2 are taken such that $d_1 = \frac{\gamma}{2}$ and $d_2 = 1-\frac{\gamma}{2}$. Therefore,

$$L_2(\mathbf{T}) = \left[t_{r:n}^\alpha - \frac{1}{\lambda_2}\ln\left(1-\frac{\gamma}{2}\right)\right]^{\frac{1}{\alpha}}, \text{ and } U_2(\mathbf{T}) = \left[t_{r:n}^\alpha - \frac{1}{\lambda_2}\ln\left(\frac{\gamma}{2}\right)\right]^{\frac{1}{\alpha}}.$$

4.3. Shortest-Length based Method

Using the fact that given $\mathbf{T}=\mathbf{t}$ the distribution of $Z = 1-e^{-\lambda_2(y^\alpha - t_{r:n}^\alpha)}$is a $Beta(s-r,n-s+1)$, we choose the constants c and d satisfying:

$$P(c < 1-e^{-\lambda_2(y^\alpha - t_{r:n}^\alpha)} < d) = 1-\gamma,$$

which is equivalent to:

$$P([t_{r:n}^{\alpha}-\frac{1}{\lambda_2}\ln{(1-c)}]^{\frac{1}{\alpha}}<Y<[t_{r:n}^{\alpha}-\frac{1}{\lambda_2}\ln{(1-d)}]^{\frac{1}{\alpha}})=1-\gamma.$$

Therefore, a$(1-\gamma)100\%$ PI of Y is given by $(L_3(\mathbf{T}),U_3(\mathbf{T}))$, such that

$$\begin{aligned} L_3(\mathbf{T}) &= \left[t_{r:n}^{\alpha}-\frac{1}{\lambda_2}\ln(1-c)\right]^{\frac{1}{\alpha}}, \\ U_3(\mathbf{T}) &= \left[t_{r:n}^{\alpha}-\frac{1}{\lambda_2}\ln(1-d)\right]^{\frac{1}{\alpha}}. \end{aligned} \tag{4.9}$$

Now, the best choices for c and d are those that minimize the PI length $U_3(\mathbf{T})-L_3(\mathbf{T})$. The optimization problem for obtaining the shortest-length (SL) $(1-\gamma)100\%$ PI can be stated as

Minimize $Length=U_3(\mathbf{T})-L_3(\mathbf{T})$
Subject to

$$B_d(s-r,n-s+1)-B_c(s-r,n-s+1)=1-\gamma.$$

The SL $(1-\gamma)100\%$ PI can be obtained by minimizing the Lagrangian function:

$$\begin{aligned} R(c,d,\omega) = &\left[t_{r:n}^{\alpha}-\frac{1}{\lambda_2}\ln(1-d)\right]^{\frac{1}{\alpha}}-\left[t_{r:n}^{\alpha}-\frac{1}{\lambda_2}\ln(1-c)\right]^{\frac{1}{\alpha}} \\ &-\omega\left[\{B_d(s-r,n-s+1)-B_c(s-r,n-s+1)\}-(1-\gamma)\right], \end{aligned}$$

where, ω is the Lagrange multiplier. By differentiating R with respect to c,d and ω, respectively, we obtain:

$$\frac{\partial R}{\partial c}=-\frac{1}{\alpha\lambda_2(1-c)}\left[t_{r:n}^{\alpha}-\frac{1}{\lambda_2}\ln(1-c)\right]^{\frac{1}{\alpha}}+\omega p(c,s-r,n-s+1)=0,$$

$$\frac{\partial R}{\partial d}=-\frac{1}{\alpha\lambda_2(1-d)}\left[t_{r:n}^{\alpha}-\frac{1}{\lambda_2}\ln(1-d)\right]^{\frac{1}{\alpha}}-\omega p(d,s-r,n-s+1)=0,$$

$$\frac{\partial R}{\partial \omega}=\left[\{B_d(s-r,n-s+1)-B_c(s-r,n-s+1)\}-(1-\gamma)\right]=0,$$

where $p(x,a,b)$ represents the PDF of $Beta(a,b)$. The above equations are equivalent to the following equations:

$$\left(\frac{1-c}{1-d}\right)\left[\frac{t_{r:n}^{\alpha}-\frac{1}{\lambda_2}\ln(1-d)}{t_{r:n}^{\alpha}-\frac{1}{\lambda_2}\ln(1-c)}\right]^{\frac{1}{\alpha}-1}=\frac{p(d,s-r,n-s+1)}{p(c,s-r,n-s+1)}, \tag{4.10}$$

and

$$B_d(s-r,n-s+1)-B_c(s-r,n-s+1)=1-\gamma. \tag{4.11}$$

Constructing the SL PI can be obtained by solving (4.10) and (4.11) numerically, and then substituting the results in (4.9). For $s=r+1$ and $s=n$, the density of Z given $\mathbf{T}=\mathbf{t}$ is decreasing and increasing, respectively. In these cases, the PIs for Y are of the forms; $(0,d_1)$ and $(0,d_2)$, where $d_1=1-\gamma^{\frac{1}{n-r}}$ and $d_2=(1-\gamma)^{\frac{1}{n-r}}$. So, the SL

PIs of $T_{r+1:n}$ and $T_{n:n}$ are respectively, obtained to be

$$L_3(\mathbf{T}) = t_{r:n}, U_3(\mathbf{T}) = \left[t_{r:n}^{\alpha} - \frac{1}{\lambda_2(n-r)} \ln \gamma \right]^{\frac{1}{\alpha}},$$

and

$$L_3(\mathbf{T}) = t_{r:n}, U_3(\mathbf{T}) = \left[t_{r:n}^{\alpha} - \frac{1}{\lambda_2} \ln(1-\gamma)^{\frac{1}{(n-r)}} \right]^{\frac{1}{\alpha}}.$$

5. Numerical Experiments and Data Analysis

Here, we perform a simulation study to assess the methods of predictions developed in the previous sections and analyze one real data set for illustrative purposes.

5.1. Numerical Experiments

In this subsection, we present a simulation study to assess the performance of the proposed predictors; which were presented in Section 3. The performances are measured in terms of the biases and their mean square prediction errors (MSPEs) of the predictors. We also compare the PIs, that are discussed in Section 4, in terms of their estimated average lengths (ALs) and coverage probabilities (CPs).
For this, a Monte Carlo (MC) simulation is performed based on different sample sizes and censoring schemes from the Weibull distribution under Simple SSALT. For particular values of n, r and s, we generate Type II censored samples from this model according to the following schemes:

Scheme 1: $\alpha = 0.75, \lambda_1 = 0.25, \lambda_2 = 1$ and $\tau = 1.5$.
Scheme 2: $\alpha = 1.1, \lambda_1 = 0.25, \lambda_2 = 1$ and $\tau = 1.5$.
Scheme 3:$\alpha = 0.75, \lambda_1 = 0.75, \lambda_2 = 1.25$ and $\tau = 0.4$.
Scheme 4: $\alpha = 1.5, \lambda_1 = 0.75, \lambda_2 = 2$ and $\tau = 0.7$.

In each case, we compute the value of the point predictors; MLP, MMLP, CMP and BUP. We also compute 95% PIs based on pivotal quantity, HCD and SL methods. Type-II censored samples from Weibull distribution were randomly generated under these four different schemes and the simulation process is repeated 2000 times. Using these randomly samples, prediction biases and MSPEs of the predictors are computed. The ALs and CPs of PIs are also reported. The so obtained results are presented in Tables 1 and 2. In Tables 3 and 4, we have reported the ALs and CPs of the PIs.

From Tables 1 and 2, we notice that the BUP is the best point predictor in the sense of the prediction bias. The biases of the CMP are smaller than those of the MLP for all of the considered cases. By considering the MSPE as an optimality criterion, it is observed that, the CMP performs better than the MLP and BUP if α is larger than 1. In this case, both predictors coincide when s gets close to r. While the MLP competes the CMP when α is smaller than 1, the MSPEs of CMP are smaller than those of the BUP. Clearly, one can easily check that the MMLP obtained by substituting the MLEs of the parameters into the PLEs, behaves well and it is computationally attractive when compared to the MLP. It is also observed that, for fixed values of n and r, the MSPEs increase as s increases for all point predictors.

It follows from Tables 3 and 4 that the SL method is more efficient than other methods for obtaining PIs by considering the AL criterion. Its performance tends to be higher when s gets large. For comparing the HCD and pivot methods, it is observed that the HCD PIs are superior to the pivot CIs in the sense of ALs when s tends to be close to r. As s approaches n, the pivot PIs behave well. By considering the CP criterion, the HCD PIs outperform the PIs obtained by SL and pivot methods. The CPs of SL and pivot PIs are very close. It is evident that the CPs of all PIs obtained by SL, HCD and pivot methods increase when s increases. In this sense, the worse CP occurs when the variable to be predicted is the right after the last observed lifetime.

Summing up, the CMP and MMLP are the best point predictors as they are computationally attractive and have good performances in terms of the MSPE criterion. For prediction interval aspect, the HCD method produces efficient PIs over other methods by considering the AL and CP criteria.

Table 1: Biases and MSPEs of the point predictors for the censored lifetimes under Schemes 1 and 2.

		Scheme 1: $\alpha = 0.75, \lambda_1 = 0.25, \lambda_2 = 1$ and $\tau = 1.5$.							
		MLP		MMLP		CMP		BUP	
(n, r)	s	Bias	MSPE	Bias	MSPE	Bias	MSPE	Bias	MSPE
	22	-0.2068	0.2566	-0.1553	0.2783	-0.0563	0.3158	0.0051	0.3388
	24	-0.2687	0.4293	-0.2461	0.4403	-0.0562	0.5097	0.0218	0.5544
$(30, 20)$	26	-0.4897	0.8165	-0.3674	0.8575	-0.1611	0.9159	-0.0538	0.9942
	28	-0.8862	1.8699	-0.5333	1.8587	-0.1919	2.1054	-0.0076	2.3867
	30	-2.4077	8.8250	-1.7954	8.3601	-0.6468	9.4091	0.0209	12.7207
	27	-0.1311	0.1414	-0.1305	0.1555	-0.0265	0.1611	0.0127	0.1702
	30	-0.1899	0.2550	-0.1660	0.2683	-0.0701	0.2871	-0.0216	0.3010
$(40, 25)$	35	-0.4285	0.7296	-0.2623	0.7365	-0.1131	0.8399	-0.0262	0.8982
	38	-0.9314	2.1653	-0.564	2.2726	-0.2297	2.5386	-0.0488	2.8209
	40	-2.3733	9.4321	-1.7064	8.7592	-0.5383	9.6898	0.1357	12.7214
	32	-0.0862	0.0996	-0.0788	0.1036	-0.0201	0.1074	0.0082	0.1120
	35	-0.1725	0.1563	-0.1191	0.1659	-0.0520	0.1701	-0.0185	0.1760
$(50, 30)$	40	-0.2316	0.3160	-0.1344	0.3431	-0.0405	0.3652	0.0071	0.3844
	45	-0.4070	0.8845	-0.2358	0.9802	-0.1029	1.0144	-0.0175	1.0803
	50	-2.2592	8.9856	-1.7275	8.5519	-0.4771	9.0405	0.1838	11.5634
		Scheme 2: $\alpha = 1.1, \lambda_1 = 0.25, \lambda_2 = 1$ and $\tau = 1.5$.							
	22	-0.1128	0.0660	-0.0985	0.0639	-0.0451	0.0617	-0.0177	0.0639
	24	-0.1564	0.1090	-0.1245	0.0991	-0.0525	0.0884	-0.0212	0.0917
$(30, 20)$	26	-0.2145	0.1788	-0.1570	0.1811	-0.0778	0.1687	-0.0397	0.1753
	28	-0.3211	0.3190	-0.2124	0.3068	-0.0941	0.2895	-0.0384	0.3066
	30	-0.8245	1.1847	-0.5035	1.0812	-0.3216	0.9782	-0.1571	1.0747
	27	-0.0511	0.0367	-0.0594	0.0381	-0.0220	0.0354	-0.0039	0.0366
	30	-0.0976	0.0694	-0.0675	0.0653	-0.0228	0.0564	-0.0019	0.0586
$(40, 25)$	35	-0.2005	0.1618	-0.1187	0.1652	-0.0709	0.1526	-0.0407	0.1574
	38	-0.3256	0.3479	-0.2092	0.3606	-0.0997	0.3390	-0.0463	0.3566
	40	-0.7654	1.1447	-0.5696	1.0177	-0.2807	1.0149	-0.1168	1.1203
	32	-0.0488	0.0266	-0.0522	0.0278	-0.0267	0.0241	-0.0132	0.0245
	35	-0.0511	0.0417	-0.0560	0.0406	-0.0140	0.0371	0.0011	0.0384
$(50, 30)$	40	-0.1002	0.0809	-0.0715	0.0775	-0.0458	0.0753	-0.0277	0.0767
	45	-0.1971	0.2003	-0.0807	0.1887	-0.0636	0.1815	-0.0355	0.1868
	50	-0.7124	1.0672	-0.5199	1.0163	-0.2310	0.8875	-0.0715	0.9749

Table 2: Biases and MSPEs of the point predictors for the censored lifetimes under Schemes 3 and 4.

Scheme 3: $\alpha = 0.75, \lambda_1 = 0.75, \lambda_2 = 1.25$ and $\tau = 0.4$.									
		MLP		MMLP		CMP		BUP	
(n, r)	s	Bias	MSPE	Bias	MSPE	Bias	MSPE	Bias	MSPE
$(30, 20)$	22	-0.1371	0.1085	-0.1290	0.1095	-0.0632	0.1344	-0.0240	0.1434
	24	-0.2019	0.1925	-0.1593	0.1951	-0.0557	0.2166	-0.0022	0.2404
	26	-0.3218	0.3794	-0.2258	0.4053	-0.0753	0.4688	0.0044	0.5319
	28	-0.6555	0.9557	-0.4474	0.9365	-0.2036	1.0860	-0.0692	1.2665
	30	-1.8073	4.9728	-1.2696	4.2856	-0.4824	5.4263	0.0288	7.7144
$(40, 25)$	27	-0.0832	0.0593	-0.0859	0.0612	-0.0229	0.0661	0.0021	0.0702
	30	-0.1233	0.1083	-0.1237	0.1138	-0.0493	0.1162	-0.0161	0.1228
	35	-0.2973	0.3750	-0.2063	0.4077	-0.0868	0.4618	-0.0242	0.5089
	38	-0.6455	1.1907	-0.3345	1.2405	-0.0808	1.4943	0.0581	1.7230
	40	-1.9936	5.7270	-1.2361	5.1217	-0.4268	6.8673	0.0937	9.7026
$(50, 30)$	32	-0.0577	0.0411	-0.0593	0.0449	-0.0219	0.0466	-0.0037	0.0486
	35	-0.1030	0.0663	-0.0808	0.0656	-0.0274	0.0723	-0.0046	0.0759
	40	-0.1775	0.1569	-0.1157	0.1638	-0.0503	0.1761	-0.0169	0.1869
	45	-0.3929	0.4901	-0.2813	0.5321	-0.1439	0.5703	-0.0835	0.6073
	50	-1.8672	5.5728	-1.1010	5.0204	-0.2777	6.5882	0.2476	8.9555
Scheme 4: $\alpha = 1.5, \lambda_1 = 0.75, \lambda_2 = 2$ and $\tau = 0.7$.									
$(30, 20)$	22	-0.0294	0.0102	-0.0352	0.0099	-0.0064	0.0098	0.0038	0.0103
	24	-0.0458	0.0155	-0.0426	0.0154	-0.0122	0.0143	-0.0012	0.0149
	26	-0.071	0.0249	-0.0554	0.0233	-0.0301	0.0231	-0.0177	0.0236
	28	-0.1118	0.0443	-0.1288	0.0486	-0.0380	0.0392	-0.0212	0.0407
	30	-0.2514	0.1287	-0.1601	0.1152	-0.0988	0.1053	-0.0554	0.1111
$(40, 25)$	27	-0.0315	0.0067	-0.0274	0.0062	-0.0119	0.0061	-0.0048	0.0062
	30	-0.0395	0.0101	-0.0252	0.0105	-0.0142	0.0094	-0.0068	0.0096
	35	-0.0642	0.0246	-0.0684	0.0249	-0.0348	0.0219	-0.0255	0.0223
	38	-0.1144	0.0518	-0.1192	0.0494	-0.0387	0.0475	-0.0234	0.0490
	40	-0.2038	0.1109	-0.0933	0.1150	-0.0636	0.1088	-0.0213	0.1175
$(50, 30)$	32	-0.0247	0.0048	-0.0253	0.0045	-0.0135	0.0043	-0.0081	0.0044
	35	-0.0252	0.0068	-0.0271	0.0063	-0.0114	0.0061	-0.0058	0.0062
	40	-0.0430	0.0119	-0.0272	0.0124	-0.0141	0.0110	-0.0079	0.0112
	45	-0.0772	0.0281	-0.0376	0.0277	-0.0140	0.0249	-0.0054	0.0255
	50	-0.2261	0.1235	-0.1985	0.1102	-0.0604	0.1089	-0.0191	0.1164

Table 3: ALs and CPs of 95% PIs of the censored lifetimes under Schemes 1 and 2.

		Scheme 1: $\alpha = 0.75, \lambda_1 = 0.25, \lambda_2 = 1$ and $\tau = 1.5$.					
		Pivotal Method		HCD Method		SL Method	
(n, r)	s	AL	CP	AL	CP	AL	CP
$(30, 20)$	22	1.386	0.670	1.248	0.659	1.247	0.650
	24	2.262	0.828	2.165	0.822	2.120	0.821
	26	3.528	0.900	3.593	0.903	3.349	0.876
	28	5.866	0.909	6.938	0.933	5.554	0.899
	30	17.272	0.945	-	0.982	16.750	0.903
$(40, 25)$	27	0.854	0.616	0.771	0.602	0.770	0.591
	30	1.607	0.810	1.545	0.801	1.526	0.802
	35	3.413	0.905	3.541	0.916	3.292	0.893
	38	6.399	0.919	7.866	0.950	6.121	0.904
	40	17.556	0.946	-	0.983	16.519	0.914
$(50, 30)$	32	0.638	0.585	0.577	0.563	0.576	0.560
	35	1.120	0.801	1.072	0.790	1.066	0.787
	40	2.035	0.877	2.027	0.879	1.974	0.875
	45	3.772	0.896	3.995	0.919	3.664	0.897
	50	17.653	0.955	-	0.981	18.119	0.934
		Scheme 2: $\alpha = 1.1, \lambda_1 = 0.25, \lambda_2 = 1$ and $\tau = 1.5$.					
$(30, 20)$	22	0.624	0.701	0.567	0.667	0.566	0.669
	24	0.997	0.843	0.959	0.836	0.944	0.808
	26	1.432	0.883	1.455	0.886	1.378	0.872
	28	2.201	0.928	2.525	0.953	2.121	0.913
	30	5.181	0.940	-	0.974	5.365	0.941
$(40, 25)$	27	0.421	0.618	0.382	0.587	0.381	0.588
	30	0.739	0.814	0.714	0.824	0.707	0.819
	35	1.402	0.896	1.446	0.905	1.367	0.887
	38	2.344	0.934	2.782	0.957	2.277	0.917
	40	5.086	0.936	-	0.973	5.540	0.940
$(50, 30)$	32	0.314	0.582	0.285	0.552	0.284	0.552
	35	0.533	0.771	0.513	0.771	0.510	0.769
	40	0.917	0.855	0.914	0.865	0.896	0.852
	45	1.497	0.916	1.571	0.931	1.469	0.908
	50	5.201	0.955	-	0.976	5.833	0.942

Table 4: ALs and CPs of 95% PIs of the censored lifetimes under Schemes 3 and 4.

		Scheme 3: $\alpha = 0.75, \lambda_1 = 0.75, \lambda_2 = 1.25$ and $\tau = 0.4$.					
		Pivotal Method		HCD Method		SL Method	
(n, r)	s	AL	CP	AL	CP	AL	CP
$(30, 20)$	22	1.366	0.706	1.231	0.680	1.229	0.681
	24	1.497	0.824	1.429	0.826	1.396	0.820
	26	2.476	0.862	2.526	0.865	2.334	0.854
	28	4.143	0.875	4.952	0.907	3.896	0.848
	30	13.410	0.904	-	0.960	12.629	0.863
$(40, 25)$	27	0.561	0.649	0.505	0.627	0.504	0.628
	30	1.031	0.802	0.990	0.800	0.976	0.796
	35	2.319	0.854	2.411	0.860	2.227	0.836
	38	4.702	0.887	5.844	0.909	4.472	0.874
	40	12.207	0.915	-	0.973	12.007	0.869
$(50, 30)$	32	0.400	0.576	0.361	0.564	0.360	0.560
	35	0.721	0.785	0.689	0.775	0.684	0.772
	40	1.360	0.848	1.355	0.848	1.316	0.838
	45	2.481	0.856	2.635	0.870	2.403	0.845
	50	13.160	0.924	-	0.964	13.246	0.901
		Scheme 4: $\alpha = 1.5, \lambda_1 = 0.75, \lambda_2 = 2$ and $\tau = 0.7$.					
$(30, 20)$	22	0.238	0.664	0.217	0.651	0.217	0.650
	24	0.373	0.822	0.360	0.821	0.355	0.819
	26	0.524	0.872	0.531	0.879	0.507	0.858
	28	0.753	0.917	0.851	0.937	0.733	0.902
	30	1.563	0.926	-	0.976	1.887	0.937
$(40, 25)$	27	0.166	0.610	0.151	0.597	0.150	0.587
	30	0.277	0.819	0.268	0.816	0.266	0.814
	35	0.496	0.879	0.509	0.885	0.486	0.875
	38	0.788	0.882	0.917	0.925	0.772	0.874
	40	1.543	0.949	-	0.966	1.779	0.933
$(50, 30)$	32	0.127	0.594	0.115	0.566	0.115	0.564
	35	0.209	0.776	0.202	0.772	0.200	0.771
	40	0.336	0.867	0.335	0.865	0.330	0.859
	45	0.536	0.889	0.560	0.903	0.529	0.887
	50	1.527	0.946	-	0.964	1.659	0.916

5.2. *Data Analysis*

To illustrate the prediction methods developed in this paper, we conduct a real data analysis. The data has been considered by Liu [16]. It represents the lifetimes (in seconds) of nanocrystalline embedded high-k device put under a specific test. Forty devices are put into a step-stress experiment with stress change time $\tau = 600$ seconds. Thirty-eight failures have been observed before the termination of the experiment. These data have been used previously by Amleh and Raqab [2]. The data are recorded as follows:

Data on the lifetimes of nanocrystalline embedded high-k device.

Stress level	Recorded data									
1	8	38	72	97	122	140	163	170	188	198
	223	256	257	265	448					
2	608	611	614	615	616	620	623	623	624	624
	631	636	646	654	660	673	675	680	684	692
	693	730	745							

It was shown by Liu [16] that Weibull distribution can be used for analyzing this data set. Suppose the life test ended when the 30-th lifetime is observed, i.e., we observe a Type-II censored sample with $n = 40, r = 30$. Our aim is to obtain point predictors of the unobserved life times $Y = T_{s:n}, s = 32, 34, 35, 37, 38, 40$ and the associated PIs. For computational ease we divide all of the values by 1000. It is not going to affect the inference procedure.

First we compute the MLEs of α by solving (2.11) numerically and it is found to be $\widehat{\alpha} = 0.7656$. From (2.8) and (2.9), we obtain $\widehat{\lambda}_1$=0.7234 and $\widehat{\lambda}_2 = 17.4605$. For predicting the future censored failures, point predictors as well as PIs are displayed in Table 5. It can be observed that the values of the MLPs and MMLPs are very close to each other. In fact, all point predictors are close to the true values of the lifetimes, moreover, the point predictors obtained are lying within all considered PIs. It can be observed that the CMP has a clear advantage over the other predictors. It can be observed also that all PIs obtained contain the true values of the future order statistics. The PIs become wider when s gets large, the reason is that the variation of $Y = T_{s:n}$ tends to be high as Y moves away from the observed lifetimes. Although all PIs are close in the sense of AL criterion, the PIs obtained by SL based method have shortest lengths.

Table 5: Point predictors and PIs for future lifetimes of $Y = T_{s:n}$.

Point predictors and PIs of $Y = T_{s:n}$					
s	True value	MLP	MMLP	CMP	BUP
32	0.675	0.6667	0.6671	0.6720	0.6744
34	0.684	0.6827	0.6842	0.6899	0.6927
35	0.692	0.6926	0.6948	0.7011	0.7042
37	0.730	0.7186	0.7226	0.7311	0.7355
38	0.745	0.7372	0.7425	0.7534	0.7588
40	——	0.8084	0.8192	0.8492	0.8665
95% PIs of $Y = T_{s:n}$					
s	True value	Pivotal PI	HCD PI	SL PI	
32	0.675	(0.6617, 0.7002)	(0.6605, 0.6946)	(0.6603, 0.6944)	
34	0.684	(0.6688, 0.7326)	(0.6677, 0.7289)	(0.6657, 0.7256)	
35	0.692	(0.6741, 0.7522)	(0.6736, 0.7506)	(0.6702, 0.7444)	
37	0.730	(0.6891, 0.8065)	(0.6912, 0.8158)	(0.6835, 0.7959)	
38	0.745	(0.7001, 0.8494)	(0.7044, 0.8756)	(0.6929, 0.8357)	
40	——	(0.7409, 1.0924)	(0.7532, ∞)	(0.660, 1.0389)	

6. Conclusion

In this paper, we have considered the prediction of future lifetimes of a simple step stress model of Weibull distribution under K-H model when the data are Type-II censored. Various point predictors are addressed including, maximum likelihood, modified maximum likelihood, conditional median, and best unbiased predictors. We have also considered prediction intervals for the future lifetimes. We have compared the performance of the predictors obtained by extensive Monte Carlo simulation study based on the biases and MSPEs for different parameter values, stress change time, and future order statistics. Prediction intervals were also compared in terms of the average lengths and coverage probabilities. It is observed that the BUP has the best performance among all point predictors in terms of prediction bias, while the MMLP and CMP are very competitive in terms of the MSPE criterion as well as they are computationally attractive. In the context of interval prediction context, it is also observed that the SL based method is the most appropriate technique for obtaining PIs of future lifetimes in the sense of AL criterion. The HCD method is an efficient method for producing PIs by considering the CP criterion. It is worth mentioning that the results of this paper were mainly obtained for Type-II censored scheme, but our methods can be performed for other censoring schemes, as Type-I, hybrid or progressive censoring. More work along these directions will be reported in the near future.

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